Estimation and comparison of signed symmetric covariation coefficient and generalized association parameter for alpha-stable dependence modeling

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ESTIMATION AND COMPARISON OF SIGNED SYMMETRIC COVARIATION COEFFICIENT AND GENERALIZED ASSOCIATION PARAMETER FOR ALPHA-STABLE DEPENDENCE MODELING

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ABSTRACT

In this paper we study the estimators of two measures of dependence: the signed symmetric covariation coefficient proposed by Garel and Kodia and the generalized association parameter put forward by Paulauskas. In the sub-Gaussian case, the signed symmetric covariation coefficient and the generalized association parameter coincide. The estimator of the signed symmetric covariation coefficient proposed here is based on fractional lower-order moments. The estimator of the generalized association parameter is based on estimation of a stable spectral measure. We investigate the relative performance of these estimators by comparing results from simulations.

1 Introduction

Many types of physical phenomena and financial data exhibit a very high variability and stable distributions are often used to model them. Since the seminal work of Mandelbrot (1963), who suggested the stable laws as possible models for the distribution of income and speculative prices, the interest in these laws has greatly increased and they are now widely applied in telecommunications and many other fields such as physics, biology, genetics and geology, see Uchaikin and Zolotarev (1999).

Stable distributions are a rich class of probability distributions, which includes the Gaussian, Cauchy and Lévy distributions in a family that allows for skewness and heavy tails. These stable

laws, characterized by Paul Lévy, are the only possible limiting laws for normalized sums of independent, identically distributed random variables. While they present many attractive theoretical properties, a major problem in working with stable laws, both univariate and multivariate, is that, with the exception of the three distributions mentioned above, their densities cannot be written in a closed form. The only available information for a stable random vector is its characteristic function. In addition to this drawback, the fact that stable non-Gaussian random vectors do not possess moments of second order has further limited their use. The concept of a correlation matrix which allows us to understand the association between the coordinates of a random vector is meaningless here. Therefore, other dependence coefficients are required.

Press (1972) developed the idea of a correlation coefficient, the so-called association parameter (a.p.), applicable to a specific class of symmetric multivariate stable laws. Inspired by Press' work, Paulauskas (1976) proposed a generalized association parameter (g.a.p.) which is applicable to general symmetric α-stable random vectors, but he did not deal with the estimation of this quantity. Kanter and Steiger (1974) showed that, under certain conditions, the conditional expectation of a stable random variable, given another, is linear and they proposed an estimator based on screened ratio for the constant of linearity. Miller (1978) proposed a new measure of dependence, called covariation, designed to replace the covariance when $1 < \alpha < 2$. The constant of linearity of conditional expectation, mentioned above, has been expressed by means of this measure and subsequently called the covariation coefficient. However, in general this coefficient is not symmetric and may be unbounded, therefore Garel and Kodia (2009) proposed a new coefficient: the signed symmetric covariation coefficient. This coefficient satisfies most properties of the classical Pearson coefficient and coincides with it when $\alpha = 2$. Here we use a slight modification of this coefficient, in order to guarantee symmetry in all cases.

In this paper we define a modified signed covariation coefficient, give its properties, and consider the estimation of this quantity. We have developed an estimator of the signed symmetric covariation coefficient based on fractional lower order moments (FLOM). We also address the question of estimating the g.a.p. In the general case, an estimator of the g.a.p. requires an estimation of the index of stability α and an estimation of the stable spectral measure.

We focus particulary on sub-Gaussian random vectors because, in this case, the g.a.p. and the signed symmetric covariation coefficient coincide. This provides a means of estimating the g.a.p.

without knowing either α or the spectral measure. Moreover, we can compare results obtained from estimators based respectively on FLOM and stable spectral measure. This also means that in the sub-Gaussian case, the matrix of signed symmetric covariation coefficients is positive semi-definite. Of course, these specificities open up new and interesting prospects.

This paper is organized as follows: Section 2 provides a reminder of basic definitions and some properties of stable random vectors, the covariation and the g.a.p. while Section 3 details the signed symmetric covariation coefficient. Garel and Kodia (2009) give an insight without proofs. That note was an introduction while this paper details all the proofs. In the definition of the signed symmetric covariation coefficient, we solve the asymmetry problem which appeared in the first definition. Other properties of this coefficient are discussed in the context of sub-Gaussian random vectors. We also give the expression of the signed symmetric covariation coefficient and the g.a.p. in the case of linear transformation of independent stable random variables. We give estimators of the signed symmetric covariation coefficient and the g.a.p. in Section 4. General performance of these estimators are discussed on the basis of simulations.

2 Stable random variables and vectors and dependence coefficients

For our purposes, we define stable random variables and vectors by their characteristic function. Following Samorodnitsky and Taqqu (1994), we denote the law of a stable random variable by $S_\alpha(\gamma, \beta, d)$, with $0 < \alpha \leq 2, \gamma \geq 0, -1 \leq \beta \leq 1$ and d a real parameter. A random variable X has a stable distribution $S_\alpha(\gamma, \beta, d)$ if its characteristic function has the form

$$\phi_X(t) = E \exp(i\, tX) = \exp\Big\{ -\gamma^\alpha |t|^\alpha \left[1 + i\, \beta \mathrm{sign}(t) w(t, \alpha)\right] + i\, dt \Big\}, \tag{1}$$

where

$$w(t, \alpha) = \begin{cases} -\tan \frac{\pi\alpha}{2} & \text{if} \quad \alpha \neq 1, \\ \frac{2}{\pi} \ln |t| & \text{if} \quad \alpha = 1, \end{cases}$$

with t a real number, and $\mathrm{sign}(t) = 1$ if $t > 0$, $\mathrm{sign}(t) = 0$ if $t = 0$ and $\mathrm{sign}(t) = -1$ if $t < 0$.

The parameter α is the characteristic exponent or index of stability, β is a measure of skewness, γ is a scale parameter and d is a location parameter. The special cases $\alpha = 2$, $\alpha = 1$ and $\alpha = 0.5$ correspond respectively to the Gaussian, Cauchy and Lévy distributions and it is only in these cases that stable laws have a closed form expression for the density. When $\beta = d = 0$, the distribution is symmetric (i. e. X and $-X$ have the same law) and is denoted $S\alpha S(\gamma)$ or for short $S\alpha S$.
Let $0 < \alpha < 2$. The characteristic function of a random vector $\mathbf{X} = (X_1, X_2)$ is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{-\int_{S_2} |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha} \left[1 + i\,\mathrm{sign}(\langle \mathbf{t}, \mathbf{s}\rangle) w(\langle \mathbf{t}, \mathbf{s}\rangle, \alpha)\right] \mathbf{\Gamma}(d\mathbf{s}) + i\,\langle \mathbf{t}, \mathbf{d}\rangle\right\}, \tag{2}$$

where $\mathbf{\Gamma}$ is a finite symmetric measure on the unit circle $S_2 = \{\mathbf{s} \in \mathbb{R}^2 : \|\mathbf{s}\| = 1\}$ and $\mathbf{d}$ is a vector in $\mathbb{R}^2$. Here $\langle \mathbf{t}, \mathbf{s}\rangle$ denotes the inner product of $\mathbb{R}^2$. The measure $\mathbf{\Gamma}$ is called the spectral measure of the α-stable random vector $\mathbf{X}$ and the pair $(\mathbf{\Gamma}, \mathbf{d})$ is unique. The vector $\mathbf{X}$ is symmetric if, and only if, $\mathbf{d} = 0$ and $\mathbf{\Gamma}$ is symmetric on S_2. In this case, its characteristic function is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{-\int_{S_2} |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha} \mathbf{\Gamma}(d\mathbf{s})\right\} = \exp\left\{-I_{\mathbf{X}}(\mathbf{t})\right\}, \tag{3}$$

where

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_2} \psi_{\alpha}(\langle \mathbf{t}, \mathbf{s}\rangle) \mathbf{\Gamma}(d\mathbf{s}), \tag{4}$$

is the exponent function, with $\psi_{\alpha}(u) = |u|^{\alpha}$. If necessary, we also denote the spectral measure of $\mathbf{X}$ by $\mathbf{\Gamma}_{\mathbf{X}}$. For any vector $\mathbf{u} \in \mathbb{R}^2$, the projection $\langle \mathbf{u}, \mathbf{X}\rangle = \sum_{k=1}^{2} u_k X_k$ has a univariate $S\alpha S$ distribution. The spectral measure determines the projection parameter function $\gamma(\mathbf{u})$ by:

$$\gamma^{\alpha}(\mathbf{u}) = \int_{S_2} |\langle \mathbf{u}, \mathbf{s}\rangle|^{\alpha} \mathbf{\Gamma}(d\mathbf{s}). \tag{5}$$

From (4) and (5) we can write

$$I_{\mathbf{X}}(\mathbf{u}) = \gamma^{\alpha}(\mathbf{u}). \tag{6}$$

The spectral measure carries essential information about the vector, in particular the dependence structure between the coordinates. So, it is not surprising that measures of dependence rely on

this spectral measure. In the sequel, unless specified otherwise, we assume $\alpha > 1$ and consider symmetric stable random variables or vectors.

Miller (1978) introduced the covariation as follows.

Definition 2.1. *Let X_1 and X_2 be jointly $S\alpha S$ and let $\mathbf{\Gamma}$ be the spectral measure of the random vector (X_1, X_2). The covariation of X_1 on X_2 is the real number defined by*

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 s_2^{\langle \alpha-1 \rangle} \mathbf{\Gamma}(d\boldsymbol{s}), \tag{7}$$

where for real numbers s and a: if $a \neq 0$, $s^{\langle a \rangle} = |s|^a \text{sign}(s)$ and if $a = 0$, $s^{\langle a \rangle} = \text{sign}(s)$. It is well known that although the covariation is linear in its first argument, it is, in general, not linear in its second argument and not symmetric in its arguments. We also have

$$[X_1, X_1]_\alpha = \int_{S_2} |s_1|^\alpha \mathbf{\Gamma}(d\mathbf{s}) = \gamma_{X_1}^\alpha, \tag{8}$$

where γ_{X_1} is the scale parameter of the $S\alpha S$ random variable X_1. The covariation norm is defined by

$$\|X_1\|_\alpha = ([X_1, X_1]_\alpha)^{1/\alpha}. \tag{9}$$

When X_1 and X_2 are independent, $[X_1, X_2]_\alpha = 0$. Proofs of these properties and other details are given in Samorodnitsky and Taqqu (1994) P. 87-97.

The covariation coefficient of X_1 on X_2 is the quantity:

$$\lambda_{X_1,X_2} = \frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha}. \tag{10}$$

It is the coefficient of the linear regression $E(X_1|X_2)$. This coefficient is not symmetric and may be unbounded. We see it easily by setting $X_2 = cX_1$, where c is a non-zero constant such that $c \neq \pm 1$. In this case we have

$$\lambda_{X_1,X_2} = \frac{[X_1, cX_1]_\alpha}{\|cX_1\|_\alpha^\alpha} = \frac{1}{c} \quad \text{and} \quad \lambda_{X_2,X_1} = \frac{[cX_1, X_1]_\alpha}{\|X_1\|_\alpha^\alpha} = c.$$

When $c \to 0$, λ_{X_1,X_2} tends to infinity.

Paulauskas (1976) introduced the generalized association parameter (g.a.p.), inspired by Press (1972). This coefficient is applicable to all symmetric stable random vectors in $\mathbb{R}^2$ and has all the properties of the ordinary correlation coefficient of a bivariate Gaussian random vector.
Let (X_1, X_2) be $S\alpha S$, $0 < \alpha \leq 2$ and $\mathbf{\Gamma}$ its spectral measure on the unit circle S_2. Let (U_1, U_2) be a random vector on S_2 with probability distribution $\widetilde{\mathbf{\Gamma}} = \mathbf{\Gamma}/\mathbf{\Gamma}(S_2)$. Due to the symmetry of $\mathbf{\Gamma}$, one has $EU_1 = EU_2 = 0$. The g.a.p. is defined as:

$$\widetilde{\rho}(X_1, X_2) = \frac{EU_1U_2}{(EU_1^2EU_2^2)^{1/2}}. \tag{11}$$

It is a measure of dependence of (X_1, X_2). For a bivariate stable vector with characteristic function (3) the g.a.p. $\widetilde{\rho}$ has the following properties valid for $0 < \alpha \leq 2$: (i) we always have $-1 \leq \widetilde{\rho} \leq 1$ and if a distribution corresponds to a random vector with independent coordinates, then $\widetilde{\rho} = 0$. (ii) $|\widetilde{\rho}(X_1, X_2)| = 1$ if, and only if, the distribution of (X_1, X_2) is concentrated on a line. (iii) For $\alpha = 2$, $\widetilde{\rho}$ coincides with the correlation coefficient of the Gaussian random vector. (iv) $\widetilde{\rho}$ is independent of α and depends only on the spectral measure $\mathbf{\Gamma}$. (v) If the characteristic function of (X_1, X_2) is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{-C(\gamma_{X_1}^2 t_1^2 + 2r\gamma_{X_1}\gamma_{X_2}t_1t_2 + \gamma_{X_2}^2 t_2^2)^{\alpha/2}\right\}, \tag{12}$$

where C is an appropriate constant, then r is the g.a.p. Paulauskas (1976) extended this concept to a d-dimensional random vector with $d > 2$.

3 Signed symmetric covariation coefficient and its properties

The signed symmetric covariation coefficient is a close relative to the covariation coefficient. However, unlike the covariation coefficient, the new coefficient is bounded. We propose here a revised version of the sign of this coefficient, avoiding the asymmetry problems which appeared in the definition given in Garel and Kodia (2009).

3.1 Definition and first properties

Definition 3.1. *Let (X_1, X_2) be a bivariate $S\alpha S$ random vector with $\alpha > 1$. The signed symmetric covariation coefficient between X_1 and X_2 is the quantity:*

$$\operatorname{scov}(X_1, X_2) = \kappa_{(X_1,X_2)} \left| \frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right|^{\frac{1}{2}}, \tag{13}$$

where

$$\kappa_{(X_1,X_2)} = \begin{cases} \text{sign}([X_1,X_2]_\alpha) & \text{if} \quad \text{sign}([X_1,X_2]_\alpha) = \text{sign}([X_2,X_1]_\alpha), \\ \\ -1 & \text{if} \quad \text{sign}([X_1,X_2]_\alpha) = -\text{sign}([X_2,X_1]_\alpha). \end{cases} \tag{14}$$

Remark: In (14) the value of sign $\kappa_{(X_1,X_2)}$ is natural in the first case. In fact, if (X_1, X_2) was a random vector with finite variance, the equality $\text{sign}([X_1,X_2]_\alpha) = \text{sign}([X_2,X_1]_\alpha)$ would always be true, because $[X_1,X_2]_2 = \frac{1}{2}\text{Cov}(X_1,X_2)$, see Samorodnitsky and Taqqu (1994) P. 87-88. But in the case of non-Gaussian random vectors, we can have $\text{sign}([X_1,X_2]_\alpha) = -\text{sign}([X_2,X_1]_\alpha)$, see Garel et al. (2004) P. 773. If it is so, we set $\kappa_{(X_1,X_2)} = \text{sign}([X_1,X_2]_\alpha \times [X_2,X_1]_\alpha) = -1$.

The following proposition shows that the signed symmetric covariation coefficient has desirable properties as does the ordinary correlation coefficient of a bivariate Gaussian random vector.

Proposition 3.2. *Let (X_1, X_2) be a bivariate $S\alpha S$ random vector with $\alpha > 1$. The signed symmetric covariation coefficient has the following properties:*

1. $-1 \leq \text{scov}(X_1,X_2) \leq 1$ *and if X_1, X_2 are independent, then* $\text{scov}(X_1,X_2) = 0$;
2. $|\text{scov}(X_1,X_2)| = 1$ *if and only $X_2 = \lambda X_1$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$;*
3. *let a and b be two non-zero reals, then*

$$\text{scov}(aX_1,bX_2) = \begin{cases} \text{sign}(ab)\text{scov}(X_1,X_2) & \text{if} \quad \text{sign}([X_1,X_2]_\alpha) = \text{sign}([X_2,X_1]_\alpha), \\ \\ \text{scov}(X_1,X_2) & \text{if} \quad \text{sign}([X_1,X_2]_\alpha) = -\text{sign}([X_2,X_1]_\alpha); \end{cases} \tag{15}$$

4. *for $\alpha = 2$,* $\text{scov}(X_1,X_2)$ *coincides with the usual correlation coefficient.*

Proof.

1. From (7) and using Holder's inequality in which $p = \alpha$ and $q = \frac{\alpha}{\alpha-1}$, we have

$$\left| \int_{S_2} s_1 |s_2|^{\alpha-1} \text{sign}(s_2) \mathbf{\Gamma}(d\mathbf{s}) \right| \leq \left(\int_{S_2} |s_1|^\alpha \mathbf{\Gamma}(d\mathbf{s}) \right)^{\frac{1}{\alpha}} \left(\int_{S_2} |s_2|^\alpha \mathbf{\Gamma}(d\mathbf{s}) \right)^{\frac{\alpha-1}{\alpha}}.$$

   Then using (9), we have $|[X_1, X_2]_\alpha| \leq \|X_1\|_\alpha \|X_2\|_\alpha^{\alpha-1}$ which implies

$$|[X_1, X_2]_\alpha| \times |[X_2, X_1]_\alpha| \leq \|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha \quad \text{and then} \quad \left| \frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha|}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right| \leq 1.$$

   Using Equation (13) we get $-1 \leq \text{scov}(X_1, X_2) \leq 1$. If X_1 and X_2 are independent, then $[X_1, X_2]_\alpha = [X_2, X_1]_\alpha = 0$, which leads to $\text{scov}(X_1, X_2) = 0$.

2. Equality $|\text{scov}(X_1, X_2)| = 1$ is equivalent to

$$\left| \int_{S_2} s_1 |s_2|^{\alpha-1} \text{sign}(s_2) \mathbf{\Gamma}(d\mathbf{s}). \int_{S_2} s_2 |s_1|^{\alpha-1} \text{sign}(s_1) \mathbf{\Gamma}(d\mathbf{s}) \right| = \int_{S_2} |s_1|^\alpha \mathbf{\Gamma}(d\mathbf{s}). \int_{S_2} |s_2|^\alpha \mathbf{\Gamma}(d\mathbf{s}).$$

   This is equivalent to equality in Holder's inequality

$$\left| \int_{S_2} s_1 s_2^{\langle \alpha-1 \rangle} \mathbf{\Gamma}(d\mathbf{s}) \right| = \left(\int_{S_2} |s_1|^\alpha \mathbf{\Gamma}(d\mathbf{s}) \right)^{\frac{1}{\alpha}} \left(\int_{S_2} |s_2|^\alpha \mathbf{\Gamma}(d\mathbf{s}) \right)^{\frac{\alpha-1}{\alpha}}$$

   which is equivalent to $s_2 = \lambda s_1$ a. e. $\mathbf{\Gamma}$, for some $\lambda \in \mathbb{R}^*$. This last relation is equivalent to $X_2 = \lambda X_1$ a.s.

3. Let a and b be two non-zero reals, then we have

$$\left| \frac{[aX_1, bX_2]_\alpha [bX_2, aX_1]_\alpha}{\|aX_1\|_\alpha^\alpha \|bX_2\|_\alpha^\alpha} \right| = \left| \frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right|.$$

   Using (14) we also have

$$\begin{aligned} \kappa_{(aX_1, bX_2)} &= \text{sign}([aX_1, bX_2]_\alpha) \qquad \text{if} \quad \text{sign}([aX_1, bX_2]_\alpha) = \text{sign}([bX_2, aX_1]_\alpha) \\ &= \text{sign}(ab)\text{sign}([X_1, X_2]_\alpha) \quad \text{if } \text{sign}([X_1, X_2]_\alpha) = \text{sign}([X_2, X_1]_\alpha), \end{aligned}$$

   because

$$\text{sign}([aX_1, bX_2]_\alpha) = \text{sign}([bX_2, aX_1]_\alpha) \Leftrightarrow \text{sign}([X_1, X_2]_\alpha) = \text{sign}([X_2, X_1]_\alpha).$$

In the same way,

$$\kappa_{(aX_1,bX_2)} = -1 \quad \text{if} \quad \text{sign}([aX_1, bX_2]_\alpha) \neq \text{sign}([bX_2, aX_1]_\alpha)$$

$$= -1 \quad \text{if} \quad \text{sign}([X_1, X_2]_\alpha) \neq \text{sign}([X_2, X_1]_\alpha).$$

Thus we have

$$\kappa_{(aX_1,bX_2)} = \begin{cases} \text{sign}(ab)\text{sign}([X_1, X_2]_\alpha) & \text{if} \quad \text{sign}([X_1, X_2]_\alpha) = \text{sign}([X_2, X_1]_\alpha), \\ -1 & \text{if} \quad \text{sign}([X_1, X_2]_\alpha) = -\text{sign}([X_2, X_1]_\alpha). \end{cases}$$

4. When $\alpha = 2$, $[X_1, X_2]_2 = [X_2, X_1]_2 = \frac{1}{2}\text{Cov}(X_1, X_2)$ and $\kappa_{(X_1,X_2)} = \text{sign}(\text{Cov}(X_1, X_2))$. Then

$$\text{scov}(X_1, X_2) = \text{sign}(\text{Cov}(X_1, X_2)) \left| \frac{[\text{Cov}(X_1, X_2)]^2}{\text{Var}(X_1)\text{Var}(X_2)} \right|^{\frac{1}{2}} = \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_1)^{\frac{1}{2}}\text{Var}(X_2)^{\frac{1}{2}}},$$

which is the usual correlation coefficient.

□

Is there a sub-family of $S\alpha S$ random vectors where the signed symmetric covariation coefficient and the generalized association parameter coincide? The answer is yes for sub-Gaussian random vectors.

3.2 Sub-Gaussian case

First recall the definition of a d-dimensional sub-Gaussian random vector, see Samorodnitsky and Taqqu (1994) P. 77-84.

Definition 3.3. *Let $0 < \alpha < 2$, let $G_1, G_2, ..., G_d$ be zero mean jointly normal random variables and let A be a positive random variable such that $A \sim S_{\alpha/2}\big(\big(\cos\frac{\pi\alpha}{4}\big)^{2/\alpha}, 1, 0\big)$, independent of $(G_1, G_2, ..., G_d)$, then $\boldsymbol{X} = A^{1/2}\boldsymbol{G} = (A^{1/2}G_1, A^{1/2}G_2, ..., A^{1/2}G_d)$ is a sub-Gaussian random vector with underlying Gaussian vector $\boldsymbol{G} = (G_1, G_2, ..., G_d)$.*

The characteristic function of $\mathbf{X}$ has the particular form:

$$\phi_{\mathbf{X}}(\mathbf{t}) = E\exp\Big\{i\sum_{m=1}^{d} t_m X_m\Big\} = \exp\Big\{-\Big|\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} t_j t_k R_{jk}\Big|^{\alpha/2}\Big\}, \tag{16}$$

where $R_{jk} = EG_jG_k$, $j,k = 1,...,d$ are the covariances of the underlying Gaussian random vector $\mathbf{G}$.

The following proposition summarizes the usefulness of the new coefficient.

Proposition 3.4. *Let* $1 < \alpha < 2$ *and* ***let*** $\boldsymbol{X}$ *be a sub-Gaussian random vector with characteristic function (16). Then the matrix of signed symmetric covariation coefficients of* $\boldsymbol{X}$ *coincides with the correlation matrix of the underlying Gaussian random vector* $\mathbf{G}$.

Proof. Let $(X_1,...,X_d)$ be a sub-Gaussian random vector with characteristic function (16), it is sufficient to state that $\forall \;\; j, \;\; k, \;\; j \neq k$, $\text{scov}(X_j, X_k) = r_{jk}$, where r_{jk} is the correlation coefficient between G_j and G_k.

First, Cambanis and Miller (1978) established that (7) is equivalent to

$$[X_1, X_2]_\alpha = \frac{1}{\alpha}\frac{\partial \gamma^\alpha(t_1,t_2)}{\partial t_1}|_{t_1=0,t_2=1}, \tag{17}$$

where t_1 and t_2 are real numbers. This second definition of the covariation is more easy to manipulate in this case.

From (3) and (5) we see that the scale parameter $\gamma(t_j,t_k)$ of $Y = t_jX_j + t_kX_k$, $j,k = 1,2$, satisfies

$$\gamma^\alpha(t_j,t_k) = 2^{-\alpha/2}(t_j^2R_{jj} + 2t_jt_kR_{jk} + t_k^2R_{kk})^{\alpha/2}.$$

Using (17), we have

$$[X_j, X_k]_\alpha = \frac{1}{\alpha}\frac{\partial \gamma^\alpha(t_j,t_k)}{\partial t_j}\Big|_{t_j=0,t_k=1} = 2^{-\alpha/2}R_{jk}R_{kk}^{(\alpha-2)/2}. \tag{18}$$

From (18) we also have

$$\gamma_{X_j} = \|X_j\|_\alpha = ([X_j, X_j]_\alpha)^{1/\alpha} = 2^{-1/2}R_{jj}^{1/2}, \quad j = 1,2. \tag{19}$$

Also we have $\kappa_{(X_j,X_k)} = \text{sign}(R_{jk})$ because $\text{sign}([X_j, X_k]_\alpha) = \text{sign}([X_k, X_j]_\alpha) = \text{sign}(R_{jk})$. Using (18) and (19) in (13), we get

$$\text{scov}(X_j, X_k) = \text{sign}(R_{jk})\Big|\frac{R_{jk}^2}{R_{jj}R_{kk}}\Big|^{\frac{1}{2}} = \frac{R_{jk}}{R_{jj}^{1/2}R_{kk}^{1/2}} = r_{jk}.$$

Thus the matrix of signed symmetric covariation coefficients of $\mathbf{X}$ coincides with the correlation matrix of the underlying Gaussian vector $\mathbf{G}$. □

This is of consequence because in the sub-Gaussian case, information about the dependance structure lies in the correlation matrix of the underlying Gaussian vector. The following lemma is a consequence of the above-mentioned proposition and states the link between the signed symmetric covariation coefficient and the g.a.p.

Lemma 3.5. *Let $1 < \alpha < 2$ and let $\boldsymbol{X}$ be a sub-Gaussian random vector with characteristic function (16). Then the matrix of signed symmetric covariation coefficients of $\boldsymbol{X}$ coincides with the matrix of generalized association parameters, called the generalized covariation matrix by Paulauskas (1976).*

Proof. Without loss of generality, let us consider the sub-Gaussian random vector $X = (X_1, X_2)$. Let us state that the g.a.p. between the components of $\mathbf{X}$ coincides with the correlation coefficient between the components of the underlying Gaussian vector $\mathbf{G}$.

The characteristic function of $\mathbf{X}$ is

$$\begin{aligned} \phi_{\mathbf{X}}(\mathbf{t}) &= \exp\left\{-\left|2^{-1}R_{11}t_1^2 + R_{12}t_1t_2 + 2^{-1}R_{22}t_2^2\right|^{\alpha/2}\right\} \\ &= \exp\left\{-\left|2^{-1}R_{11}t_1^2 + rR_{11}^{1/2}R_{22}^{1/2}t_1t_2 + 2^{-1}R_{22}t_2^2\right|^{\alpha/2}\right\} \end{aligned} \tag{20}$$

where r is the correlation coefficient between the components of the underlying Gaussian vector $\mathbf{G}$. The scale parameters of the sub-Gaussian random variables X_1 and X_2 are respectively $\gamma_{X_1} = (\frac{1}{2}R_{11})^{1/2}$ and $\gamma_{X_2} = (\frac{1}{2}R_{22})^{1/2}$. The characteristic function (20) can be written

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{-\left|\gamma_{X_1}^2t_1^2 + 2r\gamma_{X_1}\gamma_{X_2}t_1t_2 + \gamma_{X_2}^2t_2^2\right|^{\alpha/2}\right\}. \tag{21}$$

This characteristic function is equal to (12), in which $C = 1$. Thus, we establish that the correlation coefficient r between G_1 and G_2 coincides with the g.a.p. between X_1 and X_2. Using Proposition 3.4 we obtain this result. □

When $\alpha > 1$, the signed symmetric covariation coefficient coincides with the g.a.p. in the case of sub-Gaussian random vectors. However, it is not true in general. Linear transformations of independent α-stable random variables will give a counterexample in the next subsection.

When $0 < \alpha \leq 1$, the signed symmetric covariation coefficient is not defined in all cases. For a general α-stable random variable $X_j \sim S_\alpha(\gamma, \beta, d)$ with $0 < \alpha < 2$, we have

$$\begin{aligned} E|X_j|^p &< \infty \quad \text{for any value} \quad 0 < p < \alpha, \\ E|X_j|^p &= \infty \quad \text{for any value} \quad p \geq \alpha. \end{aligned} \tag{22}$$

See Samorodnitsky and Taqqu (1994) P. 18. Let $\mathbf{X} = (X_1, X_2)$ be a sub-Gaussian random vector with characteristic function (16). The following result states how the correlation coefficient between the coordinates of the underlying Gaussian vector $\mathbf{G}$ is related to $E|X_1|^p$, $E|X_2|^p$ and $E|X_1 - X_2|^p$ with $0 < p < \alpha \leq 2$.

Lemma 3.6. *Let $0 < \alpha \leq 2$. Let $\boldsymbol{X} = (X_1, X_2)$ be a sub-Gaussian random vector with underlying Gaussian vector $\boldsymbol{G}$. The correlation coefficient between G_1 and G_2 can be expressed by the relationship*

$$r = \frac{\left(E|X_1|^p\right)^{2/p} + \left(E|X_2|^p\right)^{2/p} - \left(E|X_1 - X_2|^p\right)^{2/p}}{2\Big(E|X_1|^p \; E|X_2|^p\Big)^{1/p}}, \quad 0 < p < \alpha \leq 2. \tag{23}$$

Proof. Let $\mathbf{X} = (X_1, X_2)$ be a sub-Gaussian random vector with underlying Gaussian vector $\mathbf{G}$. For the zero-mean random vector $\mathbf{G} = (G_1, G_2)$, the correlation coefficient between G_1 and G_2 is

$$\begin{aligned} r &= \frac{\mathrm{Var}(G_1) + \mathrm{Var}(G_2) - \mathrm{Var}(G_1 - G_2)}{2\mathrm{Var}(G_1)^{1/2}\mathrm{Var}(G_2)^{1/2}} \\ &= \frac{\gamma_{X_1}^2 + \gamma_{X_2}^2 - \gamma_{X_1 - X_2}^2}{2\gamma_{X_1}\gamma_{X_2}}, \end{aligned} \tag{24}$$

because $\gamma_{X_1} = \left(\frac{\mathrm{Var}(G_1)}{2}\right)^{1/2}$, $\gamma_{X_2} = \left(\frac{\mathrm{Var}(G_2)}{2}\right)^{1/2}$ and $\gamma_{X_1 - X_2} = \left(\frac{\mathrm{Var}(G_1 - G_2)}{2}\right)^{1/2}$.

The random variable $X_1 - X_2$ is symmetric because X_1 and X_2 are symmetric. Let $0 < \alpha < 2$, for $0 < p < \alpha$,

$$(E|X_j|^p)^{1/p} = c(p, \alpha)\gamma_{X_j}, \tag{25}$$

for a $S\alpha S$ random variable X_j, $c(p, \alpha)$ is a constant (See Nikias and Shao (1995) P. 32). Using (25) in (24), we finish the proof. □

From the above, the g.a.p. between the components of $\mathbf{X}$ coincides with the correlation coefficient between the components of $\mathbf{G}$, for $0 < \alpha \leq 2$.

From Lemma 3.5 and Lemma 3.6, we obviously deduce that equation (23) is another way to express the g.a.p. when $0 < \alpha \leq 2$, and the signed symmetric covariation coefficient when $\alpha > 1$.

3.3 Case of linear transformations of independent α-stable random variables

Let $1 < \alpha \leq 2$. Let X_1 and X_2 be independent random variables such as $X_k \sim S_\alpha(\gamma_{X_k}, 0, 0)$, $k = 1, 2$ and $\mathbf{X} = (X_1, X_2)$. Let $\mathbf{A} = \{a_{jk}\}$, $1 \leq j \leq k \leq 2$, be a real matrix. The random vector $\mathbf{Y} = (Y_1, Y_2) = \mathbf{AX}$, whose components are linear combinations

$$Y_j = \sum_{k=1}^{2} a_{jk} X_k, \quad j = 1, 2, \tag{26}$$

of the X_k, is α-stable with characteristic function

$$\begin{aligned} E \exp\Big\{ i \textstyle\sum_{j=1}^{2} t_j Y_j \Big\} &= \exp\Big\{ - \textstyle\sum_{k=1}^{2} \gamma_{X_k}^{\alpha} | \sum_{j=1}^{2} a_{jk} t_j |^{\alpha} \Big\} \\ &= \exp\Big\{ - \textstyle\int_{S_2} |t_1 s_1 + t_2 s_2|^{\alpha} \mathbf{\Gamma}_{\mathbf{Y}}(d\mathbf{s}) \Big\}. \end{aligned} \tag{27}$$

For simplicity, we assume $\gamma_{X_1} = \gamma_{X_2}$. Expressions of the signed symmetric covariation coefficient and g.a.p. are given below.

Proposition 3.7. *Let $1 < \alpha \leq 2$ and let $\boldsymbol{Y} = \boldsymbol{AX}$ be a bivariate $S\alpha S$ random vector with characteristic function*

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \exp\Big\{ - \gamma^{\alpha} \sum_{k=1}^{2} | \sum_{j=1}^{2} a_{jk} t_j |^{\alpha} \Big\}. \tag{28}$$

Then the signed symmetric covariation coefficient is given by:

$$\operatorname{scov}(Y_1, Y_2) = \kappa_{(Y_1, Y_2)} \frac{\Big| |a_{11}a_{21}|^{\alpha} + |a_{12}a_{22}|^{\alpha} + a_{11}a_{22}(a_{12}a_{21})^{\langle \alpha-1 \rangle} + a_{12}a_{21}(a_{11}a_{22})^{\langle \alpha-1 \rangle} \Big|^{\frac{1}{2}}}{\Big(|a_{11}a_{21}|^{\alpha} + |a_{12}a_{22}|^{\alpha} + |a_{12}a_{21}|^{\alpha} + |a_{11}a_{22}|^{\alpha} \Big)^{1/2}}, \tag{29}$$

where

$$\kappa_{(Y_1, Y_2)} = \begin{cases} \operatorname{sign}\left(\frac{a_{11}}{a_{21}} |a_{21}|^{\alpha} + \frac{a_{12}}{a_{22}} |a_{22}|^{\alpha} \right) & \text{if} \quad \operatorname{sign}([Y_1, Y_2]_\alpha) = \operatorname{sign}([Y_2, Y_1]_\alpha), \\ -1 & \text{if} \quad \operatorname{sign}([Y_1, Y_2]_\alpha) = -\operatorname{sign}([Y_2, Y_1]_\alpha). \end{cases}$$

The generalized association parameter is

$$\widetilde{\rho}(Y_1, Y_2) = \frac{a_{11}a_{21}(a_{11}^2 + a_{21}^2)^{\frac{\alpha}{2}-1} + a_{12}a_{22}(a_{12}^2 + a_{22}^2)^{\frac{\alpha}{2}-1}}{[D_1 D_2]^{\frac{1}{2}}}, \tag{30}$$

where $D_1 = a_{11}^2(a_{11}^2+a_{21}^2)^{\frac{\alpha}{2}-1} + a_{12}^2(a_{12}^2+a_{22}^2)^{\frac{\alpha}{2}-1}$ and $D_2 = a_{21}^2(a_{11}^2+a_{21}^2)^{\frac{\alpha}{2}-1} + a_{22}^2(a_{12}^2+a_{22}^2)^{\frac{\alpha}{2}-1}$.

Proof.

There is not particular difficulty to establish Equation (29). We simply use (13), taking into account X_1 and X_2 are independent (in the case of independence of random variables, the covariation is also linear in its second argument) and $\gamma_{X_1} = \gamma_{X_2}$, i. e. $\|X_1\|_\alpha = \|X_2\|_\alpha$.

Since $\mathbf{Y}$ is a linear transformation of independent α-stable random variables, its spectral measure is discrete and concentrated on 2 symmetric pairs of points of S_2 (See Samorodnitsky and Taqqu (1994) P. 69, 70). Taking into account $\gamma_{X_1} = \gamma_{X_2} = \gamma$, this spectral measure is given by

$$\begin{aligned}\mathbf{\Gamma_Y} &= \frac{1}{2}\gamma^\alpha \sum_{k=1}^{2} (a_{1k}^2 + a_{2k}^2)^{\alpha/2} \Big[\delta\Big(\frac{a_{1k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}, \frac{a_{2k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}\Big) \\ &\qquad + \delta\Big(\frac{-a_{1k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}, \frac{-a_{2k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}\Big)\Big]\end{aligned}$$

where $\delta(a, b)$ denotes the Dirac measure at the point (a, b). Let (U_1, U_2) be a random vector on S_2 with probability distribution $\widetilde{\mathbf{\Gamma}} = \mathbf{\Gamma_Y}/\mathbf{\Gamma_Y}(S_2)$. The probability distribution $\widetilde{\mathbf{\Gamma}}$ is discrete such that the symmetric points

$$\Big(\frac{a_{1k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}, \frac{a_{2k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}\Big) \text{ and } \Big(\frac{-a_{1k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}, \frac{-a_{2k}}{(a_{1k}^2+a_{2k}^2)^{1/2}}\Big) \tag{31}$$

have the same probability

$$\frac{(a_{1k}^2 + a_{2k}^2)^{\alpha/2}}{2\big[(a_{11}^2 + a_{21}^2)^{\alpha/2} + (a_{12}^2 + a_{22}^2)^{\alpha/2}\big]}, \quad k = 1, 2.$$

Then we get $E(U_1U_2)$, EU_1^2, EU_2^2 and finally (30). □

For example, when $\alpha = 1.3$ if we set $a_{11} = 12$, $a_{12} = -17$, $a_{21} = 1$ and $a_{22} = 12$, scov $= 0.60$ and $\widetilde{\rho} = 0.69$.

4 Estimators of the g.a.p. and scov

In this section, we give a natural estimator of the g.a.p., using the stable symmetric spectral measure. We propose an estimator of the signed symmetric covariation coefficient, using the fractional lower order moments (FLOM). Lemma 3.5 states that in the sub-Gaussian case, the signed symmetric covariation coefficient coincides with the generalized association parameter. This result allows us to compare all estimators. We also give an estimator of the correlation matrix of the underlying Gaussian random vector, applicable in sub-Gaussian case when $0 < \alpha < 2$.

4.1 Estimator of the g.a.p. using the stable spectral measure

As mentioned above, the generalized association parameter of a stable random vector $\mathbf{X} = (X_1, X_2)$ with spectral measure $\mathbf{\Gamma}$ is defined using a random vector $\mathbf{U} = (U_1, U_2)$ on S_2 with probability distribution $\widetilde{\mathbf{\Gamma}} = \mathbf{\Gamma}/\mathbf{\Gamma}(S_2)$. When the spectral measure $\mathbf{\Gamma}$ is discrete and concentrated on a finite number of points of the unit circle S_2, it is written

$$\mathbf{\Gamma}(\cdot) = \sum_{j=1}^{m} \sigma_j \delta_{\mathbf{s}_j}(\cdot), \tag{32}$$

where the σ_j are the weights and $\delta_{\mathbf{s}_j}$ are point masses at the points $\mathbf{s}_j = (\cos\varphi_j, \sin\varphi_j) \in S_2,\ j = 1, ..., m$. When $\mathbf{X}$ can be expressed as a linear transformation of independent α-stable random variables, its spectral measure is given by (32) (See Samorodnitsky and Taqqu (1994), P. 70, Proposition 2.3.7).

Let $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)}$ be an i.i.d. sample of symmetric α-stable bivariate random vectors with spectral measure $\mathbf{\Gamma}$ and the φ_j are unknown. Let m be the number of points $\mathbf{s}_j \in S_2$ in which we assume that the spectral measure is concentrated. For this grid we choose $\mathbf{s}_j = \big(\cos(2\pi(j-1)/m, \sin(2\pi(j-1)/m)\big),\ j = 1, .., m$. Let $\hat{\sigma}_j$ be the estimator of the weight σ_j. Then

$$\widehat{\rho}_{(X_1,X_2)} = \frac{\sum_{j=1}^{m} \hat{\sigma}_j \cos\Big(\frac{2\pi(j-1)}{m}\Big) \sin\Big(\frac{2\pi(j-1)}{m}\Big)}{\Big[\sum_{j=1}^{m} \hat{\sigma}_j \cos^2\Big(\frac{2\pi(j-1)}{m}\Big) \cdot \sum_{j=1}^{m} \hat{\sigma}_j \sin^2\Big(\frac{2\pi(j-1)}{m}\Big)\Big]^{1/2}}, \tag{33}$$

is a natural estimator of the g.a.p. between X_1 and X_2, defined in (11). This estimator is applicable to all symmetric α-stable random vectors, even if the spectral measure of (X_1, X_2) is not discrete. It is known that a general spectral measure $\mathbf{\Gamma}^*$ (not discrete and/or the location of the point masses are unknown) can be approximated by a discrete spectral measure $\mathbf{\Gamma}$, concentred in the points $\mathbf{s}_j$, $j = 1, ..., m$, such that $\mathbf{\Gamma}$ converges in the Prokhorov distance to $\mathbf{\Gamma}^*$ if $m \to \infty$. See, for instance, Garel and Massé (2009) and Davydov and Paulauskas (1999).

Now, we outline the methods we use for getting the estimates of $\hat{\sigma}_j$; these methods are detailed in Nolan et al. (2001). From (4) and (32) we can write $I_{\mathbf{X}}(\mathbf{t}) = \sum_{j=1}^{m} \psi_\alpha(\langle \mathbf{t}, \mathbf{s}_j \rangle)\sigma_j$. Further, let $(\mathbf{t}_1, ..., \mathbf{t}_m) \in \mathbb{R}^{2m}$ and define the $m \times m$ matrix

$$\Psi = \Psi_\alpha(\mathbf{t}_1, ..., \mathbf{t}_m, \mathbf{s}_1, ..., \mathbf{s}_m) = \begin{pmatrix} \psi_\alpha(\langle \mathbf{t}_1, \mathbf{s}_1 \rangle), & \cdots, & \psi_\alpha(\langle \mathbf{t}_1, \mathbf{s}_m \rangle) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \psi_\alpha(\langle \mathbf{t}_m, \mathbf{s}_1 \rangle), & \cdots, & \psi_\alpha(\langle \mathbf{t}_m, \mathbf{s}_m \rangle) \end{pmatrix}.$$

If $\vec{\sigma} = [\sigma_1, ..., \sigma_m]'$ and $\vec{I} = [I_{\mathbf{X}}(\mathbf{t}_1), ..., I_{\mathbf{X}}(\mathbf{t}_m)]'$, then

$$\vec{I} = \Psi\vec{\sigma}. \tag{34}$$

If $\mathbf{t}_1, ..., \mathbf{t}_m$ are chosen so that Ψ^{-1} exists, then $\vec{\sigma} = \Psi^{-1}\vec{I}$ gives the weights of (32). Unfortunately, in practice Ψ is ill-conditioned. To avoid the matrix inversion problem, McCulloch (2000) suggested restating the system (34) as a constrained quadratic programming problem that guarantees non negative weights:

$$\begin{aligned} &\text{minimize} \quad \|\vec{I^*} - \Psi\vec{\sigma}\|^2 \quad . \\ &\text{subject to } \vec{\sigma} \geq 0. \end{aligned} \tag{35}$$

To get the $\widehat{\sigma}_j$, $j = 1, ..., m$, we need first to estimate Ψ and $\vec{I}$, and then to solve the problem

(35). Estimating Ψ is very straightforward. We define some grids $\mathbf{t}_j = \mathbf{s}_j,\ j = 1, ..., m$. Remember that from (4) we have $\psi_\alpha(u) = |u|^\alpha$. Each element of the matrix $\widehat{\Psi}$ is got by $\widehat{\psi}_\alpha(\langle \mathbf{t}_l, \mathbf{s}_j \rangle) = |t_{l1}s_{j1} + t_{l2}s_{j2}|^{\widehat{\alpha}},\ \ j,\ l = 1, ..., m$, with $\widehat{\alpha}$ an estimator of the characteristic exponent detailed below.

Nolan et al. (2001) proposed two methods for estimating $\vec{I}$. The first uses the empirical characteristic function (ECF) of the vector. Given the i.i.d. sample $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)}$, let $\widehat{\phi}_n(\mathbf{t})$ and $\widehat{I}_n$ be the empirical counterparts of $\phi_{\mathbf{X}}$ and $I_{\mathbf{X}}$ defined in (3) and (4). Then $\widehat{\phi}_n(\mathbf{t}) = (1/n)\sum_{j=1}^{n} \exp(i\langle \mathbf{t}, \mathbf{X}^{(j)} \rangle)$ is the ECF and $\widehat{I}_n(\mathbf{t}) = -\ln \widehat{\phi}_n(\mathbf{t})$. Given $\mathbf{t}_1, ..., \mathbf{t}_m$ elements of S_2, $\vec{I}_{ECF,n} = [\widehat{I}_n(\mathbf{t}_1), ..., \widehat{I}_n(\mathbf{t}_m)]'$ is the ECF estimate of $\vec{I}$. Let $\widehat{\alpha}_j$ and $\widehat{\sigma}_j,\ j = 1,\ 2$, be the estimators of index and scale parameters of each j coordinate of the 2-dimensional data set. We use the quantile based estimators of McCulloch (1986) and estimators of Koutrouvelis (1980) to estimate these parameters. We define $\widehat{\alpha} = \alpha_{ECF} = (1/2)\sum_{j=1}^{2} \widehat{\alpha}_j$ as a consistent estimator of the joint index of stability α.

The second method for estimating $\vec{I}$ generalizes the method of McCulloch (2000). This method is called the Projection method because it is based on one-dimensional projections of the data set. First, consider a projection of the bivariate symmetric α-stable random vector $\mathbf{X}$: for any $\mathbf{u} \in S_2$, $\langle \mathbf{u}, \mathbf{X} \rangle$ is a one-dimensional symmetric α-stable random variable with characteristic function $E\exp(it\langle \mathbf{u}, \mathbf{X} \rangle) = \exp\{-I_{\mathbf{X}}(t\mathbf{u})\}$ and its scale parameter is given by (5). Now consider the sample $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)}$. Fix a grid $\mathbf{t}_1, ..., \mathbf{t}_m$ on S_2, where each $\mathbf{t}_j$ is a direction on which the data set is projected. In this way, define for each $\mathbf{t}_j$ the one-dimensional data set $\langle \mathbf{t}_j, \mathbf{X}^{(1)} \rangle, ..., \langle \mathbf{t}_j, \mathbf{X}^{(n)} \rangle$. From (6) define

$$\widehat{I}_{\mathbf{X}}(\mathbf{t}_j) = [\widehat{\gamma}(\mathbf{t}_j)]^{\widehat{\alpha}(\mathbf{t}_j)},\ j = 1, ..., m, \tag{36}$$

where the values of $\widehat{\alpha}(\mathbf{t}_j)$ and $\widehat{\gamma}(\mathbf{t}_j)$ are respectively the estimates of the index of stability and the scale parameter of the data $\langle \mathbf{t}_j, \mathbf{X}^{(1)} \rangle, ..., \langle \mathbf{t}_j, \mathbf{X}^{(n)} \rangle$, obtained using the quantile based estimators of McCulloch (1986) and estimators of Koutrouvelis (1980). Define a joint characteristic exponent by $\widehat{\alpha} = \alpha_{PROJ} = (1/m)\sum_{j=1}^{m} \widehat{\alpha}(\mathbf{t}_j)$. Then, (36) becomes $\widehat{I}_{\mathbf{X}}(\mathbf{t}_j) = [\widehat{\gamma}(\mathbf{t}_j)]^{\widehat{\alpha}},\ \ j = 1, ..., m$. Davyvod and Paulauskas (1999) proposed a different method for estimating spectral measure. We empirically observed the convergence of estimator (33) for all estimation procedures we used. Therefore we conjecture that it is a weakly consistent estimator of g.a.p. if $m \to \infty$.

4.2 Estimator of scov using Fractional Lower Order Moments

For $1 \leq p < \alpha$, consider the quantity

$$\widehat{\text{scov}}(X_1, X_2) = \widehat{\kappa}_{(X_1,X_2)} \frac{\Big|\Big(\sum_{j=1}^{n} X_{1j}\ X_{2j}^{\langle p-1\rangle}\Big)\Big(\sum_{j=1}^{n} X_{2j}\ X_{1j}^{\langle p-1\rangle}\Big)\Big|^{1/2}}{\Big[\Big(\sum_{j=1}^{n} |X_{1j}|^p\Big)\Big(\sum_{j=1}^{n} |X_{2j}|^p\Big)\Big]^{1/2}}, \tag{37}$$

where

$$\widehat{\kappa}_{(X_1,X_2)} = \begin{cases} \text{sign}\Big(\sum_{j=1}^{n} X_{1j} X_{2j}^{\langle p-1\rangle}\Big) & \text{if} \quad \text{sign}\Big(\sum_{j=1}^{n} X_{1j} X_{2j}^{\langle p-1\rangle}\Big) = \text{sign}\Big(\sum_{j=1}^{n} X_{2j} X_{1j}^{\langle p-1\rangle}\Big), \\ -1 & \text{if} \quad \text{not.} \end{cases}$$

The pairs $(X_{11}, X_{21}), ..., (X_{1n}, X_{2n})$ are independent and identical copies of (X_1, X_2).

When $p = 1$, (37) becomes

$$\widehat{\text{scov}}(X_1, X_2) = \widehat{\kappa}_{(X_1,X_2)} \frac{\Big|\Big(\sum_{j=1}^{n} X_{1j}\ \text{sign}(X_{2j})\Big)\Big(\sum_{j=1}^{n} X_{2j}\ \text{sign}(X_{1j})\Big)\Big|^{1/2}}{\Big[\Big(\sum_{j=1}^{n} |X_{1j}|\Big)\Big(\sum_{j=1}^{n} |X_{2j}|\Big)\Big]^{1/2}}, \tag{38}$$

where

$$\widehat{\kappa}_{(X_1,X_2)} = \begin{cases} \text{sign}\Big(\sum_{j=1}^{n} X_{1j}\text{sign}(X_{2j})\Big) & \text{if} \quad \text{sign}\Big(\sum_{j=1}^{n} X_{1j}\text{sign}X_{2j})\Big) = \text{sign}\Big(\sum_{j=1}^{n} X_{2j}\text{sign}(X_{1j})\Big), \\ -1 & \text{if} \quad \text{not.} \end{cases}$$

For $0 < q < \alpha \leq 2$, consider the quantity

$$\widehat{r} = \frac{\Big(\sum_{j=1}^{n} |X_{1j}|^q\Big)^{2/q} + \Big(\sum_{j=1}^{n} |X_{2j}|^q\Big)^{2/q} - \Big(\sum_{j=1}^{n} |X_{1j} - X_{2j}|^q\Big)^{2/q}}{2\Big[\Big(\sum_{j=1}^{n} |X_{1j}|^q\Big)\Big(\sum_{j=1}^{n} |X_{2j}|^q\Big)\Big]^{1/q}}. \tag{39}$$

Basic convergence properties of estimators defined by (37) and (39) are given below.

Proposition 4.1.

1. *The quantity* $\widehat{\text{scov}}$ *is a strongly consistent estimator of the signed symmetric covariation coefficient. This estimator does not require a prior estimation of the spectral measure.*

2. *In the sub-Gaussian case,* $\widehat{r}$ *is a strongly consistent estimator of the correlation coefficient between two components of the underlying Gaussian vector. When* $\alpha > 1$*, it is also a strongly consistent estimator of the signed symmetric covariation coefficient. For* $0 < \alpha \leq 2$*,* $\widehat{r}$ *is a strongly consistent estimator of the g.a.p.*

Proof. Let (X_1, X_2) be $S\alpha S$ with $\alpha > 1$. Then for all $1 \leq p < \alpha$,

$$\frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha} = \frac{EX_1X_2^{\langle p-1\rangle}}{E|X_2|^p}. \tag{40}$$

This result is established in Cambanis and Miller (1981) and d'Estampes (2003). Using (40), the signed symmetric covariation coefficient can be written, for $1 \leq p < \alpha$,

$$\text{scov}(X_1, X_2) = \kappa_{(X_1,X_2)} \left| \frac{EX_1X_2^{\langle p-1\rangle} EX_2X_1^{\langle p-1\rangle}}{E|X_1|^p E|X_2|^p} \right|^{1/2}, \tag{41}$$

where

$$\kappa_{(X_1,X_2)} = \begin{cases} \text{sign}(EX_1X_2^{\langle p-1\rangle}) & \text{if} \quad \text{sign}(EX_1X_2^{\langle p-1\rangle}) = \text{sign}(EX_2X_1^{\langle p-1\rangle}), \\ -1 & \text{if} \quad \text{not.} \end{cases}$$

Let $(X_{11}, X_{21}), ..., (X_{1n}, X_{2n})$ be independent and identical copies of (X_1, X_2). The quantity

$$\begin{aligned} \widehat{\text{scov}}(X_1, X_2) &= \widehat{\kappa}_{(X_1,X_2)} \frac{\left|\left(\frac{1}{n}\sum_{j=1}^n X_{1j}\, X_{2j}^{\langle p-1\rangle}\right)\left(\frac{1}{n}\sum_{k=1}^n X_{2k}\, X_{1k}^{\langle p-1\rangle}\right)\right|^{1/2}}{\left[\left(\frac{1}{n}\sum_{j=1}^n |X_{1j}|^p\right)\left(\frac{1}{n}\sum_{k=1}^n |X_{2k}|^p\right)\right]^{1/2}} \\ &= \widehat{\kappa}_{(X_1,X_2)} \frac{\left|\left(\sum_{j=1}^n X_{1j}\, X_{2j}^{\langle p-1\rangle}\right)\left(\sum_{k=1}^n X_{2k}\, X_{1k}^{\langle p-1\rangle}\right)\right|^{1/2}}{\left[\left(\sum_{j=1}^n |X_{1j}|^p\right)\left(\sum_{k=1}^n |X_{2k}|^p\right)\right]^{1/2}} \end{aligned} \tag{42}$$

where

$$\widehat{\kappa}_{(X_1,X_2)} = \begin{cases} \text{sign}\Big(\sum_{j=1}^n X_{1j}X_{2j}^{\langle p-1\rangle}\Big) & \text{if} \quad \text{sign}\Big(\sum_{j=1}^n X_{1j}X_{2j}^{\langle p-1\rangle}\Big) = \text{sign}\Big(\sum_{j=1}^n X_{2j}X_{1j}^{\langle p-1\rangle}\Big), \\ -1 & \text{if} \quad \text{not.} \end{cases}$$

As a consequence of the strong law of large numbers for independent copies, for $k, l = 1, 2$, $k \neq l$,

$$\frac{1}{n}\sum_{j=1}^{n} X_{kj}\, X_{lj}^{\langle p-1\rangle} \longrightarrow EX_k X_l^{\langle p-1\rangle} \ a.s. \text{ as } n \longrightarrow \infty, \tag{43}$$

and for $k = 1, 2$,

$$\frac{1}{n}\sum_{j=1}^{n} |X_{kj}|^p \longrightarrow E|X_k|^p \ a.s. \text{ as } n \longrightarrow \infty. \tag{44}$$

When X_n and Y_n respectively converge to a and b almost surely, then X_nY_n converges to ab almost surely and X_n/Y_n converges to a/b almost surely. Thus, from (43) and (44), the estimator (42) converges to $\text{scov}(X_1, X_2)$ almost surely as $n \longrightarrow \infty$.

For $0 < q < \alpha \leq 2$, the strong law of large numbers (44) implies that the estimator

$$\begin{aligned}
\widehat{r} &= \frac{\Big(\sum_{j=1}^{n} |X_{1j}|^q\Big)^{2/q} + \Big(\sum_{j=1}^{n} |X_{2j}|^q\Big)^{2/q} - \Big(\sum_{j=1}^{n} |X_{1j} - X_{2j}|^q\Big)^{2/q}}{2\Big[\Big(\sum_{j=1}^{n} |X_{1j}|^q\Big)\Big(\sum_{j=1}^{n} |X_{2j}|^q\Big)\Big]^{1/q}} \\
&= \frac{\Big(\frac{1}{n}\sum_{j=1}^{n} |X_{1j}|^q\Big)^{2/q} + \Big(\frac{1}{n}\sum_{j=1}^{n} |X_{2j}|^q\Big)^{2/q} - \Big(\frac{1}{n}\sum_{j=1}^{n} |X_{1j} - X_{2j}|^q\Big)^{2/q}}{2\Big[\Big(\frac{1}{n}\sum_{j=1}^{n} |X_{1j}|^q\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} |X_{2k}|^q\Big)\Big]^{1/q}}
\end{aligned}$$

converges to r almost surely as $n \longrightarrow \infty$. □

4.3 Some simulation results in sub-Gaussian and Linear combination cases

The performance of the estimators $\widehat{\rho}$, $\widehat{\text{scov}}$ and $\widehat{r}$ is discussed in the sub-Gaussian case. In the case of linear combinations of symmetric independent stable random variables, we discuss only the performance of the estimators $\widehat{\rho}$ and $\widehat{\text{scov}}$. We denote the scale parameters of X_1 and X_2 respectively by γ_1 and γ_2.

Now, let us describe the abbreviations used in the tables :

- scovf: this estimate of scov is obtained using Fractional Lower Order Moments, as defined in (37);

- core: this estimate of the correlation coefficient is got using (39);

- gapemc, gapeko, gappmc and gappko are the estimates of g.a.p. got using (33):

  gapemc is got when the parameters α, γ_1 and γ_2 are estimated using the quantile based estimators of McCulloch (1986) and the weights $\widehat{\sigma}_j$ are obtained by the ECF method;

  gapeko is got when the parameters α, γ_1 and γ_2 are estimated using the estimators of Koutrouvelis (1980) and the weights $\widehat{\sigma}_j$ are obtained by the ECF method;

  gappmc is got when the parameters α, γ_1 and γ_2 are estimated using the estimators of McCulloch (1986) and the weights $\widehat{\sigma}_j$ are obtained by the Projection method;

  gappko is got when the parameters α, γ_1 and γ_2 are estimated using the estimators of Koutrouvelis (1980) and the weights $\widehat{\sigma}_j$ are obtained by the Projection method;

The size of simulated samples is n and m is the number of points of the quantizer (32) and we make $u = 100$ replications. The displayed value is the mean over replications. The positive values beneath are the mean absolute deviations to the mean displayed above. We evaluated the estimations from the estimator (37) for different values of p. Comparing the mean of the estimates and the mean absolute deviations, a computationally efficient choice is $p = 1$. In (39), we computationally get the best results for the estimates and the mean absolute deviations by taking $q = \alpha/3$. We iteratively evaluate q by $q = (1/6u)\sum_{j=1}^{u}(\widehat{\alpha}_{1j} + \widehat{\alpha}_{2j})$, where $\widehat{\alpha}_{1j}$ and $\widehat{\alpha}_{2j}$ are respectively the estimates of the index of stability of X_1 and X_2 for the replication j.

We also calculate the estimations obtained from (33) for different values of m between 2 and 100. In sub-Gaussian case, the best results were obtained when $m = 10$, and in the case of linear combinations when $m = 11$. Therefore, here we report on only these values of m. In estimation of spectral measure using the ECF method, scaling the data by the median of the values $|\mathbf{X}^{(1)}|, ..., |\mathbf{X}^{(n)}|$ gives consistently good results across all scales, see Nolan et al. (2001).

Put Table 1 here.

Put Table 2 here.

When $\alpha > 1$, the estimator $\widehat{\text{scov}}$ gives very good results for estimating the signed symmetric covariation coefficient. In the sub-Gaussian case, the estimator $\widehat{r}$ is also a strongly consistent estimator of this quantity. This last estimator should be favored for sub-Gaussian random vectors because it gives the smallest mean absolute deviations and is applicable for all $0 < \alpha \leq 2$. We use an approximation on 10 or 11 points for the estimator $\widehat{\rho}$. We observed that the results do not vary very significantly when we increase the number of discretization points. This estimator, which requires a prior estimation of α and an approximate spectral measure, gives worse results compared to the previous ones. We have better results when α approaches 2. For 1,600 simulated data vectors, timing for getting **one** estimate of the scov using (38) is null. It takes 3.32 seconds for getting one estimate of the g.a.p. using (33) when the weights $\hat{\sigma}_j$, $j = 1, ..., 10$ are estimated by ECF Method, and 6.75 seconds when the weights are estimated by Projection Method.

Put Table 3 here.

In the case of linear combinations of independent and identically distributed random variables, the estimator $\widehat{\rho}$ gives consistently very good results for estimating the g.a.p. We use an approximation on 11 points for the estimator $\widehat{\rho}$. The results obtained using $\widehat{\text{scov}}$ remain satisfying.

In general, a good estimation of the spectral measure requires large samples. For example, Nolan et al. (2001) give estimates of the spectral measure for 10,000 simulated data vectors. For sub-Gaussian random vectors and linear transformation of independent α-stable random variables, we have calculated estimates of the spectral measure for 1,600 simulated data vectors and we have obtained satisfying results.

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Table 1: Estimates of scov and g.a.p. for $n = 1,600$ sub-Gaussian data vectors with $\alpha = 1.5$, $\gamma_1 = 5$, $\gamma_2 = 10$ and $m = 10$.

True value	-1.00	-0.60	-0.40	-0.20	0.00	0.10	0.30	0.50	0.90
scovf	-1.00	-0.60	-0.38	-0.20	-0.01	0.10	0.31	0.50	0.90
	0.00	0.04	0.05	0.07	0.06	0.07	0.06	0.05	0.02
core	-1.00	-0.60	-0.39	-0.20	0.00	0.10	0.31	0.50	0.90
	0.00	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.01
gapemc	-0.80	-0.55	-0.36	-0.18	-0.00	0.09	0.27	0.46	0.81
	0.00	0.02	0.02	0.03	0.02	0.02	0.02	0.02	0.00
gapeko	-0.80	-0.55	-0.36	-0.18	-0.00	0.09	0.27	0.46	0.81
	0.00	0.02	0.02	0.03	0.02	0.02	0.02	0.02	0.00
gappmc	-0.80	-0.56	-0.35	-0.17	-0.01	0.09	0.27	0.46	0.81
	0.00	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.00
gappko	-0.80	-0.55	-0.36	-0.18	-0.00	0.09	0.27	0.46	0.81
	0.00	0.02	0.02	0.03	0.03	0.03	0.02	0.02	0.00
q	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50

Table 2: Estimates of scov and gap for $n = 1,600$ sub-Gaussian data vectors with $\alpha = 0.8$, $\gamma_1 = 5$, $\gamma_2 = 10$ and $m = 11$.

True value	-1.00	-0.60	-0.40	-0.20	0.00	0.30	0.70	0.90
core	-1.00	-0.60	-0.40	-0.20	-0.00	0.30	0.70	0.90
	0.00	0.04	0.04	0.04	0.04	0.04	0.02	0.01
gapemc	-0.80	-0.50	-0.33	-0.16	-0.00	0.24	0.61	0.80
	0.01	0.05	0.05	0.05	0.04	0.04	0.05	0.02
gapeko	-0.80	-0.50	-0.32	-0.15	-0.00	0.23	0.60	0.80
	0.00	0.04	0.06	0.04	0.05	0.04	0.05	0.02
q	0.27	0.27	0.26	0.26	0.27	0.26	0.27	0.27

Table 3: Estimates of scov and g.a.p. for $n = 1,600$ linear combination data with $\alpha = 1.5$, $\gamma = 10$ and $m = 11$.

a_{11}	100	12	10	−13	−11	−12	22	9
a_{12}	16	−17	0	9	13	21	12	−17
a_{21}	−50	2	0	2	2	2	2	19
a_{22}	−8	12	10	5	4	5	20	−10
scov	−1.00	−0.60	0.00	0.11	0.30	0.50	0.71	0.91
scovf	−1.00	−0.58	0.00	0.10	0.30	0.50	0.70	0.90
	0.00	0.06	0.05	0.11	0.06	0.06	0.02	0.01
gap	−1.00	−0.65	0.00	0.27	0.35	0.53	0.56	0.83
gapemc	−0.94	−0.64	0.00	0.27	0.36	0.53	0.56	0.82
	0.00	0.02	0.02	0.02	0.03	0.03	0.02	0.01
gapeko	−0.94	−0.64	0.00	0.27	0.36	0.53	0.56	0.82
	0.00	0.02	0.02	0.02	0.02	0.03	0.02	0.01
gappmc	−0.94	−0.64	0.00	0.27	0.36	0.53	0.56	0.82
	0.00	0.02	0.03	0.03	0.04	0.05	0.02	0.01
gappko	−0.94	−0.65	0.00	0.27	0.36	0.53	0.56	0.82
	0.00	0.02	0.02	0.03	0.03	0.03	0.02	0.01