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Complement for Algebraic differential equations...

Daniel Barlet*

24/02/14

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ABSTRACT. We complete the study of some periods of polynomials in $(n+1)$ -variables with $(n+2)$ -monomials in computing the behavior of these periods in the natural parameter for such a polynomial.

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1 Introduction

This note is a complement to the study in [B.13] of period integrals of non quasi-homogeneous polynomials in $n + 1$ variables with $n + 2$ monomials. We focus here on the dependance of these period integrals on the “natural” parameter $\lambda \in \mathbb{C}^*$ which is the only “free” coefficient of such a polynomial modulo the dilatations of the variables¹

For that purpose we recall first the fact that for a polynomial function f depending polynomially of a parameter λ we may define a natural “ b -connection” on the highest (f, λ) -relative de Rham cohomology group of f which induces the derivation $\frac{\partial}{\partial \lambda}$ on period integrals. The construction for any holomorphic function depending of a holomorphic parameter is precised in the appendix.

Then we show how to compute explicitly this connection in our specific situation and we obtain a simple partial differential equation for the period integrals associated to any monomial in $\mathbb{C}[x_0, \dots, x_n]$ when we consider a polynomial of the type

$$f_\lambda(x) = \sum_{j=1}^{n+1} x^{\alpha_j} + \lambda \cdot x^{\alpha_{n+2}} \quad \text{where} \quad \alpha_j \in \mathbb{N}^{n+1}, \quad j \in [1, n+2]$$

with the following assumptions

- i) The $(n+2, n+2)$ -matrix obtained from $M := (\alpha_1, \dots, \alpha_{n+2})$ by adding a first line of 1 has rank $n+2$.
- ii) The elements $\alpha_1, \dots, \alpha_{n+1}$ form a basis of \mathbb{Q}^{n+1} .

Note that the first condition is equivalent to the fact that f is not quasi-homogeneous, and that the condition ii) is always satisfied assuming i), up to change the order of the monomials (and then to change the parameter λ to $c \cdot \lambda^m$ for some $c \in \mathbb{Q}^*$ some $m \in \mathbb{Z}^*$).

2 The λ -connection.

2.1 The general situation.

We consider here a polynomial $f \in R := \mathbb{C}[x_0, \dots, x_n][\lambda]$ depending polynomially on a parameter λ . We consider on $R \otimes \Lambda^*(\mathbb{C}^{n+1}) := \Omega_j^*$ the λ -relative de Rham complex, where $(\mathbb{C}^{n+1})^* := \bigoplus_{i=0}^n \mathbb{C} \cdot dx_i$, and we denote d_j its differential.

We shall denote by \mathcal{A} the unitary (non commutative) algebra generated by a and b with the commutation relation $a \cdot b - b \cdot a = b^2$ and by $\mathcal{A}[\lambda] := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ with its natural structure of algebra for which λ commutes with a and b .

Then the quotient $E_f := \Omega_j^{n+1} / d_j f \wedge d_j \Omega_j^{n-1}$ has a natural left $\mathcal{A}[\lambda]$ -module structure defined by

¹In our hypothesis we may assume that all monomials have coefficient 1 excepted the last one up to a linear diagonal change of variable.

- The action of a is given by the multiplication by f .
- The action of b is given by $d/f \wedge d_f^{-1}$.

Remark that for fixed λ , assuming that f_λ has an isolated singularity at 0, the b -completion $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ is the usual (formal) Brieskorn module associated to f_λ at 0. For a given monomial $\mu \in \mathbb{C}[x_0, \dots, x_n]$ the decomposition theorem of [B.13] (theorem 3.1.2) applies to the the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$ where $\tilde{\mathcal{A}}$ is the b -completion of \mathcal{A} and where $P(\mu)$ is the element in \mathcal{A} constructed in the theorem 1.2.1 of [B.13]. Then $P_d(\mu)$ is a (left-)multiple of the Bernstein element of $\tilde{\mathcal{A}}.[\mu.dx]$ in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ and it determines a finite set of possible eigenvalues for the monodromies around $s = 0$ for the period integrals (λ fixed)

$$\varphi_\lambda(s) = \int_{\gamma_{\lambda,s}} \frac{\mu.dx}{d/f}$$

for any horizontal family $\gamma_{\lambda,s}$ of compact n -dimensional cycles in the fibers of f_λ .

It is important to remark that if f_λ has a non isolated singularity at the origin, despite the fact that there is no finiteness for the $\mathbb{C}[[b]]$ -module $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$, the conclusion above is still valid because the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$, and so its image in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$, is a finite type $\mathbb{C}[[b]]$ -module. Then the product decomposition $P_d = (a - r_1.b) \dots (a - r_d.b)$, where r_1, \dots, r_d are (explicitly computable) rational numbers, gives that the set $\{e^{2i\pi.r_1}, \dots, e^{2i\pi.r_d}\}$ contains the spectrum of these monodromies (counting multiplicities).

QUESTION. Is it true that P_d is equal to the Bernstein element² of the Brieskorn module $\tilde{\mathcal{A}}.[\mu.dx]$ in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ when f_λ has an isolated singularity at the origin? \square

Proposition 2.1.1 *There exists a \mathbb{C} -linear operator $\nabla : E_f \rightarrow E_f$ with the following properties :*

1. For $\omega = d/\xi \in \Omega_f^{n+1}$ we have $\nabla([\omega]) = [d/f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} . \omega]$.
2. The map $b^{-1}.\nabla$ well defined on $b.\tilde{E}_f$ where $\tilde{E}_f := E_f/(b - \text{torsion})$, with value in \tilde{E}_f , commutes with a and b and is a λ -connection.

²For a (a,b) -module E with one generator as a $\tilde{\mathcal{A}}$ -module, the relation between its Bernstein element $P_d \in \mathcal{A}$ and its Bernstein polynomial B is given by the formula (see [B.09])

$$(-b)^d . B(-b^{-1} . a) = P_d \quad \text{where } d \text{ is the rank of } E.$$

3. If $(\gamma_{s,\lambda})_{(s,\lambda) \in S \times \Omega}$ is a horizontal family of compact n -cycles in the fibers of (f, λ) over an open set in $\mathbb{C} \times \mathbb{C}^* \setminus C(f, \lambda)$ where $C(f, \lambda)$ is the set of critical values of the map $(f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$, we have for any $\omega \in \Omega_{\gamma}^{n+1}$ the equality

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[\int_{\gamma_{s,\lambda}} \frac{\omega}{d_{\gamma} f} \right] = \int_{\gamma_{s,\lambda}} \frac{\nabla(\omega)}{d_{\gamma} f}.$$

PROOF. Remark first that ∇ is well defined because for $\xi = d_{\gamma} \eta$ we have

$$\nabla(d_{\gamma} \xi) = d_{\gamma} f \wedge \frac{\partial(d_{\gamma} \eta)}{\partial \lambda} = d_{\gamma} f \wedge d_{\gamma} \left(\frac{\partial \eta}{\partial \lambda} \right)$$

so it induces 0 in E_f .

Let $\omega = d_{\gamma} \xi \in \Omega_{\gamma}^{n+1}$ and let $d_{\gamma} \eta = d_{\gamma} f \wedge \xi$. Then we have

$$\begin{aligned} \nabla(b.[\omega]) &= \nabla(d_{\gamma} \eta) = d_{\gamma} f \wedge \frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot d_{\gamma} \eta \\ &= d_{\gamma} f \wedge \left(\frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \xi \right) = b \left[d_{\gamma} \left(\frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \xi \right) \right] \\ &= b \left[\frac{\partial(d_{\gamma} f)}{\partial \lambda} \wedge \xi + d_{\gamma} f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot d_{\gamma} \xi - d_{\gamma} \left(\frac{\partial f}{\partial \lambda} \right) \wedge \xi \right] \\ &= b \left[d_{\gamma} \left(\frac{\partial f}{\partial \lambda} \right) \cdot \xi + \nabla(d_{\gamma} \xi) - d_{\gamma} \left(\frac{\partial f}{\partial \lambda} \right) \wedge \xi \right] = b \left[\nabla(d_{\gamma} \xi) \right] \end{aligned}$$

as d_{γ} and $\frac{\partial}{\partial \lambda}$ commute. So we have $b \cdot \nabla = \nabla \cdot b$.

We have also

$$\begin{aligned} \nabla(a.[\omega]) &= \nabla(f \cdot d_{\gamma} \xi) = \nabla(d_{\gamma}(f \cdot \xi)) - \nabla(d_{\gamma} f \wedge \xi) \\ &= d_{\gamma} f \wedge \frac{\partial(f \cdot \xi)}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot f \cdot d_{\gamma} \xi - \frac{\partial f}{\partial \lambda} \cdot d_{\gamma} f \wedge \xi - \nabla(b.[\omega]) \\ &= a \cdot \nabla([\omega]) - b \cdot \nabla([\omega]). \end{aligned}$$

This implies the equality $a \cdot b^{-1} \cdot \nabla = b^{-1} \cdot \nabla \cdot a$ as \mathbb{C} -linear maps from $b \cdot \tilde{E}_f$ to \tilde{E}_f .

Note that the equalities $\nabla \cdot b = b \cdot \nabla$ and $\nabla \cdot a = (a - b) \cdot \nabla$ as \mathbb{C} -endomorphisms of E_f are more precise than the relations above.

Finally let $\varphi \in \mathbb{C}[\lambda]$ then we have

$$\begin{aligned} \nabla(\varphi \cdot d_{\gamma} \xi) &= \nabla(d_{\gamma}(\varphi \cdot \xi)) = d_{\gamma} f \wedge \frac{\partial \varphi \cdot \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \varphi \cdot d_{\gamma} \xi \\ &= \frac{\partial \varphi}{\partial \lambda} \cdot (d_{\gamma} f \wedge \xi) + \varphi \cdot \nabla(d_{\gamma} \xi) = \frac{\partial \varphi}{\partial \lambda} \cdot b[d_{\gamma} \xi] + \varphi \cdot \nabla(d_{\gamma} \xi) \end{aligned}$$

and this shows that $b^{-1}.\nabla$ is a λ -connection.

Note again that we proved the equality in E_f : $\nabla(\varphi.\omega) = \frac{\partial\varphi}{\partial\lambda}.b.\omega + \varphi.\nabla(\omega)$ valid for $\varphi \in \mathbb{C}[\lambda]$ and $\omega \in E_f$ which is more precise.

To prove the point 3. of the statement consider $\xi \in \Omega_f^n$ and let d be the total de Rham differential (in x and λ). We have

$$d\xi = d\lambda \wedge \frac{\partial\xi}{\partial\lambda} + d_f\xi \quad \text{and} \quad df = d\lambda \cdot \frac{\partial f}{\partial\lambda} + d_f f.$$

Assume we can write $d\xi = d\lambda \wedge v + d_f f \wedge u$ with $u, v \in \Omega_f^n$. Then we obtain

$$d\xi = d\lambda \wedge (v - \frac{\partial f}{\partial\lambda}.u) + d_f f \wedge u \quad \text{with} \quad u = \frac{d_f \xi}{d_f f} \quad \text{and} \quad v = \frac{\partial\xi}{\partial\lambda}.$$

If $(\gamma_{s,\lambda})$ is a horizontal family of compact n -cycles in the fibers of the map $(f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$, we shall have

$$d\left(\int_{\gamma_{s,\lambda}} \xi\right) = \left[\int_{\gamma_{s,\lambda}} (v - \frac{\partial f}{\partial\lambda}.u)\right].d\lambda + \left[\int_{\gamma_{s,\lambda}} u\right].ds.$$

So, has the chain $\cup_{s,\lambda} \gamma_{s,\lambda}$ is proper and without λ -relative boundary we obtain

$$\frac{\partial}{\partial s} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \frac{d_f \xi}{d_f f} \quad \text{and} \quad \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \left(\frac{\partial\xi}{\partial\lambda} - \frac{\partial f}{\partial\lambda} \cdot \frac{d_f \xi}{d_f f}\right)$$

Now consider $\omega \in \Omega_f^{n+1}$ and write $\omega = d_f \xi$. Then $b[\omega] = [d_f f \wedge \xi]$ and we have

$$\int_{\gamma_{s,\lambda}} \frac{b[\omega]}{d_f f} = \int_{\gamma_{s,\lambda}} \xi \quad \text{and} \quad \frac{\partial}{\partial s} \int_{\gamma_{s,\lambda}} \frac{b[\omega]}{d_f f} = \int_{\gamma_{s,\lambda}} \frac{\omega}{d_f f}.$$

So we conclude that we have

$$\frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \frac{\omega}{d_f f} = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \left(\frac{\partial\xi}{\partial\lambda} - \frac{\partial f}{\partial\lambda} \cdot \frac{d_f \xi}{d_f f}\right) = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \frac{\nabla[\omega]}{d_f f} \quad \blacksquare$$

2.2 The case of a polynomial with $n+2$ monomials in $n+1$ variables.

So we consider now the case were $f := \sum_{j=1}^{n+2} m_j$ where $m_j := x^{\alpha_j}$ $j \in [1, n+1]$ and $m_{n+2} := \lambda.x^{\alpha_{n+2}}$ with the following hypotheses (see [B. 13]) : the rank of the square matrix $M' := (\alpha_1, \dots, \alpha_{n+1})$ is $n+1$ and the rank of the square matrix \tilde{M} obtained by adding a first line of 1 to the matrix $M := (\alpha_1, \dots, \alpha_{n+2})$ is $n+2$.

Recall that if we write (with a minimal positive integer r) $r \cdot \alpha_{n+2} = \sum_{j=1}^{n+1} p_j \cdot \alpha_j$ where p_1, \dots, p_{n+1} are in \mathbb{Z} , and if we define

$$d = \inf\left\{r - \sum_{j:p_j \leq 0} p_j, \sum_{j:p_j > 0} p_j\right\} \quad \text{and} \quad d+h = \sup\left\{r - \sum_{j:p_j \leq 0} p_j, \sum_{j:p_j > 0} p_j\right\}$$

there exists an element P in $A[\lambda, \lambda^{-1}]$ of the form

$$P := P_{d+h} + c \cdot \lambda^{\pm r} \cdot P_d$$

which annihilated the class $[dx]$ in E_f , where P_{d+h} and P_d are homogeneous elements in \mathcal{A} , respectively of degree $d+h$ and d which are monic in a with rational coefficients³, and where c is in \mathbb{Q}^* . The sign in the exponent of λ will be precised in the proof of the proposition 2.2.2.

Recall also that in this situation the $\mathcal{A}[\lambda]$ module generated by the class $[dx]$ in E_f is exactly the image in E_f of $\mathbb{C}[m_1, \dots, m_{n+2}][\lambda].dx \subset \Omega_f^{n+1}$ with $m_j = x^{\alpha_j}$ with $j \in [1, n+1]$ and $m_{n+2} = \lambda \cdot x^{\alpha_{n+2}}$.

Our next result uses the following easy lemma:

Lemme 2.2.1 *Let $Q \in \mathcal{A}$ a homogeneous element in (a, b) of degree k . Then for any $\lambda \in \mathbb{C}$ we have :*

$$b.Q.b^{-1} \cdot (a - \lambda.b) = (a - (\lambda + k).b).Q.$$

PROOF. Remark first that the map $\mathcal{A} \rightarrow \mathcal{A}$ sending $x \in \mathcal{A}$ to $b.x.b^{-1}$ is well defined and bijective thanks to the following facts : b is injective and $b.\mathcal{A} = \mathcal{A}.b$. We shall prove the lemma by induction k . As the case $k = 0$ is obvious, assume that the lemma is proved for $k < k_0$ where $k_0 \geq 1$ and consider an homogeneous element Q of degree k_0 . We may assume⁴ that $Q = b.R$ or that we may find $\mu \in \mathbb{C}$ such that $Q = (a - \mu.b).R$, where R is homogeneous of degree $k_0 - 1$. In the first case we have, using the induction hypothesis :

$$b.b.R.b^{-1} \cdot (a - \lambda.b) = b.(a - (\lambda + k_0 - 1).b).R = (a - (\lambda + k_0).b).b.R = (a - (\lambda + k_0).b).Q.$$

In the second case we have, using the induction hypothesis :

$$\begin{aligned} b.(a - \mu.b).R.b^{-1} \cdot (a - \lambda.b) &= (a - (\mu + 1).b).b.R.b^{-1} \cdot (a - \lambda.b) \\ &= (a - (\mu + 1).b).(a - (\lambda + k_0 - 1).b).R \\ &= (a - (\lambda + k_0).b).(a - \mu.b).R \\ &= (a - (\lambda + k_0).b).Q. \end{aligned} \quad \blacksquare$$

³More is proved in [B.13] : P_{d+h} and P_d factorize in product of $(a - r_j.b)$ with $r_j \in \mathbb{Q}$.

⁴Recall that any homogeneous element in \mathcal{A} which is monic in a factorizes as a product of linear factors $(a - r_i.b)$, where the r_i are complex numbers; see [B.09].

Proposition 2.2.2 *Let μ be a monomial of degree k in $\mathbb{C}[x_0, \dots, x_{n+1}]$. Then we have in E_f the relation*

$$\nabla([\mu]) = \frac{-1}{\lambda} \cdot (\sigma \cdot a + (\tau - k \cdot \sigma) \cdot b)[\mu]$$

where σ, τ are defined by the relation $m_{n+2} \cdot [\mu] = (\sigma \cdot a + \tau \cdot b)[\mu]$. Moreover the value⁵ of σ is $\pm r/h$ so it does not depend on the choice of the monomial μ .

As a consequence, if we have on an open set $S \times \Omega$ in $\mathbb{C}^* \times \mathbb{C}^*$, a horizontal family $(\gamma_{s,\lambda})_{(s,\lambda) \in S \times \Omega}$ of compact n -cycles in the fibers of the map $\mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$ defined by $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$, the holomorphic function

$$(s, \lambda) \mapsto \varphi(s, \lambda) := \int_{\gamma_{s,\lambda}} \frac{\mu \cdot dx}{d/f}$$

satisfies the partial differential equation

$$-\lambda \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial s} \varphi = \sigma \cdot \frac{\partial(s \cdot \varphi)}{\partial s} + (\tau - k) \cdot \varphi$$

on $S \times \Omega$. ■

PROOF. As we have $\lambda \cdot \nabla([1]) = -m_{n+2}$ in E_f and as we know that there exist σ, τ in \mathbb{Q} such that $(\sigma \cdot a + \tau \cdot b)[1] = m_{n+2}$ for the case $\mu = 1$ the only thing to prove is the computation of σ .

Using the Cramer system with matrix $(n+2, n+2)$ obtained by adding a first line of 1 to the matrix $M := (\alpha_1, \dots, \alpha_{n+2})$, computing $a[1]$ and the $b_i[1]$ we find that σ is the coefficient $(n+2, 1)$ in the matrix \tilde{M}^{-1} . Let M' be the principal $(n+1, n+1)$ minor of \tilde{M} . This implies that

$$\sigma = (-1)^{n+1} \frac{\det(M')}{\det(\tilde{M})}.$$

But using the relation $\alpha_{n+2} = \sum_{j=1}^{n+1} \frac{p_j}{r} \cdot \alpha_j$ we obtain

$$\det(\tilde{M}) = (-1)^{n+1} \cdot \left(1 - \sum_{j=1}^{n+1} \frac{p_j}{r}\right) \cdot \det(M')$$

so we conclude that

$$\sigma = \frac{r}{r - \sum_{j=1}^{n+1} p_j}.$$

Now we have two cases :

i) $r - \sum_{p_j < 0} p_j = d + h > \sum_{p_j > 0} p_j = d$. Then $r - \sum_{j=1}^{n+1} p_j = (d + h) - d = h$.

So $\sigma = r/h$, and the exponent of λ in P is r .

⁵The sign is precised in the proof and only depends on $\alpha_1, \dots, \alpha_{n+2}$.

- ii) $\sum_{p_j > 0} p_j = d + h > r - \sum_{p_j < 0} p_j = d$. Then $r - \sum_{j=1}^{n+1} p_j = d - (d + h) = -h$.
So $\sigma = -r/h$, and the exponent of λ in P is $-r$.

Consider now the case of a degree k monomial $\mu \in \mathbb{C}[x_0, \dots, x_n]$. Then there exists again σ', τ' in \mathbb{Q} such that $(\sigma'.a + \tau'.b)[\mu] = [m_{n+2}.\mu]$ in E_f . As $a[\mu], (\beta_i + 1).b[\mu], i \in [0, n]$, where β_i is the degree in x_i of $\mu := x^\beta$, are again given from the $[m_j.\mu], j \in [1, n + 2]$ by the same Cramer system, we conclude that $\sigma' = \sigma$. To conclude the proof it is enough to apply the proposition 2.1.1. \blacksquare

Note that in the case i) above $P := P_{d+h} + c.\lambda^r.P_d$ annihilated $[\mu]$ in E_f and in the case ii) we have $P := P_{d+h} + c.\lambda^{-r}.P_d$.

The lemma 2.2.1 gives that $\lambda.\nabla(P.[\mu]) = -(\sigma.a + (\tau' - k.\sigma).b).P[\mu]$ which makes explicit the fact that $\lambda.\nabla$ is well defined on $\mathcal{A}[\lambda].[\mu] \subset E_f$.

REMARK. Recall that in [B.13] we have built in an explicit way a differential equation in $s \in S$, depending in a very simple and concrete way on $\lambda \in \mathbb{C}^*$ which is satisfied by φ . So it is easy to see that the knowledge of a formal asymptotic expansion when s goes to 0 in S^6 for a given λ_0 , of the type

$$\varphi(\lambda_0, s) \simeq \sum_{i,j} C_{i,j}.s^{\rho_i}.(Logs)^j$$

where ρ_1, \dots, ρ_I are in $-1 + \mathbb{Q}^{*+}$, $j \in [0, n]$ are integers and $C_{i,j}$ are in $\mathbb{C}[[s]]$, determines (uniquely) via the partial differential equation above, a formal expansion of the same type for each given $\lambda \in \Omega$, whose coefficients $C_{i,j}^\lambda$ are polynomials in $Log\lambda$ easily computable from the coefficients $C_{i,j}^{\lambda_0} := C_{i,j}$ of the asymptotic expansion at the initial value λ_0 of λ . This is described in the following lemma.

Lemme 2.2.3 *Let Ω be a simply connected domain in \mathbb{C}^* . Let $(\rho_i)_{i \in I}$ be a finite collection of rational numbers strictly bigger than -1 . Assume that the formal power serie*

$$\varphi_\lambda := \sum_{k=0}^N \sum_{i \in I} \sum_{m \geq 0} c_m^{i,k}(\lambda).s^{m+\rho_i}.(Logs)^k / k!$$

where $c_m^{i,k}$ are holomorphic functions in Ω , satisfies the partial differential equation

$$\lambda.\frac{\partial}{\partial \lambda} \frac{\partial}{\partial s} \varphi_\lambda = \alpha.s \frac{\partial(\varphi_\lambda)}{\partial s} + \beta.\varphi_\lambda$$

for each $\lambda \in \Omega$. Then for each i, k fixed, the function $c_m^{i,k}$ is a polynomial in $Log\lambda$ of degree $\leq m$ for each m . Moreover the collection of numbers $c_m^{i,k}(\lambda_0)$ for a given $\lambda_0 \in \Omega$ determines uniquely these polynomials.

⁶This is always the case when S contains an open sector with edge at the origin.

PROOF. The partial differential equation implies the following recursion relation for each i, k, m :

$$(m + \rho_i + 1) \cdot \lambda \cdot \frac{\partial c_{m+1}^{i,k}(\lambda)}{\partial \lambda} + \lambda \cdot \frac{\partial c_{m+1}^{i,k+1}(\lambda)}{\partial \lambda} = (\alpha \cdot (m + \rho_i) + \beta) \cdot c_m^{i,k}(\lambda) + \alpha \cdot c_m^{i,k+1}(\lambda)$$

We shall make a descending induction on k . For $k = N$ the recursion relation reduces to

$$(m + \rho_i + 1) \cdot \lambda \cdot \frac{\partial c_{m+1}^{i,N}(\lambda)}{\partial \lambda} = (\alpha \cdot (m + \rho_i) + \beta) \cdot c_m^{i,N}(\lambda)$$

and an easy induction on $m \geq 0$ gives our assertion for $k = N$.

Assuming the statement proved for $k + 1$ a simple quadrature in λ implies the case k . ■

3 Two families of examples with $d = 2$ and $h = 1$.

3.1 The family $x^{2u} + y^{2v} + z^{2w} + \lambda \cdot x^u \cdot y^v \cdot z^w$.

The condition to be in our situation is $u \cdot v \cdot w > 0$. Then we have the relation $m_4^2 = \lambda^2 \cdot m_1 \cdot m_2 \cdot m_3$ and it shows that $d = 2$ and $h = 1$.

Note that the only singularity of f in $\{f = 0\}$ is the origine.

To compute $P := P_3 + c \cdot \lambda^{-2} \cdot P_2$ which annihilates [1] is not difficult. We find

$$P = (a - (2 + \frac{u+v}{2u \cdot v}) \cdot b) (a - (1 + \frac{u+w}{2u \cdot w}) \cdot b) (a - (\frac{v+w}{2v \cdot w}) \cdot b) + \\ - 4\lambda^{-2} \cdot (a - (\frac{3}{2} + \frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w}) \cdot b) (a - \frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w} \cdot b).$$

In this case we have

$$\lambda \cdot \nabla([1]) = 2 \cdot (a - (\frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w}) \cdot b) [1] = -m_4.$$

Here we are in the case ii) above (so $\sigma = -2$).

Let me illustrate this family on a simple example : $f = x^4 + y^4 + z^2 + \lambda \cdot x^2 \cdot y^2 \cdot z$ corresponding to $u = v = 2, w = 1$. In this case we find

$$P := (a - \frac{5}{2} \cdot b) [(a - \frac{7}{4} \cdot b) (a - \frac{3}{4} \cdot b) - 4 \cdot \lambda^{-2} \cdot (a - b)] \quad \text{and} \quad \lambda \cdot \nabla([1]) = 2(a - b)[1].$$

3.2 The family $x^{2p} \cdot z^u + y^{2q} \cdot z^v + z^{u+v} + \lambda \cdot x^p \cdot y^q$.

The condition to be in our situation is $p \cdot q \cdot (u + v) > 0$. Note that the singularity at the origine is not isolated in general in these cases. We have here the equality

$$2 \cdot \alpha_4 = \alpha_1 + \alpha_2 - \alpha_3$$

The relation which determines P annihilating $[1]$ is given by $m_4^2.m_3 = \lambda^2.m_1.m_2$, so $r = 2, d = 2, h = 1$ and we are in the case i).

The computation of P gives

$$P = (a - (2 + \frac{p+q}{2p.q}).b)(a - (\frac{1}{2} + \frac{p.u+q.v+2p.q}{2p.q.(u+v)}.b)(a - (\frac{p.u+q.v+2p.q}{2p.q.(u+v)}.b) + \\ - 4\lambda^2.(a - (1 + \frac{p.u+q.v+2p.q+p.(u+v)}{2p.q.(u+v)}.b)(a - (\frac{p.u+q.v+2p.q+q.(u+v)}{2p.q.(u+v)}.b)).$$

The computation of (σ, τ) such that $(\sigma.a + \tau.b)[1] = m_4$ is easy and it gives

$$\lambda.\nabla([1]) = -(2.a - \frac{p.u+q.v+2p.q}{p.q.(u+v)}.b) = -m_4.$$

Again let me illustrated by an example : for $p = q = 2$ and $u = v = 1$ so for $f = x^4.z + y^4.z + z^2 + \lambda.x^2.y^2$. We find

$$P := (a - \frac{5}{2}.b)(a - \frac{5}{4}.b)(a - \frac{3}{4}.b) - 4\lambda^2.(a - 2b)(a - b) \quad \text{and} \quad \lambda.\nabla([1]) = -2(a - \frac{3}{4}.b)[1]$$

4 Appendix

It is interesting to remark that the proposition 2.1.1 is a special case in a specific algebraic setting of a general result on the the filtered Gauss-Manin connexion of a holomorphic function depending holomorphically of a parameter. This is the goal of this appendix to precise this point.

Let M be a complex manifold, D an open disc in \mathbb{C} and let $f : D \times M \rightarrow \mathbb{C}$ be a holomorphic function. Denote $K_j^p := Ker[d_j f \wedge; \Omega_j^p \rightarrow \Omega_j^{p+1}]$ for $p \geq 2$ and $K_j^1 := Ker[d_j f \wedge : \Omega_j^1 \rightarrow \Omega_j^2] / \mathcal{O}.d_j f$ where $d_j f$ is the λ -relative differential of f and Ω_j^p the sheaf of λ -relative holomorphic p -forms (compare with [B.08]).

Denote by (K_j^\bullet, d_j) the topological restriction of the λ -relative de Rham complex (defined above) for the map $(\lambda, x) \mapsto (\lambda, f(\lambda, x))$, to the analytic subset

$$Z := \{d_j f = 0\}$$

and let \mathcal{H}^p the p -th cohomology sheaf of this complex. Recall that these cohomogy sheaves have a natural structure of left $\mathcal{A}[\lambda]$ -modules with the action of a given by the multiplication by f and with the action of b defined by $d_j f \wedge d_j^{-1}$.

Proposition 4.0.1 *There exists a natural graded map $\nabla^\bullet : \mathcal{H}^\bullet \rightarrow \mathcal{H}^\bullet$ with the following properties :*

1. For $\omega = d_j \xi \in K_j^{p+1} \cap Ker d_j$ we have $\nabla([\omega]) = [d_j f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} . \omega]$.

2. The map $b^{-1}.\nabla$ well defined on $b.\tilde{\mathcal{H}}^\bullet$ where $\tilde{\mathcal{H}}^\bullet := \mathcal{H}^\bullet/(b\text{-torsion})$, with value in $\tilde{\mathcal{H}}^\bullet$, commutes with a and b and is a λ -connection.
3. If $(\gamma_{s,\lambda})_{(s,\lambda)\in S\times\Omega}$ is a horizontal family of compact p -cycles in the fibers of (f, λ) over an open set in $D \times M \setminus C(f, \lambda)$ where $C(f, \lambda)$ is the set of critical values of the map $(\text{id}, f) : D \times M \rightarrow D \times \mathbb{C}$, we have for any $\omega \in K_j^{p+1} \cap \text{Ker } d_j$ the equality

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[\int_{\gamma_{s,\lambda}} \frac{\omega}{d_j f} \right] = \int_{\gamma_{s,\lambda}} \frac{\nabla(\omega)}{d_j f}.$$

PROOF. First remark that if $\omega = d_j \xi \in K^{p+1}$ with $d_j f \wedge \xi = 0$, we have $d_j(\frac{\partial f}{\partial \lambda}) \wedge \xi + d_j f \wedge \frac{\partial \xi}{\partial \lambda} = 0$ so $\nabla(d_j \xi) = -d_j(\frac{\partial f}{\partial \lambda}.\xi)$ is in $d_j K^p$. Now for $\omega = d_j \xi \in K^{p+1}$ we have $d_j f \wedge \nabla(d_j \xi) = 0$ and

$$d_j(\nabla(d_j \xi)) = -d_j f \wedge d_j\left(\frac{\partial \xi}{\partial \lambda}\right) - d_j\left(\frac{\partial f}{\partial \lambda}\right) \wedge d_j \xi$$

and we obtain that

$$d_j(\nabla(d_j \xi)) = -\frac{\partial}{\partial \lambda}(d_j f \wedge \omega) = 0$$

using the fact that $\frac{\partial}{\partial \lambda}(d_j f \wedge \omega) \equiv 0$.

The proof of the other statements are analogous to the corresponding ones in proposition 2.1.1. ■

REMARKS.

1. As above we have a more precise formulation for the properties in assertion 2. of the proposition above with the following relations in \mathcal{H}^\bullet

$$\begin{aligned} \nabla(\varphi.\omega) &= \frac{\partial \varphi}{\partial \lambda}.b.\omega + \varphi.\nabla(\omega) \quad \text{for } \varphi \in \mathcal{O}_\lambda \quad \text{and } \omega \in \mathcal{H}^\bullet \\ b.\nabla &= \nabla.b \quad \text{and} \quad \nabla.a = (a - b).\nabla. \end{aligned}$$

2. The generalization of this proposition to several holomorphic parameters is immediate. □

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