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Wave simulation in heterogeneous transversely isotropic porous media with fractional attenuation: a Cartesian grid approach

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Abstract

A time-domain numerical modeling of transversely isotropic Biot poroelastic waves is proposed. The viscous dissipation occurring in the pores is described using the dynamic permeability model developed by Johnson-Koplik-Dashen (JKD). Some of the coefficients in the Biot-JKD model are proportional to the square root of the frequency. In the time-domain, these coefficients introduce shifted fractional derivatives of order $1/2$, involving a convolution product. Based on a diffusive representation, the convolution kernel is replaced by a finite number of memory variables that satisfy local-in-time ordinary differential equations, resulting in the Biot-DA (diffusive approximation) model. The properties of both the Biot-JKD and the Biot-DA model are analyzed: hyperbolicity, decrease of energy, dispersion. To determine the coefficients of the diffusive approximation, two approaches are analyzed: Gaussian quadratures and optimization methods in the frequency range of interest. The nonlinear optimization is shown to be the better way of determination. A splitting strategy is then applied to approximate numerically the Biot-DA equations. The propagative part is discretized using a fourth-order ADER scheme on a Cartesian grid, whereas the diffusive part is solved exactly. An immersed interface method is implemented to take into account heterogeneous media on a Cartesian grid and to discretize the jump conditions at interfaces. Numerical experiments are presented. Comparisons with analytical solutions show the efficiency and the accuracy of the approach, and some numerical experiments are performed to investigate wave phenomena in complex media, such as multiple scattering across a set of random scatterers.

Keywords: porous media; elastic waves; Biot-JKD model; fractional derivatives; time splitting; finite-difference methods; immersed interface method

1. Introduction

A porous medium consists of a solid matrix saturated with a fluid that circulates freely through the pores [1, 2, 3]. Such media are involved e.g. in natural rocks, engineering composites [4] and biological materials [5]. The most widely-used model describing the propagation

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15 of mechanical waves in porous media has been proposed by Biot in 1956 [1, 6]. It includes two
16 classical waves (one "fast" compressional wave and one shear wave), in addition to a second
17 "slow" compressional wave, which is highly dependent on the saturating fluid. This slow wave
18 was observed experimentally in 1981 [7], thus confirming the validity of Biot's theory.

19 Two frequency regimes have to be distinguished when dealing with poroelastic waves. In
20 the low-frequency range (LF), the flow inside the pores is of Poiseuille type [1]. The viscous
21 efforts are then proportional to the relative velocity of the motion between the fluid and the solid
22 components. In the high-frequency range (HF), modeling the dissipation is a more delicate task.
23 Biot first presented an expression for particular pore geometries [6]. In 1987, Johnson-Koplik-
24 Dashen (JKD) published a general expression for the HF dissipation in the case of random pores
25 [8], where the viscous efforts depend on the square root of the frequency. When writing the
26 evolution equations in the time domain, time fractional derivatives are introduced, which involves
27 convolution products [9].

28 Since the 90's, many time-domain simulation methods have been developed in the low-
29 frequency range for isotropic media: see [10] and the introduction of [11] for general reviews.
30 To increase the scope of such numerical methods, two significant difficulties must be tackled.
31 The first one is to take into account more realistic constitutive laws in the simulations. Indeed,
32 rocks or bones for example are anisotropic, and usually described by transversely isotropy. The
33 second difficulty concerns the high-frequency range where many applications are performed, like
34 ultrasonic imaging. Due to the presence of fractional derivatives, the past of the solution needs
35 to be stored, increasing dramatically the memory requirements and the computational time. The
36 aim of this paper is to present a numerical method for the propagation of waves in transversely
37 isotropic media and in the high-frequency range.

38 In the high-frequency range, only two numerical approaches have been proposed in the liter-
39 ature to integrate the Biot-JKD equations directly in the time-domain. The first approach consists
40 in a straightforward discretization of the fractional derivatives defined by a convolution product
41 in time [12]. In the example given in [12], the solution is stored over 20 time steps. The second
42 approach is based on the diffusive representation of the fractional derivative [13]. The convolu-
43 tion product is replaced by a continuum of memory variables satisfying local differential equa-
44 tions [14]. This continuum is then discretized using Gaussian quadrature formulae [15, 16, 17],
45 resulting in the Biot-DA (diffusive approximation) model. In the example proposed in [13], 25
46 memory variables are used, which is equivalent, in terms of memory requirement, to storing 25
47 time steps.

48 For transversely isotropic poroelastic media, the earliest work in the low-frequency range
49 is based on an operator splitting in conjunction with a Fourier pseudospectral method [18]. Re-
50 cently, a Cartesian-grid finite volume method has been developed [19]. One of the first work
51 combining anisotropic media and high-frequency range is proposed in [20]. However, the dif-
52 fusive approximation proposed in the latter article has three limitations. Firstly, the quadrature
53 formulae make the convergence towards the original fractional operator very slow. Secondly, in
54 the case of small frequencies, the Biot-DA model does not converge towards the Biot-LF model.
55 Lastly, the number of memory variables required for a given accuracy is not specified.

56 In our previous works, we focused on isotropic poroelasticity, in the low-frequency range
57 [21, 11, 22], and then in the high-frequency range [23, 24]. Here we extend this approach to
58 transversely isotropic media, introducing also an important improvement concerning the approx-
59 imation of the fractional derivatives. Thanks to a non linear optimization of the quadrature coef-
60 ficients, the number of memory variables is drastically reduced. Moreover, the quadrature coef-
61 ficients are always positive, which ensures the stability of the Biot-DA model and the stability of

62 its numerical discretization.

63 This article is organized as follows. The original Biot-JKD model is outlined in § 2 and the
64 diffusive representation of fractional derivatives is described. The energy decrease is proven, and
65 a dispersion analysis is done. In § 3, approximation of the diffusive model is presented, leading
66 to the Biot-DA system. The properties of this system are also analyzed: energy, hyperbolicity
67 and dispersion. Determination of the quadrature coefficients involved in the Biot-DA model are
68 investigated in § 3.4. Gaussian quadrature formulae and optimization methods are successively
69 proposed and compared, the latter being finally preferred. The numerical modeling of the Biot-
70 DA system is addressed in § 4, where the equations of evolution are split into two parts: the
71 propagative part is discretized using a fourth-order finite-difference scheme, and the diffusive
72 part is solved exactly. An immersed interface method is implemented to account for the jump
73 conditions and for the geometry of the interfaces on a Cartesian grid when dealing with hetero-
74 geneous media. Numerous numerical experiments are presented in § 5, validating the method
75 developed in this paper. In § 6, a conclusion is drawn and some futures lines of research are
76 suggested.

77 2. Physical modeling

78 2.1. Biot model

79 We consider a transversely isotropic porous medium, consisting of a solid matrix saturated
80 with a fluid that circulates freely through the pores [1, 2, 3]. The subscripts 1, 3 represent the x ,
81 z axes, where z is the symmetry axis. The perturbations propagate with a wavelength λ .

82 The Biot model involves 15 positive physical parameters: the density ρ_f , the dynamic vis-
83 cosity η and the bulk modulus K_f of the fluid, the density ρ_s and the bulk modulus K_s of the
84 grains, the porosity $0 \leq \phi \leq 1$, the tortuosities $\mathcal{T}_1 \geq 1$, $\mathcal{T}_3 \geq 1$, the absolute permeabilities at
85 null frequency κ_1 , κ_3 , and the symmetric definite positive drained elastic matrix \mathbf{C}

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{13} & 0 & 0 \\ c_{13} & c_{33} & 0 & 0 \\ 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & \frac{c_{11} - c_{12}}{2} \end{pmatrix}. \quad (1)$$

86 The linear Biot model is valid if the following hypotheses are satisfied [25]:

87 \mathcal{H}_1 : the wavelength λ is large in comparison with the characteristic radius of the pores r ;

88 \mathcal{H}_2 : the amplitudes of the waves in the solid and in the fluid are small;

89 \mathcal{H}_3 : the single fluid phase is continuous;

90 \mathcal{H}_4 : the solid matrix is purely elastic;

91 \mathcal{H}_5 : the thermo-mechanical effects are neglected, which is justified when the saturating fluid
92 is a liquid.

93 In the validity domain of homogenization theory (\mathcal{H}_1), two frequency ranges have to be distin-
94 guished. The frontier between the low-frequency (LF) range and the high-frequency (HF) range

95 is reached when the viscous efforts are similar to the inertial effects. The frequency transitions
 96 are given by [3]

$$f_{ci} = \frac{\eta \phi}{2\pi \mathcal{T}_i \kappa_i \rho_f} = \frac{\omega_{ci}}{2\pi}, \quad i = 1, 3. \quad (2)$$

97 Denoting \mathbf{u}_s and \mathbf{u}_f the solid and fluid displacements, the unknowns in a velocity-stress for-
 98 mulation are the solid velocity $\mathbf{v}_s = \frac{\partial \mathbf{u}_s}{\partial t}$, the filtration velocity $\mathbf{w} = \frac{\partial \mathbf{W}}{\partial t} = \frac{\partial}{\partial t} \phi (\mathbf{u}_f - \mathbf{u}_s)$, the
 99 elastic symmetric stress tensor $\underline{\sigma}$ and the acoustic pressure p . Under the hypothesis of small
 100 perturbations (\mathcal{H}_2), the symmetric strain tensor $\underline{\varepsilon}$ is

$$\underline{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T). \quad (3)$$

101 Using the Voigt notation, the stress tensor and the strain tensor are arranged into vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$

$$\begin{cases} \boldsymbol{\sigma} = (\sigma_{11}, \sigma_{33}, \sigma_{13})^T, \\ \boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{33}, 2\varepsilon_{13})^T. \end{cases} \quad (4)$$

Setting

$$\begin{cases} \xi = -\nabla \cdot \mathbf{W}, \quad \mathbf{C}^u = \mathbf{C} + m \boldsymbol{\beta} \boldsymbol{\beta}^T, & (5a) \\ \boldsymbol{\beta} = (\beta_1, \beta_1, \beta_3)^T, \quad \beta_1 = 1 - \frac{c_{11} + c_{12} + c_{13}}{3K_s}, \quad \beta_3 = 1 - \frac{2c_{13} + c_{33}}{3K_s}, & (5b) \\ K = K_s (1 + \phi (K_s/K_f - 1)), \quad m = \frac{K_s^2}{K - (2c_{11} + c_{33} + 2c_{12} + 4c_{13})/9}, & (5c) \end{cases}$$

where \mathbf{C}^u is the undrained elastic matrix and ξ is the rate of fluid, the poroelastic linear constitu-
 tive laws are

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C}^u \boldsymbol{\varepsilon} - m \boldsymbol{\beta} \xi, & (6a) \\ p = m (\xi - \boldsymbol{\beta}^T \boldsymbol{\varepsilon}). & (6b) \end{cases}$$

102 Using (5a) and (5b), we obtain equivalently

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon} - \boldsymbol{\beta} p, \\ p = m (\xi - \boldsymbol{\beta}^T \boldsymbol{\varepsilon}). \end{cases} \quad (7)$$

103 The symmetry of $\underline{\sigma}$ implies compatibility conditions between spatial derivatives of the stresses
 104 and the pressure, leading to the Beltrami-Michell equation [26, 27]

$$\begin{aligned} \frac{\partial^2 \sigma_{13}}{\partial x \partial z} &= \Theta_0 \frac{\partial^2 \sigma_{11}}{\partial x^2} + \Theta_1 \frac{\partial^2 \sigma_{33}}{\partial x^2} + \Theta_2 \frac{\partial^2 p}{\partial x^2} + \Theta_3 \frac{\partial^2 \sigma_{11}}{\partial z^2} + \Theta_0 \frac{\partial^2 \sigma_{33}}{\partial z^2} + \Theta_4 \frac{\partial^2 p}{\partial z^2}, \\ \Theta_0 &= -c_{55} \frac{c_{13}}{c_{11} c_{33} - c_{13}^2}, \quad \Theta_1 = -\frac{c_{11}}{c_{13}} \Theta_0, \quad \Theta_2 = \beta_1 \Theta_0 + \beta_3 \Theta_1, \\ \Theta_3 &= -\frac{c_{33}}{c_{13}} \Theta_0, \quad \Theta_4 = \beta_3 \Theta_0 + \beta_1 \Theta_3. \end{aligned} \quad (8)$$

105 If the medium is isotropic and in the elastic limit case ($\beta_1 = \beta_3 = 0$), we recover the usual
 106 equation of Barré de Saint-Venant.

$$\rho = \phi \rho_f + (1 - \phi) \rho_s, \quad \rho_{wi} = \frac{\mathcal{T}_i}{\phi} \rho_f, \quad i = 1, 3, \quad (9)$$

the conservation of momentum yields

$$\left\{ \begin{array}{l} \rho \frac{\partial \mathbf{v}_s}{\partial t} + \rho_f \frac{\partial \mathbf{w}}{\partial t} = \nabla \cdot \underline{\boldsymbol{\sigma}}, \end{array} \right. \quad (10a)$$

$$\left\{ \begin{array}{l} \rho_f \frac{\partial \mathbf{v}_s}{\partial t} + \text{diag}(\rho_{wi}) \frac{\partial \mathbf{w}}{\partial t} + \text{diag}\left(\frac{\eta}{\kappa_i} F_i(t)\right) * \mathbf{w} = -\nabla p, \end{array} \right. \quad (10b)$$

108 where $\text{diag}(d_i)$ denotes the 2×2 diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_3 \end{pmatrix}$, * denotes the time convolution prod-
109 uct and $F_i(t)$ are viscous operators. In LF, the flow in the pores is of Poiseuille type, and the
110 dissipation efforts in (10b) are given by

$$F_i(t) \equiv F_i^{LF}(t) = \delta(t) \iff F_i^{LF}(t) * w_i(x, z, t) = w_i(x, z, t), \quad i = 1, 3, \quad (11)$$

111 where δ is the Dirac distribution, which amounts to the Darcy's law.

112 2.2. High frequency dissipation: the JKD model

113 In HF, a Prandtl boundary layer occurs at the surface of the pores, where the effects of vis-
114 cosity are significant. Its width is inversely proportional to the square root of the frequency. Biot
115 first presented in 1956 an expression of the dissipation process for particular pore geometries
116 [6]. In 1987, a general expression of the viscous operator has been proposed by Johnson, Koplik
117 and Dashen, valid for random networks of pores with constant radii [8]. This function is the
118 most-simple one fitting the LF and HF limits and leading to a causal model. The only additional
119 parameters are the viscous characteristic length Λ_i . We take

$$P_i = \frac{4 \mathcal{T}_i \kappa_i}{\phi \Lambda_i^2}, \quad \Omega_i = \frac{\omega_{ci}}{P_i} = \frac{\eta \phi^2 \Lambda_i^2}{4 \mathcal{T}_i^2 \kappa_i^2 \rho_f}, \quad i = 1, 3, \quad (12)$$

120 where P_i are the Pride number (typically $P_i \approx 1/2$). Based on the Fourier transform in time,
121 $\widehat{F}_i(\omega) = \mathcal{F}(F_i(t)) = \int_{\mathbb{R}} F_i(t) e^{-j\omega t} dt$, the viscous operators given by the JKD model are

$$\begin{aligned} \widehat{F}_i^{JKD}(\omega) &= \left(1 + j\omega \frac{4 \mathcal{T}_i^2 \kappa_i^2 \rho_f}{\eta \Lambda_i^2 \phi^2} \right)^{1/2}, \\ &= \left(1 + j P_i \frac{\omega}{\omega_{ci}} \right)^{1/2}, \\ &= \frac{1}{\sqrt{\Omega_i}} (\Omega_i + j\omega)^{1/2}. \end{aligned} \quad (13)$$

122 Therefore, the terms $F_i(t) * w_i(x, z, t)$ involved in (6b) are

$$\begin{aligned} F_i^{JKD}(t) * w_i(x, z, t) &= \mathcal{F}^{-1} \left(\frac{1}{\sqrt{\Omega_i}} (\Omega_i + j\omega)^{1/2} \widehat{w}_i(x, z, \omega) \right), \\ &= \frac{1}{\sqrt{\Omega_i}} (D + \Omega_i)^{1/2} w_i(x, z, t). \end{aligned} \quad (14)$$

123 In the last relation of (14), $(D + \Omega_i)^{1/2}$ is an operator. $D^{1/2}$ is a fractional derivative in time of
 124 order 1/2, generalizing the usual derivative characterized by $\frac{\partial w_i}{\partial t} = \mathcal{F}^{-1}(j\omega \widehat{w}_i)$. The notation
 125 $(D + \Omega_i)^{1/2}$ accounts for the shift Ω_i in (14).

126 2.3. The Biot-JKD equations of evolution

127 The system (10) is rearranged by separating $\frac{\partial v_s}{\partial t}$ and $\frac{\partial w}{\partial t}$ and using the definitions of ε and ξ .
 128 Taking

$$\gamma_i = \frac{\eta}{\kappa_i} \frac{\rho}{\chi_i} \frac{1}{\sqrt{\Omega_i}}, \quad i = 1, 3, \quad (15)$$

one obtains the following system of evolution equations

$$\left\{ \begin{array}{l} \frac{\partial v_{s1}}{\partial t} - \frac{\rho_{w1}}{\chi_1} \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} \right) - \frac{\rho_f}{\chi_1} \frac{\partial p}{\partial x} = \frac{\rho_f}{\rho} \gamma_1 (D + \Omega_1)^{1/2} w_1 + G_{v_{s1}}, \end{array} \right. \quad (16a)$$

$$\left\{ \begin{array}{l} \frac{\partial v_{s3}}{\partial t} - \frac{\rho_{w3}}{\chi_3} \left(\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} \right) - \frac{\rho_f}{\chi_3} \frac{\partial p}{\partial z} = \frac{\rho_f}{\rho} \gamma_3 (D + \Omega_3)^{1/2} w_3 + G_{v_{s3}}, \end{array} \right. \quad (16b)$$

$$\left\{ \begin{array}{l} \frac{\partial w_1}{\partial t} + \frac{\rho_f}{\chi_1} \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} \right) + \frac{\rho}{\chi_1} \frac{\partial p}{\partial x} = -\gamma_1 (D + \Omega_1)^{1/2} w_1 + G_{w_1}, \end{array} \right. \quad (16c)$$

$$\left\{ \begin{array}{l} \frac{\partial w_3}{\partial t} + \frac{\rho_f}{\chi_3} \left(\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} \right) + \frac{\rho}{\chi_3} \frac{\partial p}{\partial z} = -\gamma_3 (D + \Omega_3)^{1/2} w_3 + G_{w_3}, \end{array} \right. \quad (16d)$$

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{11}}{\partial t} - c_{11}^u \frac{\partial v_{s1}}{\partial x} - c_{13}^u \frac{\partial v_{s3}}{\partial z} - m\beta_1 \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_{\sigma_{11}}, \end{array} \right. \quad (16e)$$

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{13}}{\partial t} - c_{55}^u \left(\frac{\partial v_{s3}}{\partial x} + \frac{\partial v_{s1}}{\partial z} \right) = G_{\sigma_{13}}, \end{array} \right. \quad (16f)$$

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{33}}{\partial t} - c_{13}^u \frac{\partial v_{s1}}{\partial x} - c_{33}^u \frac{\partial v_{s3}}{\partial z} - m\beta_3 \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_{\sigma_{33}}, \end{array} \right. \quad (16g)$$

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + m \left(\beta_1 \frac{\partial v_{s1}}{\partial x} + \beta_3 \frac{\partial v_{s3}}{\partial z} + \frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_p. \end{array} \right. \quad (16h)$$

129 The terms $G_{v_{s1}}$, $G_{v_{s3}}$, G_{w_1} , G_{w_3} , $G_{\sigma_{11}}$, $G_{\sigma_{13}}$, $G_{\sigma_{33}}$ and G_p have been introduced to model the
 130 forcing.

131 2.4. The diffusive representation

132 The shifted fractional derivatives [28] in (14) can be written ($i = 1, 3$)

$$(D + \Omega_i)^{1/2} w_i(x, z, t) = \int_0^t \frac{e^{-\Omega_i(t-\tau)}}{\sqrt{\pi(t-\tau)}} \left(\frac{\partial w_i}{\partial t}(x, z, \tau) + \Omega_i w_i(x, z, \tau) \right) d\tau. \quad (17)$$

133 The operators $(D + \Omega_i)^{1/2}$ are not local in time and involve the entire time history of w . Based
 134 on Euler's Gamma function, the diffusive representation of the totally monotone function $\frac{1}{\sqrt{\pi t}}$ is
 135 [14]

$$\frac{1}{\sqrt{\pi t}} = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\theta}} e^{-\theta t} d\theta. \quad (18)$$

136 Substituting (18) into (17) gives

$$(D + \Omega_i)^{1/2} w_i(x, z, t) = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\theta}} \psi_i(x, z, \theta, t) d\theta, \quad (19)$$

137 where the memory variables are defined as

$$\psi_i(x, z, \theta, t) = \int_0^t e^{-(\theta + \Omega_i)(t-\tau)} \left(\frac{\partial w_i}{\partial t}(x, z, \tau) + \Omega_i w_i(x, z, \tau) \right) d\tau. \quad (20)$$

For the sake of clarity, the dependence on Ω_i and w_i are omitted in ψ_i . From (20), it follows that the two memory variables ψ_i satisfy the ordinary differential equation

$$\begin{cases} \frac{\partial \psi_i}{\partial t} = -(\theta + \Omega_i) \psi_i + \frac{\partial w_i}{\partial t} + \Omega_i w_i, & (21a) \\ \psi_i(x, z, \theta, 0) = 0. & (21b) \end{cases}$$

138 The diffusive representation therefore transforms a non-local problem (17) into a continuum of
139 local problems (19). It should be emphasized at this point that no approximation have been made
140 up to now. The computational advantages of the diffusive representation will be seen in § 3 and
141 5, where the discretization of (19) and (21a) will yield a numerically tractable formulation.

142 2.5. Energy of Biot-JKD

143 Now, we express the energy of the Biot-JKD model (16).

144 **Proposition 1 (Decrease of the energy).** *Let us consider the Biot-JKD model (16) without forc-*
145 *ing, and let us denote*

$$E = E_1 + E_2 + E_3, \quad (22)$$

146 with

$$\begin{aligned} E_1 &= \frac{1}{2} \int_{\mathbb{R}^2} (\rho \mathbf{v}_s^T \mathbf{v}_s + 2 \rho_f \mathbf{v}_s^T \mathbf{w} + \mathbf{w}^T \text{diag}(\rho_{wi}) \mathbf{w}) dx dz, \\ E_2 &= \frac{1}{2} \int_{\mathbb{R}^2} \left((\boldsymbol{\sigma} + p \boldsymbol{\beta})^T \mathbf{C}^{-1} (\boldsymbol{\sigma} + p \boldsymbol{\beta}) + \frac{1}{m} p^2 \right) dx dz, \\ E_3 &= \frac{1}{2} \int_{\mathbb{R}^2} \frac{\eta}{\pi} \int_0^\infty (\mathbf{w} - \boldsymbol{\psi})^T \text{diag} \left(\frac{1}{\kappa_i \sqrt{\Omega_i \theta} (\theta + 2 \Omega_i)} \right) (\mathbf{w} - \boldsymbol{\psi}) d\theta dx dz. \end{aligned} \quad (23)$$

147 Then, E is an energy which satisfies

$$\begin{aligned} \frac{dE}{dt} &= - \int_{\mathbb{R}^2} \frac{\eta}{\pi} \int_0^\infty \left\{ \boldsymbol{\psi}^T \text{diag} \left(\frac{\theta + \Omega_i}{\kappa_i \sqrt{\Omega_i \theta} (\theta + 2 \Omega_i)} \right) \boldsymbol{\psi} \right. \\ &\quad \left. + \mathbf{w}^T \text{diag} \left(\frac{\Omega_i}{\kappa_i \sqrt{\Omega_i \theta} (\theta + 2 \Omega_i)} \right) \mathbf{w} \right\} d\theta dx dz \leq 0. \end{aligned} \quad (24)$$

148

149 Proposition 1 is proven in Appendix A. It calls for the following comments:

- 150 • the Biot-JKD model is stable;

- 151 • when the viscosity of the saturating fluid is neglected ($\eta = 0$), the energy of the system is
152 conserved;
- 153 • the terms E_1 and E_2 in (23) have a clear physical significance: E_1 is the kinetic energy,
154 and E_2 is the strain energy;
- 155 • the energy analysis is valid for continuously variable parameters.

156 2.6. Dispersion analysis

157 In this section, we derive the dispersion relation of the waves which propagate in a poroelastic
158 medium. This relation describes the frequency dependence of phase velocities and attenuations
159 of waves. For this purpose, we search for a general plane wave solution of (16)

$$\begin{cases} \mathbf{V} = (v_1, v_3, w_1, w_3)^T = \mathbf{V}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \\ \mathbf{T} = (\sigma_{11}, \sigma_{13}, \sigma_{33}, -p)^T = \mathbf{T}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \end{cases} \quad (25)$$

160 where $\mathbf{k} = k(\cos(\varphi), \sin(\varphi))^T$ is the wavevector, k is the wavenumber, \mathbf{V}_0 and \mathbf{T}_0 are the polar-
161 izations, $\mathbf{r} = (x, z)^T$ is the position, $\omega = 2\pi f$ is the angular frequency and f is the frequency.
162 On one hand, (25) is injected in (16e)-(16h). We obtain the 4×4 linear system:

$$\omega \mathbf{T} = -k \underbrace{\begin{pmatrix} c_{11}^u c_\varphi & c_{13}^u s_\varphi & \beta_1 m c_\varphi & \beta_1 m s_\varphi \\ c_{55}^u s_\varphi & c_{55}^u c_\varphi & 0 & 0 \\ c_{13}^u c_\varphi & c_{33}^u s_\varphi & \beta_3 m c_\varphi & \beta_3 m s_\varphi \\ \beta_1 m c_\varphi & \beta_3 m s_\varphi & m c_\varphi & m s_\varphi \end{pmatrix}}_{\mathbf{C}} \mathbf{V}, \quad (26)$$

163 where $c_\varphi = \cos(\varphi)$ and $s_\varphi = \sin(\varphi)$. On the other hand, substituting (25) into (16a)-(16d) gives
164 another 4×4 linear system:

$$-k \underbrace{\begin{pmatrix} c_\varphi & s_\varphi & 0 & 0 \\ 0 & c_\varphi & s_\varphi & 0 \\ 0 & 0 & 0 & c_\varphi \\ 0 & 0 & 0 & s_\varphi \end{pmatrix}}_{\mathbf{L}} \mathbf{T} = \omega \underbrace{\begin{pmatrix} \rho & 0 & \rho_f & 0 \\ 0 & \rho & 0 & \rho_f \\ \rho_f & 0 & \frac{\widehat{Y}_1^{JKD}(\omega)}{j\omega} & 0 \\ 0 & \rho_f & 0 & \frac{\widehat{Y}_3^{JKD}(\omega)}{j\omega} \end{pmatrix}}_{\mathbf{\Gamma}} \mathbf{V}, \quad (27)$$

165 where \widehat{Y}_1^{JKD} and \widehat{Y}_3^{JKD} are the viscodynamic operators [29]:

$$\widehat{Y}_i^{JKD} = j\omega \rho_{wi} + \frac{\eta}{\kappa_i} \widehat{F}_i^{JKD}(\omega), \quad i = 1, 3. \quad (28)$$

166 Since the matrix $\mathbf{\Gamma}$ is invertible, the equations (26) and (27) lead to the eigenproblem

$$\mathbf{\Gamma}^{-1} \mathbf{L} \mathbf{C} \mathbf{V} = \left(\frac{\omega}{k}\right)^2 \mathbf{V}. \quad (29)$$

167 The equation (29) is solved numerically. The two quasi-compressional waves are denoted qP_f
 168 (fast) and qP_s (slow), and the quasi-shear wave is denoted qS . The wavenumbers thus obtained
 169 depend on the frequency and on the angle φ . One of the eigenvalues is zero with multiplicity
 170 two, and the other non-zero eigenvalues correspond to the wave modes $\pm k_{pf}(\omega, \varphi)$, $\pm k_{ps}(\omega, \varphi)$
 171 and $\pm k_s(\omega, \varphi)$. Therefore three waves propagate symmetrically along the directions $\cos(\varphi)x +$
 172 $\sin(\varphi)z$ and $-\cos(\varphi)x - \sin(\varphi)z$.

173 The wavenumbers give the phase velocities $c_{pf}(\omega, \varphi) = \omega/\Re(k_{pf})$, $c_{ps}(\omega, \varphi) = \omega/\Re(k_{ps})$,
 174 and $c_s(\omega, \varphi) = \omega/\Re(k_s)$, with $0 < c_{ps} < c_{pf}$ and $0 < c_s$. The attenuations $\alpha_{pf}(\omega, \varphi) =$
 175 $-\Im(k_{pf})$, $\alpha_{ps}(\omega, \varphi) = -\Im(k_{ps})$ and $\alpha_s(\omega, \varphi) = -\Im(k_s)$ are also deduced. Both the phase
 176 velocities and the attenuations of Biot-LF and Biot-JKD are strictly increasing functions of the
 177 frequency. The high-frequency limits ($\omega \rightarrow \infty$ in (29)) of phase velocities $c_{pf}^\infty(\varphi)$, $c_{ps}^\infty(\varphi)$ and
 178 $c_s^\infty(\varphi)$ are recovered by diagonalizing the left-hand side of (16).

179 In figure 1, the physical parameters are those of medium Ω_0 (cf table 1), where the frequen-
 180 cies of transition are $f_{c1} = 25.5$ kHz, $f_{c3} = 85$ kHz. Figure 1 shows the dispersion curves in terms
 181 of the frequency at $\varphi = 0$ rad. Note that the vertical scales of the figures are radically different
 182 for the three waves. The high-frequency limit of the phase velocities of the quasi-compressional
 183 waves are $c_{pf}^\infty(0) = 5244$ m/s and $c_{ps}^\infty(0) = 975$ m/s, which justifies the denomination "fast" and
 "slow". Figure 1 calls for the following comments:

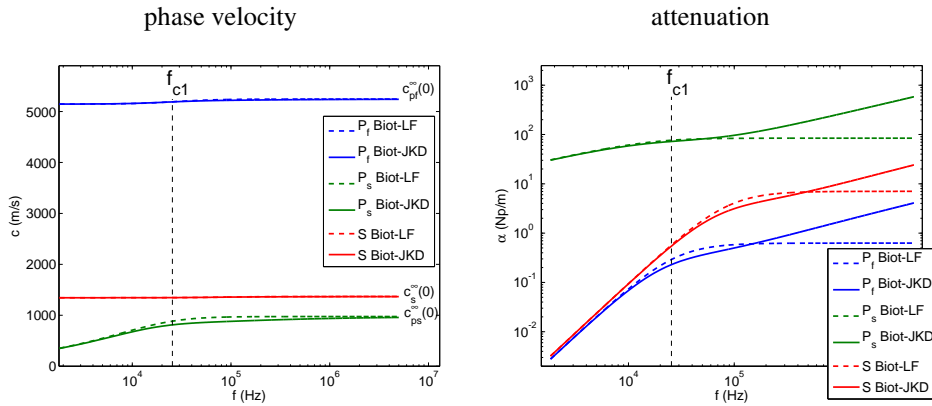


Figure 1: dispersion curves in terms of the frequency. Comparison between Biot-LF and Biot-JKD models at $\varphi = 0$ rad.

184

- 185 • when $f < f_{ci}$, the Biot-JKD and Biot-LF dispersion curves are very similar as might be
 186 expected, since $\widehat{F}_i^{JKD}(0) = \widehat{F}_i^{LF}(0) = 1$;
- 187 • the frequency evolution of the phase velocity and of the attenuation is radically different
 188 for the three waves, whatever the chosen model (LF or JKD): the effect of viscous losses
 189 is negligible on the fast wave, small on the shear wave, whereas it is very important on the
 190 slow wave;
- 191 • when $f \ll f_{ci}$, the slow compressional wave is almost static [30, 27]. When $f > f_{ci}$, the
 192 slow wave propagates but is greatly attenuated.

193 Taking

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (30)$$

194 the energy velocity vector \mathbf{V}_e is [18, 31]:

$$\begin{cases} \mathbf{V}_e = \frac{\langle \mathbf{P} \rangle}{\langle E_s + E_k \rangle} = \frac{\langle \mathbf{P} \rangle}{\langle E \rangle}, \\ \langle \mathbf{P} \rangle = -\frac{1}{2} \Re \left((\partial_x^2 (U_1 \cdot \mathbf{T})^T + \partial_z^2 (U_3 \cdot \mathbf{T})^T) \cdot \bar{\mathbf{V}} \right), \\ \langle E \rangle = \frac{1}{4} \Re \left(\left(1 + \frac{(\omega/k)^2}{|\omega/k|^2} \right) \mathbf{V}^T \mathbf{\Gamma} \bar{\mathbf{V}} \right), \end{cases} \quad (31)$$

195 where $\bar{\mathbf{V}}$ is the complex conjugate of \mathbf{V} , $\langle \mathbf{P} \rangle$ is the Umov-Poynting vector, $\langle E_k \rangle$ and $\langle E_s \rangle$ are the
 196 average kinetic and strain energy densities, and $\langle E \rangle$ is the mean energy density. The theoretical
 197 wavefronts are the locus of the end of energy velocity vector \mathbf{V}_e multiplied by the time of
 198 propagation. We will use this property in § 5 to validate the simulations.

199 3. The Biot-DA (diffusive approximation) model

200 The aim of this section is to approximate the Biot-JKD model, using a numerically tractable
 201 approach.

202 3.1. Diffusive approximation

203 The diffusive representation of fractional derivatives (19) is approximated by using a quadra-
 204 ture formula on N points, with weights a_ℓ^i and abscissae θ_ℓ^i ($i = 1, 3$):

$$\begin{aligned} (D + \Omega_i)^{1/2} w_i(x, z, t) &= \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\theta}} \psi^i(x, z, \theta, t) d\theta \\ &\simeq \sum_{\ell=1}^N a_\ell^i \psi^i(x, z, \theta_\ell^i, t), \\ &\equiv \sum_{\ell=1}^N a_\ell^i \psi_\ell^i(x, z, t). \end{aligned} \quad (32)$$

205 From (21a), the $2N$ memory variables ψ_ℓ^i satisfy the ordinary differential equations

$$\begin{cases} \frac{\partial \psi_\ell^i}{\partial t} = -(\theta_\ell^i + \Omega_i) \psi_\ell^i + \frac{\partial w_i}{\partial t} + \Omega_i w_i, \\ \psi_\ell^i(x, z, 0) = 0. \end{cases} \quad (33)$$

206 3.2. The Biot-DA first-order system

207 The fractional derivatives involved in the Biot-JKD system (16) are replaced by their diffusive
 208 approximation (32), with evolution equations (33). After some algebraic operations, the Biot-DA
 209 system is written as a first-order system in time and in space, used in the numerical simulations
 210 of § 5 ($j = 1, \dots, N$)

$$\left\{ \begin{array}{l} \frac{\partial v_{s1}}{\partial t} - \frac{\rho_{w1}}{\chi_1} \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} \right) - \frac{\rho_f}{\chi_1} \frac{\partial p}{\partial x} = \frac{\rho_f}{\rho} \gamma_1 \sum_{\ell=1}^N a_\ell^1 \psi_\ell^1 + G_{v_{s1}}, \\ \frac{\partial v_{s3}}{\partial t} - \frac{\rho_{w3}}{\chi_3} \left(\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} \right) - \frac{\rho_f}{\chi_3} \frac{\partial p}{\partial z} = \frac{\rho_f}{\rho} \gamma_3 \sum_{\ell=1}^N a_\ell^3 \psi_\ell^3 + G_{v_{s3}}, \\ \frac{\partial w_1}{\partial t} + \frac{\rho_f}{\chi_1} \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} \right) + \frac{\rho}{\chi_1} \frac{\partial p}{\partial x} = -\gamma_1 \sum_{\ell=1}^N a_\ell^1 \psi_\ell^1 + G_{w_1}, \\ \frac{\partial w_3}{\partial t} + \frac{\rho_f}{\chi_3} \left(\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} \right) + \frac{\rho}{\chi_3} \frac{\partial p}{\partial z} = -\gamma_3 \sum_{\ell=1}^N a_\ell^3 \psi_\ell^3 + G_{w_3}, \\ \frac{\partial \sigma_{11}}{\partial t} - c_{11}^u \frac{\partial v_{s1}}{\partial x} - c_{13}^u \frac{\partial v_{s3}}{\partial z} - m\beta_1 \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_{\sigma_{11}}, \\ \frac{\partial \sigma_{13}}{\partial t} - c_{55}^u \left(\frac{\partial v_{s3}}{\partial x} + \frac{\partial v_{s1}}{\partial z} \right) = G_{\sigma_{13}}, \\ \frac{\partial \sigma_{33}}{\partial t} - c_{13}^u \frac{\partial v_{s1}}{\partial x} - c_{33}^u \frac{\partial v_{s3}}{\partial z} - m\beta_3 \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_{\sigma_{33}}, \\ \frac{\partial p}{\partial t} + m \left(\beta_1 \frac{\partial v_{s1}}{\partial x} + \beta_3 \frac{\partial v_{s3}}{\partial z} + \frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z} \right) = G_p, \\ \frac{\partial \psi_j^1}{\partial t} + \frac{\rho_f}{\chi_1} \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} \right) + \frac{\rho}{\chi_1} \frac{\partial p}{\partial x} = \Omega_1 w_1 - \gamma_1 \sum_{\ell=1}^N a_\ell^1 \psi_\ell^1 - (\theta_j^1 + \Omega_1) \psi_j^1 + G_{w_1}, \\ \frac{\partial \psi_j^3}{\partial t} + \frac{\rho_f}{\chi_3} \left(\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} \right) + \frac{\rho}{\chi_3} \frac{\partial p}{\partial z} = \Omega_3 w_3 - \gamma_3 \sum_{\ell=1}^N a_\ell^3 \psi_\ell^3 - (\theta_j^3 + \Omega_3) \psi_j^3 + G_{w_3}. \end{array} \right. \quad (34)$$

211 Taking the vector of unknowns

$$U = (v_{s1}, v_{s3}, w_1, w_3, \sigma_{11}, \sigma_{13}, \sigma_{33}, p, \psi_1^1, \psi_1^3, \dots, \psi_N^1, \psi_N^3)^T, \quad (35)$$

212 and the forcing

$$G = (G_{v_{s1}}, G_{v_{s3}}, G_{w_1}, G_{w_3}, G_{\sigma_{11}}, G_{\sigma_{13}}, G_{\sigma_{33}}, G_p, G_{w_1}, G_{w_3}, G_{w_1}, G_{w_3})^T, \quad (36)$$

213 the system (34) is written in the form:

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial z} = -SU + G, \quad (37)$$

214 where A and B are the $(2N + 8) \times (2N + 8)$ propagation matrices and S is the diffusive matrix
 215 (given in Appendix B). The number of unknowns increases linearly with the number of memory
 216 variables. Only the matrix S depends on the coefficients of the diffusive approximation.

217 *3.3. Properties*

218 Some properties are stated to characterize the first-order differential system (34). First, one
 219 notes that the only difference between the Biot-LF model, the Biot-JKD model and the Biot-DA
 220 model occurs in the viscous operators

$$\widehat{F}_i(\omega) = \begin{cases} \widehat{F}_i^{LF}(\omega) = 1 & \text{Biot-LF,} \\ \widehat{F}_i^{JKD}(\omega) = \frac{1}{\sqrt{\Omega_i}} (\Omega_i + j\omega)^{1/2} & \text{Biot-JKD,} \\ \widehat{F}_i^{DA}(\omega) = \frac{\Omega_i + j\omega}{\sqrt{\Omega_i}} \sum_{\ell=1}^N \frac{a_\ell^i}{\theta_\ell^i + \Omega_i + j\omega} & \text{Biot-DA.} \end{cases} \quad (38)$$

221 The dispersion analysis of the Biot-DA model is obtained by replacing the viscous operators
 222 $\widehat{F}_i^{JKD}(\omega)$ by $\widehat{F}_i^{DA}(\omega)$ in (28). One of the eigenvalues of $\mathbf{\Gamma}^{-1} \mathcal{L} \mathbf{C}$ (29) is still zero with multiplicity
 223 two, and the other non-zero eigenvalues correspond to the wave modes $\pm k_{pf}(\omega, \varphi)$, $\pm k_{ps}(\omega, \varphi)$
 224 and $\pm k_s(\omega, \varphi)$. Consequently, the diffusive approximation does not introduce spurious wave.

225 **Proposition 2.** *The eigenvalues of the matrix $\mathbf{M} = \cos(\varphi) \mathbf{A} + \sin(\varphi) \mathbf{B}$ are*

$$sp(\mathbf{M}) = \{0, \pm c_{pf}^\infty(\varphi), \pm c_{ps}^\infty(\varphi), \pm c_s^\infty(\varphi)\}, \quad (39)$$

226 *with 0 being of multiplicity $2N + 2$.*

227 The non-zero eigenvalues do not depend on the viscous operators $\widehat{F}_i(\omega)$. Consequently, the high-
 228 frequency limits of the phase velocities $c_{pf}^\infty(\varphi)$, $c_{ps}^\infty(\varphi)$ and $c_s^\infty(\varphi)$, defined in § 2.6, are the same
 229 for both Biot-LF, Biot-JKD and Biot-DA models. An argumentation similar to [19] shows that
 230 the matrix \mathbf{M} is diagonalizable for all φ in $[0, 2\pi[$, with real eigenvalues. The three models are
 231 therefore hyperbolic.

232 **Proposition 3 (Decrease of the energy).** *An energy analysis of (34) is performed. Let us con-*
 233 *sider the Biot-DA model (34) without forcing, and let us denote*

$$E = E_1 + E_2 + E_3, \quad (40)$$

234 *where*

$$\begin{aligned} E_1 &= \frac{1}{2} \int_{\mathbb{R}^2} (\rho \mathbf{v}_s^T \mathbf{v}_s + 2\rho_f \mathbf{v}_s^T \mathbf{w} + \mathbf{w}^T \text{diag}(\rho_{wi}) \mathbf{w}) dx dz, \\ E_2 &= \frac{1}{2} \int_{\mathbb{R}^2} \left((\boldsymbol{\sigma} + p\boldsymbol{\beta})^T \mathbf{C}^{-1} (\boldsymbol{\sigma} + p\boldsymbol{\beta}) + \frac{1}{m} p^2 \right) dx dz, \\ E_3 &= \frac{1}{2} \int_{\mathbb{R}^2} \frac{\eta}{\pi} \sum_{\ell=1}^N (\mathbf{w} - \boldsymbol{\psi}_\ell)^T \text{diag} \left(\frac{a_\ell^i}{\kappa_i \sqrt{\Omega_i} \theta_\ell^i (\theta_\ell^i + 2\Omega_i)} \right) (\mathbf{w} - \boldsymbol{\psi}_\ell) dx dz. \end{aligned} \quad (41)$$

235 Then, E satisfies

$$\begin{aligned}
\frac{dE}{dt} = & - \int_{\mathbb{R}^2} \frac{\eta}{\pi} \sum_{\ell=1}^N \left\{ \psi_{\ell}^T \operatorname{diag} \left(\frac{a_{\ell}^i (\theta_{\ell}^i + \Omega_i)}{\kappa_i \sqrt{\Omega_i \theta_{\ell}^i (\theta_{\ell}^i + 2\Omega_i)}} \right) \psi_{\ell} \right. \\
& \left. + \mathbf{w}^T \operatorname{diag} \left(\frac{a_{\ell}^i \Omega_i}{\kappa_i \sqrt{\Omega_i \theta_{\ell}^i (\theta_{\ell}^i + 2\Omega_i)}} \right) \mathbf{w} \right\} dx dz.
\end{aligned} \tag{42}$$

236

237 The proof of the proposition 3 is similar to the proof of the proposition 1 and will not be repeated
238 here. Proposition 3 calls the following comments:

- 239 • the terms E_1 and E_2 are the same in both the Biot-DA and Biot-JKD models;
- 240 • E_3 and the time evolution of E are modified by the diffusive approximation;
- 241 • the abscissae θ_{ℓ}^i are always positive, as explained in § 3.4, but not necessarily the weights
242 a_{ℓ}^i . Consequently, in the general case, we cannot say that the Biot-DA model is stable.
243 However, in the particular case where the coefficients $\theta_{\ell}^i, a_{\ell}^i$ are all positive, E is an energy,
244 and $\frac{dE}{dt} < 0$: the Biot-DA model is therefore stable in this case.

245 **Proposition 4.** *Let us assume that the abscissae θ_{ℓ}^i have been sorted in increasing order*

$$\theta_1^i < \theta_2^i < \dots < \theta_N^i, \quad i = 1, 3, \tag{43}$$

246 and that the coefficients $\theta_{\ell}^i, a_{\ell}^i$ of the diffusive approximation (32) are positive. Then zero is an
247 eigenvalue with multiplicity 6 of \mathcal{S} . Moreover, the $2N + 2$ non-zero eigenvalues of \mathcal{S} (denoted s_{ℓ}^i ,
248 $\ell = 1, \dots, N + 1$) are real positive, and satisfy

$$0 < s_1^i < \theta_1^i + \Omega_i < \dots < s_N^i < \theta_N^i + \Omega_i < s_{N+1}^i, \quad i = 1, 3. \tag{44}$$

249

250 Proposition 4 is proven in Appendix C. As we will see in § 4, the proposition 4 ensures the
251 stability of the numerical method. Positivity of quadrature abscissae and weights is again the
252 fundamental hypothesis.

253 3.4. Determining the Biot-DA parameters

254 For the sake of clarity, the space coordinates and the subscripts due to the anisotropy are
255 omitted. The quadrature coefficients aim to approximate improper integrals of the form

$$(D + \Omega)^{1/2} w(t) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{\sqrt{\theta}} \psi(t, \theta) d\theta \simeq \sum_{\ell=1}^N a_{\ell} \psi(t, \theta_{\ell}). \tag{45}$$

256 Moreover, the positivity of the quadrature coefficients is crucial for the stability of the Biot-DA
257 model and its numerical implementation, as shown in propositions 3 and 4. Two approaches can
258 be employed for this purpose. While the most usual one is based on orthogonal polynomials, the
259 second approach is associated with an optimization procedure applied to the viscous operators
260 (38).

261 *3.4.1. Gaussian quadratures*

262 Various orthogonal polynomials exist to evaluate the improper integral (45). The first method,
 263 proposed in [15], is to use the Gauss-Laguerre quadrature formula, which approximates improper
 264 integrals over \mathbb{R}^+ . Slow convergence of this method is explained and corrected in [16]. It consists
 265 in replacing the Gauss-Laguerre quadrature by a Gauss-Jacobi quadrature, more suitable for
 266 functions which decrease algebraically. A last improvement, proposed in [17], consists in using
 267 a modified Gauss-Jacobi quadrature formula, recasting the improper integral (45) as

$$\frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\theta}} \psi(\theta) d\theta = \frac{1}{\pi} \int_{-1}^{+1} (1 - \tilde{\theta})^\gamma (1 + \tilde{\theta})^\delta \tilde{\psi}(\tilde{\theta}) d\tilde{\theta} \simeq \frac{1}{\pi} \sum_{\ell=1}^N \tilde{a}_\ell \tilde{\psi}(\tilde{\theta}_\ell), \quad (46)$$

268 with the modified memory variable $\tilde{\psi}$ defined as

$$\tilde{\psi}(\tilde{\theta}) = \frac{4}{(1 - \tilde{\theta})^{\gamma-1} (1 + \tilde{\theta})^{\delta+3}} \left(\frac{1 + \tilde{\theta}}{1 - \tilde{\theta}} \right) \psi \left(\left(\frac{1 - \tilde{\theta}}{1 + \tilde{\theta}} \right)^2 \right). \quad (47)$$

269 The abscissae $\tilde{\theta}_\ell$, which are the zeros of the Gauss-Jacobi polynomials, and the weights \tilde{a}_ℓ can
 270 be computed by standard routines [32]. In [17], the author proves that for fractional derivatives
 271 of order 1/2, the optimal coefficients to use are $\gamma = 1$ and $\delta = 1$. The coefficients of the diffusive
 272 approximation θ_ℓ and a_ℓ (45) are therefore related to the coefficients $\tilde{\theta}_\ell$ and \tilde{a}_ℓ (46) by

$$\theta_\ell = \left(\frac{1 - \tilde{\theta}_\ell}{1 + \tilde{\theta}_\ell} \right)^2, \quad a_\ell = \frac{1}{\pi} \frac{4 \tilde{a}_\ell}{(1 - \tilde{\theta}_\ell) (1 + \tilde{\theta}_\ell)^3}. \quad (48)$$

273 By construction, they are strictly positive.

274 *3.4.2. Optimization procedures*

275 In [23, 24], we proposed a different method to determine the coefficients θ_ℓ and a_ℓ of the
 276 diffusive approximation (45). This method is based on the frequency expressions of the viscous
 277 operators and takes into account the frequency content of the source. Our requirement is therefore
 278 to approximate the viscous operator $\widehat{F}^{JKD}(\omega)$ by $\widehat{F}^{DA}(\omega)$ (38) in the frequency range of interest
 279 $I = [\omega_{\min}, \omega_{\max}]$, centered on the central angular frequency of the source. This leads to the
 280 minimization of the quantity χ^2 with respect to the abscissae θ_ℓ and to the weights a_ℓ

$$\chi^2 = \sum_{k=1}^K \left| \frac{\widehat{F}^{DA}(\omega_k)}{\widehat{F}^{JKD}(\omega_k)} - 1 \right|^2 = \sum_{k=1}^K \left| \sum_{\ell=1}^N a_\ell \frac{(\Omega + j\omega_k)^{1/2}}{\theta_\ell + \Omega + j\omega_k} - 1 \right|^2, \quad (49)$$

281 where the angular frequencies ω_k are distributed linearly in I on a logarithmic scale of K points

$$\omega_k = \omega_{\min} \left(\frac{\omega_{\max}}{\omega_{\min}} \right)^{\frac{k-1}{K-1}}, \quad k = 1 \cdots K. \quad (50)$$

282 In [23, 24], the abscissae θ_ℓ were arbitrarily put linearly on a logarithmic scale, as (50). Only
 283 the weights a_ℓ were optimized with a linear least-squares minimization procedure of (49). Some
 284 negative weights were obtained, which represents a major drawback, at least theoretically, since
 285 the stability of the Biot-DA model can not be guaranteed.

286 To remove this drawback and improve the minimization of χ^2 , a nonlinear constrained opti-
 287 mization is developed, where both the abscissae and the weights are optimized. The coefficients θ_ℓ

288 and a_ℓ are now constrained to be positive. An additional constraint $\theta_\ell \leq \theta_{max}$ is also introduced
 289 to ensure the computational accuracy in the forthcoming numerical method (§ 4). Setting

$$\theta_\ell = (\theta'_\ell)^2, \quad a_\ell = (a'_\ell)^2, \quad (51)$$

290 the number of constraints decreases from $3N$ to N leading to the following minimization prob-
 291 lem:

$$\min_{(\theta'_\ell, a'_\ell)} \chi^2, \quad \theta'_\ell \leq \sqrt{\theta_{max}}. \quad (52)$$

292 The constrained minimization problem (52) is nonlinear and non-quadratic with respect to ab-
 293 scissae θ'_ℓ . To solve it, we implement the program SolvOpt [33, 34], used in viscoelasticity [35].
 294 Since this Shor's algorithm is iterative, it requires an initial estimate $\theta'_\ell{}^0, a'_\ell{}^0$ of the coefficients
 295 which satisfies the constraints of the minimization problem (52). For this purpose, $\theta'_\ell{}^0$ and $a'_\ell{}^0$ are
 296 initialized with the method based on the modified Gauss-Jacobi quadrature formula (48):

$$\theta'_\ell{}^0 = \frac{1 - \tilde{\theta}_\ell}{1 + \tilde{\theta}_\ell}, \quad a'_\ell{}^0 = \sqrt{\frac{1}{\pi} \frac{4 \tilde{a}_\ell}{(1 - \tilde{\theta}_\ell)(1 + \tilde{\theta}_\ell)^3}}. \quad (53)$$

297 Different initial guess have been used, derived from Gaus-Legendre and Gauss-Jacobi methods,
 298 leading to the same final coefficients θ_ℓ and a_ℓ .

299 In what follows, we always use the parameters

$$\omega_{min} = \omega_0/10, \quad \omega_{max} = 10 \omega_0, \quad \theta_{max} = 100 \omega_0, \quad K = 2N, \quad (54)$$

300 where $\omega_0 = 2\pi f_0$ is the central angular frequency of the source.

301 3.4.3. Discussion

302 To compare the quadrature methods presented in § 3.4.1 and 3.4.2, we first define the error
 303 of model ε_{mod} as

$$\varepsilon_{mod} = \left\| \frac{\widehat{F}^{DA}(\omega)}{\widehat{F}^{JKD}(\omega)} - 1 \right\|_{L_2} = \left(\int_{\omega_{min}}^{\omega_{max}} \left| \frac{\widehat{F}^{DA}(\omega)}{\widehat{F}^{JKD}(\omega)} - 1 \right|^2 d\omega \right)^{1/2}. \quad (55)$$

304 The variation of ε_{mod} in terms of the number N of memory variables, for $f_0 = 200$ kHz and
 305 $f_c = 3.84$ kHz, is represented on figure 2-a. The Gauss-Jacobi method converges very slowly,
 306 and the error is always larger than 1 % even for $N = 50$. Moreover, for values of $N \leq 10$,
 307 the error is always larger than 60 %. For both the linear and the nonlinear optimizations, the
 308 errors decrease rapidly with N . Nevertheless, the nonlinear procedure outperforms the results
 309 obtained in the linear case. For $N = 8$ for instance, the relative error of the nonlinear optimization
 310 ($\varepsilon_{mod} \simeq 7.16 \cdot 10^{-3}$ %) is 514 times smaller than the error of the linear optimization ($\varepsilon_{mod} \simeq 3.68$
 311 %). For larger values of N , the system is poorly conditioned and the order of convergence
 312 deteriorates; in practice, this is not penalizing since large values of N are not used. An example
 313 of a priori parametric determination of N in terms of both the frequency range and the desired
 314 accuracy is also given in figure 2-b for the nonlinear procedure. The case $N = 0$ corresponds to
 315 the Biot-LF model.

316 It is also important to compare the influence of the quadrature coefficient on the physical ob-
 317 servables. For that purpose, we represent on figure 3 the phase velocity and the attenuation of the

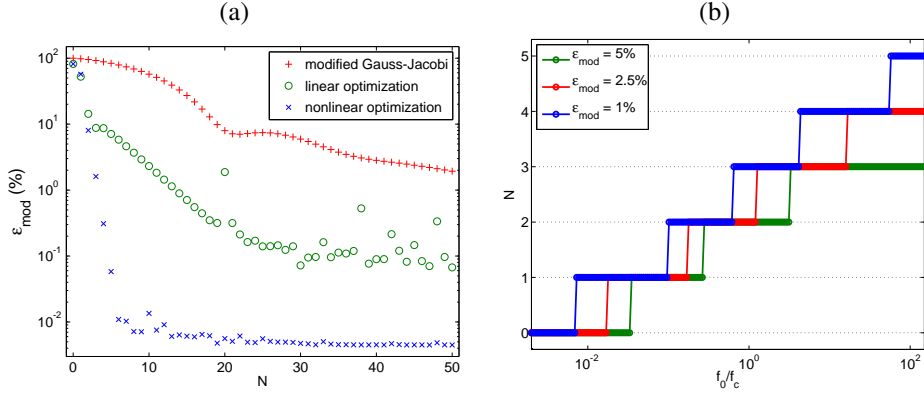


Figure 2: (a): relative error ε_{mod} in terms of N for both the modified Gauss-Jacobi quadrature and the nonlinear constrained optimization. (b): required values of N in terms of f_0/f_c and the required accuracy ε_{mod} for the nonlinear optimization.

318 slow wave of the Biot-DA model, obtained with the different quadrature methods. As expected,
 319 the results given by the Gauss-Jacobi method are extremely poor. On the contrary, the linear
 320 and non-linear procedures are able to represent very accurately the variations of these quantities
 321 on the considered range of frequencies, even for small values of N . Based on these results and
 322 the positivity requirement, the nonlinear constrained optimization is therefore considered as the
 323 better way to determine the coefficients of the diffusive approximation. This method is used in
 324 all what follows.

325 4. Numerical modeling

326 4.1. Splitting

327 In order to integrate the Biot-DA system (37), a uniform grid is introduced, with mesh size
 328 Δx , Δz and time step Δt . The approximation of the exact solution $\mathbf{U}(x_i = i \Delta x, z_j = j \Delta z, t_n =$
 329 $n \Delta t)$ is denoted by \mathbf{U}_{ij}^n , with $0 \leq i \leq N_x$, $0 \leq j \leq N_z$. If $\Delta x = \Delta z$, a straightforward
 330 discretization of (37) by an explicit time scheme typically leads to the following condition of
 331 stability

$$\Delta t \leq \min \left(\Upsilon \frac{\Delta x}{\max_{\varphi \in [0, \pi/2]} c_{pf}^{\infty}(\varphi)}, \frac{2}{R(\mathbf{S})} \right), \quad (56)$$

332 where $R(\mathbf{S})$ is the spectral radius of \mathbf{S} , and $\Upsilon > 0$ is obtained by a Von-Neumann analysis when
 333 $\mathbf{S} = \mathbf{0}$. The first term of (56), which depends of the propagation matrices \mathbf{A} and \mathbf{B} , is the
 334 classical CFL condition. The second term of (56) depends only on the diffusive matrix \mathbf{S} . From
 335 proposition 4, we deduce that the spectral radius of \mathbf{S} satisfies

$$R(\mathbf{S}) > \max_{\ell=1, \dots, N} (\theta_{\ell}^1 + \Omega_1, \theta_{\ell}^3 + \Omega_3) \quad (57)$$

336 if the coefficients θ_{ℓ}^i and a_{ℓ}^i of the diffusive approximation are positive. With highly dissipative
 337 fluids, the second term of (56) can be so small that numerical computations are intractable.

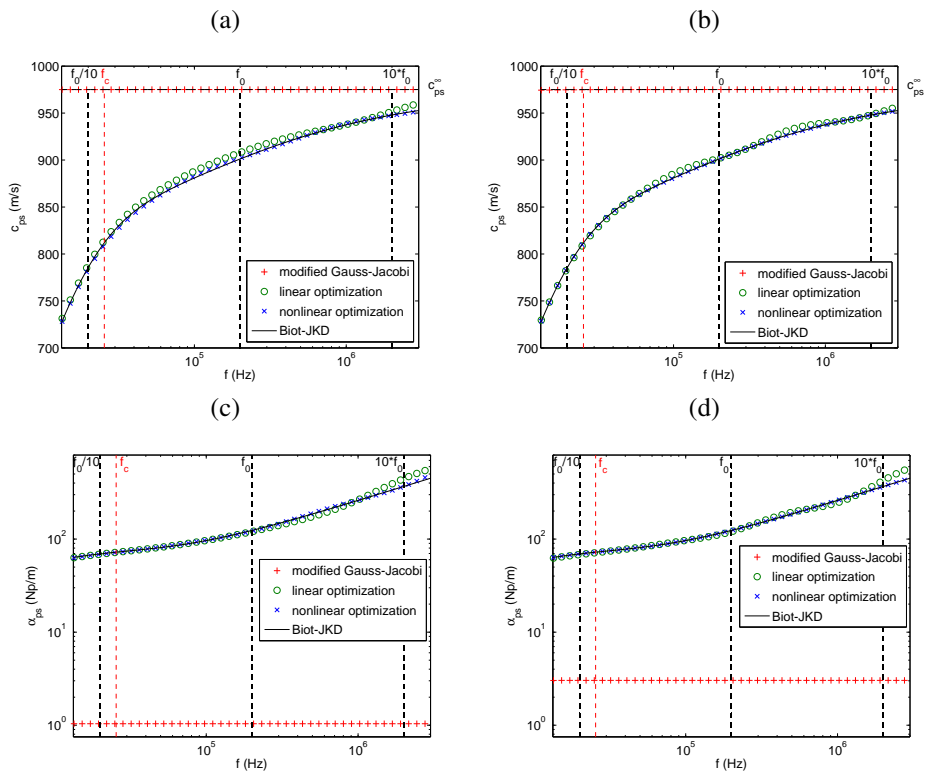


Figure 3: phase velocity (a), (b) and attenuation (c), (d) of the slow quasi-compressional wave. Comparison between the Biot-DA model and the Biot-JKD model. Left: $N = 3$, right: $N = 6$.

338 A more efficient strategy is adopted here, based on the second-order Strang splitting [36]. It
 339 consists in splitting the original system (37) into a propagative part

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = \mathbf{0}, \quad (\mathbf{H}_p) \quad (58)$$

340 and a diffusive part with forcing

$$\frac{\partial \mathbf{U}}{\partial t} = -\mathbf{S} \mathbf{U} + \mathbf{G}, \quad (\mathbf{H}_d) \quad (59)$$

341 where \mathbf{H}_p and \mathbf{H}_d are the operators associated with each part. One solves alternatively the
 342 propagative part and the diffusive part:

$$\mathbf{U}^{n+1} = \mathbf{H}_d \left(t_{n+1}, \frac{\Delta t}{2} \right) \circ \mathbf{H}_p(\Delta t) \circ \mathbf{H}_d \left(t_n, \frac{\Delta t}{2} \right) \mathbf{U}^n. \quad (60)$$

343 The discrete operator \mathbf{H}_p associated with the propagative part (58) is an ADER 4 (Arbitrary
 344 DERivatives) scheme [37]. This scheme is fourth-order accurate in space and time, is dispersive
 345 of order 4 and dissipative of order 6 [38], and has a stability limit $\Upsilon = 1$. On Cartesian grids,
 346 ADER 4 amounts to a fourth-order Lax-Wendroff scheme.

347 The solution of (59) is given by

$$\begin{aligned} \mathbf{H}_d \left(t_k, \frac{\Delta t}{2} \right) \mathbf{U}(t_0) &= e^{-\mathbf{S} \Delta t/2} \mathbf{U}(t_0) + \int_{t_0}^{t_0 + \Delta t/2} e^{-\mathbf{S}(t_0 + \Delta t/2 - \tau)} \mathbf{G}(\tau) d\tau, \\ &\simeq e^{-\mathbf{S} \frac{\Delta t}{2}} \mathbf{U}(t_0) - (\mathbf{I} - e^{-\mathbf{S} \frac{\Delta t}{2}}) \mathbf{S}^{-1} \mathbf{G}(t_k), \end{aligned} \quad (61)$$

348 with $k = n$ or $n + 1$. The exponential matrix $e^{-\mathbf{S} \Delta t/2}$ is computed numerically using the (6, 6)
 349 Padé approximation in the "scaling and squaring method" [39]. Proposition 4 ensures that the
 350 numerical integration of the diffusive step (59) is unconditionally stable [23]. Without forcing,
 351 i.e. $\mathbf{G} = \mathbf{0}$, the integration of the diffusive part (59) is exact.

352 The full algorithm is therefore stable under the optimum CFL condition of stability

$$\Delta t = \Upsilon \frac{\Delta x}{\max_{\varphi \in [0, \pi/2]} c_{pf}^{\infty}(\varphi)}, \quad \Upsilon \leq 1, \quad (62)$$

353 which is always independent of the Biot-DA model coefficients. Since the matrices \mathbf{A} , \mathbf{B} and \mathbf{S}
 354 do not commute, the order of convergence decreases from 4 to 2. Using a fourth-order ADER
 355 scheme is nevertheless advantageous, compared with the second-order Lax-Wendroff scheme:
 356 the stability limit is improved, and numerical artifacts (dispersion, attenuation, anisotropy) are
 357 greatly reduced.

358 4.2. Immersed interface method

359 Let us consider two transversely isotropic homogeneous poroelastic media Ω_0 and Ω_1 sep-
 360 arated by a stationary interface Γ , as shown in figure 4. The governing equations (37) in each
 361 medium have to be completed by a set of jump conditions. The simple case of perfect bonding
 362 and perfect hydraulic contact along Γ is considered here, modeled by the jump conditions [40]:

$$[\mathbf{v}_s] = \mathbf{0}, \quad [\mathbf{w} \cdot \mathbf{n}] = 0, \quad \left[\frac{\boldsymbol{\sigma} \cdot \mathbf{n}}{18} \right] = \mathbf{0}, \quad [p] = 0. \quad (63)$$

Figure 4: interface Γ between two poroelastic media Ω_0 and Ω_1 .

363 The discretization of the interface conditions requires special care. A straightforward stair-step
364 representation of interfaces introduces first-order geometrical errors and yields spurious numerical
365 diffractions. In addition, the jump conditions (63) are not enforced numerically if no special
366 treatment is applied. Lastly, the smoothness requirements to solve (58) are not satisfied, decreasing
367 the convergence rate of the ADER scheme.

368 To remove these drawbacks while maintaining the efficiency of Cartesian grid methods, immersed
369 interface methods constitute a possible strategy [41, 42, 11]. The latter studies can be
370 consulted for a detailed description of this method. The basic principle is as follows: at the ir-
371 regular nodes where the ADER scheme crosses an interface, modified values of the solution are
372 used on the other side of the interface instead of the usual numerical values.

373 Calculating these modified values is a complex task involving high-order derivation of jump
374 conditions (63), high-order derivation of the Beltrami-Michell equation (8) and algebraic ma-
375 nipulation, such as singular value decompositions. All these time consuming procedures can
376 be carried out during a preprocessing stage and only small matrix-vector multiplications need
377 to be performed during the simulation. After optimizing the code, the extra CPU cost can be
378 practically negligible, i.e. lower than 1% of that required by the time-marching procedure.

379 Compared with § 3-3 of [11], the modifications induced by anisotropy concern

- 380 • step 1: the derivation of the jump conditions,
- 381 • step 2: the derivation of the Beltrami-Michell equation.

382 These modifications are tedious and hence will not be repeated here. They are straightforwardly
383 deduced from the new expressions (8) and (34).

384 5. Numerical experiments

385 *Configuration*

386 In order to demonstrate the ability of the present method to be applied to a wide range of
387 applications, the numerical tests will be run on two different transversely isotropic porous media.
388 The medium Ω_0 is composed of thin layers of epoxy and glass, strongly anisotropic if the wave-
389 lengths are large compared to the thick of the layers [18]. The medium Ω_1 is water saturated
390 Berea sandstone, which is sedimentary rock commonly encountered in petroleum engineering.
391 The grains are predominantly sand sized and composed of quartz bonded by silica [18, 43].

392 The values of the physical parameters are given in table 1. The viscous characteristic lengths
393 Λ_1 and Λ_3 are obtained by setting the Pride numbers $P_1 = P_3 = 0.5$. We also report in these

394 tables some useful values, such as phase velocities, critical frequencies, and quadrature param-
 395 eters computed for each media. The central frequency of the source is $f_0 = 200$ kHz, and the
 396 quadrature coefficients $\theta_p^i, a_p^i, i = 1, 3$, are determined by nonlinear constrained optimization with
 397 $N = 3$ memory variables. The error of model ε_{mod} (55) is also given. We note that the transition
 398 frequencies f_{c1} and f_{c3} are the same for both Ω_0 and Ω_1 . In this particular case, the coefficients of
 399 the diffusive approximation are therefore also the same. In all the numerical simulations, the time
 400 step is computed from the physical parameters of the media through relations (62), setting the
 401 CFL number $\Upsilon = 0.95$. The numerical experiments are performed on an Intel Core i7 processor
 402 at 2.80 GHz.

403 In the first test, the computational domain $[-0.15, 0.15]^2$ m is discretized with $N_x = N_z =$
 404 2250 grid nodes in each direction, which amounts to 20 points per slow compressional wave-
 405 length in Ω_0 . In the other tests, the computational domain is discretized with $N_x = N_z = 1500$,
 406 which amounts also to 20 points per slow compressional wavelength in Ω_0 and in Ω_1 .

407 *Test 1: homogeneous medium*

408 In the first test, the homogeneous medium Ω_0 (table 1) is excited by a source point located at
 409 $(0 \text{ m}, 0 \text{ m})$. The only non-null component of the forcing \mathbf{F} (36) is $G_{\sigma_{13}} = g(t)h(x, z)$, where $g(t)$
 410 is a Ricker signal of central frequency f_0 and of time-shift $t_0 = 2/f_0 = 10^{-5}$ s:
 411

$$g(t) = \begin{cases} \left(2\pi^2 f_0^2 \left(t - \frac{1}{f_0} \right)^2 - 1 \right) \exp\left(-\pi^2 f_0^2 \left(t - \frac{1}{f_0} \right)^2 \right) & \text{if } 0 \leq t \leq t_0, \\ 0 & \text{otherwise,} \end{cases} \quad (64)$$

412 and $h(x, z)$ is a truncated Gaussian centered at point $(0, 0)$, of radius $R_0 = 6.56 \cdot 10^{-3}$ m and
 413 $\Sigma = 3.28 \cdot 10^{-3}$ m:

$$h(x, z) = \begin{cases} \frac{1}{\pi \Sigma^2} \exp\left(-\frac{x^2 + z^2}{\Sigma^2} \right) & \text{if } 0 \leq x^2 + z^2 \leq R_0^2, \\ 0 & \text{otherwise.} \end{cases} \quad (65)$$

414 We use a truncated gaussian for $h(x, z)$ rather than a Dirac distribution to avoid spurious numerical
 415 artifacts localized around the source point. This source generates cylindrical waves of all types:
 416 fast and slow quasi-compressional waves and quasi-shear waves, which are denoted by qP_f, qP_s
 417 and qS , respectively, in figure 5. The three waves are observed in the pressure field. Comparison
 418 with the theoretical wavefront, represented by a black dotted line in figure 5, shows that the
 419 computed waves are well positioned at this instant ($t_1 \approx 2.72 \cdot 10^{-5}$ s). No special care is applied
 420 to simulate outgoing waves (with PML, for instance), since the simulation is stopped before the
 421 waves have reached the edges of the computational domain. The cusp of the shear wave is seen
 422 in the numerical solution.

423 *Test 2: diffraction of a plane wave by a plane interface*

424 In all the following tests, the source is a plane right-going fast compressional wave, whose
 425 wavevector \mathbf{k} makes an angle $\theta = 0$ degree with the horizontal x -axis. Its time evolution is the
 426 same Ricker signal as in the first test (64). We use periodic boundary conditions at the top and at
 427 the bottom of the domain.
 428

429 In the second test, the validity of the method is checked in the particular case of heteroge-
 430 neous transversely isotropic media, where a semi-analytical solution can be obtained easily. The

	Parameters	Ω_0	Ω_1	
Saturating fluid	ρ_f (kg/m ³)	1040	1040	
	η (Pa.s)	10^{-3}	10^{-3}	
	K_f (GPa)	2.5	2.5	
Grain	ρ_s (kg/m ³)	1815	2500	
	K_s (GPa)	40	80	
Matrix	ϕ	0.2	0.2	
	\mathcal{T}_1	2	2	
	\mathcal{T}_3	3.6	3.6	
	κ_1 (m ²)	$6 \cdot 10^{-13}$	$6 \cdot 10^{-13}$	
	κ_3 (m ²)	10^{-13}	10^{-13}	
	c_{11} (GPa)	39.4	71.8	
	c_{12} (GPa)	1	3.2	
	c_{13} (GPa)	5.8	1.2	
	c_{33} (GPa)	13.1	53.4	
	c_{55} (GPa)	3	26.1	
	Λ_1 (m)	$6.93 \cdot 10^{-6}$	$2.19 \cdot 10^{-7}$	
	Λ_3 (m)	$3.79 \cdot 10^{-6}$	$1.20 \cdot 10^{-7}$	
	Dispersion	$c_{pf}^{\infty}(0)$ (m/s)	5244.40	6004.31
		$c_{pf}(f_0, 0)$ kHz (m/s)	5227.10	5988.50
		$c_{pf}^{\infty}(\pi/2)$ (m/s)	3583.24	5256.03
$c_{pf}(f_0, \pi/2)$ (m/s)		3581.42	5245.84	
$c_{ps}^{\infty}(0)$ (m/s)		975.02	1026.45	
$c_{ps}(f_0, 0)$ (m/s)		901.15	949.33	
$c_{ps}^{\infty}(\pi/2)$ (m/s)		604.41	745.59	
$c_{ps}(f_0, \pi/2)$ (m/s)		534.88	661.32	
$c_s^{\infty}(0)$ (m/s)		1368.36	3484.00	
$c_s(f_0, 0)$ (m/s)		1361.22	3470.45	
$c_s^{\infty}(\pi/2)$ (m/s)		1388.53	3522.07	
$c_s(f_0, \pi/2)$ (m/s)		1381.07	3508.05	
f_{c1} (Hz)		$2.55 \cdot 10^4$	$2.55 \cdot 10^4$	
f_{c3} (Hz)		$8.50 \cdot 10^4$	$8.50 \cdot 10^4$	
Optimization		θ_1^1 (rad/s)	$1.64 \cdot 10^5$	$1.64 \cdot 10^5$
	θ_2^1 (rad/s)	$2.80 \cdot 10^6$	$2.80 \cdot 10^6$	
	θ_3^1 (rad/s)	$3.58 \cdot 10^7$	$3.58 \cdot 10^7$	
	a_1^1 (rad ^{1/2} /s ^{1/2})	$5.58 \cdot 10^2$	$5.58 \cdot 10^2$	
	a_2^1 (rad ^{1/2} /s ^{1/2})	$1.21 \cdot 10^3$	$1.21 \cdot 10^3$	
	a_3^1 (rad ^{1/2} /s ^{1/2})	$7.32 \cdot 10^3$	$7.32 \cdot 10^3$	
	ε_{mod}^1 (%)	1.61	1.61	
	θ_1^3 (rad/s)	$3.14 \cdot 10^5$	$3.14 \cdot 10^5$	
	θ_2^3 (rad/s)	$5.06 \cdot 10^7$	$5.06 \cdot 10^7$	
	θ_3^3 (rad/s)	$4.50 \cdot 10^6$	$4.50 \cdot 10^6$	
	a_1^3 (rad ^{1/2} /s ^{1/2})	$7.57 \cdot 10^2$	$7.57 \cdot 10^2$	
	a_2^3 (rad ^{1/2} /s ^{1/2})	$8.79 \cdot 10^3$	$8.79 \cdot 10^3$	
	a_3^3 (rad ^{1/2} /s ^{1/2})	$1.38 \cdot 10^3$	$1.38 \cdot 10^3$	
	ε_{mod}^3 (%)	0.53	0.53	

Table 1: Physical parameters of the transversely isotropic media used in the numerical experiments. The phase velocities $c_{pf}(f_0, \varphi)$, $c_{ps}(f_0, \varphi)$ and $c_s(f_0, \varphi)$ are computed at $f = f_0 = 200$ kHz when the wavevector \mathbf{k} makes an angle φ with the horizontal x -axis, and $c_{pf}^{\infty}(\varphi)$, $c_{ps}^{\infty}(\varphi)$, $c_s^{\infty}(\varphi)$ denote the high-frequency limit of the phases velocities.

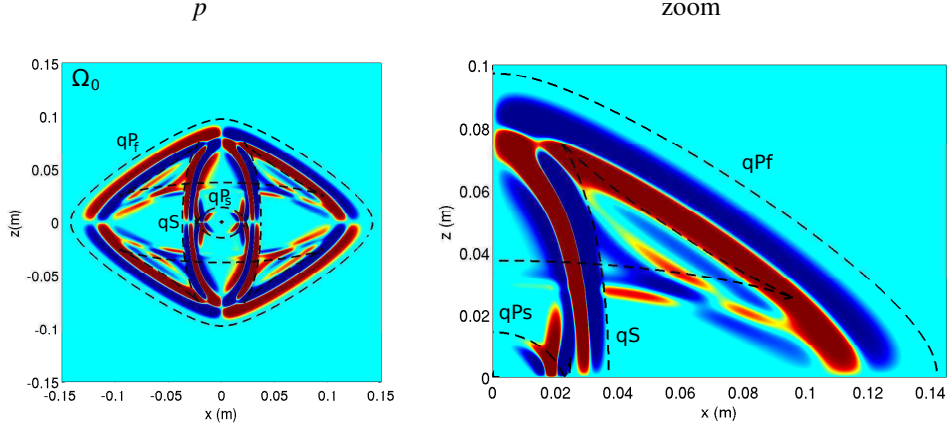


Figure 5: test 1. Fast and slow quasi-compressional waves, respectively qP_f and qP_s , and quasi-shear wave qS emitted by a source point at $(0 \text{ m}, 0 \text{ m})$. Pressure at $t_1 \approx 2.72 \cdot 10^{-5} \text{ s}$.

431 media Ω_0 and Ω_1 are separated by a vertical wave plane interface at $x = 0 \text{ m}$. The incident
 432 P_f -wave (Ip_f) propagates in the medium Ω_1 . The figure 6 shows a snapshot of the pressure
 433 at $t_1 \approx 1.48 \cdot 10^{-5} \text{ s}$, on the whole computational domain. The reflected fast and slow quasi-
 434 compressional waves, denoted respectively Rp_f and Rp_s , propagate in the medium Ω_1 ; and the
 435 transmitted fast and slow quasi-compressional waves, denoted respectively Tp_f and Tp_s , propa-
 gate in the medium Ω_0 . In this case, we compute the exact solution of Biot-DA thanks to standard

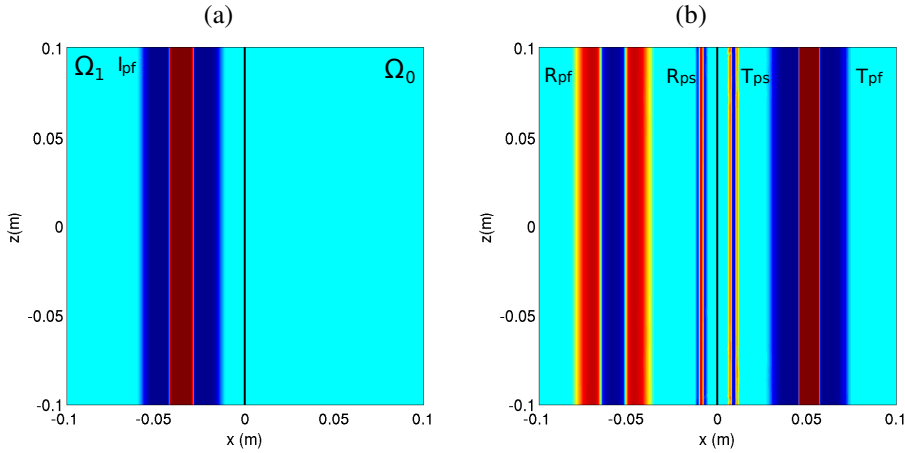


Figure 6: test 2. Snapshot of pressure at initial time (a) and at $t_1 \approx 1.48 \cdot 10^{-5} \text{ s}$ (b). The plane interface is denoted by a straight black line, separating Ω_1 (on the left) and Ω_0 (on the right).

436 tools of Fourier analysis. The figure 7 shows the excellent agreement between the analytical and
 437 the numerical values of the pressure along the line $z = 0 \text{ m}$. Despite the relative simplicity of
 438

439 this configuration (1D evolution of the waves and lack of shear waves), it can be viewed as a
 440 validation of the numerical method which is fully 2D whatever the geometrical setting.

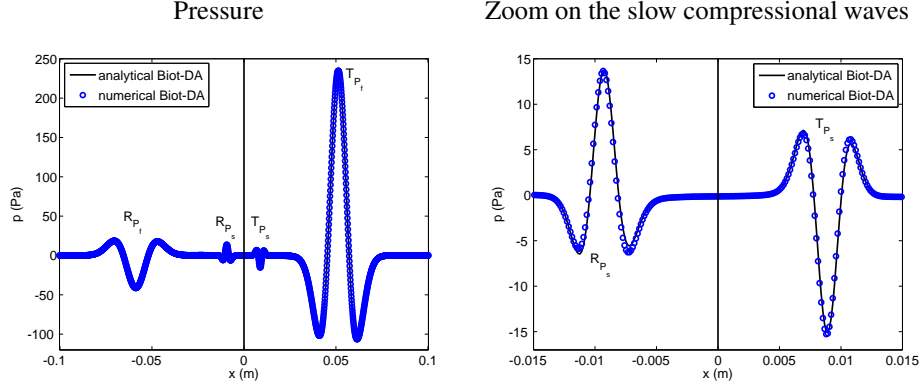


Figure 7: test2. Pressure along the line $z = 0$ m; vertical line denotes the interface. Comparison between the numerical values (circle) and the analytical values (solid line) of p at $t_1 \approx 1.48 \cdot 10^{-5}$ s.

441

442 *Test 3: diffraction of a plane wave by cylinders*

443 In the previous test, the interface was plane, but more complex geometries can be handled
 444 on a Cartesian grid thanks to the immersed interface method. As an example, we simulate the
 445 interaction of a plane wave with a cylindrical scatterer filled by medium Ω_1 , and immersed in
 446 medium Ω_0 . The cylinder, of radius 0.015 m, is centered at point (0.01 m, 0 m). The incident
 447 plane wave propagates in the medium Ω_0 . The initial conditions are illustrated in figure 8-(a),
 448 while the snapshot of p at time $t_1 \approx 1.69 \cdot 10^{-5}$ s is represented on figure 8-(b). Classical waves
 449 conversions and scattering phenomena are observed. Since the phase velocity c_{pf} in the medium
 450 Ω_1 is greater than in the medium Ω_0 , the transmitted fast compressional wave has a curved
 451 wavefront. Moreover, the shape of the reflected waves illustrates the anisotropy of the medium
 452 Ω_0 .

453

454 *Test 4: multiple ellipsoidal scatterers*

455 To illustrate the ability of the proposed numerical strategy to handle even more complex
 456 geometries, 200 ellipsoidal scatterers of medium Ω_1 , with major and minor radii of 0.025 m and
 457 0.02 m, are randomly distributed in a matrix of medium Ω_0 , leading to a concentration of 25 %.
 458 The pressure is represented at the initial time on figure 9 and at time $t_1 \approx 1.43 \cdot 10^{-4}$ s on figure
 459 10. This simulation has taken approximately 11.5 h of preprocessing and 8.5 h of time-stepping.
 460 Similar numerical experiments are also performed for a concentration of scatterer of 10 % and
 461 15 %.

462 At each time step, the components of \mathbf{U}_{ij}^n are stored inside the subdomain containing the
 463 inclusions. For this purpose, a uniform network consisting of $N_l = 800$ lines and $N_c = 25$
 464 columns of receivers is put in the subdomain. The position of the receivers is given by (x_i, z_j) ,
 465 where $i = 0, \dots, N_c - 1$ and $j = 0, \dots, N_l - 1$. The field \mathbf{U}_{ij}^n recorded on each array (each line
 466 of receivers), represented on figure 11-a, corresponds to a field propagating along one horizontal
 467 line of receivers. A main wave train is clearly visible, followed by a coda. Summing the time

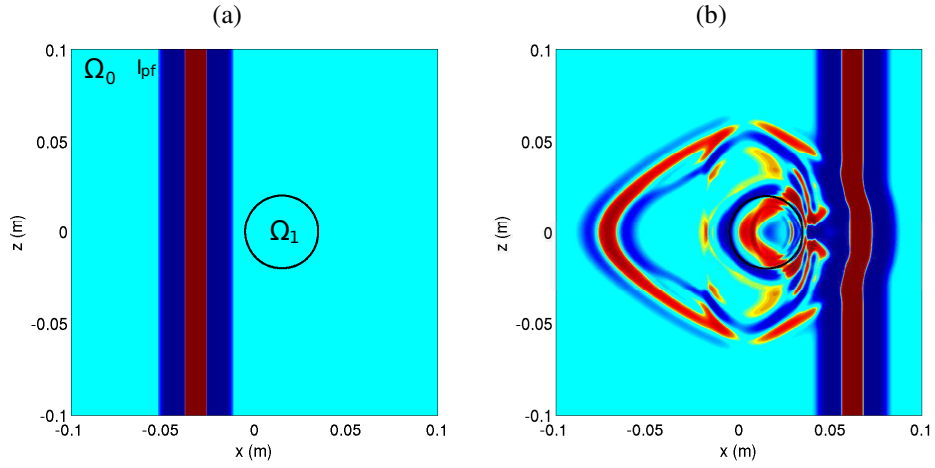


Figure 8: test 3. Snapshot of pressure at initial time (a) and at $t_1 \approx 1.69 \cdot 10^{-5}$ s (b). The cylinder is denoted by a black circle, separating the porous media Ω_0 (outside) and Ω_1 (inside).

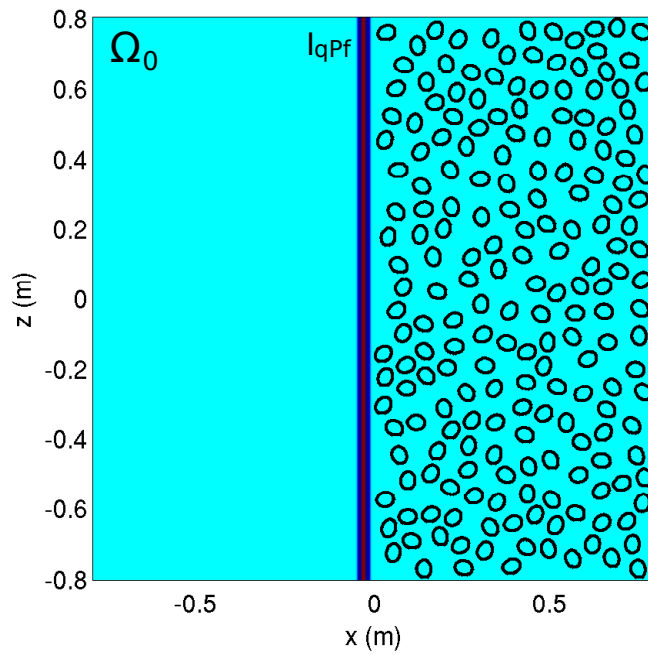


Figure 9: test 4. Multiple scattering in random media. Snapshot of p at the initial time. The matrix is Ω_0 , whereas the 200 scatterers are Ω_1 .

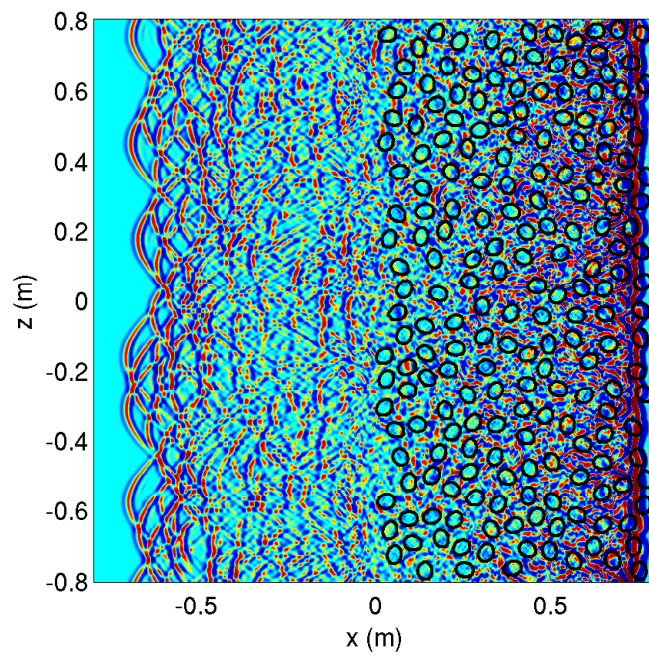


Figure 10: test 4. Multiple scattering in random media. Snapshot of p at time $t_1 \approx 1.43 \cdot 10^{-4}$ s. The matrix is Ω_0 , whereas the 200 scatterers are Ω_1 .

468 histories of these N_c arrays gives a coherent field propagating in the x direction:

$$\bar{U}_i^n = \frac{1}{N_l} \sum_{j=0}^{N_l-1} U_{ij}^n. \quad (66)$$

469 On the coherent seismogram thus obtained, represented on figure 11-b, the coda has disappeared,
 470 and the main wave train behaves like a plane wave propagating in a homogeneous (but disper-
 471 sive and attenuating) medium. The coherent phase velocity $c(\omega)$, represented in figure 12-a, is
 472 computed by applying a $p - \omega$ transform to the space-time data on the coherent field (66), where
 473 $p = 1/c$ is the slowness of the waves [44, 45]. The horizontal lines represent a simple average of
 474 the phase velocities weighted by the concentration. The coherent attenuation $\alpha(\omega)$ is estimated
 475 from the decrease in the amplitude spectrum of the coherent field during the propagation of the
 waves, see 12-b. An error estimate is also deduced, represented in figure 12 by vertical lines.

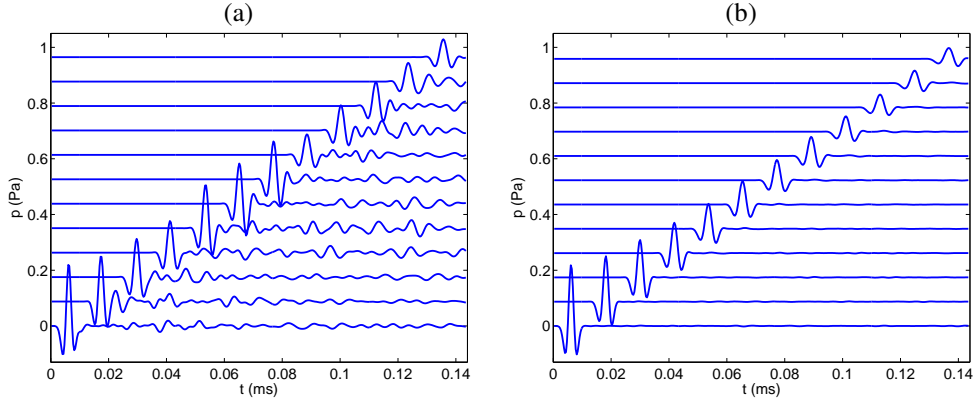


Figure 11: test 4. Incident plane qP_s -wave in a medium with 25% inclusion concentration. (a): pressure recorded along an array, (b): coherent pressure obtained after summation.

476

477 6. Conclusion

478 An explicit finite-difference method has been developed here to simulate transient poroelastic
 479 waves in the full range of validity of the Biot-JKD model, which involves order 1/2 fractional
 480 derivatives. A diffusive representation transforms the fractional derivatives, non-local in time,
 481 into a continuum of local problems, approximated by quadrature formulae. The Biot-JKD model
 482 is then replaced by an approximate Biot-DA model, much more tractable numerically. The co-
 483 efficients of the diffusive approximation are determined by a nonlinear constrained optimization
 484 procedure, leading to a small number of memory variables. The hyperbolic Biot-DA system of
 485 partial differential equations is discretized using various tools of scientific computing: Strang
 486 splitting, fourth-order ADER scheme, immersed interface method. It enables to treat efficiently
 487 and accurately the propagation of transient waves in transversely isotropic porous media.

488 Some future lines of research are suggested:

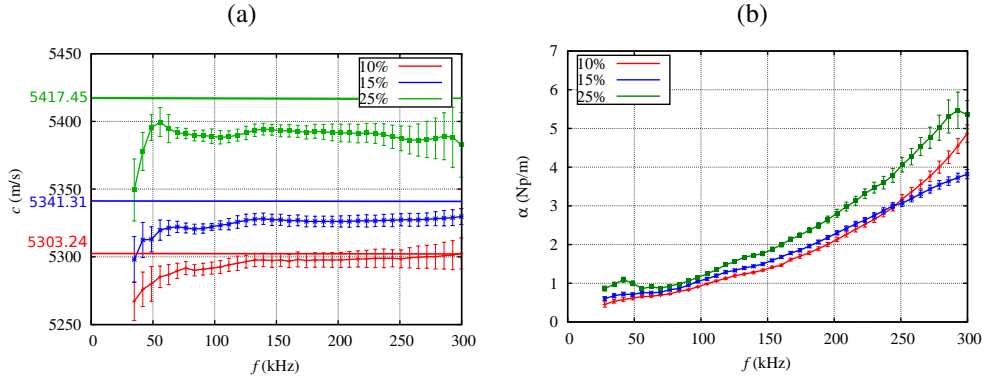


Figure 12: test 4. Effective phase velocity (a) and effective attenuation (b) at various inclusion concentrations. The vertical lines represents the error bars. The horizontal lines in (a) give the average phase velocity weighted by the concentration.

489 • *Multiple scattering.* Many theoretical methods of multiple scattering have been developed
 490 to determine the effective wavenumber of media with random scatterers; see for instance
 491 the Independent Scattering Approximation and the Waterman-Truell method [46]. The
 492 main drawback of these methods is that their validity is restricted to small concentrations
 493 of scatterers, typically less than 10 %. On the contrary, numerical methods do not suffer
 494 from such a limitation if suitable efforts are done. In particular, the errors due to the
 495 discretization (numerical dispersion, numerical dissipation, spurious diffractions on inter-
 496 faces, ...) must be much smaller than the physical quantities of interest. In [47], numerical
 497 simulations were used in the elastic case to estimate the accuracy of standard theoretical
 498 models, and also to show the improvement induced by recent models of multiple scattering
 499 [48]. As shown in test 4 of § 5, the numerical tools presented here make possible a similar
 500 study poroelastic random media and comparisons with theoretical models [49, 50].

501 However, realistic configurations would involve approximately 1500 scatterers, and sizing
 502 of the experiments leads to $N_x \times N_z = 10000^2$, and 10000 time iterations are required. Con-
 503 sequently, the numerical method has to be parallelized, for instance by Message Passing
 504 Interface (MPI).

505 • *Thermic boundary-layer.* In cases where the saturating fluid is a gas, the effects of thermal
 506 expansion of both pore fluid and the matrix have to be taken into account. In the HF
 507 regime, the thermal exchanges between fluid and solid phase occur in a small layer close
 508 to the surface of the pores. In this case, the dynamic thermal permeability is introduced
 509 [51], leading in the time-domain to an additional shifted fractional derivative of order
 510 1/2. The numerical method developed in this paper can be applied without difficulty by
 511 introducing additional memory variables.

512 • *Fractional derivatives in space.* The Biot theory is very efficient to predict the macro-
 513 scopic behavior of long-wavelength sound propagation in porous medium with relatively
 514 simple microgeometries. However, it remains far to describe correctly the coarse-grained
 515 dynamics of the medium when the microgeometry of the porous medium become more
 516 complex, for instance fractal. For rigid-framed porous media permeated by a viscothermal

517 fluid, a generalized macroscopic nonlocal theory of sound propagation has been developed
 518 to take into account not only temporal dispersion, but also spatial dispersion [52]. In this
 519 case, the coefficients depends on the frequency and on the wavenumber. In the space-time
 520 domain, it introduces not only time-fractional derivatives, but also space-fractional deriva-
 521 tives. Numerical modeling of space-fractional differential equations has been addressed
 522 by several authors [53, 54], by using a Grünwald-Letnikov approximation. The diffusive
 523 approximation of such derivatives constitutes an interesting challenge.

524 Acknowledgments

525 The authors wish to thank Dr Mathieu Chekroun (LAUM, France) for his insights about
 526 multiple scattering and for computing the coherent phase velocity and attenuation with the $p - \omega$
 527 transform in test 4.

528 Appendix A. Proof of proposition 1

529 The equation (10a) is multiplied by \mathbf{v}_s^T and integrated

$$\int_{\mathbb{R}^2} \left(\rho \mathbf{v}_s^T \frac{\partial \mathbf{v}_s}{\partial t} + \rho_f \mathbf{v}_s^T \frac{\partial \mathbf{w}}{\partial t} - \mathbf{v}_s^T (\nabla \cdot \underline{\boldsymbol{\sigma}}) \right) dx dz = 0. \quad (\text{A.1})$$

530 The first term in (A.1) is written

$$\int_{\mathbb{R}^2} \rho \mathbf{v}_s^T \frac{\partial \mathbf{v}_s}{\partial t} dx dz = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \rho \mathbf{v}_s^T \mathbf{v}_s dx dz. \quad (\text{A.2})$$

531 Integrating by part and using (7), we obtain

$$\begin{aligned} - \int_{\mathbb{R}^2} \mathbf{v}_s^T (\nabla \cdot \underline{\boldsymbol{\sigma}}) dx dz &= \int_{\mathbb{R}^2} \boldsymbol{\sigma}^T \frac{\partial \boldsymbol{\varepsilon}}{\partial t} dx dz, \\ &= \int_{\mathbb{R}^2} \boldsymbol{\sigma}^T \left(\mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} - \mathbf{C}^{-1} \boldsymbol{\beta} \frac{\partial p}{\partial t} \right) dx dz, \\ &= \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} dx dz + \int_{\mathbb{R}^2} \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\beta} \frac{\partial p}{\partial t} dx dz, \\ &= \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} (\boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} + 2 \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\beta} p) dx dz - \int_{\mathbb{R}^2} \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} \right)^T \mathbf{C}^{-1} \boldsymbol{\beta} p dx dz. \end{aligned} \quad (\text{A.3})$$

532 Equation (10b) is multiplied by \mathbf{w}^T and integrated

$$\begin{aligned} \int_{\mathbb{R}^2} \left\{ \rho_f \mathbf{w}^T \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{w}^T \text{diag}(\rho_{wi}) \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}^T \nabla p \right. \\ \left. + \mathbf{w}^T \text{diag} \left(\frac{\eta}{\kappa_i} \frac{1}{\Omega_i} (D + \Omega_i)^{1/2} \right) \mathbf{w} \right\} dx dz = 0. \end{aligned} \quad (\text{A.4})$$

533 The second term in (A.4) can be written

$$\int_{\mathbb{R}^2} \mathbf{w}^T \text{diag}(\rho_{wi}) \frac{\partial \mathbf{w}}{\partial t} dx dz = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \mathbf{w}^T \text{diag}(\rho_{wi}) \mathbf{w} dx dz. \quad (\text{A.5})$$

534 Integrating by part the third term of (A.4), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} \mathbf{w}^T \nabla p \, dx \, dz = - \int_{\mathbb{R}^2} p \nabla \cdot \mathbf{w} \, dx \, dz, \\
& = \int_{\mathbb{R}^2} p \frac{\partial \xi}{\partial t} \, dx \, dz, \\
& = \int_{\mathbb{R}^2} p \left(\frac{1}{m} \frac{\partial p}{\partial t} + \boldsymbol{\beta}^T \frac{\partial \boldsymbol{\varepsilon}}{\partial t} \right) \, dx \, dz, \\
& = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{m} p^2 \, dx \, dz + \int_{\mathbb{R}^2} p \boldsymbol{\beta}^T \left(\mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{C}^{-1} \boldsymbol{\beta} \frac{\partial p}{\partial t} \right) \, dx \, dz, \\
& = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{m} p^2 \, dx \, dz + \int_{\mathbb{R}^2} \boldsymbol{\beta}^T \mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} p \, dx \, dz + \int_{\mathbb{R}^2} \boldsymbol{\beta}^T \mathbf{C}^{-1} \boldsymbol{\beta} p \frac{\partial p}{\partial t} \, dx \, dz, \\
& = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{m} p^2 \, dx \, dz + \int_{\mathbb{R}^2} \boldsymbol{\beta}^T \mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} p \, dx \, dz + \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} \boldsymbol{\beta}^T \mathbf{C}^{-1} \boldsymbol{\beta} p^2 \, dx \, dz.
\end{aligned} \tag{A.6}$$

535 We add (A.1) and the three first terms of (A.4). Using the symmetry of \mathbf{C} , there remains

$$\int_{\mathbb{R}^2} \rho_f \left(\mathbf{v}_s^T \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}^T \frac{\partial \mathbf{v}_s}{\partial t} \right) \, dx \, dz = \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} 2 \rho_f \mathbf{v}_s^T \mathbf{w}. \tag{A.7}$$

536 Equations (19) and (A.1)-(A.7) yield

$$\frac{d}{dt} (E_1 + E_2) = - \int_{\mathbb{R}^2} \int_0^\infty \frac{\eta}{\pi \sqrt{\theta}} \mathbf{w}^T \text{diag} \left(\frac{1}{\kappa_i \sqrt{\Omega_i}} \right) \boldsymbol{\psi} \, d\theta \, dx \, dz. \tag{A.8}$$

537 To calculate the right-hand side of (A.8), equation (21a) is multiplied by \mathbf{w}^T or $\boldsymbol{\psi}^T$

$$\begin{cases} \mathbf{w}^T \frac{\partial \boldsymbol{\psi}}{\partial t} - \mathbf{w}^T \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}^T \text{diag} (\theta + \Omega_i) \boldsymbol{\psi} - \mathbf{w}^T \text{diag} (\Omega_i) \mathbf{w} = \mathbf{0}, \\ \boldsymbol{\psi}^T \frac{\partial \boldsymbol{\psi}}{\partial t} - \boldsymbol{\psi}^T \frac{\partial \mathbf{w}}{\partial t} + \boldsymbol{\psi}^T \text{diag} (\theta + \Omega_i) \boldsymbol{\psi} - \boldsymbol{\psi}^T \text{diag} (\Omega_i) \mathbf{w} = \mathbf{0}. \end{cases} \tag{A.9}$$

538 Some algebraic operations on (A.9) yield

$$\begin{aligned}
\boldsymbol{\psi}^T \text{diag} (\theta + 2 \Omega_i) \mathbf{w} &= \frac{\partial}{\partial t} \frac{1}{2} (\mathbf{w} - \boldsymbol{\psi})^T (\mathbf{w} - \boldsymbol{\psi}) \\
&+ \boldsymbol{\psi}^T \text{diag} (\theta + \Omega_i) \boldsymbol{\psi} + \mathbf{w}^T \text{diag} (\Omega_i) \mathbf{w}.
\end{aligned} \tag{A.10}$$

539 Injecting (A.10) in (A.8) leads to the relation (24)

$$\begin{aligned}
\frac{d}{dt} (E_1 + E_2 + E_3) &= - \int_{\mathbb{R}^2} \int_0^\infty \frac{\eta}{\pi \sqrt{\theta}} \left\{ \boldsymbol{\psi}^T \text{diag} \left(\frac{\theta + \Omega_i}{\kappa_i \sqrt{\Omega_i} (\theta + 2 \Omega_i)} \right) \boldsymbol{\psi} \right. \\
&+ \left. \mathbf{w}^T \text{diag} \left(\frac{\Omega_i}{\kappa_i \sqrt{\Omega_i} (\theta + 2 \Omega_i)} \right) \mathbf{w} \right\} \, d\theta \, dx \, dz.
\end{aligned} \tag{A.11}$$

540 It remains to prove that E (22) is a positive definite quadratic form. Concerning E_1 , we write

$$\rho \mathbf{v}_s^T \mathbf{v}_s + \mathbf{w}^T \text{diag}(\rho_{wi}) \mathbf{w} + 2\rho_f \mathbf{v}_s^T \mathbf{w} = \mathbf{X}_1^T \mathbf{H}_1 \mathbf{X}_1 + \mathbf{X}_3^T \mathbf{H}_3 \mathbf{X}_3, \quad (\text{A.12})$$

541 where

$$\mathbf{X}_i = (v_{si} \ w_i)^T, \quad \mathbf{H}_i = \begin{pmatrix} \rho & \rho_f \\ \rho_f & \rho_{wi} \end{pmatrix}, \quad i = 1, 3. \quad (\text{A.13})$$

542 Taking \mathcal{S}_i and \mathcal{P}_i to denote the sum and the product of the eigenvalues of matrix \mathbf{H}_i , we obtain

$$\begin{cases} \mathcal{P}_i = \det \mathbf{H}_i = \rho \rho_{wi} - \rho_f^2 = \chi_i > 0, \\ \mathcal{S}_i = \text{tr} \mathbf{H}_i = \rho + \rho_w > 0. \end{cases} \quad (\text{A.14})$$

543 The eigenvalues of \mathbf{H}_i are therefore positive. This proves that E_1 is a positive definite quadratic
544 form. The terms E_2 , E_3 and $-\frac{dE}{dt}$ are obviously positive definite quadratic form because the
545 involved matrices are definite positive. \square

546 Appendix B. Matrices of propagation and dissipation

547 The matrices in (37) are

$$\mathbf{A} = \begin{pmatrix} \mathbf{0}_{4,4} & \mathbf{A}_1 & \mathbf{0}_{4,2N} \\ \mathbf{A}_2 & \mathbf{0}_{4,4} & \mathbf{0}_{4,2N} \\ \mathbf{0}_{2N,4} & \mathbf{A}_3 & \mathbf{0}_{2N,2N} \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} \frac{\rho_f}{\chi_1} & 0 & 0 & \frac{\rho}{\chi_1} \\ 0 & \frac{\rho_f}{\chi_3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\rho_f}{\chi_1} & 0 & 0 & \frac{\rho}{\chi_1} \\ 0 & \frac{\rho_f}{\chi_3} & 0 & 0 \end{pmatrix}, \quad (\text{B.1})$$

$$\mathbf{A}_1 = \begin{pmatrix} -\frac{\rho_{w1}}{\chi_1} & 0 & 0 & -\frac{\rho_f}{\chi_1} \\ 0 & -\frac{\rho_{w3}}{\chi_3} & 0 & 0 \\ \frac{\rho_f}{\chi_1} & 0 & 0 & \frac{\rho}{\chi_1} \\ 0 & \frac{\rho_f}{\chi_3} & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} -c_{11}^u & 0 & -\beta_1 m & 0 \\ 0 & -c_{55}^u & 0 & 0 \\ -c_{13}^u & 0 & -\beta_3 m & 0 \\ \beta_1 m & 0 & m & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{0}_{4,4} & \mathbf{B}_1 & \mathbf{0}_{4,2N} \\ \mathbf{B}_2 & \mathbf{0}_{4,4} & \mathbf{0}_{4,2N} \\ \mathbf{0}_{2N,4} & \mathbf{B}_3 & \mathbf{0}_{2N,2N} \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} 0 & \frac{\rho_f}{\chi_1} & 0 & 0 \\ 0 & 0 & \frac{\rho_f}{\chi_3} & \frac{\rho}{\chi_3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{\rho_f}{\chi_1} & 0 & 0 \\ 0 & 0 & \frac{\rho_f}{\chi_3} & \frac{\rho}{\chi_3} \end{pmatrix}, \quad (\text{B.2})$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -\frac{\rho_{w1}}{\chi_1} & 0 & 0 \\ 0 & 0 & -\frac{\rho_{w3}}{\chi_3} & -\frac{\rho_f}{\chi_3} \\ 0 & \frac{\rho_f}{\chi_1} & 0 & 0 \\ 0 & 0 & \frac{\rho_f}{\chi_3} & \frac{\rho}{\chi_3} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & -c_{13}^u & 0 & -\beta_1 m \\ -c_{55}^u & 0 & 0 & 0 \\ 0 & -c_{33}^u & 0 & -\beta_3 m \\ 0 & \beta_3 m & 0 & m \end{pmatrix},$$

549 and \mathbf{S} is the diffusive matrix

$$\mathbf{S} = \begin{pmatrix} \mathbf{0}_{4,4} & \mathbf{0}_{4,4} & \mathbf{S}_1 \\ \mathbf{0}_{4,4} & \mathbf{0}_{4,4} & \mathbf{0}_{4,2N} \\ \mathbf{S}_3 & \mathbf{0}_{2N,4} & \mathbf{S}_2 \end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix} 0 & 0 & -\Omega_1 & 0 \\ 0 & 0 & 0 & -\Omega_3 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\Omega_1 & 0 \\ 0 & 0 & 0 & -\Omega_3 \end{pmatrix}, \quad (\text{B.3})$$

$$\mathbf{S}_1 = \begin{pmatrix} -\frac{\rho_f}{\rho} \gamma_1 a_1^1 & 0 & \cdots & -\frac{\rho_f}{\rho} \gamma_1 a_N^1 & 0 \\ 0 & -\frac{\rho_f}{\rho} \gamma_3 a_1^3 & \cdots & 0 & -\frac{\rho_f}{\rho} \gamma_3 a_N^3 \\ \gamma_1 a_1^1 & 0 & \cdots & \gamma_1 a_N^1 & 0 \\ 0 & \gamma_3 a_1^3 & \cdots & 0 & \gamma_3 a_N^3 \end{pmatrix},$$

$$\mathbf{S}_2 = \begin{pmatrix} \gamma_1 a_1^1 + (\theta_1^1 + \Omega_1) & 0 & \cdots & \gamma_1 a_N^1 & 0 \\ 0 & \gamma_3 a_1^3 + (\theta_1^3 + \Omega_3) & \cdots & 0 & \gamma_3 a_N^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1 a_1^1 & 0 & \cdots & \gamma_1 a_N^1 + (\theta_N^1 + \Omega_1) & 0 \\ 0 & \gamma_3 a_1^3 & \cdots & 0 & \gamma_3 a_N^3 + (\theta_N^3 + \Omega_3) \end{pmatrix}.$$

550 **Appendix C. Proof of proposition 4**

551 We denote $\mathbf{P}_{\mathcal{B}}$ the change-of-basis matrix satisfying

$$\mathbf{U} = \mathbf{P}_{\mathcal{B}} (\mathbf{U}_1, \mathbf{U}_3, \boldsymbol{\sigma}, p)^T, \quad (\text{C.1})$$

552 with

$$\mathbf{U}_i = (v_{si}, w_i, \psi_1^i, \dots, \psi_N^i)^T, \quad i = 1, 3. \quad (\text{C.2})$$

553 The matrix $\mathbf{P}_{\mathcal{B}}$ is thus invertible, and the matrices \mathbf{S} (Appendix B) and $\mathbf{S}_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}}^{-1} \mathbf{S} \mathbf{P}_{\mathcal{B}}$ are similar.

554 The matrix $\mathbf{S}_{\mathcal{B}}$ writes

$$\mathbf{S}_{\mathcal{B}} = \begin{pmatrix} \tilde{\mathbf{S}}_1 & \mathbf{0}_{N+2, N+2} & \mathbf{0}_{N+2, 3} & \mathbf{0}_{N+2, 1} \\ \mathbf{0}_{N+2, N+2} & \tilde{\mathbf{S}}_3 & \mathbf{0}_{N+2, 3} & \mathbf{0}_{N+2, 1} \\ \mathbf{0}_{3, N+2} & \mathbf{0}_{3, N+2} & \mathbf{0}_{3, 3} & \mathbf{0}_{3, 1} \\ \mathbf{0}_{1, N+2} & \mathbf{0}_{1, N+2} & \mathbf{0}_{1, 3} & 0 \end{pmatrix} \quad (\text{C.3})$$

555 with ($i = 1, 3$)

$$\tilde{\mathbf{S}}_i = \begin{pmatrix} 0 & 0 & -\frac{\rho_f}{\rho} \gamma_i a_1^i & -\frac{\rho_f}{\rho} \gamma_i a_2^i & \cdots & -\frac{\rho_f}{\rho} \gamma_i a_N^i \\ 0 & 0 & \gamma_i a_1^i & \gamma_i a_2^i & \cdots & \gamma_i a_N^i \\ \hline 0 & -\Omega_i & \gamma_i a_1^i + (\theta_1^i + \Omega_i) & \gamma_i a_2^i & \cdots & \gamma_i a_N^i \\ 0 & -\Omega_i & \gamma_i a_1^i & \gamma_i a_2^i + (\theta_2^i + \Omega_i) & \cdots & \gamma_i a_N^i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\Omega_i & \gamma_i a_1^i & \gamma_i a_2^i & \cdots & \gamma_i a_N^i + (\theta_N^i + \Omega_i) \end{pmatrix}. \quad (\text{C.4})$$

556 The characteristic polynomial of \mathbf{S} is

$$P_{\mathbf{S}}(s) = s^4 P_{\tilde{\mathbf{S}}_1}(s) P_{\tilde{\mathbf{S}}_3}(s), \quad (\text{C.5})$$

557 where $P_{\tilde{\mathbf{S}}_i}(s)$ denotes the characteristic polynomial of the matrix $\tilde{\mathbf{S}}_i$, i.e. $\tilde{\mathbf{S}}_i(s) = \det(\tilde{\mathbf{S}}_i - s \mathbf{I}_{N+2})$
 558 with \mathbf{I}_{N+2} the $(N + 2)$ -identity matrix. This $(N + 2)$ -determinant is expanded along the first
 559 column. The line I and the column J of the $(N + 1)$ -determinant thus obtained are denoted L_I
 560 and C_J , respectively ($0 \leq I, J \leq N$). The following algebraic manipulations are then performed
 561 successively:

$$\begin{aligned} (i) \quad & L_\ell \leftarrow L_\ell - L_0, \quad \ell = 1, \dots, N, \\ (ii) \quad & C_0 \leftarrow C_0 \prod_{\ell=1}^N (\theta_\ell^i + \Omega_i - s), \\ (iii) \quad & C_0 \leftarrow C_0 - (s - \Omega_1) \prod_{\substack{k=1 \\ k \neq \ell}}^N (\theta_k^i + \Omega_i - s) C_\ell, \quad \ell = 1, \dots, N. \end{aligned} \quad (\text{C.6})$$

562 One deduces

$$P_{\tilde{\mathcal{S}}_i}(s) = -s Q_i(s) = s^2 \prod_{\ell=1}^N (\theta_\ell^i + \Omega_i - s) + \gamma_i s (s - \Omega_i) \sum_{\ell=1}^N a_\ell^i \prod_{\substack{k=1 \\ k \neq \ell}}^N (\theta_k^i + \Omega_i - s). \quad (\text{C.7})$$

563 From equation (C.7), one has $P_{\tilde{\mathcal{S}}_i}(0) \neq 0$ while $Q_i(0) \neq 0$, therefore 0 is an eigenvalue of the
564 matrix $\tilde{\mathcal{S}}_i$ with multiplicity 1. In what follows, the positivity of the coefficients θ_ℓ^i, a_ℓ^i of the
565 diffusive approximation is used. In the limit $s \rightarrow 0^+$, then asymptotically

$$P_{\tilde{\mathcal{S}}_i}(s) \underset{s \rightarrow 0^+}{\sim} -\gamma_i \Omega_i s \sum_{\ell=1}^N a_\ell^i \prod_{\substack{k=1 \\ k \neq \ell}}^N (\theta_k^i + \Omega_i) \Rightarrow \text{sgn}(P_{\tilde{\mathcal{S}}_i}(0^+)) = -1. \quad (\text{C.8})$$

566 Moreover, using (43), then at the quadrature abscissae one has for all $\ell = 1, \dots, N$

$$P_{\tilde{\mathcal{S}}_i}(\theta_\ell^i + \Omega_i) = \gamma_i \theta_\ell^i (\theta_\ell^i + \Omega_i) a_\ell^i \prod_{\substack{k=1 \\ k \neq \ell}}^N (\theta_k^i - \theta_\ell^i) \Rightarrow \text{sgn}(P_{\tilde{\mathcal{S}}_i}(\theta_\ell^i + \Omega_i)) = (-1)^{\ell+1}. \quad (\text{C.9})$$

567 Finally, the following limit holds

$$P_{\tilde{\mathcal{S}}_i}(s) \underset{s \rightarrow +\infty}{\sim} (-1)^N s^{N+2} \Rightarrow \text{sgn}(P_{\tilde{\mathcal{S}}_i}(+\infty)) = (-1)^N. \quad (\text{C.10})$$

568 We introduce the following intervals

$$I_N^i =]\theta_N^i + \Omega_i, +\infty[, \quad I_\ell^i =]\theta_\ell^i, \theta_{\ell+1}^i + \Omega_i], \quad \text{for } \ell = 1, \dots, N-1, \quad I_0^i =]0, \theta_1^i + \Omega_i]. \quad (\text{C.11})$$

569 The real-valued continuous function $P_{\tilde{\mathcal{S}}_i}$ changes of sign on each interval I_ℓ^i . Consequently,
570 according to the intermediate value theorem, $P_{\tilde{\mathcal{S}}_i}$ has at least one zero in each interval. Since $P_{\tilde{\mathcal{S}}_i}$
571 has at the most $N+1$ distinct zeros in $]0, +\infty[$, we deduce that $\exists! s_\ell^i \in I_\ell^i / P_{\tilde{\mathcal{S}}_i}(s_\ell^i) = 0, \quad \ell =$
572 $1, \dots, N+1$. Using equation (C.5), the characteristic polynomial of \mathcal{S} (C.7) is therefore

$$P_{\mathcal{S}}(s) = s^6 \prod_{\ell=1}^{N+1} (s - s_\ell^1) (s - s_\ell^3), \quad (\text{C.12})$$

573 which concludes the proof. \square

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