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Twin relationships in Parsimonious Games: some results*

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Abstract

In a vintage paper concerning Parsimonious games, a subset of constant sum homogeneous weighted majority games, Isbell introduced a twin relationship based on transposition properties of the incidence matrices upon minimal winning coalitions of such games. A careful investigation of such properties allowed the discovery of some results on twin games presented in this paper. In detail we show that a) twin games have the same minimal winning quota and b) each Parsimonious game admits a unique balanced lottery on minimal winning coalitions, whose probabilities are given by the individual weights of its twin game.

Keywords

Homogeneous weighted majority games, incidence matrices, twin relationships, minimal winning quota, balanced lottery.

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1 Introduction

In this paper we present some results concerning twin relationships in Parsimonious games (henceforth P games). P games are the subset of constant sum homogeneous weighted majority games characterized by the parsimony property to have, for any given number $n (> 3)$ ¹ of non dummy players in the game, the smallest number, i.e. exactly n , of minimal winning coalitions. P games have been defined and studied by Isbell in a vintage paper ([3], 1956) where the special properties of this class of games are described. Among other things Isbell introduced a twin relationship on P games, but without deepening the point, which at the best of our knowledge did not receive any further attention in the relevant literature.

*A very preliminary version of the paper has been presented at the 2013 Workshop of the Central European Program in Economic Theory, which took place in Udine (20-21 June) and may be found in CEPET working papers [7].

¹We will consider here P games with $n > 3$. Indeed, for $n = 3$ there is a unique P game in which all players share the same weight, while all P games with $n > 3$ have at least two types of players. See [3], p. 185.

To understand the twin relationship it is convenient to recall that its premise is the existence of a general rule that drives the one to one correspondence that obviously exists between the set of players and the set of minimal winning coalitions of any P game. In turn this correspondence comes out from the following idea: keeping account of the minimal homogeneous representation of a given P game, divide the players in h groups ($2 \leq h \leq n - 2$) so as all members of the same group (of the same type) share the same individual weight. Order players $j = 1, \dots, n$ and types $t = 1, \dots, h$ according to a non decreasing (for players) or a strictly increasing (for types) weight convention. In particular, players of the group with minimum weight (type 1) may be called “peones”, those of type $h - 1$ “vice-top” and the player (of group h) with greatest weight “top player”. In any P game there are lower bounds on the number of peones and of vice top and a binding constraint on the top class: indeed there is just one top player.

Let us shortly call odd (respectively even) players, those whose type is odd (even). After that, the one to one correspondence is described by the following rules: to any non top player there is associated the minimal winning coalition made by that player and all players of alternative parity and greater weight, whereas the coalition made by all odd players (which may include or not the top player) is associated to the top player. Note that, in this way, also the set of minimal winning coalitions is divided in h groups and, matching the type order of players and associated coalitions, that the numerosness x_t of type t players is equal to the one of type t minimal winning coalitions. All these results imply a special structure of the incidence matrix upon minimal winning coalitions of a P game, i.e. the square n dimensional binary matrix A whose elements a_{ij} are 1 if column player j belongs to the row minimal winning coalition i (or S_i), and 0 otherwise. In particular the square submatrix M , obtained from A by deleting the last row and column, is block diagonal upper triangular. The diagonal blocks of M are x_t dimensional diagonal square matrices, rectangular blocks over the diagonal alternate matrices with all elements equal to one to null matrices, while by definition all the entries under the diagonal blocks are zero.

This special structure inspired Isbell in recognizing that the transposed A^T of the incidence matrix A of any “primal” game G should still be the incidence matrix of a P game \overline{G} to be called twin (dual in Isbell terminology) of G . In other words a couple of P games are twins if and only if each incidence matrix of the couple is obtained by transposition of the incidence matrix of the twin².

Our contribution in this paper is embedded in a couple of theorems regarding twins. The first result says that twins have the same minimal winning quota, and is a straightforward corollary of a theorem regarding the determinant of the incidence matrix of any P game. The theorem states that the absolute value of the determinant is the minimal winning quota of the game.

The second theorem regards the connection between the balanced lottery on minimal winning coalitions of a P game and the individual weights of its twin. To understand the point suppose that the following three-stage mechanism is adopted to fix a fair and stable result for a P game, avoiding lengthy and may be unsatisfying negotiations between players. In the first stage the probabilities driving a lottery on the set of minimal winning coalitions are fixed. In the second stage the lottery mechanism makes the choice of one minimal winning coalition and in the third the (normalized to one) global reward of the game is divided within players of the chosen minimal winning coalition. In order to grant ex post fairness, in this stage individual rewards are proportional to individual weights.

We define balanced³ a lottery that gives to all players the same (ex ante) probability to be a member

²As we shall see later (Prop. 3.12) an alternative fully equivalent definition of twins could be based on the symmetry of the free type representations of twin games.

³Generally speaking, a given set of m coalitions $\mathbf{S} = (S_1, \dots, S_i, \dots, S_m)$ is balanced if there exists a positive m -dimensional vector \mathbf{d} such that for any (non dummy) player j it is $\sum_{i:j \in S_i} d_i = 1$. Then, except for a normal-

of the chosen minimal winning coalition. Clearly a balanced lottery adds ex ante to ex post fairness: indeed it is easy to check that (given the ex post mechanism of reward) only under a balanced lottery the expected gain of each player is exactly his (normalized) individual weight. A question immediately arises: any P game admits (may be just one) balanced lotteries? The answer is positive: our second fundamental theorem shows that for any P game there exists just one balanced lottery which is given by the normalized and properly reordered set of individual weights of its twin. Precisely the balanced probabilities of type t coalitions in the game G are, for any t but the top one, the normalized weights of type $h-t$ players in the twin \overline{G} , while the probability of the coalition associated to the top player is just the normalized weight of the top player in \overline{G} . The transposition properties of incidence matrices play a decisive role also in the proof of this result.

The plan of the paper is as follows. In section 2 a short recall of the basics of constant sum homogeneous weighted majority games is offered; section 3, divided in three subsections (one to one correspondence between players and minimal winning coalitions; the special structure of the incidence matrix of a P game; the transposition approach to the twin relationships in P games) resumes fundamental results on P games⁴. Section 4 (minimal winning quota in twin games) and 5 (balanced lotteries and twin games) give proofs of our main theorems on twin games. Some examples are offered in section 6. Conclusions follow in the final section 7.

2 Basics on homogeneous weighted majority games

Let us recall some well known basic definitions.

As usual $N = \{1, \dots, n\}$ denotes the set of all (non dummy) players.

A coalition S is a subset of the set N . A game G in coalitional function form is defined by the coalitional function of the game, that is the real function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$.

A game in coalitional function form is simple if its v function has values in $\{0, 1\}$. A coalition S is winning if $v(S) = 1$, losing if $v(S) = 0$.

A simple game is constant sum if, for any $S \in \mathcal{P}(N)$, $v(S) + v(\tilde{S}) = 1$.

A coalition S is said to be minimal winning if $v(S) = 1$ and, for any $T \subsetneq S$, $v(T) = 0$. The set of minimal winning coalitions is denoted by WM . A player j who does not belong to any minimal winning coalition is said dummy; at the other extreme, it is a dictator if it is $v(j) = 1$. We will consider games free of dictator and dummies.

A simple, weighted majority game is described by a representation $(q; \mathbf{w})$ where \mathbf{w} is a vector (w_1, w_2, \dots, w_n) of positive weights, $q > \frac{1}{2} \cdot w(N) = \frac{1}{2} \cdot \sum_{j \in N} w_j$ is the winning quota and $v(S) = 1 \Leftrightarrow w(S) = \sum_{j \in S} w_j \geq q$.

A representation $(q; \mathbf{w})$ of a weighted majority game is homogeneous if $w(S) = q$ for any S of WM .

Homogeneous weighted majority games are games for which (at least) a homogeneous representation exists.

Consider now the class of all simple, constant sum n person homogeneous weighted majority games⁵.

ization, a balanced lottery on a P game has the property to balance the set of minimal winning coalitions of the game.

⁴Such results may be found in Isbell paper or are straightforward consequences of his work.

⁵At the origins of game theory, homogeneous weighted majority games (h.w.m.g.) have been introduced in [11] by Von Neumann-Morgenstern and have been studied mainly under the constant sum condition. Subsequent treatments in the absence of the constant sum condition (with deadlocks) may be found e.g. in [6] by Ostmann, who gave the proof that any h.w.m.g. (including non constant sum ones) has a unique minimal homogeneous representation, and in [10]. Generally speaking, the homogeneous minimal representation is to be thought in a broader sense but hereafter the restrictive application concerning the constant sum case is used.

Proposition 2.1. *All games of such a class admit a minimal homogeneous representation, that is a (homogeneous) representation $(q; \mathbf{w})$ such that all weights are integers, there are players with (minimum) weight 1 and $q = \frac{1+w(N)}{2}$, which implies for any $S \in WM$, $w(S) - w(\tilde{S}) = 1$.*

Hereafter we will suppose that in such a representation the vector \mathbf{w} is ordered according to the convention $w_1 = 1, w_j \leq w_{j+1}$ for any j .

Proposition 2.2. *The cardinality of the WM set of our class may be either greater or equal (but not lower) than n . See [3], p. 185.*

Definition 2.1. *We call Parsimonious games (hereafter P games) the subset of constant sum homogeneous weighted majority games characterized by the parsimony property to have for any given number n of non dummy players in the game the smallest number, i.e. exactly n , of minimal winning coalitions.*

As said before, general properties of P games have been studied by Isbell. A recall of such properties is given in the next section divided in three subsections devoted respectively to:

- a The one to one correspondence between players and minimal winning coalitions
- b The incidence matrix upon winning minimal coalitions of a P game
- c The transposition approach to the twin relationship in P games

3 Main results on Parsimonious games

3.1 One to one correspondence between players and minimal winning coalitions

Let us recall that in any n person P game there are h ($2 \leq h \leq n - 2$) types; players of type t share, in the minimal homogeneous representation, a type weight w_t and the ordering of types satisfies $w_t < w_{t+1}$. As said in the introduction, odd (even) players are those of type t odd (even). Let us denote by x_t the numerousness of players of type t and give the following:

Definition 3.1. *Let G be a P game with h types; the type representation of G is the vector $\mathbf{x} = (x_1, \dots, x_t, \dots, x_h)$, while the free type representation ${}_f\mathbf{x} = (x_1, \dots, x_t, \dots, x_{h-1})$ is the one obtained by deleting its last component x_h .*

The main Isbell result ([3], p. 185, penultimate indent) was that

Proposition 3.1. *In all P games the one to one correspondence between players and minimal winning coalitions is described by a unique general rule. Precisely, to any non top player (of type $t < h$) there corresponds the coalition made by that player and all players of alternative parity and greater weight; to the top player the coalition made by all odd players (which may or not include the top player).*

Keeping account that all minimal winning coalitions share the same minimal winning quota and hence the same sum of individual weights, the following consequences of the general rule are straightforward for the sequence of type weights:

$$w_1 = 1 \quad \text{initial condition} \tag{3.1a}$$

$$w_2 = x_1 \cdot w_1 = x_1 \tag{3.1b}$$

$$w_t = x_{t-1} \cdot w_{t-1} + w_{t-2}, \quad t = 3, \dots, h - 1 \tag{3.1c}$$

$$w_h = (x_{h-1} - 1) \cdot w_{h-1} + w_{h-2} \tag{3.1d}$$

Hence

Proposition 3.2. *In any P game with h types the weights of all types are unequivocally given as the recursive implicit function (3.1) of the free type representation $f\mathbf{x} = (x_1, \dots, x_t, \dots, x_{h-1})$.*

Moreover

Proposition 3.3. *To preserve strict monotony of the sequence of type weights, the type representation needs to satisfy, in addition to the binding constraint $x_h = 1$, the lower bounds: $x_1 > 1$ (otherwise $w_2 = w_1 = 1$) and $x_{h-1} > 1$ (otherwise $w_h = w_{h-2}$). No other constraints hold ([3], p. 185).*

In the next section we will exploit also the following relations ([3], p. 186, first indent) between type weights of the players:

$$\text{for } t \text{ odd: } w_t = 1 + \sum_{s \text{ even} < t} x_s \cdot w_s \quad (3.2a)$$

$$\text{for } t \text{ even: } w_t = \sum_{s \text{ odd} < t} x_s \cdot w_s \quad (3.2b)$$

Formula (3.2a) comes from the minimal winning character of the coalitions made by one of the peones and all even players and, respectively, by one player of type t (odd) and all even players of greater weight; formula (3.2b) from the minimal winning character of the coalitions made by all odd players and, respectively, by one player of type t even and all odd players of greater weight. Hence

Proposition 3.4.

$$q = 1 + \sum_{s \text{ even}} x_s \cdot w_s = \sum_{s \text{ odd}} x_s \cdot w_s \quad (3.3)$$

$$w(N) = 1 + 2 \cdot \sum_{s \text{ even}} x_s \cdot w_s = 2 \cdot \sum_{s \text{ odd}} x_s \cdot w_s - 1 \quad (3.4)$$

3.2 The structure of the incidence matrix of a P game

The incidence matrix A upon minimal winning coalitions of a P game is the square binary $n \times n$ matrix obtained through an association of players to columns (in order of non decreasing weights), and of minimal winning coalitions to rows (in the order induced by the one to one correspondence explained in the previous subsection) and putting $a_{ij} = 1$ if player j is a member of the minimal winning coalition i (or S_i associated to player i), and $a_{ij} = 0$ otherwise.

The square submatrix M obtained from A by deleting the last row and the last column turns out to be a matrix with a special structure.

Denoting by $B_{r,c}$, with $r, c = 1, \dots, h-1$ the block (dimension $x_r \times x_c$) associated to coalitions of type r and players of type c , it turns out that:

Proposition 3.5. *Diagonal blocks ($r = c$) are square identity matrices;*

Proposition 3.6. *Those with $c < r < h$, i.e. under the diagonal, are rectangular null matrices;*

Proposition 3.7. *Those with $r < c < h$ are rectangular with elements identically equal to 1(0) if $(r + c)$ is odd (even).*

Moreover keeping in consideration also the last row and column of A there are other blocks with one or both r, c equal to h :

Proposition 3.8. *For $c = h$ the elements of the blocks are still identically equal to 1 (0) if $(r + c)$ is odd (even);*

Proposition 3.9. *For $r = h$ the elements are identically equal to 1 (0) for c odd (even).*

As we shall see in next sections, the incidence matrix is a helpful tool to study a P game and in particular the connections between twin P games.

3.3 Twin relationships in P games

The special structure of the incidence matrices of P games inspired Isbell ([3], p. 185, last indent) in recognizing an important property embedded in the following:

Proposition 3.10. *The transposed A^T of the incidence matrix of any (primal) P game G is still the incidence matrix of a P game \overline{G} , to be seen as the twin (dual in Isbell terminology) of G .*

Remark 3.1. *It is important to underline that in A^T the association of all non top players to columns and of the corresponding minimal winning coalitions to rows holds just in the reverse order (from the last but most powerful in the game to the least powerful) than the standard one (which goes from the least powerful to the last but most powerful). In more detail in A^T for any $t = 1, \dots, h - 1$ and $r < h$ the block indexed $B_{r,t}$ is associated to the set of players of type $h - t$ and to coalitions of type $h - r$ in \overline{G} ; only the last column still remains associated to the top player (for $r < h$ $B_{r,h}$ concerns coalitions of type $h - r$ and the top player in \overline{G}) and the last row to the coalition corresponding to the top player (for $r < h$ $B_{h,r}$ concerns players of type $h - r$ and the coalition of type h in \overline{G}).*

Remark 3.2. *For all $t < h$ the number of players (and of the associated coalitions) of type t in G is the number of players (and of the associated coalitions) of type $h - t$ in \overline{G} .*

Hence, more formally:

Proposition 3.11. *M^T , the transposed matrix of the submatrix M (of A), should be the incidence submatrix \overline{M} of another P game \overline{G} , but now with \overline{M} block diagonal lower triangular with ordering of coalitions and players inverted (i.e. non increasing) with respect to that of M . To obtain \overline{A} complete the transposition of the last row and column.*

If we now wish to recover the standard ordering in \overline{A} , we should transform A^T in the following way to obtain a modified transposed A^τ : at first rewrite all rows of A^T inverting the order of all but the last entry in each row, then rewrite all columns inverting the order of all but the last entry in each column. It turns out that:

Result 3.1.

$$\begin{aligned}
 a_{i,j}^T &= a_{n-i,n-j}^\tau \quad \text{for any } i, j < n \\
 a_{i,j}^T &= a_{n-i,n}^\tau \quad \text{for any } i < n \\
 a_{n,j}^T &= a_{n,n-j}^\tau \quad \text{for any } j < n \\
 a_{n,n}^T &= a_{n,n}^\tau
 \end{aligned} \tag{3.5}$$

After these modifications A^τ turns out to be the incidence matrix \overline{A} of \overline{G} coherent with the standard ordering convention of players and coalitions. To understand the point see also the examples in section 6.

We signal that an alternative, easier and more immediate understanding of the connection between a couple of twin P games, is given⁶ through their free type representations and is resumed by:

Proposition 3.12. *For any $t = 1, \dots, h - 1$, $\overline{x}_t = x_{h-t}$.*

The proposition says that a couple of twins share the same value of n and h and that their free type representation vectors are each other symmetric.

4 The minimal winning quota in twin P games

In this section we will show that:

Theorem 4.1. *Let G, \overline{G} be any couple of twin P games and q and \overline{q} their minimal winning quotas; then $q = \overline{q}$.*

Theorem 4.1 is a straightforward corollary of the following theorem linking the minimal winning quota and the determinant of the incidence matrix of any P game:

Theorem 4.2. *The incidence matrix A of any P game G with h types and minimal winning quota q is non singular and its determinant is $|A| = (q) \cdot (-1)^{h+1}$.*

Proof. Let us denote by a_i , $i = 1, \dots, n$ the row vectors of A . Recalling that the submatrix M is upper triangular, we wish to transform A in a triangular matrix $T(A)$ with elements t_{ij} by adding to its last row a proper linear combination $\sum_{i=1, \dots, n-1} z_i a_i$ of the rows of M . This would clearly imply that:

Proposition 4.1.

$$|T(A)| = |A| = t_{nn}$$

We prove now:

Proposition 4.2. *The coefficients of such a linear combination are given by*

$$z_i = (-1)^t \cdot w_i$$

if the coalition i (of type t) is associated to a player i of type t .

□

To check that Prop. 4.2 holds we must verify that the elements of the last row of T satisfy:

$$t_{nj} = \sum_{i=1, \dots, n-1} (-1)^t w_i \cdot a_{ij} + a_{nj} = 0 \quad \text{for any } j = 1, \dots, n - 1 \quad (4.1a)$$

$$t_{nn} = \sum_{i=1, \dots, n-1} (-1)^t w_i \cdot a_{in} + a_{nn} = (q) \cdot (-1)^{h+1} \quad \text{for } j = n \quad (4.1b)$$

We now give the proof of equation (4.1a).

⁶As suggested by a rather cryptic sentence in [3], last row, p. 185, on the point see also [8], sect. 3, p. 7.

Proof. As a consequence of Propositions 3.5, 3.6, 3.7, 3.9 of section 3.2, a null vector for the first $n - 1$ components of the row vector t_n is obtained if and only if the following relations hold:

$$\sum_{0 < s \text{ even} < t} x_s \cdot z_s + z_t + 1 = 0 \quad \text{for columns of type } t < h \text{ odd} \quad (4.2a)$$

$$\sum_{0 < s \text{ odd} < t} x_s \cdot z_s + z_t + 0 = 0 \quad \text{for columns of type } t < h \text{ even} \quad (4.2b)$$

Then, starting from $t = 1$, recursively:

$$z_1 = -1 = -w_1 \quad \text{by (3.1a)}$$

for $t = 2$:

$$x_1 \cdot z_1 + z_2 = 0; \quad z_2 = x_1 \cdot w_1 = w_2 \quad \text{by (3.1b)}$$

for $t = 3$:

$$x_2 \cdot z_2 + z_3 + 1 = 0; \quad z_3 = -(1 + x_2 \cdot w_2) = -w_3 \quad \text{by (3.2a)}$$

for $t = 4$:

$$x_1 \cdot z_1 + x_3 \cdot z_3 + z_4 = 0; \quad z_4 = (x_1 \cdot w_1 + x_3 \cdot w_3) = w_4 \quad \text{by (3.1b)}$$

and, by immediate induction on t , $z_t = -w_t$ for t odd and $z_t = w_t$ for t even. \square

We give now the proof of equation (4.1b).

Proof. Suppose (4.1a) holds and consider at first an even h ; then (by Proposition 3.8) $a_{in} = 1$ for all coalitions i of odd type (odd t) and 0 for all coalitions of even type, while (by Proposition 3.9) $a_{nn} = 0$, and we have

$$t_{nn} = \sum_{i=1, \dots, n-1} (-1)^t w_i \cdot a_{in} + a_{nn} = -(x_1 \cdot w_1 + x_3 \cdot w_3 + \dots + x_{h-1} \cdot w_{h-1}) \quad (4.3)$$

Formula (4.3) gives t_{nn} as the opposite of the sum of the weights of all odd players, which is the total weight q of the minimal winning coalition associated to the top player.

Hence $t_{nn} = -q = (q) \cdot (-1)^{h+1}$.

On the other side, if h is odd $a_{nn} = 1$ and $a_{in} = 1$ for all even t and we have:

$$t_{nn} = \sum_{i=1, \dots, n-1} (-1)^t \cdot w_i \cdot a_{in} + a_{nn} = (x_2 \cdot w_2 + x_4 \cdot w_4 + \dots + x_{h-1} \cdot w_{h-1}) + 1 \quad (4.4)$$

Then (4.4) gives t_{nn} as the sum of the weights of all even players and one of the players with minimum weight, which is the total weight q of a minimal winning coalition associated to one peone.

Hence $t_{nn} = q = (q) \cdot (-1)^{h+1}$ and the proof of Theorem 4.2 has been completed. \square

After that Theorem 4.1 follows immediately from the chain:

$$(q) \cdot (-1)^{h+1} = |A| = |A^T| = |\bar{A}| = (\bar{q}) \cdot (-1)^{h+1}$$

5 Balanced lotteries and twin relationships in P games

Definition 5.1. A lottery on a n person P game G is a probability distribution on the minimal winning coalitions of G , i.e. a column vector $\mathbf{p} = (p_1, \dots, p_j, \dots, p_n)$ with p_j the probability assigned to the minimal winning coalition S_j .

Definition 5.2. A balanced lottery on G is a lottery \mathbf{p} which assigns to all players the same probability π to be a member of the minimal winning coalition selected by the lottery. Hence and more formally a balanced lottery should satisfy:

$$\mathbf{p}^T \cdot A = \pi \cdot \mathbf{1}^T$$

The following results connect balanced lotteries and twin relationships in P games.

Theorem 5.1. Let G and \overline{G} be a couple of twin games. There exists just one balanced lottery \mathbf{p} on G ; its probabilities are given by the normalized individual weights of the twin \overline{G} according to the following rule:

$$p_j = \overline{w}_{n-j} / \overline{w}(N) \quad \text{for } j = 1, \dots, n-1 \quad (5.1a)$$

$$p_n = \overline{w}_n / \overline{w}(N) \quad (5.1b)$$

while

$$\pi = \overline{q} / \overline{w}(N) = q / w(N) \quad (5.2)$$

Proof. By definition 5.2 a balanced lottery \mathbf{p} on G must satisfy

$$\mathbf{p}^T \cdot A = \pi \cdot \mathbf{1}^T \quad (5.3)$$

while by Prop. 3.10:

$$\overline{A}\overline{\mathbf{w}} = A^T\overline{\mathbf{w}} = \overline{q}\mathbf{1} \quad (5.4)$$

Exploiting non singularity and hence invertibility of A , multiply (5.3) and (5.4) respectively by A^{-1} and $(A^T)^{-1}$ to obtain:

$$\mathbf{p}^T = \pi \mathbf{1}^T A^{-1} \quad \text{or} \quad \mathbf{p} = \pi (A^{-1})^T \mathbf{1} \quad (5.5)$$

$$\overline{\mathbf{w}} = \overline{q} (A^T)^{-1} \mathbf{1} \quad (5.6)$$

Being $(A^{-1})^T = (A^T)^{-1}$ it is:

$$\pi^{-1} \mathbf{p} = \overline{q}^{-1} \overline{\mathbf{w}} \quad (5.7)$$

so as:

$$\mathbf{p} = \frac{\pi}{\overline{q}} \overline{\mathbf{w}} \quad (5.8)$$

Premultiplying both sides of (5.8) by $\mathbf{1}^T$ we obtain

$$\mathbf{1}^T \mathbf{p} = 1 = \frac{\pi}{\overline{q}} \mathbf{1}^T \overline{\mathbf{w}} = \frac{\pi}{\overline{q}} \overline{w}(N) \quad (5.9)$$

or

$$\frac{\pi}{\overline{q}} = \frac{1}{\overline{w}(N)} \quad (5.10)$$

and finally, by Theorem 4.1

$$\pi = \frac{\overline{q}}{\overline{w}(N)} = \frac{q}{w(N)} \quad (5.11)$$

which proves (5.2) and

$$\mathbf{p} = \frac{1}{\bar{w}(N)} \bar{\mathbf{w}} \quad (5.12)$$

which gives the probability vector of the balanced lottery, which is unique thanks to the non singularity of A . \square

To understand the behaviour of the first $n - 1$ components of the vector \mathbf{p} , given in Theorem 5.1, recall that by Prop. 3.11 the ordering of coalitions and players in A^T (in \bar{G}) is, except for the coalition associated to the top player, reversed respect to the one in A (in G); this implies that, for such coalitions, the probability assigned to a coalition of type t (in G) is the (normalized) weight of a player of type $h - t$ in \bar{G} .

We conclude this section with:

Proposition 5.1. *If the division of payoffs among members of a minimal winning coalition is proportional to their individual weights (in such a coalition), the balanced lottery gives to each player in the game G an expected payoff proportional to her individual weight in G (and not in \bar{G} !).*

We wish to underline here that the rationality of the formation of a minimal winning coalition, with division of payoff among its members proportional to their individual weights, goes back to the early stage of n person game theory in coalitional form; in particular, for P games the set of n “imputations” generated by this logic are a stable set solution *a la* Von Neumann-Morgenstern of the P game G^7 .

The proof of the Prop. 5.1 is immediate as the expected reward $E(j)$ for any player j is given, keeping account of (5.10), by:

$$E(j) = (w_j/q)\pi = (w_j/q)(q/w(N)) = (w_j/w(N)) \quad (5.13)$$

In this way both expected payoffs and actual payoff division are proportional to the weights of the minimal homogeneous representation of the game. We could say that *ex ante* and *ex post* fairness are reconciled. We recall that this result is similar to the one proposed by Montero [4] and Montero-Vidal Puga [5] as the outcome of a more sophisticated model of bargaining.

6 Some examples

Ex. 1 Let G be the nine person P game with minimal homogeneous representation given by:

$$26; 1, 1, 1, 3, 4, 4, 11, 11, 15$$

⁷As well known [11] a stable set solution is a set V of imputations such that: there is no dominance among imputations of V , and any imputation not in V is dominated by at least one imputation of V .

The incidence matrix A of G is then:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The transposition of A gives the following incidence matrix \bar{A} of the game \bar{G} , the twin of G , whose minimal homogeneous representation is $26; 1, 1, 2, 2, 5, 7, 7, 7, 19$,

$$\bar{A} = A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Note that in this matrix the ordering of all players, but the top, on columns and of the corresponding coalitions on rows is reversed with respect to the standard one; for example, the first block of three rows corresponds to the three coalitions of type $t = 4$ in the game \bar{G} ; such coalitions are formed by a vice top player of type 4 and by the top player.

If we wish to have \bar{A} in the form coherent with the standard ordering of players and coalitions we modify A^T into the modified transposed A^τ :

$$\bar{A} = A^\tau = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is immediate to check that G and \overline{G} have the same minimal winning quota $q = \overline{q} = 26$. The balanced lottery on G is given, according to (5.1a) and (5.1b) by the vector:

$$(7/51, 7/51, 7/51, 5/51, 2/51, 2/51, 1/51, 1/51, 19/51)$$

Then, for example, the balanced lottery gives probability $7/51$ to the coalition formed by one of the peones, the player with weight 3 and both the last but top players.

Conversely, the balanced lottery on \overline{G} is given by

$$(11/51, 11/51, 4/51, 4/51, 3/51, 1/51, 1/51, 1/51, 15/51)$$

Note the extreme low probability ($1/51$) given in both games to the coalitions formed by the top and by one of the last but top players.

Ex. 2 Let G be the nine person P game with minimal homogeneous representation given by:

$$25; 1, 1, 1, 3, 4, 7, 7, 7, 18$$

The incidence matrix A of G is then:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The transposition of A gives the following incidence matrix \overline{A} of the game \overline{G} , the twin of G , whose minimal homogeneous representation is still $25; 1, 1, 1, 3, 4, 7, 7, 7, 18$:

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Note that G and \overline{G} have the same minimal homogeneous representation, which reveals that they are the same game, even if, at first sight, it seems different from A . Really the difference is not substantial as it comes merely from the reversion of the order of non top players and associated coalitions.

The identity between G and \overline{G} is recovered once we compute A^τ .

$$A^\tau = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is immediately checked that the A^τ version of \overline{A} is equal to A , which confirms that $\overline{G} = G$, as revealed also by the bilateral symmetry of the representation vector ${}_f\mathbf{x}$ or by the coincidence of ${}_f\mathbf{x}$ and ${}_f\overline{\mathbf{x}}$ both equal to $(3,1,1,3)$.

To resume we could say that

Proposition 6.1. *G and \overline{G} are coincident (are the same game) if there is equality between: a) A and the A^τ version of \overline{A} or b) between the free type representations ${}_f\mathbf{x}$ and ${}_f\overline{\mathbf{x}}$ or c) between the minimal homogeneous representations (q, \mathbf{w}) and $(\overline{q}; \overline{\mathbf{w}})$.*

On the contrary, the formal equality between A and the A^τ version of \overline{A} is not a necessary condition for the coincidence between G and \overline{G} , except for games G with only two types ($h = 2$).

Concerning this point the following result holds:

Proposition 6.2. *The incidence matrix A of any P game G with $h = 2$ types of players satisfies $A = A^T = A^\tau$. Hence for such games surely $G = \overline{G}$.*

The proof is immediate keeping account that no reversion of the order of non top players (and associated coalitions) is possible when $h - 1 = 1$, while all but the last entries of the last row and the last column are 1; the same information comes from the bilateral symmetry of ${}_f\mathbf{x}$ which implies equality of ${}_f\mathbf{x}$ and ${}_f\overline{\mathbf{x}}$.

Finally the balanced lotteries on both G and \overline{G} are given by :

$$7/25, 7/25, 7/25, 4/25, 3/25, 1/25, 1/25, 1/25, 18/25$$

More generally it is obvious that:

Proposition 6.3. *A pair G and \overline{G} of identical twin games have the same balanced lottery, while not identical twins G and \overline{G} have the same minimal winning quota but different balanced lotteries.*

7 Conclusions

In his smart treatment of Parsimonious games, going back to the early stage of game theory development, Isbell introduced a twin relationship, which at least at our knowledge, did not find any further investigation in relevant literature.

Looking carefully at the properties of twin games and signalling preliminarily that twins have free type representation vectors which are each other symmetric, we show in this paper that a) twin games have the same minimal winning quota and b) any P game has just one balanced lottery, given by the (properly normalized and reordered) vector of individual weights of its twin.

Our paper is purely theoretical and we do not discuss here any application of our results; yet we feel confident that, keeping account of the prominent role played by symmetry in the design of the universe, applications of symmetric properties in hard as well as in social sciences may be found so that this could be a promising road for future research⁸.

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⁸There is a huge literature concerning symmetry in hard sciences; let us recall here some prominent sentences: “Symmetry is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection” [12, p. 5]; “Symmetry considerations dominate modern fundamental physics both in quantum theory and in relativity” [1, p. ix preface]; “Symmetry plays an essential role in science” [9, editor foreword].