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# Internal and subspace correction approximations of implicit variational inequalities

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## Keywords

Evolution inequalities, implicit inequalities, contact problems, existence results, internal approximation, subspace correction methods

## Abstract

The aim of this paper is to study the existence of solutions and some approximations for a class of implicit evolution variational inequalities that represents a generalization of several quasistatic contact problems in elasticity. Using appropriate estimates for the incremental solutions, the existence of a continuous solution and convergence results are proved for some corresponding internal approximation and backward difference scheme. To solve the fully discrete problems, general additive subspace correction algorithms are considered, for which global convergence is proved and some error estimates are established.

## 1 Introduction

This paper concerns the mathematical and numerical analysis of a system of evolution variational inequalities that represents a generalization of several quasistatic frictional contact problems in small deformation elasticity.

The results presented here are based on a unified approach, which contains in particular the contact problems studied in [1], [2], and can also be applied to various quasistatic problems, including unilateral or bilateral contact with nonlocal friction, normal compliance conditions with friction or more complex

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interaction laws, as, for example, interface laws coupling unilateral contact, adhesion and nonlocal friction between two elastic bodies [3].

The uniqueness of the solution does not hold in general, as was shown in [4] for a quasistatic frictional contact problem, even for an arbitrarily small coefficient of friction.

In this paper, we consider a class of implicit evolution variational inequalities involving a nonlinear operator, for which approximation and existence results are proved by using a semi-discrete scheme. The case of a linear operator was investigated in [1]. Also, for a problem which represents a particular case of the one considered here, some similar results were announced in [5].

To solve the general quasi-variational inequalities of the second kind that are obtained by the previous incremental procedure, two additive subspace correction algorithms are introduced, for which global convergence is proved and error estimates are established.

In [6] (see also [7] and [8]), one- and two-level multiplicative methods have been introduced for variational and quasi-variational inequalities of the second kind. Also, their application to contact problems with friction has been analyzed. The additive methods, even if they are a little slower than the multiplicative ones, have the advantage of being totally parallelizable. In the present paper, we use the same techniques as in the multiplicative case, but the convexity of the functionals plays a more important role here. Some details in these proofs are omitted, but they can be found in [9], [6] or [7]. In a subsequent paper, we shall use these results to prove the convergence of the one- and two-level additive methods for contact problems with friction.

The paper is organized as follows. In Section 2, the formulation of a general system of evolution inequalities and some examples are presented. Using an implicit time discretization scheme and some estimates, strong convergences are proved and an existence result is established.

In Section 3, convergence results are proved for a method based on a semi-discrete internal approximation.

In Section 4, two subspace correction algorithms, of additive type, are proposed for the solution of the problem discretized in time. These algorithms are introduced in a general framework and in a Hilbert space. We suppose here an assumption on the convex set and the correction subspaces, which will be essential in the proof of the convergence of the algorithms. Mainly, this hypothesis refers to the decomposition of the elements in the convex set, and introduces a constant  $C_0$  which will play an important role in the writing of the convergence rate. Another hypothesis is made on the non differentiable term in the inequality. Under these assumptions, we prove that the two subspace correction algorithms are convergent and give an estimation of the convergence rate. As for the multiplicative algorithms introduced in [6], the

introduced assumptions are satisfied in the particular case of the frictional contact problems if the coefficient of friction is sufficiently small.

## 2 Existence results for a system of evolution inequalities

Let  $(V, \langle \cdot, \cdot \rangle)$ ,  $(H, (\cdot, \cdot)_H)$  be two real Hilbert spaces with the associated norms  $\|\cdot\|$  and  $\|\cdot\|_H$ , respectively. Let  $K$  be a closed convex cone contained in  $V$  with its vertex at 0 and  $(K(g))_{g \in V}$  be a family of nonempty closed subsets of  $K$  satisfying the following conditions:  $0 \in K(0)$  and

$$\text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n) \text{ and } v_n \rightarrow v \text{ in } V \text{ then } v \in K(g). \quad (1)$$

Consider a functional  $F : V \rightarrow \mathbb{R}$  differentiable on  $V$ , and assume that its derivative  $F' : V \rightarrow V$  is strongly monotone and Lipschitz continuous, that is there exist two constants  $\alpha, \beta > 0$  for which

$$\alpha \|v - u\|^2 \leq \langle F'(v) - F'(u), v - u \rangle \quad (2)$$

and

$$\|F'(v) - F'(u)\|_V \leq \beta \|v - u\| \quad (3)$$

for all  $u, v \in V$ .

Using the relations

$$\begin{aligned} F(v) - F(u) &= \int_0^1 \langle F'(u + r(v - u)), v - u \rangle dr \\ &= \langle F'(u), v - u \rangle + \int_0^1 \langle F'(u + r(v - u)) - F'(u), v - u \rangle dr \end{aligned}$$

and (2), (3), it is easily seen that for all  $u, v \in V$  it results

$$\begin{aligned} \langle F'(u), v - u \rangle + \frac{\alpha}{2} \|v - u\|^2 &\leq F(v) - F(u) \\ &\leq \langle F'(u), v - u \rangle + \frac{\beta}{2} \|v - u\|^2. \end{aligned} \quad (4)$$

We remark that since  $F$  satisfies (4), it follows that  $F$  is strictly convex and sequentially weakly lower semicontinuous on  $V$ .

Let  $\gamma : V \times K \rightarrow H$  be an operator such that  $\gamma(0, 0) = 0$ ,

$$\begin{aligned} & \text{if } g_n \rightarrow g \text{ in } V, v_n \in K \text{ and } v_n \rightarrow v \text{ in } V \\ & \text{then } \gamma(g_n, v_n) \rightarrow \gamma(g, v) \text{ in } H \end{aligned} \quad (5)$$

and for all  $g_i \in V, v_i \in K, i = 1, 2$ ,

$$\|\gamma(g_1, v_1) - \gamma(g_2, v_2)\|_H \leq k_1(\|g_1 - g_2\| + \|v_1 - v_2\|), \quad (6)$$

where  $k_1$  is a positive constant.

Let  $j : V \times K \times V \rightarrow \mathbb{R}$  be a functional satisfying the following conditions:

$$j(g, v, \cdot) \text{ is sequentially weakly continuous on } V \quad \forall g \in V, v \in K, \quad (7)$$

$$j(g, v, \cdot) \text{ is sub-additive for all } g \in V, v \in K, \text{ that is} \quad (8)$$

$$j(g, v, w_1 + w_2) \leq j(g, v, w_1) + j(g, v, w_2) \quad \forall g, w_{1,2} \in V, v \in K,$$

$$j(g, v, \cdot) \text{ is positively homogeneous for all } g \in V, v \in K, \quad (9)$$

that is  $j(g, v, \theta w) = \theta j(g, v, w) \quad \forall g, w \in V, v \in K, \theta \geq 0$ ,

$$j(0, 0, w) = 0 \quad \forall w \in V, \quad (10)$$

and there exists  $k_2 > 0$  such that

$$\begin{aligned} & |j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)| \\ & \leq k_2(\|g_1 - g_2\| + \|\gamma(g_1, v_1) - \gamma(g_2, v_2)\|_H)\|w_1 - w_2\| \\ & \quad \forall g_i, w_i \in V, v_i \in K, i = 1, 2. \end{aligned} \quad (11)$$

We assume that  $k_1$  and  $k_2$  satisfy the following property:

$$k_1 k_2 < \alpha. \quad (12)$$

For each  $g \in V$ , let  $b(g, \cdot, \cdot) : K(g) \times V \rightarrow \mathbb{R}$  be a functional such that

$$\forall v \in K(g), b(g, v, \cdot) \text{ is linear on } V, \quad (13)$$

$$|b(g, v, w)| \leq k_3(\|g\| + \|v\|)\|w\| \quad \forall v \in K(g), \forall w \in V, \quad (14)$$

where  $k_3$  is a positive constant,

$$\begin{aligned} & \text{if } g_n \rightarrow g \text{ in } V, v_n \in K(g_n), v_n \rightarrow v \text{ in } V \\ & \text{and } w_n \rightarrow w \text{ in } V, \text{ then } b(g_n, v_n, w_n) \rightarrow b(g, v, w). \end{aligned} \quad (15)$$

From the above properties of  $F$ ,  $j$  and  $K$  and by a classical argument, it follows that for all  $g \in V$ ,  $d \in K$ ,  $w \in K(g)$  the elliptic variational inequality

$$u \in K \quad \langle F'(u), v - u \rangle + j(g, w, v - d) - j(g, w, u - d) \geq 0 \quad \forall v \in K$$

has a unique solution, so that we can define the mapping  $S_{g,d} : K(g) \rightarrow K$  by  $S_{g,d}(w) = u$ . We assume that for all  $g \in V$ ,  $d \in K$

$$K(g) \text{ is stable under } S_{g,d} \text{ i.e. } S_{g,d}(K(g)) \subset K(g). \quad (16)$$

For all  $g \in V$ ,  $d \in K$ , we consider the following problems:

$$(\tilde{P}) \quad \begin{cases} u \in K(g) & \langle F'(u), v - u \rangle + j(g, u, v - d) - j(g, u, u - d) \\ & \geq b(g, u, v - u) \quad \forall v \in V, \\ b(g, u, z - u) \geq 0 & \forall z \in K, \end{cases}$$

$$(\tilde{Q}) \quad u \in K(g) \quad \langle F'(u), v - u \rangle + j(g, u, v - d) - j(g, u, u - d) \geq 0 \quad \forall v \in K,$$

and we assume that

$$\text{if } u \text{ is a solution of } (\tilde{Q}), \text{ then } u \text{ is a solution of } (\tilde{P}). \quad (17)$$

**Remark 2.1.** *If  $u$  satisfies  $(\tilde{P})$ , then  $u$  obviously satisfies  $(\tilde{Q})$ .*

**Remark 2.2.** *i) From (11) and (6) it follows that*

$$\begin{aligned} & |j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)| \\ & \leq ((k_1 + 1)k_2\|g_1 - g_2\| + k_1k_2\|v_1 - v_2\|)\|w_1 - w_2\| \\ & \quad \forall g_i, w_i \in V, v_i \in K, i = 1, 2. \end{aligned} \quad (18)$$

*ii) Since, by (9)  $j(\cdot, \cdot, 0) = 0$ , from (11), for  $w_2 = 0$ ,  $w_1 = w$ , it results that*

$$\begin{aligned} & |j(g_2, v_2, w) - j(g_1, v_1, w)| \\ & \leq k_2(\|g_1 - g_2\| + \|\gamma(g_1, v_1) - \gamma(g_2, v_2)\|_H)\|w\| \\ & \leq ((k_1 + 1)k_2\|g_1 - g_2\| + k_1k_2\|v_1 - v_2\|)\|w\| \\ & \quad \forall g_i, w \in V, v_i \in K, i = 1, 2. \end{aligned} \quad (19)$$

*iii) As  $j$  satisfies (8), (9),  $j(g, v, \cdot)$  is convex, and from (10) and (18), with  $g_2 = g$ ,  $v_2 = v$ ,  $g_1 = v_1 = 0$ , it follows that*

$$\begin{aligned} & |j(g, v, w_1) - j(g, v, w_2)| \leq ((k_1 + 1)k_2\|g\| + k_1k_2\|v\|)\|w_1 - w_2\| \\ & \quad \forall g \in V, v \in K, \forall w_i \in V, i = 1, 2. \end{aligned}$$

iv) Using (5) and (19) we have

$$\begin{aligned} & \text{if } g_n \rightarrow g \text{ in } V, v_n \in K \text{ and } v_n \rightharpoonup v \text{ in } V \\ & \text{then } j(g_n, v_n, w) \rightarrow j(g, v, w) \quad \forall w \in V. \end{aligned} \quad (20)$$

Let  $f \in W^{1,2}(0, T; V)$  be given. Using the hypotheses (2), (3), (16) and (18), it follows that  $S_{f(0),0} : K(f(0)) \rightarrow K(f(0))$  is a contraction if the condition (12) holds. Thus, the following implicit elliptic variational inequality has a unique solution  $u_0 \in K(f(0))$  (see, e.g. [10]):

$$\langle F'(u_0), w - u_0 \rangle + j(f(0), u_0, w) - j(f(0), u_0, u_0) \geq 0 \quad \forall w \in K.$$

We consider the following evolution problem involving an implicit variational inequality.

**Problem P:** Find  $u \in W^{1,2}(0, T; V)$  such that

$$(P) \quad \begin{cases} u(0) = u_0, u(t) \in K(f(t)) \quad \forall t \in ]0, T[, \\ \langle F'(u(t)), v - \dot{u}(t) \rangle + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \quad \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall v \in V \text{ a.e. on } ]0, T[, \\ b(f(t), u(t), w - u(t)) \geq 0 \quad \forall w \in K, \forall t \in ]0, T[. \end{cases}$$

The quasistatic unilateral contact problems with nonlocal friction in linearized elasticity studied in [2], [1] can be considered as particular cases of problem P corresponding to a quadratic functional  $F$ .

We shall briefly present two examples involving nonlinear operators  $F'$ .

We consider an elastic body occupying an open, bounded, connected set  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  with a Lipschitz continuous boundary  $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ , where  $\Gamma_1, \Gamma_2, \Gamma_3$  are open and disjoint parts of  $\Gamma$  with  $\text{meas}(\Gamma_1) > 0$ . Assume the small deformation hypothesis and that the inertial effects are negligible.

We denote by  $\mathbf{u}$  the displacement field, by  $\boldsymbol{\varepsilon}$  the infinitesimal strain tensor and by  $\boldsymbol{\sigma}$  the stress tensor, with the components  $u = (u_i)$ ,  $\varepsilon = (\varepsilon_{ij})$  and  $\sigma = (\sigma_{ij})$ , respectively. We use the classical decompositions  $\mathbf{u} = u_N \mathbf{n} + \mathbf{u}_T$ ,  $u_N = \mathbf{u} \cdot \mathbf{n}$ ,  $\boldsymbol{\sigma} \mathbf{n} = \sigma_N \mathbf{n} + \boldsymbol{\sigma}_T$ ,  $\sigma_N = (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n}$ , where  $\mathbf{n}$  is the outward normal unit vector to  $\Gamma$  with the components  $n = (n_i)$ . Also, the usual summation convention will be used for  $i, j, k, l = 1, \dots, d$ .

Assume that in  $\Omega$  a body force  $\boldsymbol{\varphi}_1 \in W^{1,2}(0, T; [L^2(\Omega)]^d)$  is prescribed, on  $\Gamma_1$  the displacement vector equals zero and on  $\Gamma_2$  a traction  $\boldsymbol{\varphi}_2 \in W^{1,2}(0, T; [L^2(\Gamma_2)]^d)$  is applied.

The displacement fields will be assumed to belong to the following space:

$$V_0 := \{\mathbf{v} \in [H^1(\Omega)]^d; \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}.$$

The body can be in contact with a support on  $\Gamma_3$ , so that the displacements and the stress vector on this part of the boundary will satisfy some frictional contact conditions. The initial gap between the contact surface and the support is assumed to be zero.

*Example 1.* (Unilateral contact with nonlocal friction for a nonlinear Hencky material)

Consider an elastic body satisfying the following nonlinear Hencky-Mises constitutive equation (see [11], [12]):

$$\boldsymbol{\sigma}(\mathbf{u}) = \hat{\boldsymbol{\psi}}(\mathbf{u}) = \left(k - \frac{2}{3}\hat{\mu}(\hat{\gamma}(\mathbf{u}))\right)(\text{tr } \boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I} + 2\hat{\mu}(\hat{\gamma}(\mathbf{u})) \boldsymbol{\varepsilon}(\mathbf{u}),$$

where  $k$  is the constant bulk modulus,  $\hat{\mu}$  is a continuously differentiable function in  $[0, +\infty[$  satisfying

$$0 < \hat{\mu}_0 \leq \hat{\mu}(r) \leq \frac{3}{2}k, \quad 0 < \hat{\mu}_1 \leq \hat{\mu}(r) + 2 \frac{\partial \hat{\mu}(r)}{\partial r} r \leq \hat{\mu}_2, \quad \forall r \geq 0,$$

$$\hat{\gamma}(\mathbf{u}) := \hat{\gamma}(\mathbf{u}, \mathbf{u}), \quad \hat{\gamma}(\mathbf{u}, \mathbf{v}) = -\frac{2}{3} \vartheta(\mathbf{u}) \vartheta(\mathbf{v}) + 2 \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}),$$

with

$$\vartheta(\mathbf{u}) := \text{tr } \boldsymbol{\varepsilon}(\mathbf{u}) = \text{div } \mathbf{u} \quad \forall \mathbf{u}, \mathbf{v} \in V_0.$$

Assume Signorini contact conditions with a nonlocal friction law on  $\Gamma_3$ .

The classical formulation of the quasistatic contact problem is as follows.

**Problem  $\mathbf{P}_c^1$ :** Find a displacement field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  which satisfies the initial condition  $\mathbf{u}(0) = \mathbf{u}_0$  in  $\Omega$  such that for all  $t \in ]0, T[$

$$(P_c^1) \quad \left\{ \begin{array}{l} \text{div } \boldsymbol{\sigma}(\mathbf{u}) = -\boldsymbol{\varphi}_1 \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \hat{\boldsymbol{\psi}}(\mathbf{u}) \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \\ \boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\varphi}_2 \quad \text{on } \Gamma_2, \\ u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0 \quad \text{on } \Gamma_3, \\ |\boldsymbol{\sigma}_T| \leq \mu |(\mathcal{R}\boldsymbol{\sigma})_N| \quad \text{on } \Gamma_3 \\ \text{and} \quad \left\{ \begin{array}{l} |\boldsymbol{\sigma}_T| < \mu |(\mathcal{R}\boldsymbol{\sigma})_N| \Rightarrow \dot{\mathbf{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = \mu |(\mathcal{R}\boldsymbol{\sigma})_N| \Rightarrow \exists \lambda \geq 0, \dot{\mathbf{u}}_T = -\lambda \boldsymbol{\sigma}_T, \end{array} \right. \end{array} \right.$$

where  $\mu \in L^\infty(\Gamma_3)$  with  $\mu \geq 0$  a.e. on  $\Gamma_3$  is the coefficient of friction and  $\mathcal{R}\boldsymbol{\sigma}$  is a regularization of the stress tensor that will be presented below.



To obtain a variational formulation for this problem, we use the following notations:

$$\begin{aligned} K_0 &:= \{\mathbf{v} \in V_0; v_N \leq 0 \text{ a.e. on } \Gamma_3\}, \quad (\cdot, \cdot) := (\cdot, \cdot)_{[H^1(\Omega)]^d}, \\ H^{\frac{1}{2}}(\Gamma_3) &:= \{w : \Gamma_3 \rightarrow \mathbb{R}; w \in H^{\frac{1}{2}}(\Gamma), w = 0 \text{ a.e. on } \Gamma_1\}, \\ \forall \mathbf{L} \in V_0 \quad S_{\mathbf{L}} &:= \{\mathbf{v} \in V_0; \int_{\Omega} \hat{\boldsymbol{\psi}}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\phi}) dx = (\mathbf{L}, \boldsymbol{\phi}) \\ &\quad \forall \boldsymbol{\phi} \in V_0 \text{ such that } \boldsymbol{\phi} = \mathbf{0} \text{ a.e. on } \Gamma_3\}. \end{aligned}$$

We assume that  $\mathcal{R} : [L^2_{\text{Sym}}(\Omega)]^{d^2} \rightarrow [H^1(\Omega)]^{d^2}$  is a regularization operator satisfying  $\mathcal{R}\mathbf{0} = \mathbf{0}$  and

$$\text{if } \mathbf{v}_n \in K_0 \text{ and } \mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } V_0 \text{ then } \mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}_n) \rightarrow \mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}) \text{ in } [H^1(\Omega)]^{d^2},$$

$$\|\mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}_1) - \mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}_2)\|_{[H^1(\Omega)]^{d^2}} \leq k_4 \|\mathbf{v}_1 - \mathbf{v}_2\|_{[H^1(\Omega)]^d} \quad \forall \mathbf{v}_i \in K_0, i = 1, 2,$$

where  $k_4$  is a positive constant. Similar regularizations were described in [13], [14] for a viscoelastic body. For each  $\mathbf{L} \in V_0$  and  $\mathbf{v} \in S_{\mathbf{L}}$ , we define the stress vector  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n} \in ([H^{\frac{1}{2}}(\Gamma_3)]^d)'$  by

$$\forall \mathbf{w} \in [H^{\frac{1}{2}}(\Gamma_3)]^d \quad \langle \boldsymbol{\sigma}(\mathbf{v})\mathbf{n}, \mathbf{w} \rangle_{\Gamma_3} = \int_{\Omega} \hat{\boldsymbol{\psi}}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{w}}) dx - (\mathbf{L}, \bar{\mathbf{w}}),$$

where  $\langle \cdot, \cdot \rangle_{\Gamma_3}$  denotes the duality pairing on  $([H^{\frac{1}{2}}(\Gamma_3)]^d)' \times [H^{\frac{1}{2}}(\Gamma_3)]^d$ ,  $\bar{\mathbf{w}} \in V_0$  satisfies  $\bar{\mathbf{w}} = \mathbf{w}$  a.e. on  $\Gamma_3$ , and we define the normal component of the stress vector  $\sigma_N(\mathbf{v}) \in (H^{\frac{1}{2}}(\Gamma_3))'$  by

$$\forall w \in H^{\frac{1}{2}}(\Gamma_3) \quad \langle \sigma_N(\mathbf{v}), w \rangle_{\Gamma_3} = \int_{\Omega} \hat{\boldsymbol{\psi}}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{w}}) dx - (\mathbf{L}, \bar{\mathbf{w}}),$$

where  $\langle \cdot, \cdot \rangle_{\Gamma_3}$  denotes also the duality pairing on  $(H^{\frac{1}{2}}(\Gamma_3))' \times H^{\frac{1}{2}}(\Gamma_3)$ ,  $\bar{\mathbf{w}} \in V_0$  satisfies  $\bar{\mathbf{w}}_T = \mathbf{0}$  a.e. on  $\Gamma_3$ ,  $\bar{\mathbf{w}}_N = w$  a.e. on  $\Gamma_3$ . It is easy to verify that for all  $\mathbf{v} \in S_{\mathbf{L}}$  the definitions of  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$  and of  $\sigma_N(\mathbf{v})$  do not depend on the choices of  $\bar{\mathbf{w}}$  having the above properties, respectively.

Let  $J_1 : K_0 \times V_0 \rightarrow \mathbb{R}$  be a functional defined by

$$J_1(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_3} \mu |(\mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}))_N| |\mathbf{w}_T| ds \quad \forall \mathbf{v} \in K_0, \mathbf{w} \in V_0,$$

and  $\mathbf{L} \in V_0$  be given by the relation

$$(\mathbf{L}, \mathbf{v}) = (\boldsymbol{\varphi}_1, \mathbf{v})_{[L^2(\Omega)]^d} + (\boldsymbol{\varphi}_2, \mathbf{v})_{[L^2(\Gamma_2)]^d} \quad \forall \mathbf{v} \in V_0. \quad (21)$$

We have the following variational formulation of  $(P_c^1)$ .

**Problem  $P_v^1$ :** Find  $\mathbf{u} \in W^{1,2}(0, T; V_0)$  such that

$$(P_v^1) \begin{cases} \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}(t) \in K_0 \quad \forall t \in ]0, T[, \\ \int_{\Omega} \hat{\boldsymbol{\psi}}(\mathbf{u}(t)) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)) dx + J_1(\mathbf{u}(t), \mathbf{v}) - J_1(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{L}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + \langle \sigma_N(\mathbf{u}(t)), v_N - \dot{u}_N(t) \rangle_{\Gamma_3} \quad \forall \mathbf{v} \in V_0 \text{ a.e. on } ]0, T[, \\ \langle \sigma_N(\mathbf{u}(t)), z_N - u_N(t) \rangle_{\Gamma_3} \geq 0 \quad \forall \mathbf{z} \in K_0, \forall t \in ]0, T[. \end{cases}$$

We define the functional  $F : V_0 \rightarrow \mathbb{R}$  by

$$F(\mathbf{v}) = \frac{1}{2} k \int_{\Omega} \vartheta^2(\mathbf{v}) dx + \frac{1}{2} \int_{\Omega} \left( \int_0^{\hat{\gamma}(\mathbf{v})} \hat{\mu}(r) dr \right) dx \quad \forall \mathbf{v} \in V_0.$$

One can verify, see, e.g. [11], Ch. 8, that  $F$  is differentiable on  $V_0$  and for all  $\mathbf{u}, \mathbf{v} \in V_0$

$$(F'(\mathbf{u}), \mathbf{v}) = \int_{\Omega} \left[ \left( k - \frac{2}{3} \hat{\mu}(\hat{\gamma}(\mathbf{u})) \right) \vartheta(\mathbf{u}) \vartheta(\mathbf{v}) + 2 \hat{\mu}(\hat{\gamma}(\mathbf{u})) \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \right] dx.$$

Taking  $V = V_0$ ,  $K = K_0$ ,  $H = L^2(\Gamma_3)$ ,  $f = \mathbf{L}$ ,  $K(f) = K_0 \cap S_{\mathbf{L}}$ ,

$$\begin{aligned} \gamma(\mathbf{L}, \mathbf{v}) &= |(\mathcal{R}\hat{\boldsymbol{\psi}}(\mathbf{v}))_N|, \quad j(\mathbf{L}, \mathbf{v}, \mathbf{w}) = J_1(\mathbf{v}, \mathbf{w}) - (\mathbf{L}, \mathbf{w}), \\ b(\mathbf{L}, \mathbf{v}, \mathbf{w}) &= \langle \sigma_N(\mathbf{v}), w_N \rangle_{\Gamma_3}, \end{aligned}$$

it results that  $(P_v^1)$  can be written in the form  $(P)$ , where  $\mathbf{L}$  is defined by (21). It is easily seen that if the coefficient of friction is sufficiently small then the condition (12) is satisfied.

*Example 2.* (Frictional contact with normal compliance)

We consider a linearly elastic body and we denote by  $\boldsymbol{\mathcal{E}}$  the elasticity tensor, with the Cartesian coordinates  $\boldsymbol{\mathcal{E}} = (a_{ijkl})$  satisfying the usual properties of symmetry and ellipticity.

Let us define  $a : V_0 \times V_0 \rightarrow \mathbb{R}$  by

$$a(\mathbf{v}, \mathbf{w}) = \int_{\Omega} a_{ijkl} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{kl}(\mathbf{w}) dx = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{w}) dx \quad \forall \mathbf{v}, \mathbf{w} \in V_0.$$

We suppose that the frictional contact between  $\Gamma_3$  and the support is described by a normal compliance law, see, e.g. [15] and references therein for more general laws.

The classical formulation of the contact problem is as follows.

**Problem  $P_c^2$ :** Find a displacement field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  which satisfies the initial condition  $\mathbf{u}(0) = \mathbf{u}_0$  in  $\Omega$  such that for all  $t \in ]0, T[$

$$(P_c^2) \left\{ \begin{array}{l} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = -\boldsymbol{\varphi}_1 \text{ in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \boldsymbol{\mathcal{E}} \boldsymbol{\varepsilon}(\mathbf{u}) \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1, \\ \boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\varphi}_2 \text{ on } \Gamma_2, \\ \sigma_N = -C_N(u_N)_+ \text{ on } \Gamma_3 \text{ with } C_N > 0, \\ |\boldsymbol{\sigma}_T| \leq \mu |\sigma_N| \text{ on } \Gamma_3 \\ \text{and } \begin{cases} |\boldsymbol{\sigma}_T| < \mu |\sigma_N| \Rightarrow \dot{\mathbf{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = \mu |\sigma_N| \Rightarrow \exists \lambda \geq 0, \dot{\mathbf{u}}_T = -\lambda \boldsymbol{\sigma}_T. \end{cases} \end{array} \right.$$

Let  $J_2 : V_0 \times V_0 \rightarrow \mathbb{R}$  and  $p_N : V_0 \times V_0 \rightarrow \mathbb{R}$  be two functionals defined by

$$J_2(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_3} \mu C_N(v_N)_+ |\mathbf{w}_T| ds \quad \forall \mathbf{v}, \mathbf{w} \in V_0,$$

$$p_N(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_3} C_N(v_N)_+ w_N ds \quad \forall \mathbf{v}, \mathbf{w} \in V_0.$$

A variational formulation of  $(P_c^2)$  is as follows.

**Problem  $P_v^2$ :** Find  $\mathbf{u} \in W^{1,2}(0, T; V_0)$  such that  $\mathbf{u}(0) = \mathbf{u}_0$  and

$$(P_v^2) \left\{ \begin{array}{l} a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + p_N(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + J_2(\mathbf{u}(t), \mathbf{v}) - J_2(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{L}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) \quad \forall \mathbf{v} \in V_0 \text{ a.e. on } ]0, T[, \end{array} \right.$$

where  $\mathbf{L} \in V_0$  is given by the relation (21).

Let us define the functional  $F : V_0 \rightarrow \mathbb{R}$  by

$$F(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) + \frac{C_N}{2} \int_{\Gamma_3} (v_N)_+^2 ds \quad \forall \mathbf{v} \in V_0.$$

$F$  is differentiable on  $V_0$  and for all  $\mathbf{u}, \mathbf{v} \in V_0$

$$(F'(\mathbf{u}), \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + p_N(\mathbf{u}, \mathbf{v}).$$

Taking  $V = K = K(f) = V_0$ ,  $H = L^2(\Gamma_3)$ ,  $f = \mathbf{L}$ ,

$$\gamma(\mathbf{L}, \mathbf{v}) = C_N(v_N)_+, \quad j(\mathbf{L}, \mathbf{v}, \mathbf{w}) = J_2(\mathbf{v}, \mathbf{w}) - (\mathbf{L}, \mathbf{w}),$$

$$b(\mathbf{L}, \mathbf{v}, \mathbf{w}) \equiv 0,$$

it results that  $(P_v^2)$  can be written in the form  $(P)$ .

We remark that  $C_N$  can be chosen arbitrarily positive and that if the coefficient of friction is sufficiently small then the condition (12) is satisfied.

Now, we shall prove the existence of a solution to problem P by using an implicit time discretization scheme and its convergence properties.

For  $\nu \in N^*$ , we set  $\Delta t := T/\nu$ ,  $t_i := i \Delta t$  and  $K^i := K(f(t_i))$ ,  $i = 0, 1, \dots, \nu$ . If  $\theta$  is a continuous function of  $t \in [0, T]$  valued in some vector space, we use the notations  $\theta^i := \theta(t_i)$  unless  $\theta = u$ , and if  $\zeta^i$ ,  $\forall i \in \{0, 1, \dots, \nu\}$ , are elements of some vector space, then we set

$$\partial \zeta^i := \frac{\zeta^{i+1} - \zeta^i}{\Delta t} \quad \forall i \in \{0, 1, \dots, \nu - 1\}.$$

We denote  $u^0 := u_0$  and we approximate  $(P)$  using the following sequence of incremental problems  $(P_\nu^i)_{i=0,1,\dots,\nu-1}$ .

**Problem  $P_\nu^i$ :** Find  $u^{i+1} \in K^{i+1}$  such that

$$(P_\nu^i) \quad \begin{cases} \langle F'(u^{i+1}), v - \partial u^i \rangle + j(f^{i+1}, u^{i+1}, v) - j(f^{i+1}, u^{i+1}, \partial u^i) \\ \geq b(f^{i+1}, u^{i+1}, v - \partial u^i) \quad \forall v \in V, \\ b(f^{i+1}, u^{i+1}, w - u^{i+1}) \geq 0 \quad \forall w \in K. \end{cases}$$

By (17) and Remark 2.1 for  $g = f^{i+1}$ ,  $d = u^i$ , it is easily seen that for all  $i \in \{0, 1, \dots, \nu - 1\}$  the problem  $P_\nu^i$  is equivalent to the following quasi-variational inequality.

**Problem  $Q_\nu^i$ :** Find  $u^{i+1} \in K^{i+1}$  such that

$$(Q_\nu^i) \quad \begin{cases} \langle F'(u^{i+1}), w - u^{i+1} \rangle + j(f^{i+1}, u^{i+1}, w - u^i) \\ - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \geq 0 \quad \forall w \in K. \end{cases}$$

From the hypotheses (2), (12), (16) and (18), it follows that  $S_{f^{i+1}, u^i} : K^{i+1} \rightarrow K^{i+1}$  is a contraction. Therefore  $(Q_\nu^i)$  has a unique solution which is equally the unique solution of  $(P_\nu^i)$ , for all  $i \in \{0, 1, \dots, \nu - 1\}$ .

**Remark 2.3.** *i) Since  $K$  is a cone with the vertex at 0, the solutions  $u$  of  $(P)$  and  $u^{i+1}$  of  $(P_\nu^i)$  obviously satisfy*

$$b(f(t), u(t), u(t)) = 0 \quad \text{on } [0, T]$$

and

$$b(f^{i+1}, u^{i+1}, u^{i+1}) = 0 \quad \forall i \in \{0, 1, \dots, \nu - 1\}.$$

*ii) It follows also that (see, e.g. [1])*

$$b(f(t), u(t), \dot{u}(t)) = 0 \quad \text{a.e. on } [0, T].$$

Let us define the following functions:

$$\left\{ \begin{array}{l} u_\nu(0) = \hat{u}_\nu(0) = u^0, \quad f_\nu(0) = f^0 \quad \text{and} \\ \forall i \in \{0, 1, \dots, \nu - 1\}, \quad \forall t \in ]t_i, t_{i+1}], \\ u_\nu(t) = u^{i+1}, \\ \hat{u}_\nu(t) = u^i + (t - t_i)\partial u^i, \\ f_\nu(t) = f^{i+1}. \end{array} \right.$$

Then for all  $\nu \in N^*$  the sequence of inequalities  $(P_\nu^i)_{i=0,1,\dots,\nu-1}$  is equivalent to the following incremental formulation: for almost every  $t \in [0, T]$

$$(P_\nu) \left\{ \begin{array}{l} u_\nu(t) \in K(f_\nu(t)), \quad \langle F'(u_\nu(t)), v - \frac{d}{dt}\hat{u}_\nu(t) \rangle + j(f_\nu(t), u_\nu(t), v) \\ -j(f_\nu(t), u_\nu(t), \frac{d}{dt}\hat{u}_\nu(t)) \geq b(f_\nu(t), u_\nu(t), v - \frac{d}{dt}\hat{u}_\nu(t)) \quad \forall v \in V, \\ b(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq 0 \quad \forall w \in K. \end{array} \right.$$

Also, the sequence  $(Q_\nu^i)_{i=0,1,\dots,\nu-1}$  implies the following inequality: for almost every  $t \in [0, T]$

$$(R_\nu) \quad \langle F'(u_\nu(t)), w - u_\nu(t) \rangle + j(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq 0 \quad \forall w \in K,$$

which, by (4), is clearly equivalent to the following inequality: for almost every  $t \in [0, T]$

$$(\hat{R}_\nu) \quad F(w) - F(u_\nu(t)) + j(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq \frac{\alpha}{2} \|w - u_\nu(t)\|^2 \quad \forall w \in K.$$

Using straightforward generalizations of the linear case presented in [1] one can prove the following two lemmas.

**Lemma 2.1.** *For all  $\nu \in N^*$  we have*

$$\|u_\nu(t)\| \leq M_0 \|F'(0)\| + M_1 \|f\|_{C([0,T];V)} \quad \forall t \in [0, T], \quad (22)$$

$$\|u_\nu(s) - u_\nu(t)\| \leq M_1 \int_s^{\min\{t+\Delta t, T\}} \|\dot{f}(\tau)\| d\tau \quad \forall s, t \in [0, T], \quad s < t,$$

$$\|u_\nu(t) - \hat{u}_\nu(t)\| \leq \frac{T}{\nu} \left\| \frac{d}{dt}\hat{u}_\nu(t) \right\| \quad a.e. \quad t \in [0, T],$$

$$\|u_\nu - \hat{u}_\nu\|_{L^2(0,T;V)} = \frac{T}{\nu\sqrt{3}} \left\| \frac{d}{dt}\hat{u}_\nu \right\|_{L^2(0,T;V)}, \quad (23)$$

$$\left\| \frac{d}{dt} \hat{u}_\nu \right\|_{L^2(0,T;V)} \leq M_1 \|\dot{f}\|_{L^2(0,T;V)},$$

where

$$M_0 = \frac{1}{\alpha - k_1 k_2}, \quad M_1 = \frac{(k_1 + 1)k_2}{\alpha - k_1 k_2}.$$

**Lemma 2.2.** *There exist a subsequence of  $(u_\nu, \hat{u}_\nu)_\nu$ , denoted by  $(u_{\nu_p}, \hat{u}_{\nu_p})_p$ , and an element  $u \in W^{1,2}(0, T; V)$  such that*

$$u_{\nu_p}(t) \rightharpoonup u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (24)$$

$$\hat{u}_{\nu_p} \rightharpoonup u \quad \text{in } W^{1,2}(0, T; V),$$

$$\frac{d}{dt} \hat{u}_{\nu_p} \rightharpoonup \dot{u} \quad \text{in } L^2(0, T; V). \quad (25)$$

Also, we have

$$\liminf_{p \rightarrow \infty} \int_0^T j(f_{\nu_p}(t), u_{\nu_p}(t), \frac{d}{dt} \hat{u}_{\nu_p}(t)) dt \geq \int_0^T j(f(t), u(t), \dot{u}(t)) dt. \quad (26)$$

Note that since  $F'$  is nonlinear, it is not possible to use a simple extension of the linear case presented in [1] in order to pass to the limit for the terms involving  $F'$  or  $F$ .

Now, we prove the following strong convergence and existence result.

**Theorem 2.1.** *Under the assumptions (1)–(3), (5)–(17) every convergent subsequence of  $(u_\nu, \hat{u}_\nu)_\nu$ , still denoted by  $(u_\nu, \hat{u}_\nu)_\nu$ , and its limit  $u \in W^{1,2}(0, T; V)$ , given by lemma 2.2, satisfy the following properties:*

$$u_\nu(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (27)$$

$$\hat{u}_\nu \rightarrow u \quad \text{in } L^2(0, T; V), \quad (28)$$

and  $u$  is a solution of problem  $P$ .

*Proof.* Let  $(u_\nu)_\nu$  be the subsequence given by lemma 2.2 and  $u$  its weak limit.

Taking  $w = u(t)$  in  $(\hat{R}_\nu)$ , we obtain for all  $t \in [0, T]$

$$F(u(t)) - F(u_\nu(t)) + j(f_\nu(t), u_\nu(t), u(t) - u_\nu(t)) \geq \frac{\alpha}{2} \|u(t) - u_\nu(t)\|^2.$$

Since  $F$  is sequentially weakly lower semicontinuous, by using the previous relation, (19), (24), (5) and (7), we have that for all  $t \in [0, T]$

$$\begin{aligned}
& \limsup_{\nu \rightarrow \infty} \frac{\alpha}{2} \|u(t) - u_\nu(t)\|^2 \\
& \leq F(u(t)) + \limsup_{\nu \rightarrow \infty} (-F(u_\nu(t))) \\
& \quad + \lim_{\nu \rightarrow \infty} (j(f_\nu(t), u_\nu(t), u(t) - u_\nu(t)) - j(f(t), u(t), u(t) - u_\nu(t))) \\
& \quad + \lim_{\nu \rightarrow \infty} j(f(t), u(t), u(t) - u_\nu(t)) \\
& \leq F(u(t)) - \liminf_{\nu \rightarrow \infty} F(u_\nu(t)) \\
& + \lim_{\nu \rightarrow \infty} k_2(\|f_\nu(t) - f(t)\| + \|\gamma(f_\nu(t), u_\nu(t)) - \gamma(f(t), u(t))\|_H) \|u(t) - u_\nu(t)\| \\
& \quad + \lim_{\nu \rightarrow \infty} j(f(t), u(t), u(t) - u_\nu(t)) \leq 0,
\end{aligned}$$

which proves (27). By applying Lebesgue's dominated convergence theorem, it follows from (27) and (22) that

$$u_\nu \rightarrow u \quad \text{in} \quad L^2(0, T; V), \quad (29)$$

and using (23) we obtain (28).

It remains to prove that the limit  $u$  is a solution of problem P.

First, by (1) we obtain that  $u(t) \in K(f(t)) \quad \forall t \in ]0, T[$  and by (15) one can pass to the limit in the second inequality of  $(P_\nu)$  for all  $t \in ]0, T[$  so that the second inequality of  $(P)$  is satisfied. Second, integrating both sides in the first inequality of  $(P_\nu)$  over  $[0, T]$  and passing to the limit, by the relations (27), (29), (25), (3), (20), (26) and Remark 2.3 i), we have for all  $v \in L^2(0, T; V)$

$$\begin{aligned}
& \int_0^T \langle F'(u(t)), v(t) - \dot{u}(t) \rangle dt + \int_0^T j(f(t), u(t), v(t)) dt \\
& \quad - \int_0^T j(f(t), u(t), \dot{u}(t)) dt \geq \int_0^T b(f(t), u(t), v(t)) dt. \quad (30)
\end{aligned}$$

By Lebesgue's theorem and Remark 2.3 ii), it follows that  $u$  is a solution of the first inequality of problem P.  $\square$

### 3 Internal approximation and convergence results

In this section we shall consider a semi-discrete approximation of  $(P)$ , which extends the classical internal approximations as presented in, e.g. [16], [17]. We shall prove a convergence result for a method based on an internal approximation and a backward difference scheme that represents a generalization of the case studied in [1], where  $F'$  is linear and there is no internal approximation for the functional  $j$ , which is a limitation on the numerical realization of the discrete problem.

For a positive parameter  $h$  converging to 0, let  $(V_h)_h$  be an internal approximation of  $V$ , that is a family of finite-dimensional subspaces of  $V$  which satisfies:

$$\begin{aligned} & \text{there exists } U \subset V \text{ such that } \overline{U} = V \text{ and} \\ & \forall v \in U, \exists v_h \in V_h \text{ for each } h, \text{ such that } v_h \rightarrow v \text{ in } V. \end{aligned} \quad (31)$$

Let  $(K_h)_h$  be a family of closed convex cones with their vertices at 0 such that  $K_h \subset V_h$  for all  $h$  and  $(K_h)_h$  is an internal approximation of  $K$ , i.e.

$$\text{if } v_h \in K_h \text{ for all } h \text{ and } v_h \rightarrow v \text{ then } v \in K, \quad (32)$$

$$\forall v \in K, \exists v_h \in K_h \text{ for each } h, \text{ such that } v_h \rightarrow v \text{ in } V. \quad (33)$$

Let  $(K_h(g))_{g \in V}$  be a family of nonempty closed subsets of  $K_h$  such that  $0 \in K_h(0)$  for all  $h$ , satisfying the following conditions:

$$\text{if } g_n \rightarrow g \text{ in } V, v_{hn} \in K_h(g_n) \text{ and } v_{hn} \rightarrow v_h \text{ in } V_h \text{ then } v_h \in K_h(g), \quad (34)$$

$$\text{if } v_h \in K_h(g) \text{ for all } h \text{ and } v_h \rightarrow v \text{ then } v \in K(g) \quad \forall g \in V. \quad (35)$$

We assume that there exists an operator  $\gamma_h : V \times K_h \rightarrow H$  such that  $\gamma_h(0, 0) = 0$  and for all  $g_i \in V, v_{hi} \in K_h, i = 1, 2$ ,

$$\|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H \leq k_1(\|g_1 - g_2\| + \|v_{h1} - v_{h2}\|). \quad (36)$$

Let  $j_h : V \times K_h \times V_h \rightarrow \mathbb{R}$  be a functional satisfying the following conditions for all  $g \in V$ :

$$\begin{aligned} & \text{if } v_h \in K_h \text{ for all } h, v_h \rightarrow v \text{ in } V \text{ and } w_h \rightarrow w \text{ in } V \\ & \text{then } \lim_{h \rightarrow 0} j_h(g, v_h, w_h) = j(g, v, w), \end{aligned} \quad (37)$$

$$\text{for all } h \text{ and } v_h \in K_h \quad j_h(g, v_h, \cdot) \text{ is sub-additive,} \quad (38)$$



for all  $h$  and  $v_h \in K_h$   $j_h(g, v_h, \cdot)$  is positively homogeneous,

$$j_h(0, 0, w_h) = 0 \quad \forall w_h \in V_h, \quad (39)$$

and

if  $v_h(t) \in K_h$  for all  $h$  and  $t \in [0, T]$ ,  $v_h \rightharpoonup v$  in  $W^{1,2}(0, T; V)$

$$\text{then } \liminf_{h \rightarrow 0} \int_0^T j_h(g(t), v_h(t), \dot{v}_h(t)) dt \geq \int_0^T j(g(t), v(t), \dot{v}(t)) dt \quad (40)$$

for all  $g \in C([0, T]; V)$ ,

$$\begin{aligned} & |j_h(g_1, v_{h1}, w_{h2}) + j_h(g_2, v_{h2}, w_{h1}) - j_h(g_1, v_{h1}, w_{h1}) - j_h(g_2, v_{h2}, w_{h2})| \\ & \leq k_2(\|g_1 - g_2\| + \|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H) \|w_{h1} - w_{h2}\| \quad (41) \\ & \quad \forall g_i \in V, v_{hi} \in K_h, w_{hi} \in V_h, i = 1, 2. \end{aligned}$$

From the properties of  $F$ ,  $j_h$  and  $K_h$ , it follows that for all  $g \in V$ ,  $d_h \in K_h$ ,  $w_h \in K_h(g)$ , the elliptic variational inequality:  $u_h \in K_h$

$$\langle F'(u_h), v_h - u_h \rangle + j_h(g, w_h, v_h - d_h) - j_h(g, w_h, u_h - d_h) \geq 0 \quad \forall v_h \in K_h$$

has a unique solution. Hence we can define a mapping  $S_{g, d_h}^h : K_h(g) \rightarrow K_h$  by  $S_{g, d_h}^h(w_h) = u_h$ . We suppose that for all  $g \in V$ ,  $d_h \in K_h$

$$S_{g, d_h}^h(K_h(g)) \subset K_h(g). \quad (42)$$

For all  $g \in V$ ,  $d_h \in K_h$ , we consider the following problems:

$$(\tilde{P}_h) \begin{cases} u_h \in K_h(g), \quad \langle F'(u_h), v_h - u_h \rangle + j_h(g, u_h, v_h - d_h) \\ \quad - j_h(g, u_h, u_h - d_h) \geq b(g, u_h, v_h - u_h) \quad \forall v_h \in V_h, \\ b(g, u_h, z_h - u_h) \geq 0 \quad \forall z_h \in K_h, \end{cases}$$

and

$$(\tilde{Q}_h) \begin{cases} u_h \in K_h(g), \quad \langle F'(u_h), v_h - u_h \rangle + j_h(g, u_h, v_h - d_h) \\ \quad - j_h(g, u_h, u_h - d_h) \geq 0 \quad \forall v_h \in K_h. \end{cases}$$

Assume that

$$\text{if } u_h \text{ is a solution of } (\tilde{Q}_h), \text{ then } u_h \text{ is a solution of } (\tilde{P}_h). \quad (43)$$

**Remark 3.1.** If  $u_h$  satisfies  $(\tilde{P}_h)$ , then  $u_h$  obviously satisfies  $(\tilde{Q}_h)$ .

**Remark 3.2.** *Using similar arguments as in Remark 2.2, one can easily prove the following properties:*

$$\begin{aligned} & |j_h(g_1, v_{h1}, w_{h2}) + j_h(g_2, v_{h2}, w_{h1}) - j_h(g_1, v_{h1}, w_{h1}) - j_h(g_2, v_{h2}, w_{h2})| \\ & \leq ((k_1 + 1)k_2\|g_1 - g_2\| + k_1k_2\|v_{h1} - v_{h2}\|)\|w_{h1} - w_{h2}\| \\ & \forall g_i, w_{hi} \in V_h, v_{hi} \in K_h, i = 1, 2, \end{aligned}$$

$$\begin{aligned} & |j_h(g_2, v_{h2}, w_h) - j_h(g_1, v_{h1}, w_h)| \\ & \leq ((k_1 + 1)k_2\|g_1 - g_2\| + k_1k_2\|v_{h1} - v_{h2}\|)\|w_h\| \\ & \forall g_i \in V, w_h \in V_h, v_{hi} \in K_h, i = 1, 2, \end{aligned}$$

$$\begin{aligned} & |j_h(g, v_h, w_{h1}) - j_h(g, v_h, w_{h2})| \\ & \leq ((k_1 + 1)k_2\|g\| + k_1k_2\|v_h\|)\|w_{h1} - w_{h2}\| \\ & \forall g \in V, v_h \in K_h, \forall w_{hi} \in V_h, i = 1, 2, \end{aligned}$$

if  $g_n \rightarrow g$  in  $V$ ,  $v_{hn} \in K_h$  and  $v_{hn} \rightarrow v_h$  in  $V_h$   
then  $j_h(g_n, v_{hn}, w_h) \rightarrow j_h(g, v_h, w_h) \quad \forall w_h \in V_h$ .

We introduce the following semi-discrete approximation of problem P.

**Problem P<sub>h</sub>:** Find  $u_h \in W^{1,2}(0, T; V_h)$  such that

$$(P_h) \begin{cases} u_h(0) = u_{0h}, \quad u_h(t) \in K_h(f(t)) \quad \forall t \in ]0, T[, \\ \langle F'(u_h(t)), v_h - \dot{u}_h(t) \rangle + j_h(f(t), u_h(t), v_h) - j_h(f(t), u_h(t), \dot{u}_h(t)) \\ \geq b(f(t), u_h(t), v_h - \dot{u}_h(t)) \quad \forall v_h \in V_h \text{ a.e. on } ]0, T[, \\ b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall z_h \in K_h, \quad \forall t \in ]0, T[, \end{cases}$$

where  $u_{0h} \in K_h(f(0))$  is the unique solution of the variational inequality

$$\begin{aligned} & \langle F'(u_{0h}), w_h - u_{0h} \rangle + j_h(f(0), u_{0h}, w_h) \\ & - j_h(f(0), u_{0h}, u_{0h}) \geq 0 \quad \forall w_h \in K_h. \end{aligned}$$

By (4), it follows that  $(P_h)$  implies the following relation:

$$(\hat{R}_h) \begin{cases} F(w_h) - F(u_h(t)) + j_h(f(t), u_h(t), w_h - u_h(t)) \\ \geq \frac{\alpha}{2}\|w_h - u_h(t)\|^2 \quad \forall w_h \in K_h \text{ a.e. on } ]0, T[. \end{cases}$$

The full discretization of  $(P_h)$  is obtained by using an implicit scheme as for  $(P)$ . For  $u_h^0 := u_{0h}$  and  $i \in \{0, 1, \dots, \nu - 1\}$ , we define  $u_h^{i+1}$  as the solution of the following problem.

**Problem  $P_{h\nu}^i$ :** Find  $u_h^{i+1} \in K_h^{i+1}$  such that

$$(P_{h\nu}^i) \quad \begin{cases} \langle F'(u_h^{i+1}), v_h - \partial u_h^i \rangle + j_h(f^{i+1}, u_h^{i+1}, v_h) - j_h(f^{i+1}, u_h^{i+1}, \partial u_h^i) \\ \geq b(f^{i+1}, u_h^{i+1}, v_h - \partial u_h^i) \quad \forall v_h \in V_h, \\ b(f^{i+1}, u_h^{i+1}, z_h - u_h^{i+1}) \geq 0 \quad \forall z_h \in K_h, \end{cases}$$

where  $K_h^{i+1} := K_h(f^{i+1})$ .

As in Section 2, it follows that for all  $i \in \{0, 1, \dots, \nu - 1\}$  the problem  $(P_{h\nu}^i)$  is equivalent to the following variational inequality:

find  $u_h^{i+1} \in K_h^{i+1}$  such that

$$(Q_{h\nu}^i) \quad \begin{cases} \langle F'(u_h^{i+1}), w_h - u_h^{i+1} \rangle + j_h(f^{i+1}, u_h^{i+1}, w_h - u_h^i) \\ - j_h(f^{i+1}, u_h^{i+1}, u_h^{i+1} - u_h^i) \geq 0 \quad \forall w_h \in K_h. \end{cases}$$

From (2), (36), (41), (12) and (42) it results that the mapping  $S_{f^{i+1}, u_h^i}^h : K_h^{i+1} \rightarrow K_h^{i+1}$  is a contraction, so that  $(Q_{h\nu}^i)$  has a unique solution which is also the unique solution of  $(P_{h\nu}^i)$ , for all  $i \in \{0, 1, \dots, \nu - 1\}$ .

If we define the functions

$$\begin{cases} u_{h\nu}(0) = \hat{u}_{h\nu}(0) = u_{0h} \quad \text{and} \\ \forall i \in \{0, 1, \dots, \nu - 1\}, \quad \forall t \in ]t_i, t_{i+1}], \\ u_{h\nu}(t) = u_h^{i+1}, \\ \hat{u}_{h\nu}(t) = u_h^i + (t - t_i)\partial u_h^i, \end{cases}$$

then for all  $\nu \in N^*$  the sequence of inequalities  $(P_{h\nu}^i)_{i=0,1,\dots,\nu-1}$  is equivalent to the following incremental formulation: for almost every  $t \in [0, T]$

$$(P_{h\nu}) \quad \begin{cases} u_{h\nu}(t) \in K_h(f_\nu(t)), \quad \langle F'(u_{h\nu}(t)), v_h - \frac{d}{dt}\hat{u}_{h\nu}(t) \rangle + j_h(f_\nu(t), u_{h\nu}(t), v_h) \\ - j_h(f_\nu(t), u_{h\nu}(t), \frac{d}{dt}\hat{u}_{h\nu}(t)) \geq b(f_\nu(t), u_{h\nu}(t), v_h - \frac{d}{dt}\hat{u}_{h\nu}(t)) \quad \forall v_h \in V_h, \\ b(f_\nu(t), u_{h\nu}(t), w_h - u_{h\nu}(t)) \geq 0 \quad \forall w_h \in K_h. \end{cases}$$

We have the following analogue to theorem 2.1 in the finite-dimensional case.

**Theorem 3.1.** *Assume that (2), (3), (12)–(15), (34), (36), (38), (39), (41)–(43) hold. Then there exists a subsequence of  $(u_{h\nu}, \hat{u}_{h\nu})_\nu$ , still denoted by  $(u_{h\nu}, \hat{u}_{h\nu})_\nu$ , such that*

$$u_{h\nu}(t) \rightarrow u_h(t) \quad \text{in } V \quad \forall t \in [0, T],$$

$$\hat{u}_{h\nu} \rightarrow u_h \quad \text{in } L^2(0, T; V),$$

where  $u_h$  is a solution of  $(P_h)$ .

By similar arguments as in lemma 2.1 and passing to the limit by using the previous theorem, we find the following a priori estimates for the solutions of  $(P_h)$  which are limits of subsequences of  $(u_{h\nu})_\nu$ .

**Lemma 3.1.** *If  $u_h$  is a solution of  $(P_h)$  then*

$$\|u_h(t)\| \leq M_0 \|F'(0)\| + M_1 \|f\|_{C([0, T]; V)} \quad \forall t \in [0, T], \quad (44)$$

$$\|u_h(s) - u_h(t)\| \leq M_1 \int_s^t \|\dot{f}(\tau)\| d\tau \quad \forall s, t \in [0, T], \quad s < t,$$

$$\|u_h\|_{W^{1,2}(0, T; V)} \leq M_2,$$

where

$$M_2 = \sqrt{2 M_0^2 T \|F'(0)\|^2 + 2 M_1^2 T \|f\|_{C([0, T]; V)}^2 + M_1^2 \|\dot{f}\|_{L^2(0, T; V)}^2}.$$

We have the following convergence and existence result.

**Theorem 3.2.** *Under the assumptions (2), (3), (12)–(15), (31)–(43) there exists a subsequence of  $(u_h)_h$  such that*

$$u_h(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (45)$$

$$u_h \rightarrow u \quad \text{in } L^2(0, T; V), \quad (46)$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad \text{in } L^2(0, T; V), \quad (47)$$

where  $u$  is a solution of  $(P)$ .

*Proof.* From lemma 3.1 it follows that there exists a subsequence of  $(u_h)_h$  and an element  $u \in W^{1,2}(0, T; V)$  such that

$$u_h(t) \rightharpoonup u(t) \quad \text{in } V \quad \forall t \in [0, T], \quad (48)$$

$$u_h \rightharpoonup u \quad \text{in } W^{1,2}(0, T; V). \quad (49)$$

which implies (47).

As one can easily prove that

$$u_{0h} \rightarrow u_0 \text{ in } V,$$

from (48) it follows that  $u(0) = u_0$ .

For each  $t \in [0, T]$ , by (33) one can find  $(\hat{u}_h)_h$  such that  $\hat{u}_h \in K_h$  and  $\hat{u}_h \rightarrow u(t)$  in  $V$ . Taking  $w_h = \hat{u}_h$  in  $(\hat{R}_h)$ , we have for all  $h$

$$F(\hat{u}_h) - F(u_h(t)) + j_h(f(t), u_h(t), \hat{u}_h - u_h(t)) \geq \frac{\alpha}{2} \|\hat{u}_h - u_h(t)\|^2.$$

Since  $F$  is strongly continuous and sequentially weakly lower semicontinuous, by using the previous relation, (48) and (37), we obtain

$$\begin{aligned} & \limsup_{h \rightarrow 0} \frac{\alpha}{2} \|\hat{u}_h - u_h(t)\|^2 \\ & \leq \lim_{h \rightarrow 0} F(\hat{u}_h) - \liminf_{h \rightarrow 0} F(u_h(t)) + \lim_{h \rightarrow 0} j_h(f(t), u_h(t), \hat{u}_h - u_h(t)) \\ & \leq F(u(t)) - F(u(t)) + j(f(t), u(t), 0) = 0, \end{aligned}$$

which proves (45).

From (44) and (45), by applying Lebesgue's dominated convergence theorem, we obtain (46).

Now, we shall prove that the limit  $u$  is a solution of problem P.

Using (33), (45), (35) and passing to the limit in the second inequality of  $(P_h)$  for all  $t \in ]0, T[$ , we obtain that  $u(t) \in K(f(t)) \forall t \in [0, T]$  and the second inequality of  $(P)$ .

Let  $\pi_h v$  be the projection of  $v \in V$  on  $V_h$  defined by  $\langle \pi_h v, w_h \rangle = \langle v, w_h \rangle \forall w_h \in V_h$ . Thus if  $v \in L^2(0, T; V)$  then  $\pi_h v \in L^2(0, T; V_h)$  and using (33) we have  $\pi_h v(t) \rightarrow v(t)$  in  $V$  a.e. on  $[0, T]$ . For all  $v \in L^2(0, T; V)$ , integrating both sides in the first inequality of  $(P_h)$  over  $[0, T]$  with  $v_h = \pi_h v$  and passing to the limit, by the relations (3), (45), (46), (49), (37) and (40) it follows that (30) is satisfied, which, by Lebesgue's theorem, implies that  $u$  is a solution to the first inequality of problem P.  $\square$

Using theorems 3.1 and 3.2, we obtain the following main approximation result.

**Theorem 3.3.** *Under the assumptions of theorem 3.2, there exists a subsequence of  $(u_{h\nu})_{h\nu}$  such that*

$$u_{h\nu}(t) \rightarrow u(t) \text{ in } V \quad \forall t \in [0, T], \quad (50)$$

$$\dot{u}_{h\nu} \rightharpoonup \dot{u} \text{ in } L^2(0, T; V), \quad (51)$$

where  $u \in W^{1,2}(0, T; V)$  is a solution of  $(P)$ .

Furthermore every cluster point of  $(u_{h\nu})_{h\nu}$  is a solution of  $(P)$ .

## 4 Subspace correction approximation

The aim of this section is to give, for a fixed time step  $i$ , two additive subspace correction algorithms for problem  $Q_\nu^i$ , prove their global convergence and estimate the convergence rate.

The methods we deal here generalize the projected multilevel relaxation method suggested in [18] and [19] for complementarity problems. This method was later developed in [20]–[22] and named as monotone multigrid method. On the other hand, the application of this method to other types of convex sets in general abstract spaces and monotone minimizing functionals have been investigated in [23], [24] and [25], for instance. Also, the case where the inequality contains extra terms which do not stem from the minimization of a functional has been investigated in [26]. Additional non-linear terms have also to be considered in the case of quasi-variational, or implicit, inequalities.

As in the previous section, we consider a Hilbert space  $V$  and let  $V_1, \dots, V_m$ ,  $m \geq 2$ , be some closed subspaces. We also consider a closed convex subset  $\mathcal{K} \subset V$  and assume that the following assumption is satisfied.

**ASSUMPTION 4.1.** *There exists a constant  $C_0 > 0$  such that for each  $w, v \in \mathcal{K}$  there exist  $v_\iota \in V_\iota$ ,  $\iota = 1, \dots, m$ , which satisfy*

$$v_\iota \in \mathcal{K} - w \text{ for } \iota = 1, \dots, m, \quad (52)$$

$$v - w = \sum_{\iota=1}^m v_\iota, \quad (53)$$

and

$$\sum_{\iota=1}^m \|v_\iota\| \leq C_0 \|v - w\|. \quad (54)$$

This assumption is satisfied for various convex sets in Sobolev spaces and will be used in the proofs, where  $v$  is the exact solution and  $w$  is the current approximation.

Now, we consider a functional  $\varphi : V \times V \rightarrow \mathbb{R}$  and we assume that it is convex and lower semicontinuous with respect to the second variable, and

$$\begin{aligned} & |\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \\ & \leq k_1 k_2 \|v_1 - v_2\| \|w_1 - w_2\| \quad \forall v_1, v_2, w_1, w_2 \in V. \end{aligned} \quad (55)$$

Also, we suppose

**ASSUMPTION 4.2.**

$$\sum_{\iota=1}^m \varphi(u, w + v_\iota) \leq (m - 1)\varphi(u, w) + \varphi(u, v) \quad (56)$$

for each  $u \in \mathcal{K}$  and for  $v, w \in \mathcal{K}$  and  $v_\iota \in V_\iota$ ,  $\iota = 1, \dots, m$ , as in Assumption 4.1.

This assumption has been introduced for technical proof reasons.

As in the previous section, let  $F : V \rightarrow \mathbb{R}$  be a differentiable functional satisfying (2), (3) and (12), and we consider the problem of finding  $\hat{u} \in \mathcal{K}$ , the solution of the following quasi-variational inequality

$$\langle F'(\hat{u}), v - \hat{u} \rangle + \varphi(\hat{u}, v) - \varphi(\hat{u}, \hat{u}) \geq 0 \quad \forall v \in \mathcal{K}. \quad (57)$$

Since  $\varphi$  satisfies (55), with similar arguments as for problem  $Q_\nu^i$ , we can prove that problem (57) admits a unique solution.

Evidently, since  $\varphi$  is convex with respect to the second variable,  $F$  is differentiable and satisfies (4), problem (57) is equivalent to the problem

$$\hat{u} \in \mathcal{K} \quad F(\hat{u}) + \varphi(\hat{u}, \hat{u}) \leq F(v) + \varphi(\hat{u}, v) \quad \forall v \in \mathcal{K}.$$

Also, in view of (4), we see that the solution  $\hat{u}$  of (57) satisfies

$$\frac{\alpha}{2} \|v - \hat{u}\|^2 \leq F(v) - F(\hat{u}) + \varphi(\hat{u}, v) - \varphi(\hat{u}, \hat{u}) \quad \forall v \in \mathcal{K}. \quad (58)$$

A first algorithm corresponding to the subspaces  $V_1, \dots, V_m$  and the convex set  $\mathcal{K}$  is the following.

**ALGORITHM 4.1.** *We start the algorithm with an arbitrary  $u^0 \in \mathcal{K}$ . At iteration  $n+1$ , having  $u^n \in \mathcal{K}$ ,  $n \geq 0$ , we simultaneously solve the inequalities*

$$\begin{aligned} w_\iota^{n+1} &\in V_\iota \cap (\mathcal{K} - u^n) \\ \langle F'(u^n + w_\iota^{n+1}), v_\iota - w_\iota^{n+1} \rangle + \varphi(u^n + w_\iota^{n+1}, u^n + v_\iota) \\ &\quad - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1}) \geq 0 \quad \forall v_\iota \in V_\iota \cap (\mathcal{K} - u^n), \end{aligned} \quad (59)$$

for  $\iota = 1, \dots, m$ , and then we update  $u^{n+1} = u^n + \frac{r}{m} \sum_{\iota=1}^m w_\iota^{n+1}$ , where  $r$  is a fixed constant such that  $0 < r \leq 1$ .

A simplified variant of Algorithm 4.1 can be written as

**ALGORITHM 4.2.** *We start the algorithm with an arbitrary  $u^0 \in \mathcal{K}$ . At iteration  $n+1$ , having  $u^n \in \mathcal{K}$ ,  $n \geq 0$ , we solve the inequalities*

$$\begin{aligned} w_\iota^{n+1} &\in V_\iota \cap (\mathcal{K} - u^n) \\ \langle F'(u^n + w_\iota^{n+1}), v_\iota - w_\iota^{n+1} \rangle + \varphi(u^n, u^n + v_\iota) \\ &\quad - \varphi(u^n, u^n + w_\iota^{n+1}) \geq 0 \quad \forall v_\iota \in V_\iota \cap (\mathcal{K} - u^n), \end{aligned} \quad (60)$$

for  $\iota = 1, \dots, m$ , and then we update  $u^{n+1} = u^n + \frac{r}{m} \sum_{\iota=1}^m w_\iota^{n+1}$ , where  $r$  is a fixed constant such that  $0 < r \leq 1$ .

The following theorem proves that if  $k_1 k_2$  is small enough in comparison with  $\alpha$ , then Algorithms 4.1 and 4.2 are convergent.

**Theorem 4.1.** *Let us assume that Assumptions 4.1 and 4.2 are satisfied. If  $\hat{u}$  is the solution of problem (57),  $u^n$ ,  $n \geq 0$ , are its approximations obtained from one of Algorithms 4.1 or 4.2, there exists a constant  $\chi_0$  satisfying  $1 > \chi_0 > 0$ , such that if*

$$\frac{k_1 k_2}{\alpha} \leq \chi_0 \quad (61)$$

then the two algorithms are globally convergent and we have the following error estimations

$$\begin{aligned} & F(u^n) + \varphi(\hat{u}, u^n) - F(\hat{u}) - \varphi(\hat{u}, \hat{u}) \\ & \leq \left( \frac{C_1}{C_1 + 1} \right)^n [F(u^0) + \varphi(\hat{u}, u^0) - F(\hat{u}) - \varphi(\hat{u}, \hat{u})], \end{aligned} \quad (62)$$

$$\|u^n - \hat{u}\|^2 \leq \frac{2}{\alpha} \left( \frac{C_1}{C_1 + 1} \right)^n [F(u^0) + \varphi(\hat{u}, u^0) - F(\hat{u}) - \varphi(\hat{u}, \hat{u})], \quad (63)$$

where the constant  $C_1 > 0$  depends on  $\alpha$ ,  $\beta$ ,  $k_1$ ,  $k_2$ , the number of subspaces  $m$  and on the constant  $C_0$  introduced in Assumption 4.1.

**Remark 4.1.** *Condition (12) is an existence and uniqueness condition for the solution of problems  $P_\nu^i$  and  $Q_\nu^i$  in Section 2. Theorem 4.1 proves that the algorithms introduced in this section for problem  $Q_\nu^i$  are convergent if condition (61) is satisfied. We see that the two conditions are similar, but the convergence condition is more restrictive than the existence and uniqueness one. This fact seems to be natural, and, roughly speaking, these conditions hold for frictional contact problems if the coefficient of friction is sufficiently small.*

*Proof.* Since the proof of the theorem is almost the same for both algorithms, we prove the theorem only for Algorithm 4.1. Also, in view of (58), we notice that (63) can be obtained from (62).

Now, we prove (62). Let us denote

$$\tilde{u}^{n+1} = u^n + \sum_{\iota=1}^m w_\iota^{n+1}. \quad (64)$$



Applying Assumption 4.1 for  $w = u^n$  and  $v = \hat{u}$ , we get a decomposition  $u_1^n, \dots, u_m^n$  of  $\hat{u} - u^n$ . From (52), we can replace  $v_i$  by  $u_i^n$  in (59), and in view of (4) and the convexity of  $F$ , we obtain

$$\begin{aligned}
& F(u^{n+1}) - F(\hat{u}) + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) + \frac{r}{m} \frac{\alpha}{2} \|\hat{u} - \tilde{u}^{n+1}\|^2 \\
& \leq (1 - \frac{r}{m}) [F(u^n) - F(\hat{u})] + \frac{r}{m} \left[ F(\tilde{u}^{n+1}) - F(\hat{u}) + \frac{\alpha}{2} \|\hat{u} - \tilde{u}^{n+1}\|^2 \right] \\
& \quad + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) \\
& \leq (1 - \frac{r}{m}) [F(u^n) - F(\hat{u})] + \frac{r}{m} \langle F'(\tilde{u}^{n+1}), \tilde{u}^{n+1} - \hat{u} \rangle + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) \\
& \leq (1 - \frac{r}{m}) [F(u^n) - F(\hat{u})] + \frac{r}{m} \sum_{i=1}^m \langle F'(u^n + w_i^{n+1}) - F'(\tilde{u}^{n+1}), u_i^n - w_i^{n+1} \rangle \\
& \quad + \frac{r}{m} \sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] \\
& \quad + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& F(u^{n+1}) - F(\hat{u}) + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) + \frac{r}{m} \frac{\alpha}{2} \|\hat{u} - \tilde{u}^{n+1}\|^2 \\
& \leq (1 - \frac{r}{m}) [F(u^n) - F(\hat{u}) + \varphi(\hat{u}, u^n) - \varphi(\hat{u}, \hat{u})] \\
& \quad + \frac{r}{m} \sum_{i=1}^m \langle F'(u^n + w_i^{n+1}) - F'(\tilde{u}^{n+1}), u_i^n - w_i^{n+1} \rangle \\
& \quad + \frac{r}{m} \sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] \\
& \quad + \frac{r}{m} [\varphi(\hat{u}, u^n) - \varphi(\hat{u}, \hat{u})] + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, u^n). \tag{65}
\end{aligned}$$

Using (3), (54) and the Hölder inequality, we get (see (3.21) in [9], for  $p = q = 2$ ),

$$\begin{aligned}
& \sum_{i=1}^m \langle F'(u^n + w_i^{n+1}) - F'(\tilde{u}^{n+1}), u_i^n - w_i^{n+1} \rangle \\
& \leq \beta m \left[ 1 + C_0 \left( 1 + \frac{1}{2\varepsilon_1} \right) \right] \sum_{i=1}^m \|w_i^{n+1}\|^2 + \beta C_0 \frac{\varepsilon_1}{2} \|\hat{u} - \tilde{u}^{n+1}\|^2 \tag{66}
\end{aligned}$$

for each  $\varepsilon_1 > 0$ . In view of the convexity of  $\varphi$  in the second variable, we have

$$\varphi(\hat{u}, u^{n+1}) \leq (1 - r)\varphi(\hat{u}, u^n) + \frac{r}{m} \sum_{i=1}^m \varphi(\hat{u}, u^n + w_i^{n+1}).$$

From this equation, by (56), (55) and (54), we obtain

$$\begin{aligned}
& \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n + u_\iota^n) - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})] \\
& \quad + \frac{r}{m} [\varphi(\hat{u}, u^n) - \varphi(\hat{u}, \hat{u})] + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, u^n) \\
& \leq \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n + u_\iota^n) - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})] \\
& \quad + \frac{r}{m} \sum_{\iota=1}^m \varphi(\hat{u}, u^n + w_\iota^{n+1}) - \frac{r}{m} [(m-1)\varphi(\hat{u}, u^n) + \varphi(\hat{u}, \hat{u})] \\
& \leq \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n + u_\iota^n) - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})] \\
& \quad + \frac{r}{m} \sum_{\iota=1}^m [\varphi(\hat{u}, u^n + w_\iota^{n+1}) - \varphi(\hat{u}, u^n + u_\iota^n)] \\
& \leq \frac{r}{m} k_1 k_2 \sum_{\iota=1}^m \|u^n + w_\iota^{n+1} - \hat{u}\| \|w_\iota^{n+1} - u_\iota^n\| \\
& \leq \frac{r}{m} k_1 k_2 \left[ \|\tilde{u}^{n+1} - \hat{u}\| + \sum_{\iota=1}^m \|w_\iota^{n+1}\| \right] \sum_{\iota=1}^m (\|w_\iota^{n+1}\| + \|u_\iota^n\|) \\
& \leq \frac{r}{m} k_1 k_2 \left[ \|\tilde{u}^{n+1} - \hat{u}\| + \sum_{\iota=1}^m \|w_\iota^{n+1}\| \right] \\
& \quad \cdot \left[ C_0 \|\tilde{u}^{n+1} - \hat{u}\| + (1 + C_0) \sum_{\iota=1}^m \|w_\iota^{n+1}\| \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n + u_\iota^n) - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})] \\
& \quad + \frac{r}{m} [\varphi(\hat{u}, u^n) - \varphi(\hat{u}, \hat{u})] + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, u^n) \\
& \leq \frac{r}{m} k_1 k_2 [C_0 + (1 + 2C_0) \frac{\varepsilon_2}{2}] \|\tilde{u}^{n+1} - \hat{u}\|^2 \\
& \quad + r k_1 k_2 [1 + C_0 + \frac{1 + 2C_0}{2\varepsilon_2}] \sum_{\iota=1}^m \|w_\iota^{n+1}\|^2 \tag{67}
\end{aligned}$$

for each  $\varepsilon_2 > 0$ . Consequently, from (65)–(67), we have

$$\begin{aligned}
& F(u^{n+1}) - F(\hat{u}) + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) \\
& + \left\{ \frac{\alpha}{2} - \beta C_0 \frac{\varepsilon_1}{2} - k_1 k_2 [C_0 + (1 + 2C_0) \frac{\varepsilon_2}{2}] \right\} \|\hat{u} - \tilde{u}^{n+1}\|^2 \\
& \leq \frac{m-r}{r} [F(u^n) - F(u^{n+1}) + \varphi(\hat{u}, u^n) - \varphi(\hat{u}, u^{n+1})] \\
& + \left\{ \beta m \left[ 1 + C_0 \left( 1 + \frac{1}{2\varepsilon_1} \right) \right] + k_1 k_2 m \left[ 1 + C_0 + \frac{1 + 2C_0}{2\varepsilon_2} \right] \right\} \\
& \cdot \sum_{\iota=1}^m \|w_\iota^{n+1}\|^2
\end{aligned} \tag{68}$$

for every  $\varepsilon_1, \varepsilon_2 > 0$ .

From (59) and (4), we get that, for each  $n \geq 0$  and  $\iota = 1, \dots, m$ ,

$$\begin{aligned}
& F(u^n) - F(u^n + w_\iota^{n+1}) + \varphi(u^n + w_\iota^{n+1}, u^n) \\
& - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1}) \geq \frac{\alpha}{2} \|w_\iota^{n+1}\|^2.
\end{aligned}$$

Using the convexity of  $F$  and the above relation, we have

$$\begin{aligned}
& F(u^{n+1}) \leq (1-r)F(u^n) + \frac{r}{m} \sum_{\iota=1}^m F(u^n + w_\iota^{n+1}) \leq F(u^n) \\
& - \frac{r}{m} \frac{\alpha}{2} \sum_{\iota=1}^m \|w_\iota^{n+1}\|^2 + \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n) - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})].
\end{aligned}$$

Thus, it results

$$\begin{aligned}
& \frac{r}{m} \frac{\alpha}{2} \sum_{\iota=1}^m \|w_\iota^{n+1}\|^2 \leq F(u^n) - F(u^{n+1}) \\
& + \varphi(\hat{u}, u^n) - \varphi(\hat{u}, u^{n+1}) + \frac{r}{m} \sum_{\iota=1}^m [\varphi(u^n + w_\iota^{n+1}, u^n) \\
& - \varphi(u^n + w_\iota^{n+1}, u^n + w_\iota^{n+1})] - \varphi(\hat{u}, u^n) + \varphi(\hat{u}, u^{n+1}).
\end{aligned} \tag{69}$$

Similarly with (67), we obtain

$$\begin{aligned}
& \frac{r}{m} \sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] \\
& \quad - \varphi(\hat{u}, u^n) + \varphi(\hat{u}, u^{n+1}) \\
& \leq \frac{r}{m} \sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] \\
& \quad + \frac{r}{m} \sum_{i=1}^m [\varphi(\hat{u}, u^n + w_i^{n+1}) - \varphi(\hat{u}, u^n)] \\
& \leq \frac{r}{m} k_1 k_2 \sum_{i=1}^m \|u^n + w_i^{n+1} - \hat{u}\| \|w_i^{n+1}\| \\
& \leq \frac{r}{m} k_1 k_2 \left( \sum_{i=1}^m \|w_i^{n+1}\| + \|\tilde{u}^{n+1} - \hat{u}\| \right) \sum_{i=1}^m \|w_i^{n+1}\| \\
& \leq \frac{r}{m} k_1 k_2 \left(1 + \frac{1}{2\varepsilon_3}\right) m \sum_{i=1}^m \|w_i^{n+1}\|^2 + \frac{r}{m} k_1 k_2 \frac{\varepsilon_3}{2} \|\tilde{u}^{n+1} - \hat{u}\|^2 \tag{70}
\end{aligned}$$

for each  $\varepsilon_3 > 0$ . In view of (69) and (70), we get

$$\begin{aligned}
& \left[ \frac{\alpha}{2} - k_1 k_2 \left(1 + \frac{1}{2\varepsilon_3}\right) m \right] \sum_{i=1}^m \|w_i^{n+1}\|^2 \\
& \leq \frac{m}{r} [F(u^n) - F(u^{n+1}) + \varphi(\hat{u}, u^n) - \varphi(\hat{u}, u^{n+1})] \\
& \quad + k_1 k_2 \frac{\varepsilon_3}{2} \|\tilde{u}^{n+1} - \hat{u}\|^2 \tag{71}
\end{aligned}$$

for each  $\varepsilon_3 > 0$ . If we write

$$\begin{aligned}
C_{1\varepsilon} &= \frac{m-r}{r} + C_{4\varepsilon} \frac{m}{r}, \\
C_{2\varepsilon} &= \frac{\alpha}{2} - k_1 k_2 \left(1 + \frac{1}{2\varepsilon_3}\right) m, \\
C_{3\varepsilon} &= \frac{\alpha}{2} - \beta C_0 \frac{\varepsilon_1}{2} - k_1 k_2 \left( C_0 + \frac{1+2C_0}{2} \varepsilon_2 \right) - k_1 k_2 \frac{\varepsilon_3}{2} C_{4\varepsilon}, \\
C_{4\varepsilon} &= \frac{m}{C_{2\varepsilon}} \left[ \beta \left(1 + C_0 \left(1 + \frac{1}{2\varepsilon_1}\right)\right) + k_1 k_2 \left(1 + C_0 + \frac{1+2C_0}{2\varepsilon_2}\right) \right], \tag{72}
\end{aligned}$$

then, from (68) and (71), on the condition  $C_{2\varepsilon} > 0$ , we get

$$\begin{aligned} & F(u^{n+1}) - F(\hat{u}) + \varphi(\hat{u}, u^{n+1}) - \varphi(\hat{u}, \hat{u}) + C_{3\varepsilon} \|\hat{u} - \tilde{u}^{n+1}\|^2 \\ & \leq C_{1\varepsilon} [F(u^n) - F(u^{n+1}) + \varphi(\hat{u}, u^n) - \varphi(\hat{u}, u^{n+1})]. \end{aligned} \quad (73)$$

We can easily see that  $C_{3\varepsilon}$ , as a function of  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$ , reaches its maximum value for

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{k_1 k_2 m}{\frac{\alpha}{2} - k_1 k_2 m}, \quad (74)$$

and this is

$$\begin{aligned} C_{3max} &= \frac{\alpha}{2} - k_1 k_2 C_0 - [\beta C_0 + k_1 k_2 (1 + 2C_0)] \frac{k_1 k_2 m}{\frac{\alpha}{2} - k_1 k_2 m} \\ &\quad - (1 + C_0)(\beta + k_1 k_2) \frac{k_1 k_2^2 m^2}{(\frac{\alpha}{2} - k_1 k_2 m)^2}. \end{aligned}$$

Condition  $C_{3max} \geq 0$  is satisfied if

$$\left(\frac{1}{2} - C_0 \frac{k_1 k_2}{\alpha}\right) \frac{\alpha}{\beta} \geq (1 + 3C_0) \frac{\frac{k_1 k_2}{\alpha} m}{\frac{1}{2} - \frac{k_1 k_2}{\alpha} m} + 2(1 + C_0) \frac{(\frac{k_1 k_2}{\alpha})^2 m^2}{(\frac{1}{2} - \frac{k_1 k_2}{\alpha} m)^2}.$$

Writing  $\chi = \frac{k_1 k_2}{\alpha}$ , we see that equation

$$\left(\frac{1}{2} - C_0 \chi\right) \frac{\alpha}{\beta} = (1 + 3C_0) \frac{\chi m}{\frac{1}{2} - \chi m} + 2(1 + C_0) \frac{\chi^2 m^2}{(\frac{1}{2} - \chi m)^2}, \quad (75)$$

which corresponds to the above inequality, has at least solution in  $(0, \frac{1}{2C_0})$ , and if  $\chi_0$  is the smallest one, then by taking  $\frac{k_1 k_2}{\alpha} \leq \chi_0$ , we have  $C_{3max} \geq 0$ .

The value of  $C_{2\varepsilon}$  for  $\varepsilon_3$  in (74) is

$$C_{2max} = \frac{1}{2} \left( \frac{\alpha}{2} - k_1 k_2 m \right).$$

Since we can always take  $C_0 \geq m$  and the above solution  $\chi_0$  of equation (75) satisfies  $\chi_0 < \frac{1}{2m}$ , then  $C_{2max} > 0$  if  $\frac{k_1 k_2}{\alpha} \leq \chi_0$ .

Now, (62) is obtained from (73), where

$$C_1 = \frac{2m}{\frac{\alpha}{2} - k_1 k_2 m} \left\{ \beta \left[ 1 + \frac{C_0}{2k_1 k_2 m} \left( \frac{\alpha}{2} + k_1 k_2 m \right) \right] + \frac{1}{2} \left[ \frac{\alpha}{2} (2C_0 + 1) + k_1 k_2 m \right] \right\}$$

is the value of  $C_{1\varepsilon}$  for  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_3$  given in (74).

Constant  $C_1$  has a similar expression in the case of Algorithm 4.2.  $\square$

The above general result will be used in a subsequent paper to prove the convergence of the one- and two-level methods for contact problems with nonlocal friction. In this case, the constant  $C_0$  will be explicitly written in terms of the mesh parameters.

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