On Salem numbers, expansive polynomials and Stieltjes continued fractions

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Résumé. On considère une réciproque à la Construction de Salem (1945) de suites convergentes de nombres de Salem en termes d'association entre polynômes de Salem et de quotients d'Hurwitz par l'intermédiaire de polynômes expansifs de petite mesure de Mahler. Cette association utilise le Théorème A (1995) de Bertin-Boyd d'entrecroisement de conjugués sur le cercle unité ; dans ce contexte, un nombre de Salem est produit et codé par un m-uplet de nombres rationnels strictement positifs caractérisant la fraction continue de Stieltjes (SITZ) du quotient (alternant) d'Hurwitz correspondant. Le sous-ensemble des fractions continues de Stieltjes au-dessus d'un polynôme de Salem à racines simples, ne s'annulant pas en ± 1, provenant de polynômes expansifs unitaires de terme constant égal à leur mesure de Mahler, admet une structure de semi-groupe. Cette structure de semi-groupe se transporte sur les ensembles de nombres de Garsia généralisés correspondants.

Abstract. A converse method to the Construction of Salem (1945) of convergent families of Salem numbers is investigated in terms of an association between Salem polynomials and Hurwitz quotients via expansive polynomials of small Mahler measure. This association makes use of Bertin-Boyd's Theorem A (1995) of interlacing of conjugates on the unit circle; in this context, a Salem number β is produced and coded by an m-tuple of positive rational numbers characterizing the (SITZ) Stieltjes continued fraction of the corresponding Hurwitz quotient (alternant). The subset of Stieltjes continued fractions over a Salem polynomial having simple roots, not cancelling at ± 1, coming from monic expansive polynomials of constant term equal to their Mahler measure, has a semigroup structure. The sets of corresponding generalized Garsia numbers inherit this semi-group structure.

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CONTENTS

1. Introduction

A Salem number is an algebraic integer $\theta > 1$ such that the Galois conjugates $\theta^{(i)}$ of θ satisfy: $|\theta^{(i)}| \leq 1$ with at least one conjugate of modulus 1. The set of Salem numbers is denoted by T. A Pisot number is a real algebraic integer, all of whose other conjugates have modulus strictly less than 1. The set of Pisot numbers is traditionally denoted by S.

An open problem is the characterization of the set $\overline{\mathrm{T}}$ of limit points of T. A first Conjecture of Boyd [Bo0] asserts that the union $\mathrm{S} \cup \mathrm{T}$ is closed. A second Conjecture of Boyd ([Bo], p 327) asserts that the first derived set of $\mathrm{S} \cup \mathrm{T}$ is S. At least, S is included in $\overline{\mathrm{T}}$ ([B-S], Theorem 6.4.1). In 1945, Salem ([Sa], Theorem IV) developped the so-called *Construction of Salem* to show that every Pisot number is a limit point of convergent sequences of Salem numbers from both sides. Siegel [Si] proved that the smallest Pisot number is $\theta_0 = 1.32\ldots$, dominant root of $X^3 - X - 1$, implying that the number of Salem numbers in any interval $(1, M)$, with $M \geq \theta_0$, is infinite. However, if 1 is a limit point of T, then T would be everywhere dense in $[1, +\infty)$ since $\theta \in T$ implies $\theta^m \in T$ for all positive integer m by [Sa0], p. 106, [Pi], p. 46. That T were everywhere dense in $[1, +\infty)$ is probably false [Bo0]. If the Conjecture of Lehmer were true, then we would have the existence of a smallest Salem number > 1. Converse methods

to the Construction of Salem to describe interesting sequences of algebraic numbers, eventually Salem numbers, converging to a given Salem number are more difficult to establish.

Association theorems between Pisot polynomials and Salem polynomials were introduced by Boyd ([Bo] Theorem 4.1), Bertin and Pathiaux-Delefosse ([BB], [BPD] pp 37–46, [B-S] chapter 6), to investigate the relations between infinite collections of Pisot numbers and a given Salem number.

In the scope of studying interlacing on the unit circle and $\overline{\mathrm{T}}$, McKee and Smyth [MKS] recently used new interlacing theorems, different from those introduced (namely Theorem A and Theorem B) by Bertin and Boyd [BB] [BPD]. These interlacing theorems, and their limit-interlacing versions, reveal to be fruitful. Theorem 5.3 (in [MKS]) shows that all Pisot numbers are produced by a suitable (SS) interlacing condition, supporting the second Conjecture of Boyd; similarly Theorem 7.3 (in [MKS]), using Boyd's association theorems, shows that all Salem numbers are produced by interlacing and that a classification of Salem numbers can be made.

In the present note, we reconsider the interest of the interlacing Theorems of [BB], as potential tools for this study of limit points, as alternate analogues of those of McKee and Smyth; we focus in particular on the Theorem A of [BB] (recalled as Theorem 2.1 below).

The starting point is the observation that expansive polynomials are basic ingredients in the proof of Theorem A and that the theory of expansive polynomials recently, and independently, received a strong impulse [Bu]. The idea is then to bring back to Theorem A a certain number of results from this new theory: Hurwitz polynomials, alternants, their coding in continued fractions... In §2 Theorem A and McKee and Smyth's interlacing modes of conjugates on the unit circle are recalled (to fix the notations); association theorems between Salem polynomials and expansive polynomials are obtained. In §3 a Stieltjes terminated continued fraction with positive rational numbers is shown to code analytically not only a Salem number but also all its interlacing conjugates. The subset of such Stieltjes continued fractions, over a given Salem number, is a discrete point set and has a semigroup structure (Proposition 3.2), when reducing monic expansive polynomials to those having a positive constant coefficient.

The arithmetico-analytic deformation and limit properties of this coding function will be reported elsewhere.

The theory of expansive polynomials is fairly recent [Bu]. The terminology "expansive polynomial" appeared in the study of (CNS) canonical number systems, for instance in Kovács [Ko], in Akiyama and Gjini [AG] for self-affine attractors in $\mathbb{R}^n$. Then expansive polynomials were associated canonically with Hurwitz polynomials by Burcsi [Bu] to obtain an

exhaustive classification and their properties. The method of coding Hurwitz polynomials by finite sets of positive rational integers in continued fractions (Henrici [H], chapter 12) is transported to expansive polynomials. In the present note, we continue further this coding towards Salem numbers using Theorem A.

In §2 we recall the two related subclasses A_q and B_q of Salem numbers which arise from the interlacing Theorems A and B ([BB], [BPD] p. 129 and 133). Since the set T decomposes as

$$T = \bigcup_{q \geq 2} A_q = \bigcup_{q \in \mathbb{N}} B_q \tag{1.1}$$

investigating the limit points of T as only limit points of Salem numbers in the subclasses A_q, by the coding by continued fractions as presently, and their deformations, does not result in a loss of generality.

Notations :

Definition. A Salem polynomial is a monic polynomial with integer coefficients having exactly one zero (of multiplicity 1) outside the unit circle, and at least one zero lying on the unit circle.

A Salem polynomial is not necessarily irreducible. If it cancels at $\theta > 1$, and it is reducible, then, by Kronecker's Theorem [Kr], it is the product of cyclotomic polynomials by the minimal polynomial of θ. Let θ be a Salem number. The minimal polynomial $T(z)$ of θ has an even degree 2n, $n \geq 2$, with simple roots. $T(z)$ has exactly one zero θ of modulus > 1, one zero $\frac{1}{\theta}$ of modulus < 1 and $2n-2$ zeros on the unit circle, as pairs $(\alpha_j, \overline{\alpha_j})$ of complex-conjugates. The notation 'T' for Salem polynomials is the same as for the set of Salem numbers, since it presents no ambiguity in the context.

Definition. An expansive polynomial is a polynomial with coefficients in a real subfield of $\mathbb{C}$, of degree ≥ 1, such that all its roots in $\mathbb{C}$ have a modulus strictly greater than 1.

An expansive polynomial is not necessarily monic.

Definition. Let $P(z)$ be a polynomial $\in \mathbb{Z}[z]$, and $n = \deg(P)$. The reciprocal polynomial of $P(z)$ is $P^*(z) = z^n P(\frac{1}{z})$. A polynomial P is a reciprocal polynomial if $P^*(z) = P(z)$. A polynomial P is an antireciprocal polynomial if $P^*(z) = -P(z)$.

Definition. If $P(X) = a_0 \prod_{j=1}^{n}(X - \alpha_j)$ is a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$, and roots α_j, the Mahler measure of P is

$$\mathrm{M}(P) := |a_0| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

In the case where expansive polynomials are irreducible, the following definition extends the classical one of Garsia [Br0] [Br1] [HP].

Definition. A generalized Garsia number is an algebraic integer for which the minimal polynomial is a monic (irreducible) expansive polynomial with absolute value of the constant term greater than or equal to 2. A generalized Garsia polynomial P is a monic irreducible expansive polynomial with integer coefficients such that $\mathrm{M}(P) \geq 2$. A Garsia number is a generalized Garsia number of Mahler measure equal to 2.

Definition. The n-th cyclotomic polynomial, with integer coefficients, is denoted by $\Phi_n(X), n \geq 1$, with $\Phi_1(X) = X-1, \Phi_2(X) = X+1$ and $\deg(\Phi_n)$ even as soon as $n > 2$. The degree $\varphi(n) = n \prod_{p \,\mathrm{prime},\, p|n}(1 - 1/p)$ of $\Phi_n(X)$ is the Euler's indicator function of n.

2. Bertin-Boyd Interlacing Theorem A

2.1. The Q-construction of a reciprocal (or an anti-reciprocal) polynomial from a polynomial P by an algebraic function. Let $\mathbb{K}$ be a real subfield of $\mathbb{C}$, $P(X) \in \mathbb{K}[X], \deg(P) = n \geq 1$, and z the complex variable. With $\epsilon = \pm 1$, the polynomial defined by

$$Q(z) = zP(z) + \epsilon P^*(z) \tag{2.1}$$

satisfies $Q^*(z) = \epsilon\, Q(z)$. It is either a reciprocal (if $\epsilon = +1$), or an anti-reciprocal (if $\epsilon = -1$) polynomial. The algebraic function obtained by the related polynomial :

$$Q(z,t) = zP(z) + \epsilon t P^*(z) \tag{2.2}$$

defines an affine algebraic curve over $\mathbb{C}$:

$$\{(z,t) \in \mathbb{C}^2 \mid Q(z,t) = 0\}. \tag{2.3}$$

For $0 \leq t \leq 1$ the equation $Q(z,t) = 0$ over $\mathbb{C}$ defines an algebraic curve $z = Z(t)$ with n+1 branches. $Z(0)$ is the set of the zeros of $zP(z)$ and $Z(1)$ is the set of the zeros of $Q(z)$.

By Q-construction from P, over $\mathbb{K}$, we mean the couple $(Q(z), Q(z,t))$ given by the reciprocal or anti-reciprocal polynomial Q and its associated algebraic function $Q(z,t)$, both having specific properties arising from those of P and the sign of ϵ, as described below.

On $|z| = 1$, let us remark that $|P(z)| = |P^*(z)|$. Then $Q(z,t)$ has no zeros on $|z| = 1$ for $0 \leq t < 1$. Each branch of the algebraic curve $z = Z(t)$ is

(i) either included in $|z| \leq 1$; when it starts from a zero of $zP(z)$ in $|z| < 1$,

(ii) or included in $|z| \geq 1$; when it starts from a zero of $zP(z)$ in $|z| > 1$.

Then a zero of $Q(z)$ on the unit disc $|z| = 1$ is

(i) either stemming from a branch included in the unit disc; then it is called an *exit*,
(ii) or stemming from a branch outside the unit disc; then it is called an *entrance*.

The example of the Lehmer polynomial and the smallest known Salem number, Lehmer's number, is given in Figure 1.

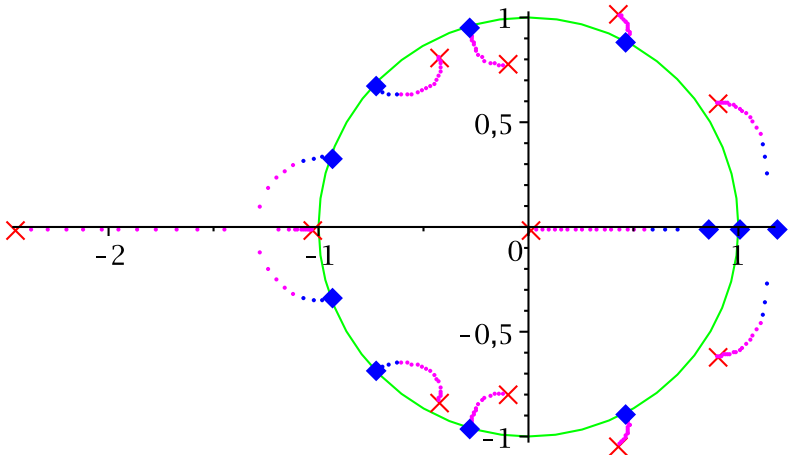

FIGURE 1. Branches of the algebraic curve obtained with the monic nonexpansive polynomial $P(z) = z^{10}+2z^9+3z^7+z^6+2z^5+2z^4+z^3+4z^2+z+2$ (crosses), producing by the Q-construction the anti-reciprocal polynomial $Q(z) = (z-1)(z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1)$ (diamonds), which has 5 entrances, 4 exits and 1 zero of modulus > 1, the Lehmer number : $\theta \approx 1.17628....$

2.2. Expansive polynomials of Mahler measure q. Classes A_q. In the particular case where $\mathbb{K} = \mathbb{Q}$ and $P(X)$ is monic and expansive with integer coefficients, $zP(z)$ has one zero in the open unit disc and n zeros outside the closed unit disc. The algebraic curve $z = Z(t)$ has at most one exit and n entrances. Therefore $Q(z)$ has at most one zero inside $|z| < 1$.

Since $Q(z)$ is a reciprocal polynomial, if $Q(z)$ has no zero in $|z| < 1$, then all his zeros are on the unit circle. And if $Q(z)$ has exactly one zero α in $|z| < 1$, it has exactly one zero θ in $|z| > 1$ and $n-2$ zeros on the unit circle. Then, if $n \geq 4$, $\theta = \frac{1}{\alpha}$ is a Salem number and Q is a Salem polynomial. We say that θ is produced by the Q-construction from P.

Definition. Let $q \in \mathbb{N}^*$ be a nonzero integer. The class A_q is the set of Salem numbers produced by the Q-construction (over $\mathbb{K} = \mathbb{Q}$) from monic

expansive polynomials $P(X) \in \mathbb{Z}[X]$ having a constant term equal to $\pm q$ and $\epsilon = -\mathrm{sgn}P(0)$.

Remark. The sets A_0 and A_1 are empty since all the zeros α_i of any monic expansive polynomial P are in $|z| > 1$, so that we have $q = |\prod \alpha_i| > 1$.

Definition. Let q be an integer ≥ 2. The set of monic expansive polynomials $P(z) \in \mathbb{Z}[z]$ such that $|P(0)| = \mathrm{M}(P) = q$ producing a Salem number by the Q-construction $Q(z) = zP(z) + \epsilon z^n P(\frac{1}{z})$ with $\epsilon \in \{-1; +1\}$, is denoted by E_q.

Theorem 2.1 (Bertin, Boyd [BB], Theorem A)**.** *Suppose that θ is a Salem number with minimal polynomial T. Let $q \in \mathbb{N} \setminus \{0, 1\}$. Then θ is in A_q if and only if there is a cyclotomic polynomial K with simple roots and $K(1) \neq 0$ and a reciprocal polynomial $L(X) \in \mathbb{Z}[X]$ with the following properties :*

(a) $L(0) = q - 1$,
(b) $\deg(L) = \deg(KT) - 1$,
(c) $L(1) \geq -K(1)T(1)$,
(d) *L has all its zeros on $|z| = 1$ and they interlace the zeros of KT on $|z| = 1$ in the following sense : let $e^{i\psi_1}, ...e^{i\psi_m}$ the zeros of L on $\{\mathrm{Im}z \geq 0\} \setminus \{z = -1\}$ with $0 < \psi_1 < ... < \psi_m < \pi$ and let $e^{i\phi_1}, ...e^{i\phi_m}$ the zeros of KT on $\{\mathrm{Im}z \geq 0\}$ with $0 < \phi_1 < ... < \phi_m \leq \pi$, then $0 < \psi_1 < \phi_1 < ... < \psi_m < \phi_m$.*

The construction of the polynomials L and KT is explicit ([BPD], pp 129–133), as follows :

- if $\epsilon = -1$ then $Q(1) = 0$. Then P_1 and Q_1 are chosen as $P_1(z) = P(z)$ and $Q_1(z) = \dfrac{Q(z)}{z-1}$,
- else if $\epsilon = +1$, then P_1 and Q_1 are chosen as $P_1(z) = (z-1)P(z)$ and $Q_1(z) = Q(z)$.

In both cases, the polynomial $Q_1(z)$ is a reciprocal polynomial which satisfies the equation:

$$(z-1)Q_1(z) = zP_1(z) - P_1^*(z). \tag{2.4}$$

We say that P_1 lies "over Q_1". As $Q_1(\theta) = 0$, Q_1 is the product of the minimal polynomial of θ by a product of cyclotomic polynomials as

$$K(z)T(z) = Q_1(z), \tag{2.5}$$

and the polynomial L, reciprocal by construction, is given by

$$L(z) = P_1(z) - Q_1(z). \tag{2.6}$$

In §2.4.1, §2.4.2 and §2.4.3 we establish existence theorems for the polynomials P_1 associated to a given Salem polynomial Q_1 by the equation

(2.4), focusing on the case $\epsilon = -1$, that is with $P(0) = P_1(0) > 0$ and $\deg(P)$ even. The methods of Geometry of Numbers used call for nondegenerated polyhedral cones in Euclidean spaces of dimension half the degree of the Salem polynomial. These theorems are called *association theorems*. In §2.4.4 the second case of association theorem, with $\epsilon = +1$, is briefly shown to call for similar methods, after a suitable factorization of (2.4) and sign changes. The algorithmic search of an expansive polynomial over a Salem polynomial is considered in §2.4.5 from a practical viewpoint.

The example of the interlacing of roots of L and KT associated to the Lehmer number is given in Figure 2.

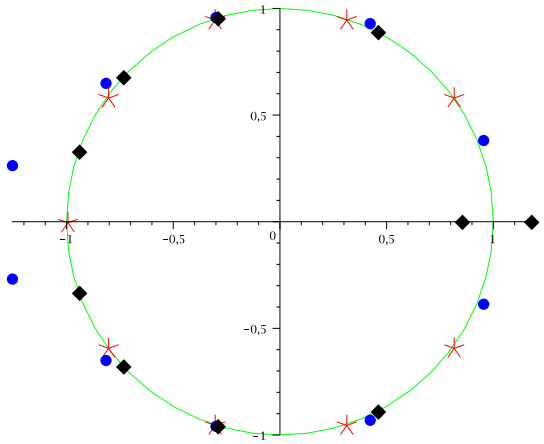

FIGURE 2. Interlacing of the zeros of L = P-T (asterisks) and those of KT = T (diamonds) on the unit circle obtained with the monic expansive polynomial $P(z) = z^{10} + 2z^9 + z^8 - z^7 - z^6 - z^4 - z^3 + 2z + 2$ (circles), producing by the Q-construction the anti-reciprocal Salem polynomial $Q(z) = (z-1)(z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1) = (z-1)T(z)$ with the Lehmer number $\theta \approx 1.176\ldots$ as dominant root of T.

In §2.3 the type of interlacing provided by Theorem 2.1 is revisited in the more general context of interlacing modes on the unit circle proposed by McKee and Smyth [MKS].

The classes (B_q) of Salem numbers are defined by a similar construction with a polynomial P which has a single zero in $|z| > 1$. We refer to [BB] ([BPD], p. 133) for their definition. All the Salem numbers are generated by the classes A_q and B_q, giving rise to (1.1). The distribution of the small Salem numbers in the classes A_q in intervals was studied by Boyd [Bo] and Bertin and Pathiaux-Delefosse [BPD]. Bertin and Boyd [BB] proved that:

for $q \geq 2$ and $k \geq 1$, $A_2 \subset A_q$ and $A_q \subset A_{kq-k+1}$. The distribution in the other classes remains obscure.

Conjecture (Local Density Conjecture)**.** For all $c > 1$, there exists $M > 0$ such that $T \cap [1; c]$ is contained in a finite union $\bigcup_{2 \leq q \leq M} A_q$.

2.3. Interlacing on the unit circle and McKee-Smyth interlacing quotients. Following McKee and Smyth [MKS] [MKS1] three types of interlacing conditions, CC, CS and SS, are relevant.

Definition. Suppose that C_1 and C_2 are coprime polynomials with integer coefficients and positive top coefficients. We say that C_1 and C_2 satisfy the **CC-interlacing** condition (CC for Cyclotomic-Cyclotomic) or C_1/C_2 is a CC-interlacing quotient if

- C_1 and C_2 have all their roots in the unit circle.
- all the roots of C_1 and C_2 are simple,
- the roots of C_1 and C_2 interlace on $|z| = 1$.

Remark. (i) As C_1 and C_2 have the same number of roots, C_1 and C_2 have the same degree; (ii) as the non-real zeros of C_1 and C_2 are conjugated in complex sense two by two, the reals -1 and $+1$ must be in the set of their roots to ensure the interlacing on the unit circle; (iii) one polynomial among C_1 and C_2 is a reciprocal polynomial, the other being an anti-reciprocal polynomial, having $(z-1)$ in its factorization; (iv) the terminology CC for "Cyclotomic-Cyclotomic" is abusing. Indeed, C_1 and C_2 are not necessary monic, so they are not necessary cyclotomic polynomials.

A complete classification of all pairs of cyclotomic polynomials whose zeros interlace on the unit circle is reported in [MKS1].

Definition. Suppose that C and S are coprime polynomials with integer coefficients and positive top coefficients. We say that C and S satisfy the **CS-interlacing condition** (CS for Cyclotomic-Salem) or C/S is a CS-interlacing quotient if

- S is reciprocal and C is antireciprocal,
- C and S have the same degree,
- all the roots of C and S are simple, except perhaps at $z = 1$,
- $z^2 - 1 | C$,
- C has all its roots in $|z| = 1$,
- S has all but two roots in $|z| = 1$, with these two being real, positive, $\neq 1$,
- the roots of C and S interlace on $\{|z| = 1\} \setminus \{1\}$.

Definition. Suppose that S_1 and S_2 are coprime polynomials with integer coefficients and positive top coefficients. We say that S_1 and S_2 satisfy the **SS-interlacing condition** (SS for Salem-Salem) or S_2/S_1 is a SS-interlacing quotient if

- one of S_1 and S_2 is reciprocal polynomial, the other is an anti-reciprocal polynomial,
- S_1 and S_2 have the same degree,
- all the roots of S_1 and S_2 are simple,
- S_1 and S_2 have all but two roots in $|z| = 1$, with these two being real, positive, $\neq 1$,
- the roots of S_1 and S_2 interlace on $\{|z| = 1\} \setminus \{1\}$.

Remark. There are two types of SS-interlacing condition : if the largest real roots of S_1S_2 is a root of S_1, then S_2/S_1 is called a 1-SS-interlacing quotient, and S_1/S_2 is called a 2-SS-interlacing quotient.

Theorem 2.1 provides interlacing on the unit circle; more precisely, referring to (2.4) and denoting $n := \deg(Q_1) = \deg(P_1)$, let us show that if n is

(i) even, the quotient $(z-1)L/KT$ is a CS-interlacing quotient,
(ii) odd, no CS-interlacing condition is satisfied.

Indeed, if n is even, from (2.6), with $L = P_1 - Q_1$,

$$L(-1) = P_1(-1) - \frac{-P_1(-1) - (-1)^n P_1(\frac{1}{-1})}{(-1-1)} = P_1(-1) - \frac{-2P(-1)}{-2} = 0.$$

Then the factor $(z+1)$ divides L and we can take $C = (z-1)L$ and $S = KT$. As L and KT are reciprocal polynomials, $(z-1)L$ is an anti-reciprocal polynomial. Moreover, $\deg(L) = \deg(KT) - 1$, so $(z-1)L$ and KT have the same degree. Finally, by definition, $(z^2-1)|C$. The roots of C and S are simple and all but two roots of S interlace the roots of C on $\{|z| = 1\} \setminus \{z = 1\}$, S having two real roots being positive inverse and $\neq 1$.

On the contrary, if n is odd,

$$Q_1(-1) = \frac{-P_1(-1) - (-1)^n P_1(\frac{1}{-1})}{(-1-1)} = 0.$$

Then the factor $(z+1)$ divides the Salem polynomial $Q_1 = KT$; -1 cannot be a zero of L. The item (z^2 - 1) | C in the CS-interlacing condition is then missing.

Let us turn to the Salem numbers associated with by these interlacing modes. McKee and Smyth [MKS] use the variant $z \to x = \sqrt{z} + 1/\sqrt{z}$ of the Tchebyshev transformation, instead of the more usual one $z \to x = z + 1/z$, to study the Salem numbers produced by the above different cases of interlacing quotients.

Theorem 2.2 ([MKS], Theorem 3.1)**.** *Let C_2/C_1 be a CC-interlacing quotient with C_1 monic, of degree ≥ 4. By the map $x = \sqrt{z} + \frac{1}{\sqrt{z}}$ the function $\dfrac{\sqrt{z}C_2(z)}{(z-1)C_1(z)}$ is transformed into the real interlacing quotient $\dfrac{c_2(x)}{c_1(x)}$ with c_1*

and c_2 coprime polynomials in $\mathbb{Z}[x]$. If $\lim_{x->2^+} \frac{c_2(x)}{c_1(x)} > 2$ then the solutions of the equation

$$\frac{C_2(z)}{(z-1)C_1(z)} = 1 + \frac{1}{z} \tag{2.7}$$

are a Salem number, its conjugates and roots of unity.

Theorem 2.3 ([MKS], Theorem 5.1)**.** *Let C/S be a CS-interlacing quotient with S monic, of degree ≥ 4. The solutions of the equation*

$$\frac{C(z)}{(z-1)S(z)} = 1 + \frac{1}{z} \tag{2.8}$$

are a Salem number, its conjugates and roots of unity.

Theorem 2.1 provides a CS-interlacing quotient if the common degree $n = \deg(Q_1) = \deg(P_1)$ is even, as mentioned above; then the quotient $(z-1)L/KT$ is a CS-interlacing quotient, with KT a monic polynomial. As such, we can now apply Theorem 2.3 to this quotient. This theorem offers the construction of an other Salem polynomial T_2 :

$$T_2(z) = zL(z) - (z+1)K(z)T(z). \tag{2.9}$$

Denote θ the dominant root of T and θ_2 the dominant root of T_2. Remark that θ_2 is always different from θ. Otherwise, if $\theta_2 = \theta$ then $\theta L(\theta) = T_2(\theta) + (z+1)K(\theta)T(\theta) = 0$, which is impossible because all zeros of L lies on the unit circle.

For exemple, with the smallest Salem known number $\theta \approx 1.1762808$ of degree 10, root of $T(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$, we obtain by this construction, with the expansive polynomial $P(z) = x^{10} + 2x^9 + x^8 + x^2 + 2x + 2$ over T, the Salem number $\theta_2 \approx 1.5823471$ of degree 6, root of the Salem polynomial $(z+1)(z^2+z+1)(z^2-z+1)(z^6-z^4-2z^3-z^2+1)$. Remark that this sequence depends on the choice of the polynomial P over T.

Theorem 2.4 ([MKS], Theorem 5.2)**.** *Let S_2/S_1 be a SS-interlacing quotient with S_1 monic. If $\lim_{z->1+} \frac{S_2(z)}{(z-1)S_1(z)} < 2$ then the solutions of the equation*

$$\frac{S_2(z)}{(z-1)S_1(z)} = 1 + \frac{1}{z} \tag{2.10}$$

are a Salem number, its conjugates and roots of unity.

Under the assumption that Lehmer's Conjecture is true, Theorem 9.2 in [MKS] shows that the smallest Salem number θ is such that there exists a

type 2 SS-interlacing quotient S_1/S_2, with two monic polynomials S_1 and S_2 satisfying

$$\frac{S_2(z)}{(z-1)S_1(z)} = \frac{2}{1+z} \tag{2.11}$$

for which the only solutions of (2.11) are $z = \theta$, its conjugates and perhaps some roots of unity. The other small Salem numbers are probably produced by type 2 SS-interlacing quotients with a condition of type (2.11) as well; comparing with the Local Density Conjecture in §2.2, they are obtained from expansive polynomials of small Mahler measure equal or close to 2.

Extension of the field of coefficients: CC-interlacing quotients were extended by Lakatos and Losonczi [LL] to classes of reciprocal polynomials having coefficients in $\mathbb{R}$.

2.4. Association Theorems between expansive polynomials and Salem polynomials. The classes A_q of Salem numbers, in Theorem 2.1, call for two disjoint classes of monic expansive polynomials P: those for which the constant term $P(0)$ is positive (case $\epsilon = -1$), those for which it is negative (case $\epsilon = +1$). Below we focus on the existence of expansive polynomials over a Salem polynomial when the Mahler measure $\mathrm{M}(P)$ is equal to $P(0)$ (case $\epsilon = -1$). In §2.4.4 we indicate how the previous construction can be adapted to deduce existence theorems in the second case $\epsilon = +1$.

2.4.1. *A criterium of expansivity.*

Theorem 2.5. *Let T be an irreducible Salem polynomial of degree $m \geq 4$. Denote by $\beta > 1$ its dominant root. Let P be a polynomial $\in \mathbb{R}[z]$ of degree m such that*

$$(z-1)T(z) = zP(z) - P^*(z). \tag{2.12}$$

P is an expansive polynomial if and only if

- *$P(1)T(1) < 0$, and,*
- *for all pairs of zeros $(\alpha, \overline{\alpha})$ of T of modulus 1,*

$$(\alpha - 1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0.$$

Remark. The equation (2.12) implies that P is monic and denoting $P(z) = p_m z^m + p_{m-1}z^{m-1} + .. + p_1 z + p_0$, then $p_i \in \mathbb{Z}$ with $|p_0| > p_m = 1$. Since the rhs of (2.12) cancels at $z = 1$, the factorization of the lhs of (2.12) by $z - 1$ is natural. Denote $Q(z) := (z-1)T(z)$.

Proof. The conditions are necessary: as T is reciprocal, Q is anti-reciprocal of degree $m + 1$; Q has exactly one zero β outside $\overline{D(0;1)}$, one zero $\frac{1}{\beta}$ in

$D(0;1)$, and $m-1$ zeros on $C(0;1)$, which are $z=1$ and $\frac{m}{2}-1$ pairs of complex conjugates $(\alpha_i, \overline{\alpha_i})$. Let Q_t be the parametric polynomial

$$Q_t(z) = zP(z) - tP^*(z). \tag{2.13}$$

On the unit circle $C(0;1)$, we have $|zP(z)| = |z|.|P(z)| = |P(z)| = |P^*(z)|$, and $|P(z)| \neq 0$ because P is an expansive polynomial. Then, for $0 < t < 1$, $|Q_t(z) - zP(z)| = |-tP^*(z)| < |P^*(z)| = |zP(z)|$. The theorem of Rouché implies that the polynomials $Q_t(z)$ and $zP(z)$ have the same number of zeros in the compact $\overline{D(0;1)}$. So $Q_t(z)$ has exactly one zero in $\overline{D(0;1)}$ and m zeros outside the closed unit disc.

By (2.13), for $t > 1$, $Q_{\frac{1}{t}}(\frac{1}{z}) = \frac{-1}{tz^{m+1}} Q_t(z)$. Then, if α is a zero of Q_t, then $\frac{1}{\alpha}$ is a zero of $Q_{\frac{1}{t}}(\frac{1}{z})$. Moreover, $\frac{1}{\overline{\alpha}}$ is a zero of $Q_{\frac{1}{t}}(\frac{1}{z})$ as well because $Q_{\frac{1}{t}} \in \mathbb{R}(z)$ (thus we obtain the zeros of $Q_{\frac{1}{t}}$ from those of Q_t by an inversion of centre 0 of radius 1). Let $f(z) := \frac{Q(z)}{P^*(z)}$. Then, by (2.13),

$$\frac{Q_t(z)}{P^*(z)} = f(z) + (1-t). \tag{2.14}$$

For $\alpha \in \{\frac{1}{\beta}, \beta, 1, \alpha_1, \overline{\alpha_1}, \alpha_2, \overline{\alpha_2}, ..., \alpha_{\frac{m}{2}-1}, \overline{\alpha_{\frac{m}{2}-1}}\}$, when z lies in a neighboorhood of α, the equation $Q_t(z) = 0$ is equivalent to the equation $f(z) = t-1$. As Q has simple zeros, $Q(\alpha) = 0$ and $Q'(\alpha) \neq 0$. Then $f(\alpha) = 0$ and $f'(\alpha) = \frac{Q'(\alpha)P^*(\alpha) - Q(\alpha)(P^*)'(\alpha)}{P^*(\alpha)^2} = \frac{Q'(\alpha)}{P^*(\alpha)} \neq 0$. By the local inversion theorem, in the neighboorhood of $t=1$, there exist an analytic function h_α such that the equation $Q_t(z) = 0$ is equivalent to $z = h_\alpha(t)$, with $h_\alpha(1) = \alpha$ and $h'_\alpha(1) \neq 0$. Then, $\{h_\alpha(t); \alpha \in \{\frac{1}{\beta}, \beta, 1, (\alpha_i)_{1 \leq i < m}\}\}$ are the zeros of $Q_t(z)$. In the neighboorhood of $t=1$, we have :

$$h_\alpha(t) = h_\alpha(1) + (t-1)h'_\alpha(1) + ... \tag{2.15}$$

By the inversion property of Q_t, if $h_\alpha(t)$ is a zero of $Q_t(z)$ then $1/\overline{h_\alpha(t)}$ is a zero of $Q_{\frac{1}{t}}$: there exist $\widetilde{\alpha} \in \{\beta^{-1}, \beta, 1, (\alpha_i)_{1 \leq i < m}\}$ such that

$$\frac{1}{\overline{h_\alpha(t)}} = h_{\widetilde{\alpha}}(\frac{1}{t}). \tag{2.16}$$

When $t=1$, we obtain: $1/\overline{h_\alpha(1)} = 1/\overline{\alpha} = \alpha/|\alpha|^2$ and $h_{\widetilde{\alpha}}(1) = \widetilde{\alpha}$. In particular, for any α of modulus 1, we obtain $\widetilde{\alpha} = \alpha$ and $1/\overline{h_\alpha(t)} = h_\alpha(\frac{1}{t})$, that is

$$h_\alpha(t)\overline{h_\alpha(\frac{1}{t})} = 1. \tag{2.17}$$

In this case, we denote $h_\alpha(t) = X(t)+iY(t)$. Then (2.17) becomes $(X(t)+iY(t))(X(1/t) - iY(1/t)) = 1$. The imaginary part of this equation is $Y(t)X(1/t)-X(t)Y(1/t) = 0$. By derivation, we obtain for $t = 1$: $Y'(1)X(1)-Y(1)X'(1) = 0$. Thus $X'(1)/X(1) = Y'(1)/Y(1)$. Let $\lambda \in \mathbb{R}$ be this quotient. Thus $h'_\alpha(t) = X'(t) + iY'(t) = \lambda(X(t) + iY(t)) = \lambda h_\alpha(t)$, with $h'_\alpha(1) = \lambda\alpha$ for $t = 1$. For any α on the unit circle, equation (2.15) gives

$$h_\alpha(t) = \alpha\left[1 + (t-1)\lambda + ...\right]. \tag{2.18}$$

Since $\alpha \neq \dfrac{1}{\beta}$, then $|h_\alpha(t)| > 1 = |\alpha|$ for $0 < t < 1$, implying $\lambda < 0$. As h_α satisfies the equation (2.14), we have :

$$Q(h_\alpha(t)) + (1-t)P^*(h_\alpha(t)) = 0. \tag{2.19}$$

Deriving (2.19) at $t = 1$, we obtain: $h'_\alpha(1)Q'(\alpha) - P^*(h_\alpha(1)) = 0$. We deduce $h'_\alpha(1) = P^*(\alpha)/Q'(\alpha)$. Then,

$$0 > \lambda = \frac{h'_\alpha(1)}{\alpha} = \frac{P^*(\alpha)}{\alpha Q'(\alpha)} = \frac{\alpha^m P(\frac{1}{\alpha})}{\alpha Q'(\alpha)} = \frac{\alpha^{m-1}P(\overline{\alpha})P(\alpha)}{P(\alpha)Q'(\alpha)} = \frac{|P(\alpha)|^2}{\alpha^{1-m}P(\alpha)Q'(\alpha)}.$$

We deduce $\alpha^{1-m}P(\alpha)Q'(\alpha) < 0$, for $\alpha = 1$ or any root of T of modulus 1. Let us transform these inequalities as a function of T. For $\alpha = 1$, $Q'(1) = T(1)$, since $Q'(z) = T(z) + (z-1)T'(z)$, and we readily obtain :

$$P(1)T(1) < 0. \tag{2.20}$$

And if $\alpha \neq 1$ is a root of T of modulus 1, since $Q'(\alpha) = (\alpha-1)T'(\alpha)$,

$$(\alpha-1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0. \tag{2.21}$$

Remark that $(\alpha-1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0 \Leftrightarrow (\overline{\alpha}-1)\overline{\alpha}^{1-m}P(\overline{\alpha})T'(\overline{\alpha}) < 0$ since both quantities are real: the condition (2.21) is related to the pair $(\alpha, \overline{\alpha})$ for any α of modulus 1. Hence the claim.

The conditions are sufficient: first let us show that P has no zero of modulus 1. Suppose the contrary; there exist $\alpha, |\alpha| = 1$, such that $P(\alpha) = 0$. Then, as P is in $\mathbb{R}[z]$, $\overline{\alpha}$ is a zero of P. Then, $Q(\alpha) = \alpha P(\alpha) - (\alpha)^m P(\overline{\alpha}) = 0$. So α would be a zero of modulus 1 of $Q(z) = (z-1)T(z)$. The only possibilities are $z = 1$ and the zeros of T of modulus 1. If $\alpha = 1$, then the condition $0 = P(1)T(1) < 0$ leads to a contradiction. Similarly, if $\alpha \neq 1$, then $0 = (\alpha-1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0$ would also be impossible.

Let us show that $zP(z)$ has one zero in $D(0;1)$ (which is $z = 0$) and $m-1$ zeros outside $\overline{D(0;1)}$. Let α be a zero of T of modulus 1. Let h_α defined as in (2.15) with (2.18):

$$h_\alpha(t) = \alpha + h'_\alpha(1)(t-1) + ...$$

The assumption $(\alpha-1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0$ (or $P(1)T(1) < 0$ if $\alpha = 1$) implies that $|h_\alpha(t)| > |\alpha| = 1$ for t in the neighboorhood of 1, $t < 1$. Thus, in this neighboorhood, Q_t has at least m-2 zeros outside $\overline{D(0;1)}$. These

zeros belong to the algebraic branches which end at the zeros α of Q of modulus 1.

Moreover, as $|P(z)| = |P^*(z)|$, for $|z| = 1$, we have : $|zP(z)| = |P(z)| = |P^*(z)| > |tP^*(z)|$ for all t, $0 < t < 1$. Hence Q_t has no zero on the unit circle, for all t, $0 < t < 1$. By continuity of the algebraic curves defined by $Q_t(z) = 0$, the branch ending at β is included in $\mathbb{C} \setminus \overline{D(0;1)}$: this branch originates from a root of $zP(z)$ which lies outside $\overline{D(0;1)}$. In the same way, the branch ending at $1/\beta$ originates from a root of $zP(z)$ which is inside $D(0;1)$. Therefore P is expansive. □

The above Criterium of expansivity, i.e. Theorem 2.5, only involves conditions at the roots of Q of modulus 1. However, though the existence of expansive polynomials P satisfying (2.12) is only proved below in §2.4.2, the following Proposition shows that two extra inequalities at the Salem number β and its inverse β^{-1} should also be satisfied.

Proposition 2.1. *Let T be an irreducible Salem polynomial of degree $m \geq 4$. Denote by $\beta > 1$ its dominant root. Let P be an expansive polynomial $\in \mathbb{R}[z]$, of degree m, such that*

$$(z-1)T(z) = zP(z) - P^*(z). \tag{2.22}$$

Then, the polynomial P satisfies the two equivalent properties:

$$\text{(i) } P(1/\beta)T'(1/\beta) < 0 \qquad \text{and} \qquad \text{(ii) } P(\beta)T'(\beta) > 0.$$

Proof. Let Q_t and h_α defined as above in (2.13) and (2.15) respectively. For $\alpha \in \{\frac{1}{\beta}, \beta\}$, the relation (2.16) gives : $\widetilde{\alpha} = \frac{1}{\alpha}$ and then $1/\overline{h_\alpha(t)} = h_{\frac{1}{\alpha}}(\frac{1}{t})$.

(i) *Case $\alpha = 1/\beta$:* for $0 \leq t \leq 1$, $\deg(Q_t(z)) = m + 1 \geq 5$ is odd. As $Q_t(z) \in \mathbb{R}[z]$, Q_t has at least one real root and pairs of complex-conjugated roots. Since it admits only one root in $D(0;1)$ this zero is real for symmetry reasons; and the branch starting at $z = 0$ and ending at $z = 1/\beta$ is included in $\mathbb{R}$ (i.e : $h_{\frac{1}{\beta}}(t) \in \mathbb{R}$ for $0 < t < 1$). Then, $h'_{\frac{1}{\beta}}(1) = P^*(\frac{1}{\beta})/Q'(\frac{1}{\beta}) > 0$ implying $P(\frac{1}{\beta})Q'(\frac{1}{\beta}) > 0$. Since $Q'(\frac{1}{\beta}) = (\frac{1}{\beta} - 1)T'(\frac{1}{\beta})$ we readily obtain: $P(\frac{1}{\beta})T'(\frac{1}{\beta}) < 0$.

(ii) *Case $\alpha = \beta$:* we have $h_\beta(t) = 1/\overline{h_{\frac{1}{\beta}}(\frac{1}{t})} = 1/h_{\frac{1}{\beta}}(\frac{1}{t})$. By derivation we have: $h'_\beta(t) = -\frac{-1}{t^2}h'_{\frac{1}{\beta}}(\frac{1}{t})/(h_{\frac{1}{\beta}}(\frac{1}{t}))^2$; therefore, for $t = 1$: $h'_\beta(1) = h'_{\frac{1}{\beta}}(1)/(h_{\frac{1}{\beta}}(1))^2 = \beta^2\frac{P^*(\frac{1}{\beta})}{Q'(\frac{1}{\beta})} = \beta^2 h'_{\frac{1}{\beta}}(1)$. The two nonzero real numbers $h'_\beta(1)$ and $h'_{\frac{1}{\beta}}(1)$ are simultaneously positive or negative. They are positive.

As $h'_\beta(1) = P^*(\beta)/Q'(\beta) = \beta P(\beta)/(\beta-1)T'(\beta) = \dfrac{\beta}{\beta-1}\dfrac{P(\beta)^2}{P(\beta)T'(\beta)}$, we deduce: $P(\beta)T'(\beta) > 0$. □

2.4.2. *Existence of an expansive polynomial.*

Theorem 2.6. *Let T be an irreducible Salem polynomial of degree $m \geq 4$. It exist a monic expansive polynomial P of degree m such that :*

$$(z-1)T(z) = zP(z) - P^*(z). \tag{2.23}$$

Proof. Denote $T(z) := z^m + t_1 z^{m-1} + ... t_{\frac{m}{2}-1} z^{\frac{m}{2}+1} + t_{\frac{m}{2}} z^{\frac{m}{2}} + t_{\frac{m}{2}-1} z^{\frac{m}{2}-1} ... + t_1 z + t_0$ and $P(z) := z^m + p_{m-1}z^{m-1} + ... + p_1 z + p_0$. Though P is nonreciprocal, only half of the coefficient vector $(p_i)_{i=0,...,m-1}$ entirely determines P: indeed, using the fact that T is reciprocal, the polynomial identity (2.23) gives the following $m/2$ relations between the coefficients:

$$p_{m-i} = p_{i-1} + t_i - t_{i-1}, \qquad 1 \leq i \leq \frac{m}{2}. \tag{2.24}$$

The problem of the existence of P is then reduced to finding a $m/2$-tuple of integers $(p_i)_{i=0,...,m/2-1} \in \mathbb{Z}^{m/2}$, characterizing the "point" P in the lattice $\mathbb{Z}^{m/2}$, satisfying the $m/2$ conditions of the Criterium of expansivity of Theorem 2.5. In terms of the coefficient vectors these conditions are linear, each of them determining an affine hyperplane in $\mathbb{R}^{m/2}$. The Criterium of expansivity means that the "point" P should lie in the intersection of the $m/2$ open half-spaces defined by these hyperplanes. Let us call admissible cone this intersection. We will show that this facetted polyhedral cone is nonempty; hence it will contain infinitely many points of the lattice $\mathbb{Z}^{m/2}$. In other terms the number of monic expansive polynomials P lying over T will be shown to be infinite.

Let us make explicit the equations of the delimiting hyperplanes of the cone, from Theorem 2.5, using (2.24). *The half-space given by $P(1)T(1) < 0$:* the condition at $z = 1$ is

$$P(1)T(1) = \sum_{i=0}^{m} p_i \sum_{i=0}^{m} t_i = \sum_{i=0}^{\frac{m}{2}-1} 2(2\sum_{k=0}^{\frac{m}{2}-1} t_k + t_{\frac{m}{2}})p_i + t_{\frac{m}{2}}(2\sum_{k=0}^{\frac{m}{2}-1} t_k + t_{\frac{m}{2}}) < 0. \tag{2.25}$$

By identification this inequation can be written: $\lambda_0 + \sum_{i=0}^{\frac{m}{2}-1} a_{0,i} p_i < 0$ with $a_{0,0}, a_{0,1}, \ldots, a_{0,\frac{m}{2}-1}, \lambda_0 \in \mathbb{Z}$ only functions of the coefficients t_i of T.

The half-space given by $(\alpha-1)\alpha^{1-m}P(\alpha)T'(\alpha) < 0$: the condition at $z = \alpha \neq 1, |\alpha| = 1$, a zero of T, is $(\alpha-1)\alpha^{1-m}(\sum_{i=0}^{m} p_i\alpha^i)(\sum_{i=1}^{m} it_i\alpha^{i-1}) < 0$,

i.e.

$$(\alpha-1)(\alpha^{\frac{m}{2}}+\alpha^{\frac{m}{2}-1}+t_{\frac{m}{2}})(\frac{m}{2}t_{\frac{m}{2}}+\sum_{k=0}^{\frac{m}{2}-1}(k\alpha^{k-\frac{m}{2}}+(m-k)\alpha^{\frac{m}{2}-k}))$$

(2.26)

$$+\sum_{i=0}^{\frac{m}{2}-1}(\alpha^{i-\frac{m}{2}}+\alpha^{\frac{m}{2}+1-i})(\alpha-1)(\frac{m}{2}t_{\frac{m}{2}}+\sum_{k=0}^{\frac{m}{2}-1}t_k(k\alpha^{k-\frac{m}{2}}+(m-k)\alpha^{\frac{m}{2}-k}))p_i<0.$$

By identification this inequation can be written: $\lambda_\alpha+\sum_{i=0}^{\frac{m}{2}-1}a_{\alpha,i}p_i<0$ with $a_{\alpha,0},a_{\alpha,1},\ldots,a_{\alpha,\frac{m}{2}-1},\lambda_\alpha$ real in the algebraic number field which is the splitting field of the polynomial T. The set of the solutions of the following linear system of inequations in $\mathbb{R}^{\frac{m}{2}}$:

$$\begin{cases}\lambda_0+a_{0,0}p_0+a_{0,1}p_1+\ldots a_{0,\frac{m}{2}-1}p_{\frac{m}{2}-1} & <0,\\ \lambda_1+a_{1,0}p_0+a_{1,1}p_1+\ldots a_{1,\frac{m}{2}-1}p_{\frac{m}{2}-1} & <0,\\ \ldots & \\ \lambda_{\frac{m}{2}-1}+a_{\frac{m}{2}-1,0}p_0+a_{\frac{m}{2}-1,1}p_1+\ldots a_{\frac{m}{2}-1,\frac{m}{2}-1}p_{\frac{m}{2}-1} & <0,\end{cases}$$

is the admissible cone C^+. Let us show that it is nonempty. We have just to show that the face hyperplanes H_i defined by the equations $\lambda_i+a_{i,0}p_0+a_{i,1}p_1+\ldots a_{i,\frac{m}{2}-1}p_{\frac{m}{2}-1}=0$ intersect at a unique point. The linear system of equations corresponding to $\bigcap_{i=0}^{\frac{m}{2}-1}H_i$ is equivalent to

$$\begin{cases}P(1)T(1) & =0,\\ (\alpha_1-1)\alpha_1^{1-m}P(\alpha_1)T'(\alpha_1) & =0,\\ \ldots & \\ (\alpha_{\frac{m}{2}-1}-1)\alpha_{\frac{m}{2}-1}^{1-m}P(\alpha_{\frac{m}{2}-1})T'(\alpha_{\frac{m}{2}-1}) & =0,\end{cases}$$

where α_i are the simple zeros of T of modulus 1. Obviously this system is equivalent to $P(1)=P(\alpha_1)=\ldots=P(\alpha_{\frac{m}{2}-1})=0$, since T, being irreducible, has simple roots and that $T(1)\neq 0$. The only monic polynomial P in $\mathbb{R}[z]$, of degree m, which cancels at $\{1,\alpha_1,\ldots,\alpha_{\frac{m}{2}-1}\}$ is of the form $P_0(z)=(z-1)(z-r)\prod_{i=1}^{\frac{m}{2}-1}(z^2-(\alpha_i+\overline{\alpha_i})z+1)$. From (2.12), P_0 satisfies $P_0(\beta)=\beta^{m-1}P_0(1/\beta)$, thus $r=\frac{1}{2}(\beta+1/\beta)$. The half-coefficient vector $\mathcal{M}_0={}^t(p_i)_{0\leq i\leq\frac{m}{2}-1}$ of P_0 is the unique solution of this system of equations (the notation $\mathcal{M}_t$ is that of §2.4.5). Remark that $p_0=\frac{1}{2}(\beta+1/\beta)$. The point $\mathcal{M}_0$ is the summit vertex of the admissible cone C^+. Since the admissible cone is nonempty, it contains infinitely many points of $\mathbb{Z}^{m/2}$. Hence the claim. □

Proposition 2.2. *Let $q \geq 2$ be an integer and T an irreducible Salem polynomial of degree m. The subset of E_q of the monic expansive polynomials P over T, of degree m, such that $P(0) = q$, is finite.*

Proof. In [Bu], Burcsi proved that the number of expansive polynomials of the form $z^m + p_{m-1}z^{m-1} + ... + p_1 z + p_0 \in \mathbb{Z}[z]$ with $|p_0| = q$ is finite and it is at most $\prod_{k=1}^{m-1}(2B(p_0, m, k) + 1)$ with $B(p_0, m, k) = \binom{m-1}{k-1} + |p_0|\binom{m-1}{k}$. □

2.4.3. *Existence of an expansive polynomial over a Salem polynomial and a product of cyclotomic polynomials.* We now extend the existence Theorem 2.6 to monic expansive polynomials lying over nonirreducible Salem polynomials and even products of cyclotomic polynomials only. The proofs, of similar nature as before, are left to the reader: they include a Criterium of expansivity, as a first step, then imply the nonemptyness of a certain admissible open cone of solutions in a suitable Euclidean space. Let us remark that the construction method which is followed calls for expansive polynomials of even degrees, and for a lhs term in (2.27) (as in (2.12)) which has factors of multiplicity one in its factorization (case of simple roots).

Theorem 2.7. *Let $2 < n_1 < n_2 < ... < n_s$ be a increasing sequence of s integers, $s \geq 1$, and denote by $\xi_{n_i,j}$ the zeros of Φ_{n_i}. Let T be either the trivial constant polynomial equal to 1 ($m = 0$), or an irreducible Salem polynomial of degree $m \geq 4$, with dominant root $\beta > 1$. Let P be a polynomial $\in \mathbb{R}[z]$ of degree $m + \sum_{i=1}^{s} \varphi(n_i)$ such that*

$$(z-1)\,T(z)\Big(\prod_{i=1}^{s}\Phi_{n_i}(z)\Big) = zP(z) - P^*(z). \tag{2.27}$$

Then P is an expansive polynomial if and only if

(i) $P(1)T(1)\Big(\prod_{i=1}^{s}\Phi_{n_i}(1)\Big) < 0$,

(ii) *for all pairs of conjugates $(\xi_{n_i,j}, \overline{\xi_{n_i,j}})$,*

$$(\xi_{n_i,j} - 1)\xi_{n_i,j}^{1-m}P(\xi_{n_i,j})T(\xi_{n_i,j})\Big(\prod_{k=1,k\neq i}^{s}\Phi_{n_k}(\xi_{n_i,j})\Big)\Phi'_{n_i}(\xi_{n_i,j}) < 0, \text{ and,}$$

(iii) *if $T \not\equiv 1$, for all pairs of zeros $(\alpha, \overline{\alpha})$ of T of modulus 1,*

$$(\alpha - 1)\alpha^{1-m}P(\alpha)T'(\alpha)\prod_{i=1}^{s}\Phi_{n_i}(\alpha) < 0.$$

Theorem 2.8. *Let $2 < n_1 < n_2 < ... < n_s$ be a increasing sequence of s integers, $s \geq 1$. Let T be either the trivial constant polynomial equal to 1 ($m = 0$), or an irreducible Salem polynomial of degree $m \geq 4$, with*

dominant root $\beta > 1$. There exist a monic expansive polynomial $P(z) \in \mathbb{R}[z]$ of degree $m + \sum_{i=1}^{s} \varphi(n_i)$ such that

$$(z-1)\,T(z)\Big(\prod_{i=1}^{s} \Phi_{n_i}(z)\Big) = zP(z) - P^*(z). \tag{2.28}$$

2.4.4. *The second class of Salem numbers in A_q.* Given an integer $q \geq 2$, the second class of Salem numbers β in A_q is provided by expansive polynomials P for which $P(0) = -\mathrm{M}(P) = -q$ lying above the minimal polynomial T of β, referring to Theorem 2.1. From (2.4) this case corresponds to $\epsilon = +1$, with $P_1(z) = (z-1)P(z)$, and then to

$$Q(z) = Q_1(z) = zP(z) + P^*(z). \tag{2.29}$$

Since the rhs of (2.29) cancels at $z = -1$ when $\deg(P)$ is even, a natural factorization of Q in (2.29) is $Q(z) = (z+1)Q_2(z)$, with Q_2 a Salem polynomial of even degree. Then, instead of the equation (2.4), this second class of Salem numbers calls for association theorems between Q_2 and P using

$$(z+1)Q_2(z) = zP(z) + P^*(z). \tag{2.30}$$

The association Theorem 2.9 between a Salem polynomial and an expansive polynomial of even degree we obtain in this case can be deduced from the preceeding ones, by suitable sign changes, as analogues of Theorem 2.6 and Theorem 2.8.

Theorem 2.9. *Let $2 < n_1 < n_2 < ... < n_s$ be a increasing sequence of s integers, $s \geq 1$. Let T be either the trivial constant polynomial equal to 1 ($m = 0$), or an irreducible Salem polynomial of degree $m \geq 4$, with dominant root $\beta > 1$. There exist a monic expansive polynomial $P(z) \in \mathbb{R}[z]$ of degree $m + \sum_{i=1}^{s} \varphi(n_i)$ such that*

$$(z+1)\,T(z)\Big(\prod_{i=1}^{s} \Phi_{n_i}(z)\Big) = zP(z) + P^*(z). \tag{2.31}$$

The claim is also valid for $m \geq 4$ and $s = 0$ (i.e. with no cylotomic factor).

Notation: given a Salem polynomial $T(z) = t_m z^m + t_{m-1} z^{m-1} + \ldots + t_1 z + t_0$, of degree m, with $t_{m/2+1} = t_{m/2-1}, \ldots, t_{m-1} = t_1, t_0 = t_m = 1$, we denote by

$$\omega_1 : {}^t(t_0, t_1, \ldots, t_{m-1}) \;\mapsto\; {}^t(p_0, p_1, \ldots, p_{m-1}) \tag{2.32}$$

the ("choice") mapping sending the coefficient vector of T to that of the monic expansive polynomial $P(z) = p_m z^m + p_{m-1} z^{m-1} + \ldots + p_1 z + p_0$ over it, $p_m = 1$, with $P(0) = p_0$ negative or positive, which is chosen such that the half coefficient vector lies on the lattice $\mathbb{Z}^{m/2}$ inside the admissible cone

in $\mathbb{R}^{m/2}$. In this definition we except the dominant coefficients, equal to 1. The converse mapping

$$\Omega_1 : {}^t(p_0, p_1, \ldots, p_{m-1}) \ \mapsto \ {}^t(t_0, t_1, \ldots, t_{m-1}), \tag{2.33}$$

defined by $(z-1)T(z) = zP(z) - P^*(z)$ if $P(0) = p_0 > 0$, or $(z+1)T(z) = zP(z) + P^*(z)$ if $P(0) = p_0 < 0$, allows to sending a monic polynomial P with real coefficients to a monic polynomial T. For any Salem polynomial T with coefficients t_i as above, the identity $\Omega_1(\omega_1({}^t(t_i))) = {}^t(t_i)$ holds.

2.4.5. *Algorithmic determination of expansive polynomials.* Let T be an irreducible Salem polynomial of degree $m \geq 4$. Denote by A the invertible matrix $A = (a_{i,j})_{0 \leq i,j \leq \frac{m}{2}-1}$ where the coefficients $a_{i,j}$ are given by the equations of the face hyperplanes of the admissible cone in $\mathbb{R}^{m/2}$, namely (2.25) and (2.26). Denote by Λ be the column vector of $\mathbb{R}^{\frac{m}{2}}$ defined by $\Lambda = {}^t(\lambda_0, \lambda_1, \ldots, \lambda_{\frac{m}{2}-1})$, where λ_j is the constant term of the equation of the j-th face hyperplane, given by (2.25) and (2.26). The polyhedral admissible cone is defined by the pair (A, Λ); the pair (A, Λ) is said *associated with* T.

For any vector V in $\mathbb{R}^{m/2}$ let us denote by round(V) the vector of $\mathbb{Z}^{m/2}$ the closest to V, componentwise (i.e. for each component x of V in the canonical basis of $\mathbb{R}^{m/2}$, the integer the closest to x is selected).

Proposition 2.3. *Let $m \geq 4$ be an even integer. Let $T(z) = \sum_{i=0}^{m} t_{m-i} z^i$ be an irreducible Salem polynomial of degree m, with associated pair $(A = (a_{i,j}), \Lambda = (\lambda_i))$. Let*

$$b := \frac{\sqrt{m}}{2\sqrt{2}} \max_{0 \leq i \leq \frac{m}{2}-1} \Big(\sum_{k=0}^{\frac{m}{2}-1} a_{i,k}^2 \Big)^{1/2}. \tag{2.34}$$

For $t \geq b$ let $\mathcal{M}_t = \text{round}(A^{-1}({}^t(-t, -t, ..., -t) - \Lambda)) := {}^t(M_0, \ldots, M_{\frac{m}{2}-1})$. Then

(i) the polynomial $P(z) = z^m + p_{m-1}z^{m-1} + ... + p_1 z + p_0$ defined by

$$\begin{cases} p_i = M_i, & 0 \leq i \leq \frac{m}{2} - 1, \\ p_{m-i} = M_{i-1} + t_i - t_{i-1}, & 1 \leq i \leq \frac{m}{2} - 1, \end{cases} \tag{2.35}$$

is an expansive polynomial which satisfies $(z-1)T(z) = zP(z) - P^*(z)$;

(ii) the Salem number β of minimal polynomial T belongs to the class A_{q_b} with

$$q_b = M_0 = pr_0 \mathcal{M}_b \tag{2.36}$$

possibly to other classes A_q with $q < q_b$.

Proof. (i) For $t \in \mathbb{R}^+$, let $\mathcal{M}_t = {}^t(M_0, M_1, ..., M_{\frac{m}{2}-1})$, with real coordinates in the canonical basis of $\mathbb{R}^{m/2}$, be the unique vector solution of the equation

$$A\mathcal{M}_t + \Lambda = {}^t(-t, -t, \ldots, -t). \tag{2.37}$$

If $t \neq 0$, $\mathcal{M}_t$ lies in the open admissible cone and, if $t = 0$, $\mathcal{M}_0$ is the summit vertex of the cone. Let d_2 be the usual Euclidean distance on $\mathbb{R}^{m/2}$ and d_∞ the distance defined by $d_\infty(V, W) = \max_{i=0,...,m/2-1} |V_i - W_i|$. Denote by $\overline{B_\infty}$ the closed balls (cuboids) for this distance.

For $0 \leq i \leq \frac{m}{2} - 1$ denote by H_i the i-th face hyperplane of the admissible cone, by $\mathcal{N}_i = {}^t(a_{i,0}, a_{i,1}, ..., a_{i,\frac{m}{2}-1})$ the vector orthogonal to the hyperplane H_i and by $\mathrm{pr}_{H_i}(\mathcal{M}_t)$ the orthogonal projection of $\mathcal{M}_t$ on H_i. For all $t \geq b$, the scalar product $\|\mathcal{N}_i\|^{-1}(\mathcal{N}_i \cdot \mathcal{M}_t)$ is equal to $d_2(\mathcal{M}_t, \mathrm{pr}_{H_i}(\mathcal{M}_t)) =$

$$\frac{|a_{i,0}M_0 + a_{i,1}M_1 + ...a_{i,\frac{m}{2}-1}M_{\frac{m}{2}-1} + \lambda_i|}{\sqrt{a_{i,0}^2 + a_{i,1}^2 + ... + a_{i,\frac{m}{2}-1}^2}} > \frac{|-t|}{b\frac{2\sqrt{2}}{\sqrt{m}}} \geq \frac{\sqrt{m}}{2\sqrt{2}}.$$

Therefore the cuboid $\overline{B}_\infty(\mathcal{M}_t, \frac{1}{2})$, which is a fundamental domain of the lattice $\mathbb{Z}^{\frac{m}{2}}$, does not contain the point $\mathrm{pr}_{H_i}(\mathcal{M}_t)$. This cuboid is included in the (open) admissible cone C^+. Consequently the vector $(p_i)_{0 \leq i \leq \frac{m}{2}-1} = \mathrm{round}(\mathcal{M}_t)$ lies in $\mathbb{Z}^{\frac{m}{2}} \cap C$; it is the half-coefficient vector of an expansive polynomial.

(ii) The above method allows to compute explicitely the expansive polynomial P the closest to the summit vertex $\mathcal{M}_0$ of the cone: by solving (2.37) with $t = b$ and using (2.35). Since the integer component M_0 of $\mathcal{M}_b$ is equal to the Mahler measure q of P and that P produces β, we deduce the claim. □

Similar explicit bounds q_b for q are easy to establish if T is a nonirreducible Salem polynomial (as in §2.4.3) or in the second case of Salem numbers in A_q (as in §2.4.4) using the equations of the face hyperplanes deduced from the Criteria of expansivity. We do not report them below.

We have seen that, using Theorem 2.1 and Theorem 2.3, we can construct sequences of Salem polynomials and Salem numbers, depending upon the choice of the expansive polynomial over the Salem polynomial. With the method of Theorem 2.3, we compute iteratively for instance the following sequence of Salem numbers from the Lehmer number: $\theta \approx 1.1762808$, $\theta_2 \approx 8.864936672$, $\theta_3 \approx 21.56809548$, $\theta_4 \approx 45.44171097$, $\theta_5 \approx 87.36603624$, $\theta_6 \approx 155.3214209$, $\theta_7 \approx 261.2942782$, $\theta_8 \approx 423.2784753$, $\theta_9 \approx 668.2676464$, $\theta_{10} \approx 1037.261121$.

The Schur-Cohn-Lehmer Theorem may be used to compute an expansive polynomial P over a Salem polynomial T with dominant root in A_q for a given $q \leq q_b$. It may occur that this component-by-component search of the coefficient vector of P fails.

Theorem 2.10 (Schur-Cohn-Lehmer)**.** *Let $P(z) = p_m z^m + p_{m-1}z^{m-1} + .. + p_1 z + p_0$, $p_m p_0 \neq 0$, be a polynomial of degre m with real coefficients. Let $\mathcal{T}$ be the transformation $\mathcal{T} : P(z) \mapsto \mathcal{T}(P)(z) = p_0 P(z) - p_m P^*(z)$.*

Let $\mathcal{T}^1 := \mathcal{T}$ and $\mathcal{T}^k := \mathcal{T}(\mathcal{T}^{k-1})$ for $k \geq 2$. Then $P(z)$ is an expansive polynomial if and only if

$$\mathcal{T}^k(P)(0) > 0 \qquad \text{for all } 1 \leq k \leq m.$$

Proof. Lehmer [L], Marden [Mn], pp 148-151. □

Remark that the first step of the algorithm leads to $\mathcal{T}(P)(0) = p_0^2 - p_m^2 = p_0^2 - 1 > 0$, for a monic expansive polynomial P, so that the inequality $q = |p_0| = \mathrm{M}(P) \geq 2$ holds (cf Remark 2.2).

The coefficient vector (p_i) of P is constructed recursively: for $k > 0$, knowing $(p_i)_{i \leq k-1}$ and $(p_i)_{i \geq 2n-k+1}$, we choose p_k in the interval given by the quadratic equation in p_k : $\mathcal{T}^{k+1}(P)(0) > 0$ and we compute $p_{m-k} = -t_{k-1} + t_k + p_{k-1}$. For some initial values $p_0 = q$ not large enough, we obtain no solution.

The first 20 smallest Salem numbers, with their minimal polynomials and associated expansive polynomials are reported in Figure 3.

Remark. Let $T(z)$ be a Salem polynomial, of degree m. Denote by k the maximal real subfield of the splitting field of T. The admissible cone C^+ is defined by the associated pair $(A = (a_{i,j}), \Lambda = (\lambda_i)) \in GL(\frac{m}{2}, k) \times k^{m/2}$, i.e. the affine equations of its facets, in $\mathbb{R}^{m/2}$, which are functions of the roots of T. Every point, say $\not{P}$, of $C^+ \cap \mathbb{Z}^{m/2}$ has coordinates ${}^t(p_0, p_1, \ldots, p_{m/2-1})$ which constitute the half coefficient vector of a unique monic expansive polynomial, say P, by (2.24): there is a bijection between $\not{P}$ and P. Denote by $\widehat{\not{P}}$ the second half of the coefficient vector of P. The set of monic expansive polynomials P, defined by their coefficient vectors, is an infinite subset of $\mathbb{Z}^m$, in bijection with $C^+ \cap \mathbb{Z}^{m/2}$. The map ω_1 decomposes into two parts:

$$\omega_{1,C^+} : \mathbb{Z}^m \mapsto C^+ \cap \mathbb{Z}^{m/2}, {}^t(t_i)_{i=0,\ldots,m-1} \mapsto {}^t(p_i)_{i=0,\ldots,\frac{m}{2}-1}$$

and, once the image of ω_{1,C^+} is fixed, using (2.24),

$$\widehat{\omega_{1,C^+}} : \mathbb{Z}^{m/2} \mapsto \mathbb{Z}^{m/2}, {}^t(p_i)_{i=0,\ldots,\frac{m}{2}-1} \mapsto (p_{\frac{m}{2}+i})_{i=0,\ldots,\frac{m}{2}-1}.$$

By abuse of notation, we denote:

$$\omega_{1,C^+}(T) = \not{P},\ \widehat{\omega_{1,C^+}}(\not{P}) = \widehat{\not{P}}, \text{ and } \omega_1(T) = (\not{P}, \widehat{\not{P}}) = P.$$

2.4.6. *Condition on an expansive polynomial P to obtain a Salem number.* Given a monic expansive polynomial $P(X) \in \mathbb{Z}[X]$ there are two cases for the geometry of the roots of Q by the Q-construction (cf §2.1): (i) either all the zeroes of Q are on $|z| = 1$: Q is then a product of cyclotomic polynomials, or (ii) Q are all but two zeroes on $|z| = 1$, and we obtain a Salem number β.

N°	D°		Coefficients of polynomials (* have been computed by Marie-José Bertin)	
1	10	T	1 1 0 -1 -1 -1 -1 -1 0 1 1	
		P	1 2 2 0 -2 -3 -3 -2 1 3 2	*
			1 2 2 2 3 4 4 3 3 3 2	*
			1 2 1 -1 -1 -1 -1 -1 0 2 2	*
			1 2 1 0 0 0 0 0 1 2 2	*
2	18	T	1 -1 1 -1 0 0 -1 1 -1 1 -1 1 -1 0 0 -1 1 -1 1	
		P	1 0 2 0 1 0 -1 0 -2 0 -2 0 -2 0 0 0 2 0 2	*
			1 0 1 0 0 0 -1 1 -1 1 -1 1 -1 0 0 -1 2 -1 2	
3	14	T	1 0 0 -1 -1 0 0 1 0 0 -1 -1 0 0 1	
		P	1 1 1 0 0 1 1 2 1 1 0 0 1 1 2	
4	14	T	1 0 -1 0 0 0 0 -1 0 0 0 0 -1 0 1	
		P	1 1 0 0 0 0 0 -1 0 0 0 0 -1 1 2	
5	10	T	1 0 0 0 -1 -1 -1 0 0 0 1	
		P	1 1 1 1 -1 -1 -1 0 1 1 2	*
			1 1 1 2 2 2 2 3 2 1 2	*
			1 1 1 1 -1 -1 -1 0 1 1 2	*
			1 1 2 2 1 1 1 2 2 2 2	*
			1 1 0 1 -1 -1 -1 0 1 0 2	*
6	18	T	1 -1 0 0 0 0 0 0 -1 1 -1 0 0 0 0 0 0 -1 1	
		P	1 0 1 0 0 0 0 0 -1 0 -2 0 0 0 0 0 0 0 2	*
			1 0 0 1 0 0 0 0 0 -1 1 -1 0 0 0 0 0 1 -1 2	*
			1 0 0 0 0 0 0 0 0 -1 1 -1 0 0 0 0 0 0 -1 2	
7	10	T	1 0 0 -1 0 -1 0 -1 0 0 1	
		P	1 1 1 0 1 0 1 0 1 1 2	
			1 1 1 0 1 1 2 0 1 1 2	*
			1 1 1 0 0 -2 -1 -1 1 1 2	*
			1 1 0 -1 0 -1 0 -1 0 0 2	*
8	20	T	1 -1 0 0 0 -1 1 0 0 -1 1 -1 0 0 1 -1 0 0 0 -1 1	
		P	1 0 0 0 0 -1 1 0 0 -1 1 -1 0 0 1 -1 0 0 0 -1 2	
9	22	T	1 0 -1 -1 0 0 0 1 1 0 -1 -1 -1 0 1 1 0 0 0 -1 -1 0 1	
		P	1 1 0 -1 0 0 0 1 1 0 -1 -1 -1 0 1 1 0 0 0 -1 -1 1 2	
10	16	T	1 -1 0 0 0 0 0 0 -1 0 0 0 0 0 0 0 -1 1	
		P	1 0 1 0 0 0 0 0 0 -1 0 0 0 0 0 0 0 2	*
11	26	T	1 0 -1 0 0 -1 0 0 -1 0 1 0 0 1 0 0 1 0 -1 0 0 -1 0 0 -1 0 1	
		P	1 1 0 0 0 -1 0 0 -1 0 1 0 0 1 0 0 1 0 -1 0 0 -1 0 0 -1 1 2	
12	12	T	1 -1 1 -1 0 0 -1 0 0 -1 1 -1 1	
		P	1 0 1 1 0 2 0 1 2 -1 3 -1 2	*
			1 0 1 -1 0 0 -1 0 0 -1 1 -1 2	*
			1 0 2 0 1 0 -1 0 0 0 2 0 2	
13	18	T	1 0 0 0 0 0 -1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 1	
		P	1 1 1 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 2	
14	20	T	1 0 -1 0 0 -1 0 0 0 0 0 0 0 0 0 -1 0 0 -1 0 1	
		P	1 1 0 0 0 -1 0 1 1 0 0 0 0 1 1 -1 0 0 -1 1 2	
15	14	T	1 0 -1 -1 0 1 0 -1 0 1 0 -1 -1 0 1	
		P	1 1 0 -1 0 2 1 -1 0 2 1 -1 -1 1 2	
16	18	T	1 -1 0 0 -1 1 0 0 0 -1 0 0 0 1 -1 0 0 -1 1	
		P	1 0 1 0 -1 0 -1 0 0 0 1 0 0 0 -2 0 0 0 2	*
		P	1 0 0 0 -1 1 0 0 0 -1 0 0 0 1 -1 0 0 -1 2	
17	24	T	1 -1 0 0 -1 1 0 -1 1 -1 0 1 -1 1 0 -1 1 -1 0 1 -1 0 0 -1 1	
		P	1 0 0 0 -1 1 0 -1 1 -1 0 1 -1 1 0 -1 1 -1 0 1 -1 0 0 -1 2	
18	22	T	1 -1 0 -1 1 0 0 0 -1 1 -1 1 -1 1 -1 0 0 0 1 -1 0 -1 1	
		P	1 0 0 -1 1 0 0 0 -1 1 -1 1 -1 1 -1 0 0 0 1 -1 0 -1 2	
19	10	T	1 0 -1 0 0 -1 0 0 -1 0 1	
		P	1 1 0 -1 -1 0 1 -1 -2 1 2	*
			1 1 0 1 2 0 1 2 0 1 2	*
			1 1 0 1 1 0 1 1 0 1 2	*
20	26	T	1 -1 0 0 0 0 -1 0 0 0 0 0 0 1 0 0 0 0 0 0 -1 0 0 0 0 -1 1	
		P	1 1 2 1 1 1 0 1 0 0 0 0 0 0 -1 0 0 0 0 0 0 1 1 1 1 1 3	

FIGURE 3. Minimal polynomials T of the 20 smallest Salem numbers β and some of the associated monic expansive polynomials P over T. The degree D is $\deg(T) = \deg(P)$. The coefficients of P are such that the constant term, on the right, is the integer $q \geq 2$ for which $\beta \in A_q$.

To ensure that Q has a root outside the closed unit disk Boyd ([Bo] p.318, Corollary 3.2) proposes a criterium counting the number of entrances of Q. Below we give a criterium on P to discriminate the two cases.

Proposition 2.4. *Let P be a monic expansive polynomial, with integer coefficients, of even degree $m \geq 4$. Let T be a polynomial having simple roots such that $(z-1)T(z) = zP(z) - P^*(z)$ holds. If the following condition*

$$P(-1)((1-m)P(1) + 2P'(1)) < 0 \tag{2.38}$$

holds then T is a Salem polynomial which does not cancel at ± 1. If, instead, T satisfies the condition $(z+1)T(z) = zP(z) + P^*(z)$, then T is a Salem polynomial which does not cancel at ± 1 if

$$P(1)((1-m)P(-1) - 2P'(-1)) < 0. \tag{2.39}$$

Proof. First, T is necessarily monic, reciprocal, with integer coefficients. Now, deriving $(z-1)T(z) = zP(z) - P^*(z)$, we obtain: $(z-1)T'(z) + T(z) = P(z) + zP'(z) - mz^{m-1}P(\frac{1}{z}) + z^{m-2}P'(\frac{1}{z})$. Hence $T(1) = (1-m)P(1) + 2P'(1)$. On one hand, $T(-1) = P(-1) \neq 0$ since P is expansive. Let us show that $T(1) \neq 0$. Let us assume the contrary. Since T has even degree and that any cyclotomic polynomial Φ_n has even degree as soon as $n > 2$, an even power of $(z-1)$ should divide $T(z)$. Since T has simple roots we obtain a contradiction. By the theorem of intermediate values, T as a unique real root in $(-1, 1)$ if and only if the condition $T(1)T(-1) < 0$, equivalently (2.38), holds.

The second case, with (2.39), can be proved using similar arguments. □

2.4.7. *Semi-group structure.* Let $q \geq 2$ be an integer. Let $m \geq 4$ be an even integer. Let T be a Salem polynomial of degree m, cancelling at a Salem number β, having simple roots, such that $T(\pm 1) \neq 0$. Denote by $E^+_{T,q} \subset E_q$ the set of monic expansive polynomials over T of degree m, defined by their coefficient vectors, as a point subset of the lattice $\mathbb{Z}^m$, satisfying (2.12), with $P(0) = +q$, resp. $E^-_{T,q} \subset E_q$ the set of monic expansive polynomials over T of degree m satisfying (2.30), with $P(0) = -q$. Denote by $\mathcal{P}^+_T := \bigcup_{q \geq 2} E^+_{T,q}$, resp. $\mathcal{P}^-_T := \bigcup_{q \geq 2} E^-_{T,q}$ and

$$\mathcal{P}_T := \mathcal{P}^+_T \cup \mathcal{P}^-_T = \bigcup_{q \geq 2} \Big(E^+_{T,q} \cup E^-_{T,q} \Big).$$

Theorem 2.11. *Let $P \in E^+_{T,q}$ and $P' \in E^+_{T,q'}$ be two monic expansive polynomials over T of degree m. Then the sum $P + P' - T$ is a monic expansive polynomial of degree m over T, in $E^+_{T,q+q'-1}$. The internal law $(P, P') \mapsto P \oplus P' := P + P' - T$ defines a commutative semi-group structure on $\mathcal{P}^+_T \cup \{T\}$, where T is the neutral element.*

Proof. Let us give two proofs. (1) Using McKee Smyth interlacing quotients: by Theorem 2.1, the polynomials $L := P - T$ and $L' := P' - T$ are monic, reciprocal and satisfy the conditions:

- $L(0) = q - 1$ and $L'(0) = q' - 1$,
- $\deg(L) = \deg(L') = \deg(T) - 1$,
- $L(1) \leq -T(1)$ and $L'(1) \leq -T(1)$,
- the zeroes of L interlace the zeroes of T on the half unit circle and similarly for the zeroes of L'.

After Definition 2.3 in §2.3, with $K = K' = 1$, $\dfrac{(z-1)L}{T}$ and $\dfrac{(z-1)L'}{T}$ are CS-interlacing quotients. As in the proof of McKee and Smyth's Proposition 6 in [MKS0], we can transform these quotients into real quotients by the Tchebyshev transformation $x = z + \frac{1}{z}$ and we obtain in the real world the rational function $f(x) = \gamma \dfrac{\prod(x - \delta_i)}{\prod(x - \alpha_i)}$ and respectively $f'(x) = \gamma' \dfrac{\prod(x - \delta'_i)}{\prod(x - \alpha_i)}$. These rational functions can be written $f(x) = \sum \dfrac{\lambda_i}{(x - \alpha_i)}$ and $f'(x) = \sum \dfrac{\lambda'_i}{(x - \alpha_i)}$ with $\lambda_i > 0$ and $\lambda'_i > 0$. Then $(f + f')(x) = \sum \dfrac{\lambda_i + \lambda'_i}{(x - \alpha_i)}$ has strictly positive coefficients. Its derivative is everywhere negative. Then there is exactly one zero between two successive poles. Then, back to the complex world, $\dfrac{(z-1)(L + L')}{T}$ is a CS-interlacing quotient. Recall that $T(1) < 0$ since $T(1/\beta) = T(\beta) = 0$. Finally, we have

- $(L + L')(0) = (q + q' - 1) - 1$,
- $\deg(L + L') = \deg(T) - 1$ because L and L' are both monic,
- $(L + L')(1) \leq -2T(1) \leq -T(1)$ because $T(1) < 0$,
- the zeroes of $L + L'$ and the zeroes of T interlace on the half unit circle.

Thus $L + L' + T \in E^+_{T,q+q'-1}$.

(2) Using the equations of the face hyperplanes of the admissible cone C^+ associated with T: using the notations of §2.4.5 and Remark 2.4.5, all the components of the vectors $A\mathcal{P} + \Lambda$ and $A\mathcal{P}' + \Lambda$ are strictly negative and

$$A\mathcal{T} + \Lambda = {}^t((T(1))^2, 0, \ldots, 0).$$

Then all the components of $A\big(\mathcal{P} + \mathcal{P}' - \mathcal{T}\big) + \Lambda$ are strictly negative. Hence $\mathcal{P} + \mathcal{P}' - \mathcal{T}$ belongs to C^+. Thus $L + L' + T \in E^+_{T,q+q'-1}$. □

The semi-group law $\oplus$ satisfies:

- $T \oplus T = T$,
- $P \oplus P' = P + L' = P' + L = L + L' + T$,

- $(P \oplus P') - T = (P - T) + (P' - T)$.

Note that the opposite (additive inverse) of $P \in E^+_{T,q}$ is $2T - P$, which is outside the set $\mathcal{P}^+_T$ since $2T(0) - P(0) = 2 - q \leq 0$. Thus $(\mathcal{P}^+_T \cup \{T\}, \oplus)$ is a semi-group and not a group.

The structure of the set of positive real generalized Garsia numbers, in particular for those of Mahler measure equal to 2 (i.e. Garsia numbers in the usual sense), is of interest for many purposes [HP]. In the present context of association between Salem numbers and generalized Garsia numbers, the above semi-group structure can be transported on sets of generalized Garsia numbers and generalized Garsia polynomials as follows.

Corollary 2.1. *If P and P' are two generalized Garsia polynomials of the same degree $m \geq 4$ over a given Salem polynomial T, such that $P(0) = +\mathrm{M}(P), P'(0) = +\mathrm{M}(P')$, satisfying the assumption that $P(0) + P'(0) - 1$ is a prime number, then $P + P' - T$ is a generalized Garsia polynomial of degree m and Mahler measure equal to $\mathrm{M}(P) + \mathrm{M}(P') - 1$.*

3. From expansive polynomials to Hurwitz polynomials, Hurwitz alternants, and continued fractions

3.1. Hurwitz polynomials.

Definition. A real Hurwitz polynomial $H(X)$ is a polynomial in $\mathbb{R}[X]$ of degree ≥ 1 whose roots have all a negative real part.

The constructions involved in Theorem 2.1 require monic expansive polynomials with coefficients in $\mathbb{Z}$. Since the conformal representation $z \mapsto Z = \dfrac{z+1}{z-1}$ transforms the open unit disk $|z| < 1$ into to the half plane $Re(Z) < 0$, it transforms the set of zeros of the reciprocal polynomial P^* of an expansive polynomial P into the set of zeros of a Hurwitz polynomial. In other terms the polynomial $P(z) = p_0 + p_1 z + \ldots + p_m z^m$ is expansive if and only if the polynomial $H(z) = (z-1)^m P^*(\dfrac{z+1}{z-1})$ is a real Hurwitz polynomial. In this equivalence, note that $z = 0$ is never a root of P^*, implying that H never cancels at $z = -1$. The analytic transformation $z \to Z$, resp. $Z \to z = (1+Z)/(Z-1)$, corresponds to a linear transformation of the coefficient vector of P, resp. of H, of $\mathbb{R}^{m+1}$, as follows, the proof being easy and left to the reader.

Proposition 3.1. (i) *If $P(z) = p_0 + p_1 z + ... + p_m z^m$ is an expansive polynomial then $H(z) = (z+1)^m P(\dfrac{z-1}{z+1})$ is a real Hurwitz polynomial and, denoting $H(z) = h_0 + h_1 z + ... + h_m z^m$, then $h_0 \neq 0$*

and

$$h_i = \sum_{j=0}^{m} \sum_{k=0}^{i} (-1)^{j-k} \binom{j}{k} \binom{m-j}{i-k} p_j, \qquad 0 \leq i \leq m. \tag{3.1}$$

(ii) *If* $H(z) = h_0 + h_1 z + ... + h_m z^m$ *is a real Hurwitz polynomial such that* $H(-1) \neq 0$, *then* $P(z) = (1-z)^m H(\frac{z+1}{1-z})$ *is an expansive polynomial and, if we denote* $P(z) = p_0 + p_1 z + ... + p_m z^m$, *then*

$$p_i = \sum_{j=0}^{m} \sum_{k=0}^{i} (-1)^{i-k} \binom{j}{k} \binom{m-j}{i-k} h_j, \qquad 0 \leq i \leq m. \tag{3.2}$$

Notation: using the formulae (3.1), since $h_0 = \sum_{i=0}^{m} (-1)^i p_i = P(-1) \neq 0$, we denote by

$$\omega_2 : {}^t(p_0, p_1, \ldots, p_{m-1}) \mapsto {}^t\Big(\frac{h_1}{P(-1)}, \frac{h_2}{P(-1)}, \ldots, \frac{h_m}{P(-1)}\Big) \tag{3.3}$$

the mapping which sends the coefficient vector of the monic polynomial $P(z) = p_0 + p_1 z + ... + z^m$ to that of $H(z) = (P(-1))^{-1}\big(h_0 + h_1 z + ... + h_m z^m\big)$ where $P(-1) = \sum_{i=0}^{m-1} (-1)^i p_i + 1$ here; let us remark that the constant term $h_0/P(-1)$ of this polynomial is equal to 1.

Conversely, using (3.2), since $p_m = \sum_{i=0}^{m} (-1)^i h_i = H(-1) \neq 0$, we denote by

$$\Omega_2 : {}^t(h_1, h_2, \ldots, h_m) \mapsto {}^t\Big(\frac{p_0}{H(-1)}, \frac{p_1}{H(-1)}, \ldots, \frac{p_{m-1}}{H(-1)}\Big) \tag{3.4}$$

the mapping which sends the coefficient vector of the Hurwitz polynomial $H = 1 + h_1 z + ... + h_m z^m$, having constant coefficient equal to 1, to that of the polynomial $P(z) = (H(-1))^{-1}\big(p_0 + p_1 z + \ldots + p_m z^m\big)$ where $H(-1) = 1 + \sum_{i=1}^{m} (-1)^i h_i$ here; let us remark that the dominant coefficient $p_m/H(-1)$ is equal to 1. Reversing the order of the terms, we denote by

$$\Omega_2^* : {}^t(h_1, h_2, \ldots, h_m) \mapsto {}^t\Big(\frac{p_{m-1}}{H(-1)}, \frac{p_{m-2}}{H(-1)}, \ldots, \frac{p_0}{H(-1)}\Big) \tag{3.5}$$

the mapping which sends the coefficient vector of the Hurwitz polynomial $H = 1 + h_1 z + ... + h_m z^m$ to that of the polynomial $P^*(z) = 1 + (H(-1))^{-1}\big(p_{m-1} z + \ldots + p_0 z^m\big)$ with $H(-1) = 1 + \sum_{i=1}^{m} (-1)^i h_i$.

Hence, the transformation $\Omega_2 \, o \, \omega_2$ identifies with the identity transformation on the set of monic expansive polynomials P.

Remark. The transformations (3.1) and (3.2) are $\mathbb{Z}$-linear. Hence, for any real field $\mathbb{K}$ and any expansive polynomial $P(z) \in \mathbb{K}[z]$, the projective classes $\mathbb{K}\big({}^t(p_i)\big)$ and $\mathbb{K}\big({}^t(h_i)\big)$ are in bijection. This observation justifies the definitions of ω_2 and Ω_2. Indeed, these mappings suitably send a representant of $\mathbb{K}\big({}^t(p_i)\big)$ to a representant of $\mathbb{K}\big({}^t(h_i)\big)$, what allows to fix one

coefficient in the coefficient vector: in the first case the dominant coefficient p_m equal to 1, in the second case the constant term h_0 equal to 1.

Theorem 3.1 (Stodola, 1893)**.** *If a polynomial $H(z) = h_0 + h_1 z + h_2 z^2 + .. + h_m z^m \in \mathbb{R}[z]$ is a Hurwitz polynomial then $(h_i)_{0 \leq i \leq m}$ are all together positive or all together negative.*

Proof. Let $r_1, r_2, .., r_p$ be the real zeros of a real polynomial $H(z)$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n, \overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_n}$ be its nonreal zeros. Then we have the factorization :

$$H(z) = h_m \prod_{1 \leq k \leq p} (z - r_k) \prod_{1 \leq j \leq n} (z - \alpha_j)(z - \overline{\alpha_j})$$

$$H(z) = h_m \prod_{1 \leq k \leq p} (z - r_k) \prod_{1 \leq j \leq n} (z^2 - 2Re(\alpha_j) z + |\alpha_j|^2)$$

Now, $H(z)$ is a Hurwitz polynomial if $r_k < 0$, for all $1 \leq k \leq r$, and $Re(\alpha_j) < 0$, for all $1 \leq j \leq n$. Then all the coefficients of all the factors are positive. Thus, if $h_m > 0$, all the coefficients of $H(z)$ are positive and if $h_m < 0$, then they are all negative. □

Remark. From now on, we consider that the coefficients h_i of the real Hurwitz polynomials are all (strictly) positive (otherwise, we will consider $-H(z)$).

3.2. Hurwitz alternants and Stieltjes continued fractions.

Definition. The quotient (alternant) of Hurwitz $h(z)$ associated with a polynomial $H(z) = h_m x^m + h_{m-1} x^{m-1} + \ldots + h_1 x + h_0 \in \mathbb{R}_{>0}[z]$ is the rational fraction

$$h(z) := \frac{h_1 + h_3 z + h_5 z^2 + ..}{h_0 + h_2 z + h_4 z^2 + ..} = \frac{\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} h_{2i+1} z^i}{\sum_{i=0}^{\lfloor m/2 \rfloor} h_{2i} z^i}. \tag{3.6}$$

The continued fraction of the quotient of Hurwitz $h(z)$ of (3.6) is

$$h(z) = \frac{h_1 + h_3 z + h_5 z^2 + ..}{h_0 + h_2 z + h_4 z^2 + ..} = \frac{\frac{h_1}{h_0}(1 + \frac{h_3}{h_1} z + \frac{h_5}{h_1} z^2 + ...)}{1 + \frac{h_2}{h_0} z + \frac{h_4}{h_0} z^2 + ...}$$

$$= \frac{h_1/h_0}{1 + z \left[\frac{(h_1 h_2 - h_0 h_3) + (h_1 h_4 - h_0 h_5) z + (h_1 h_6 - h_0 h_7) z^2 ...}{h_0 h_1 + h_0 h_3 z + h_0 h_5 z^2 + ...} \right]}$$

$$= \cfrac{\frac{h_1}{h_0}}{1+\cfrac{\frac{h_1h_2-h_0h_3}{h_0h_1}\,z}{1+z\,[...]}} = \cfrac{f_1}{1+\cfrac{f_2z}{1+\cfrac{f_3z}{1+\cfrac{...}{1+...}}}} \quad \text{written} =: \big[f_1/f_2/f_3/\ldots\big](z)$$

for short, with $f_1 = h_1/h_0$, $f_2 = (h_1h_2 - h_0h_3)/(h_0h_1)$, ... Stieltjes [St] (1894) has extensively studied these continued fractions, their Padé approximants and the expressions of the numerators f_j as Hankel determinants (Henrici [H], Chap. 12, p 549 and pp 555–559, where they are called SITZ continued fractions). For H as in Definition 3.2, denote $D_0 := 1, D_{-1} := h_0^{-1}, D_{-2} := h_0^{-2}$,

$$D_1 = h_1,\ D_2 = \begin{vmatrix} h_1 & h_3 \\ h_0 & h_2 \end{vmatrix},\ D_3 = \begin{vmatrix} h_1 & h_3 & h_5 \\ h_0 & h_2 & h_4 \\ 0 & h_1 & h_3 \end{vmatrix}$$

and generally

$$D_j := \det\big(h_j^{(1)}, h_j^{(3)}, \ldots, h_j^{(2j-1)}\big),$$

where $h_j^{(r)}$ denotes the j-dimensional column vector whose components are the first j elements of the sequence $h_r, h_{r-1}, h_{r-2}, \ldots$ ($h_r := 0$ for $r > m$ and for $r < 0$). Then

$$f_j = \frac{D_j}{D_{j-1}}\frac{D_{j-3}}{D_{j-2}}, \qquad j = 1, 2, \ldots, m. \tag{3.7}$$

Definition. A rational fraction $F(z) \in \mathbb{R}(z)$ is a m-terminated continued fraction if it exist $f_1, f_2, \ldots, f_m \in \mathbb{R}, f_m \neq 0$, such that $F(z) = \big[f_1/f_2/\ldots/f_m\big](z)$.

Theorem 3.2 (Hurwitz, 1895)**.** *The polynomial $H(z) \in \mathbb{R}[z]$, of degree $m \geq 1$, has all his roots on the left half plane $Re(z) < 0$ if and only if the Hurwitz alternant $h(z)$ of $H(z)$ can be represented by an m-terminated continued fraction $\big[f_1/f_2/\ldots/f_m\big](z)$ in which every number f_k is (strictly) positive.*

Proof. Henrici [H], Theorem 12.7c. □

If $h(z)$ is a m-terminated rational fraction, given by (3.6), then it is irreducible, i.e. the numerator $\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} h_{2i+1}z^i$ and the denominator $\sum_{i=0}^{\lfloor m/2 \rfloor} h_{2i}z^i$ of $h(z)$ have no common zeros ([H], Theorem 12.4a). Two real Hurwitz polynomials H_1 and H_2 such that $H_1/H_2 \in \mathbb{R}^*$ have the same Hurwitz quotient by (3.6). The real Hurwitz polynomials H obtained by ω_2 (Proposition 3.1) from expansive polynomial P in $\mathbb{Z}[z]$ have coefficients in $\mathbb{Z}$ and even degrees. Therefore their Hurwitz alternants are represented

by m-terminated continued fractions $\big[f_1/f_2/\ldots/f_m\big](z)$ with $f_j \in \mathbb{Q}_{>0}$ for $1 \leq j \leq m$, such that $h(1) \neq 1$ since $H(-1) \neq 0$. Recall that $h(x) > 0$ for $x \geq 0$ in general, with $h(\{\mathrm{Im}(z) > 0\}) \subset \{\mathrm{Im}(z) < 0\}, h(\{\mathrm{Im}(z) < 0\}) \subset \{\mathrm{Im}(z) > 0\}$ ([H], p 553).

In the case where all the f_js are positive rational numbers, following Stieltjes [St], the m-terminated continued fraction $\big[f_1/f_2/\ldots/f_m\big](z)$ can also be written with integers, say $e_0, e_1, e_2, \ldots, e_m$, as follows

$$\cfrac{e_1}{e_0 + \cfrac{e_2 z}{e_1 + \cfrac{e_3 z}{e_{m-2} + \cfrac{\ldots}{e_{m-2} + e_m z}}}} = \cfrac{\cfrac{e_1}{e_0}}{1 + \cfrac{\cfrac{e_2}{e_0 e_1} z}{1 + \cfrac{\ldots}{1 + \cfrac{e_m}{e_{m-2}} z}}}$$

with $f_1 = \dfrac{e_1}{e_0}$, $f_m = \dfrac{e_m}{e_{m-2}}$ and $f_i = \dfrac{e_i}{e_{i-2} e_{i-1}}$ for $2 \leq i \leq m-1$. A Hurwitz polynomial $H(z)$ associated with $h(z)$ is given by the $m+1$ positive integers $e_0, e_1, e_2, ..., e_m$ with $\gcd(e_0, e_1, \ldots, e_m)$=1.

Example. The Salem number $\beta = 1.582347\ldots \in A_2$ of degree six, whose minimal polynomial is $T(x) = z^6 - z^4 - 2z^3 - z^2 + 1$, admits the polynomial $P(z) = z^6 + z^5 + z^4 + z^2 + 2z + 2$ as expansive polynomial lying over T by $(z-1)T(z) = zP(z) - P^*(z)$. By the mapping $z \mapsto Z$ and ω_2 (Proposition 3.1) we obtain the Hurwitz polynomial $H(Z) = 4Z^6 + 5Z^5 + 29Z^4 + 10Z^3 + 14Z^2 + Z + 1$. $H(z)$ is represented by the continued fraction of the Hurwitz alternant: $[f_1/f_2/.../f_6](z) = [1/4/4/5/\dfrac{4}{5}/\dfrac{1}{5}](z)$ with $e_0 = e_1 = 1, e_2 = 4, e_3 = 16, e_4 = 320, e_5 = 4096, e_6 = 64$.

However, though simple and suggested in [H] as one of the basic transformations of Stieltjes ([St], pp J1-J5), the coding of the Hurwitz alternants in integers is not realistic from a numerical viewpoint since the trouble is that we obtain quickly very large integers e_i for many Salem numbers. For instance, for the Lehmer number $\theta = 1.176\ldots$ of minimal polynomial $T(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^3 + z + 1$, of degree 10, the two expansive polynomials $P_1(z) = z^{10} + 2z^9 + z^8 + z^2 + 2z + 2$ and $P_2(z) = z^{10} + z^9 + z^8 - z^7 - z^6 - z^5 - z^4 - z^3 + 2z + 2$ lie over T, producing θ. The Hurwitz alternant h_1 of $H_1 = \omega_2(P_1)$, resp. h_2 of $H_2 = \omega_2(P_2)$, is

$$h_1(z) = \frac{10z^4 + 120z^3 + 252z^2 + 120z + 10}{9z^5 + 269z^4 + 770z^3 + 434z^2 + 53z + 1},$$

$$\text{resp.} \qquad h_2(z) = \frac{4z^4 + 112z^3 + 280z^2 + 112z + 4}{3z^5 + 261z^4 + 798z^3 + 426z^2 + 47z + 1}.$$

For h_1, $e_0 = 1, e_1 = 10, e_2 = 410, e_3 = 8320, e_4 \approx 2,272\,10^7, e_5 \approx 1,772\,10^{11}$, $e_6 \approx 1,944\,10^{18}, e_7 \approx 2,594\,10^{29}, e_8 \approx 1,414\,10^{18}, e_9 \approx 1,151\,10^{47}, e_{10} \approx 1,528\,10^{64}$. For h_2, $e_0 = 1, e_1 = 4, e_2 = 76, e_3 = 2816, e_4 = 3329024, e_5 = 6335496192$, $e_6 \approx 3,189\,10^{16}, e_7 \approx 8,641\,10^{15}, e_8 \approx 10^{26}, e_9 \approx 1,755\,10^{41}$, $e_{10} \approx 3,758\,10^{55}$.

Notation: putting $H(z) = 1 + h_1 z + h_2 z^2 + \ldots + h_m z^m$, a real Hurwitz polynomial, we denote by

$$\omega_3 : {}^t(h_1, h_2, \ldots, h_m) \mapsto {}^t(f_1, f_2, \ldots, f_m) \tag{3.8}$$

the mapping which sends the coefficient vector of H to the m-tuple (f_j) given by (3.7) where each f_j, strictly positive, belongs to the real field $\mathbb{Q}(h_1, \ldots, h_m)$ generated by the coefficients of H. We consider the m-tuple $(f_1, \ldots, f_m)$ as the set of coordinates of a point of the mth-Euclidean space $\mathbb{R}^m$, in the canonical basis, generically denoted by F. Conversely (since Salem numbers have even degrees, together with the Salem, expansive and Hurwitz polynomials we consider, we only report the case where m is even), for $m \geq 2$ even, we denote by

$$\Omega_3 : \left(\mathbb{R}^+\right)^m \mapsto \mathbb{R}^m,\ {}^t(f_1, f_2, \ldots, f_m) \mapsto {}^t(h_1, h_2, \ldots, h_m) \tag{3.9}$$

the mapping which sends the m-tuple (f_j) of positive real numbers to the coefficient vector of $H(z) = 1 + h_1 z + h_2 z^2 + \ldots + h_m z^m$ by the relations

$$\text{for } 0 < i \leq \tfrac{m}{2} : \qquad h_{2i} = \sum_{\{(l_1, l_2, \ldots, l_i)\}} \left(\prod_{j=1}^{i} f_{l_j}\right)$$

where the sum is taken over all i-tuples $(l_1, \ldots, l_i)$ such that $l_s \in \{2, \ldots, m\}$ for $s = 1, \ldots, i$, and $l_r - l_s \geq 2$ for all $r > s$,

$$\text{for } 0 \leq i \leq \tfrac{m}{2} - 1 : \qquad h_{2i+1} = f_1 \sum_{\{(l_1, l_2, \ldots, l_i)\}} \left(\prod_{j=1}^{i} f_{l_j}\right)$$

where the sum is taken over i-tuples $(l_1, \ldots, l_i)$ such that $l_s \in \{3, \ldots, m\}$ for $s = 1, \ldots, i$, and $l_r - l_s \geq 2$ for all $r > s$.

The map $\omega_3 \circ \Omega_3$ is the identity transformation on $\left(\mathbb{R}^+\right)^m$ and $\Omega_3 \circ \omega_3$ identifies with the identity mapping on the set of real Hurwitz polynomials of degree m having a constant term equal to 1.

3.3. Salem numbers from m-terminated continued fractions. Let β be a Salem number. Let $m \geq \deg \beta$ be an even integer. Let $T(X)$ be a Salem polynomial cancelling at β, of degree m, having simple roots such

that $T(\pm 1) \neq 0$. In the sequel, by abuse of notation, we denote

$$\begin{array}{llll} P = \omega_1(T) & \text{instead of } (2.32); & T = \Omega_1(P) & \text{instead of } (2.33); \\ H = \omega_2(P) & \text{instead of } (3.3); & P = \Omega_2(H) & \text{instead of } (3.4); \\ F = \omega_3(H) & \text{instead of } (3.8); & H = \Omega_3(F) & \text{instead of } (3.9), \end{array}$$

and

$$P^* = \Omega_2^*(H) \quad \text{instead of } (3.5),$$

where P (expansive) is sytematically monic, and H (real Hurwitz) and P^* always have a constant term equal to 1. The Hurwitz alternant of H is $h(z) = [f_1/f_2/\ldots/f_m](z)$ with $F = {}^t(f_1, f_2, \ldots, f_m)$. If β belongs to the class $A_q, q \geq 2$, then $\mathrm{M}(P) = q$, and Theorem 2.1 amounts to the following:

$$(z-1)T(z) = \big(z(\Omega_2 \circ \Omega_3) - (\Omega_2^* \circ \Omega_3)\big)(F), \tag{3.10}$$

$$P(z) = \omega_1(T)(z) = (\Omega_2 \circ \Omega_3)(F), \tag{3.11}$$

with $L(X) = P(X) - T(X) = \omega_1(T)(X) - T(X)$ the reciprocal polynomial whose roots, all of modulus 1, interlace with those of T on $|z| = 1$.

Definition. We call *interlacing conjugates of* β, with respect to T and P, all on the unit circle, the roots of L and the roots of modulus 1 of T.

The set of interlacing conjugates of β is the union of the subcollection of the Galois conjugates of β of modulus 1, and the subcollection of the roots of L and the roots of unity which are zeroes of T. There are finitely many interlacing configurations on the unit circle by Proposition 2.2.

Once T is fixed, $m \geq 4$ even, with the notations of §2.4.7, denote

$$\mathcal{F}_T^+ := \left\{F = \omega_3 \circ \omega_2(P) \mid P \in \mathcal{P}_T^+\right\}, \ \mathcal{F}_T^- := \left\{F = \omega_3 \circ \omega_2(P) \mid P \in \mathcal{P}_T^-\right\},$$
$$\mathcal{F}_T := \mathcal{F}_T^+ \cup \mathcal{F}_T^- \subset \left(\mathbb{Q}^{+*}\right)^m.$$

Proposition 3.2. *The point set $\mathcal{F}_T$ has the following properties: (i) it is discrete and has no accumulation point in $\left(\mathbb{R}^{+*}\right)^m$; (ii) its affine dimension is equal to m; (iii) the set $\mathcal{F}_T^+ \cup \{0\}$ is a semi-group; (iv) the intersection of the hypersurface defined by $\{X = {}^t(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \big(\Omega_3(X)\big)(-1) = 0\}$ with $\mathcal{F}_T$ is empty.*

Proof. (i) The set $\mathcal{P}_T$ of monic expansive polynomials P over T, viewed as a point subset of $\mathbb{Z}^m$ by their coefficients vectors, is uniformly discrete: it has a minimal interpoint distance equal to 1. Considering $\mathcal{P}_T$ and $\mathcal{F}_T$ embedded in $\mathbb{C}^m$, the maps ω_2 and ω_3, defined on $\mathbb{C}^m$ by (3.3) and (3.8) respectively, are analytical. Invoking the Theorem of the open image for the analytic mapping $\omega_3 \circ \omega_2$, every open disk centered at a point P in $\mathcal{P}_T$ is sent to an open neighboorhood of its image point $F = \omega_3 \circ \omega_2(P)$. Therefore $\mathcal{F}_T$ is a discrete point set as union of isolated points, with eventually accumulation points on the boundary of the octant $(\mathbb{R}^{+,*})^m$.

(ii) Suppose that the affine dimension is less than m. Then there would exist an affine hyperplane $y(x_1, \dots, x_m) = 0$ containing $\mathcal{F}_T$ in $\mathbb{R}^m$, and in particular $\mathcal{F}_T^+$. The image of this affine hyperplane by $\Omega_2 \circ \Omega_3$ would be an hypersurface of $\mathbb{R}^m$ and the set $\mathcal{P}_T^+$ of monic expansive polynomials P over T would be included in it. But there is a bijection between P and $\not{P}$ and the discrete sector $C^+ \cap \mathbb{Z}^{m/2}$ has affine dimension $m/2$. Contradiction.

(iii) By transport of the internal law $\oplus$ on $\mathcal{P}_T^+ \cup \{T\}$, we obtain a semi-group structure on $\mathcal{F}_T^+ \cup \{\omega_3 \circ \omega_2(T)\}$, where the neutral element is $\omega_3 \circ \omega_2(T)$. Let us now show that

$$\omega_3 \circ \omega_2(T)(z) = \frac{h_1 + h_3 z + h_5 z^2 + ..}{h_0 + h_2 z + h_4 z^2 + ..} = 0.$$

In fact, $h_{2i+1} = \displaystyle\sum_{j=0}^{m} \sum_{k=0}^{2i+1} (-1)^{j-k} \binom{j}{k} \binom{m-j}{2i+1-k} t_j =$

$$\sum_{j=0}^{\frac{m}{2}-1} \sum_{k=0}^{2i+1} (-1)^{j-k} \binom{j}{k} \binom{m-j}{2i+1-k} t_j +$$

$$\sum_{j=\frac{m}{2}}^{m} \sum_{k=0}^{2i+1} (-1)^{j-k} \binom{j}{k} \binom{m-j}{2i+1-k} t_{m-j}$$

$$= \sum_{j=0}^{\frac{m}{2}-1} \sum_{k=0}^{2i+1} (-1)^{j-k} \binom{j}{k} \binom{m-j}{2i+1-k} t_j +$$

$$\sum_{j=0}^{\frac{m}{2}-1} \sum_{l=0}^{2i+1} (-1)^{m-j-2i-1+l} \binom{m-j}{2i+1-l} \binom{j}{l} t_j$$

$$= \sum_{j=0}^{\frac{m}{2}-1} \sum_{k=0}^{2i+1} ((-1)^{j-k} + (-1)^{-j-1+k}) \binom{j}{k} \binom{m-j}{2i+1-k} t_j = 0.$$

Since $T(-1) \neq 0$, the coefficient h_0 is not equal to 0. The image of T by ω_2 is of type ${}^t(h_0, 0, h_2, 0, h_4, 0, \dots, 0, h_m)/h_0$. It admits a Hurwitz alternant h equal to $[0/0/\dots/0](z)$, where 0 is counted m times, corresponding to the point $F = 0$ in $\mathbb{R}^m$. Indeed, all the coefficients $f_j, j = 1, \dots, m$, are equal to 0, by (3.7), since the first lines of the determinants D_j are all made of zeros.

(iv) The real Hurwitz polynomials H obtained as images of elements of $\mathcal{P}_T$ by ω_2 never cancel at $z = -1$. The image by ω_3 of the hyperplane of $\mathbb{R}^m$ of equation $\{{}^t(h_1, h_2, \dots, h_m) \in \mathbb{R}^m \mid 1 - h_1 + h_2 - \dots + h_m = 0\}$ is an hypersurface which does not contain any point F of $\mathcal{F}_T$. □

Once T is fixed, and P chosen over T, the geometrical coding

$$(3.12) \qquad (\beta, T, P) \quad \longleftrightarrow \quad F = \omega_3 \circ \omega_2 \circ \omega_1(T) = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in (\mathbb{Q}_{>0})^m$$

shows that the Stieltjes continued fraction F entirely controls β and simultaneously all the interlacing conjugates of β, by (3.10) and (3.11). When P runs over the complete set of monic expansive polynomials over T, F runs over $\mathcal{F}_T$. If F_1 and $F_2 \neq F_1$ are in $\mathcal{F}_T$, they give rise to equivalent codings of the *same* Salem number β.

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