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## TERMINATION OF LINEAR BOUNDED TERM REWRITING SYSTEMS

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**ABSTRACT.** For the whole class of linear term rewriting systems and for each integer  $k$ , we define *k-bounded rewriting* as a restriction of the usual notion of rewriting. We show that the problem of the existence of an infinite  $k$ -bounded derivation, called the  $k$ -bounded termination problem, is decidable. The *k-bounded* class (BO( $k$ )) is, by definition, the set of linear systems for which every derivation can be replaced by a  $k$ -bounded derivation. In general, the  $k$ -bounded termination problem for BO( $k$ ) systems is not equivalent to the termination problem. This leads us to define more restricted classes for which those problems are equivalent: the classes BOLP( $k$ ) of *k-bounded systems that have the length preservation property*. By definition, a system is BOLP( $k$ ) if every derivation of length  $n$  can be replaced by a  $k$ -bounded derivation of length  $n$ . We define the class BOLP of *bounded systems that have the length preservation property* as the union of all the BOLP( $k$ ) classes. The class BOLP contains (strictly) several already known classes of systems: the inverse left-basic semi-Thue systems, the linear growing term rewriting systems, the inverse Linear-Finite-Path-Ordering systems, the strongly bottom-up systems.

### 1. Introduction

**General context.** A Term-Rewriting System (TRS in short) is said *terminating* when it does not admit any infinite derivation. Such a property is part of the definition of a *complete* TRS, which is a useful algebraic notion. This property is also pertinent for TRSs which are models of functional programs or any kind of computational process. The *termination-problem* is the problem to know whether a given TRS  $\mathcal{R}$  is terminating or not. It is well-known that this problem is undecidable for general finite TRS ([7]) and even for quite restricted subclasses of TRS (see [2],[10] for example). Nevertheless, because of its importance, many techniques have been developed in order to prove termination of TRSs (see in particular [3],[9, section 1.3], [14, chap. 6]) or even to decide automatically termination, but for specific classes of TRS.

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Contents. The present paper follows the last trend of research quoted above:

- 1- we show that termination is decidable for a particular strategy that we call bounded rewriting,
- 2- we deduce from this decision procedure that the usual termination problem is decidable for some classes of TRS.

We define a new rewriting strategy for linear systems called *bounded* rewriting. Let  $k \in \mathbb{N}$ . Intuitively, a derivation is said to be  $k$ -*bounded* ( $\text{bo}(k)$ ) if when a rewriting rule is applied, the parts of the substitution located at a depth greater than  $k$  are not used further in the derivation, i.e. do not match a left-handside of a rule applied further. A rewriting system  $\mathcal{R}$  will be said to be  $k$ -*bounded* ( $\text{bo}(k)$ ) if for any derivation  $s \rightarrow_{\mathcal{R}}^* t$ , there exists a  $k$ -bounded derivation  $s \text{bo}(k) \rightarrow_{\mathcal{R}}^* t$ . The class of  $k$ -bounded systems is denoted by  $\text{BO}(k)$ , and the class of bounded systems  $\text{BO}$  is  $\bigcup_{k \in \mathbb{N}} \text{BO}(k)$ . A rewriting system will be said to  $\text{bo}(k)$ -*terminates* if there is no infinite  $\text{bo}(k)$ -derivation. The main result of this paper is the decidability of the  $\text{bo}(k)$ -termination problem. This rewriting strategy is closely related to the *bottom-up* strategy introduced in [4]: every bottom-up system is bounded, and for every bounded system, there is an equivalent system which is bottom-up. Both strategies are defined using marking tools, but the definition of the bounded strategy is simpler and more intuitive. For every linear system  $(\mathcal{R}, \mathcal{F})$  and every integer  $k$ , there is a system  $(\mathcal{R}', \mathcal{F})$  such that  $\forall n \in \mathbb{N}, \rightarrow_{\mathcal{R}}^n = \rightarrow_{\mathcal{R}'}^n$  and  $\text{bo}(k) \circ \rightarrow_{\mathcal{R}}^n = \text{bo}(0) \circ \rightarrow_{\mathcal{R}'}^n$ . Thus, it is sufficient to prove that the  $\text{bo}(0)$ -termination problem is decidable to obtain the decidability of the  $\text{bo}(k)$ -termination problem, for  $k \in \mathbb{N}$ . Following the idea developed for the bottom-up strategy, we use a ground rewriting system  $\mathcal{S} \cup \mathcal{A}$  to simulate  $\text{bo}(0)$ -derivations. This construction is made in such a way that the existence of an infinite  $\text{bo}(0)$ -derivation in  $\mathcal{R}$  is equivalent to the existence of an infinite derivation in  $\mathcal{S} \cup \mathcal{A}$ . It follows from the decidability of the termination problem for ground systems that the  $\text{bo}(0)$ -termination problem is decidable. The system  $\mathcal{A}$  has rules which allow to replace any subterm of a term  $t$  located at an internal node by a leaf labeled by the constant symbol  $\#$ , and the system  $\mathcal{S}$  consists of a set of rules of the form  $l\sigma \rightarrow r\sigma$  where  $l \rightarrow r \in \mathcal{R}$  and  $\sigma$  is a substitution that maps variables to an element of  $\mathcal{F}_0 \cup \{\#\}$ . A  $\text{bo}(0)$ -step  $C[l\sigma] \rightarrow C[r\sigma]$  in  $\mathcal{R}$  is simulated in two steps : first, using  $\mathcal{A}$ , we reduce  $C[l\sigma]$  to  $C[l\sigma']$  where  $l\sigma' \in \text{LHS}(\mathcal{S})$ , and then we apply the rule  $l\sigma' \rightarrow r\sigma' \in \mathcal{S}$ . We define a subclass of  $\text{BO}(k)$ , the length preservation bottom-up class  $\text{BOLP}(k)$ , for which the termination problem and the  $k$ -bounded termination problem are equivalent. A  $\text{BO}(k)$  system is  $\text{BOLP}(k)$  iff for every derivation  $s \rightarrow_{\mathcal{R}}^* t$  there is a  $\text{bo}(k)$  derivation of same length. The class of length preservation bounded system  $\text{BOLP}$  is  $\bigcup_{k \in \mathbb{N}} \text{BOLP}(k)$ . This class contains several already known systems: the inverse left-basic semi-Thue systems [12], the linear growing term rewriting systems [8], the inverse Linear-Finite-Path-Overlapping systems [13], and the strongly bottom-up systems [4].

## 2. Preliminaries

### 2.1. Words and Terms

The set  $\mathbb{N}$  is the set of positive integers. A finite *word* over an alphabet  $A$  is a map  $u : [0, \ell - 1] \rightarrow A$ , for some  $\ell \in \mathbb{N}$ . The integer  $\ell$  is the *length* of the word  $u$  and is denoted by  $|u|$ . The set of words over  $A$  is denoted by  $A^*$  and endowed with the usual *concatenation* operation  $u, v \in A^* \mapsto u \cdot v \in A^*$ . The *empty* word is denoted by  $\varepsilon$ . A word  $u$  is a *prefix* of a word  $v$  iff there exists some  $w \in A^*$  such that  $v = u \cdot w$ . We denote by  $u \preceq v$  the fact that  $u$  is a prefix of  $v$ . Assuming a total order on  $A$ , we denote by  $\preceq_{\text{Lex}}$  the lexicographic order on words.

We assume the reader familiar with terms. We call *signature* a set  $\mathcal{F}$  of symbols with fixed arity  $ar : \mathcal{F} \rightarrow \mathbb{N}$ . The subset of symbols of arity  $m$  is denoted by  $\mathcal{F}_m$ .

As usual, a set  $P \subseteq \mathbb{N}^*$  is called a *tree-domain* (or, *domain*, for short) iff for every  $u \in \mathbb{N}^*$ ,  $i \in \mathbb{N}$ :

$$(u \cdot i \in P \Rightarrow u \in P) \ \& \ (u \cdot (i + 1) \in P \Rightarrow u \cdot i \in P).$$

We call  $P' \subseteq P$  a *subdomain* of  $P$  iff,  $P'$  is a domain and, for every  $u \in P, i \in \mathbb{N}$ :

$$(u \cdot i \in P' \ \& \ u \cdot (i + 1) \in P) \Rightarrow u \cdot (i + 1) \in P'.$$

A (first-order) *term* on a signature  $\mathcal{F}$  is a partial map  $t : \mathbb{N}^* \rightarrow \mathcal{F}$  whose domain is a non-empty tree-domain and which respects the arities. We denote by  $\mathcal{T}(\mathcal{F}, \mathcal{V})$  the set of first-order terms built upon the signature  $\mathcal{F} \cup \mathcal{V}$ , where  $\mathcal{F}$  is a finite signature and  $\mathcal{V}$  is a denumerable set of variables of arity 0.

The domain of  $t$  is also called its set of *positions* and denoted by  $\mathcal{Pos}(t)$ . The set of variables of  $t$  is denoted by  $\mathcal{Var}(t)$ . The root symbol of  $t$ ,  $t(\epsilon)$  is also denoted by  $\text{root}(t)$ . The set of variable positions (resp. non variable positions) of a term  $t$  is denoted by  $\mathcal{Pos}_{\mathcal{V}}(t)$  (resp.  $\mathcal{Pos}_{\overline{\mathcal{V}}}(t)$ ). The set of *leaves* of  $t$  is the set of positions  $u \in \mathcal{Pos}(t)$  such that  $u \cdot \mathbb{N} \cap \mathcal{Pos}(t) = \emptyset$ . It is denoted by  $Lv(t)$ . We write  $\mathcal{Pos}^+(t)$  for  $\mathcal{Pos}(t) \setminus \{\epsilon\}$ . Given  $v \in \mathcal{Pos}^+(t)$ , its *father*  $\text{fth}(v)$  is the position  $u$  such that  $v = u \cdot w$  and  $|w| = 1$ . Given a term  $t$  and  $u \in \mathcal{Pos}(t)$  the *subterm of  $t$  at  $u$*  is denoted by  $t/u$  and defined by  $\mathcal{Pos}(t/u) = \{w \mid u \cdot w \in \mathcal{Pos}(t)\}$  and  $\forall w \in \mathcal{Pos}(t/u), t/u(w) = t(u \cdot w)$ . A term which does not contain twice the same variable is called *linear*. Given a linear term  $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ ,  $x \in \mathcal{Var}(t)$ , we shall denote by  $\text{pos}(t, x)$  the position of  $x$  in  $t$ . The *depth* of a term  $t$  is inductively defined by:

- $\text{dpt}(t) := 0$  if  $t \in \mathcal{V}$ ,
- $\text{dpt}(t) := 1$  if  $t \in \mathcal{F}_0$ ,
- $\text{dpt}(t) := 1 + \max(\{\text{dpt}(t/i), i \in \{0, \dots, n\}\})$  if  $\text{root}(t) \in \mathcal{F}_n$ .

A term containing no variable is called *ground*. The set of ground terms is  $\mathcal{T}(\mathcal{F})$ . Among all the variables, there is a special one  $\square$ . A term containing exactly one occurrence of  $\square$  is called a *context*. A context is usually denoted as  $C[\square]$ . If  $v$  is the position of  $\square$  in  $C[\square]$ ,  $C[t]$  denotes the term  $C[\square]$  where  $t$  has been substituted at position  $v$ . We also denote by  $C[\square]_v$  such a context and by  $C[t]_v$  the result of the substitution. We denote by  $|t| := \text{Card}(\mathcal{Pos}(t))$  the *size* of a term  $t$ . A *substitution*  $\sigma$  is a mapping from  $\mathcal{V}$  to  $\mathcal{T}(\mathcal{F}, \mathcal{V})$ . The substitution  $\sigma$  extends uniquely to a morphism  $\sigma : \mathcal{T}(\mathcal{F}, \mathcal{V}) \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$ , where  $\sigma(f(t_1, \dots, t_n)) = f(\sigma(t_1), \dots, \sigma(t_n))$ , for each  $f \in \mathcal{F}$ ,  $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ . Let  $t$  be a linear term and  $\mathcal{Pos}_{\mathcal{V}}(t) = \{u_1, \dots, u_n\}$ , where the  $u_i$  are given in lexicographic order. The term  $t$  is said to be *standardized* if for all  $i, 1 \leq i \leq n, t/u_i = x_i$ .

## 2.2. Term rewriting systems

A *rewrite rule* built upon the signature  $\mathcal{F}$  is a pair  $l \rightarrow r$  of terms in  $\mathcal{T}(\mathcal{F}, \mathcal{V})$  which satisfies  $\mathcal{Var}(r) \subseteq \mathcal{Var}(l)$ . We call  $l$  (resp.  $r$ ) the *left-handside* (resp. *right-handside*) of the rule (*lhs* and *rhs* for short). A rule is *linear* if both its left and right-hand sides are linear. A rule is *left-linear* if its left-hand side is linear. Given a set of rules  $\mathcal{R}$ , we denote by  $\text{LHS}(\mathcal{R})$  the set  $\{l \mid l \rightarrow r \in \mathcal{R}\}$ . A *term rewriting system* (*system* for short) is a pair  $(\mathcal{R}, \mathcal{F})$  where  $\mathcal{F}$  is a signature and  $\mathcal{R}$  a set of rewrite rules built upon the signature  $\mathcal{F}$  such that  $\text{LHS}(\mathcal{R}) \cap \mathcal{V} = \emptyset$ . When  $\mathcal{F}$  is clear from the context or contains exactly the symbols of  $\mathcal{R}$ , we may omit  $\mathcal{F}$  and write simply  $\mathcal{R}$ . Given a rewriting system  $(\mathcal{R}, \mathcal{F})$ , and two terms  $t_1, t_2$ , we say that there exists a  *$\mathcal{R}$ -rewriting step* between  $t_1$  and  $t_2$  in  $\mathcal{R}$  and write  $t_1 \rightarrow_{\mathcal{R}} t_2$  if there exists a context  $C[\square]$ , a rule  $l \rightarrow r \in \mathcal{R}$ , and a substitution  $\sigma$  such that  $t_1 = C[l\sigma]$  and  $t_2 = C[r\sigma]$ . The term  $l\sigma$  is called a *redex* of  $t_1$ , and  $r\sigma$  is called a *contractum* of  $t_1$ . Given some  $n \geq 0$ , a derivation in  $\mathcal{R}$  of length  $n$  from  $s$  to  $t$  is a sequence of the form  $s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n$ . The relation  $\rightarrow_{\mathcal{R}}^n$  is defined as follows:  $s \rightarrow_{\mathcal{R}}^n t$  if there exists a

derivation of length  $n$  from  $s$  to  $t$ . The relation  $\rightarrow_{\mathcal{R}}^*$  ( $\rightarrow_{\mathcal{R}}^+$ ) is defined by:  $s \rightarrow^* t$  ( $s \rightarrow^+ t$ ) if there is some  $n \geq 0$  ( $n > 0$ ) such that  $s \rightarrow_{\mathcal{R}}^n t$ . More generally, the notation defined in [9] will be used in proofs.

A system is *linear* (resp. *left-linear*) if each of its rules is linear (resp. left-linear). A system  $\mathcal{R}$  is *growing* [8] if for every rule  $l \rightarrow r \in \mathcal{R}$ , all variables in  $\mathcal{V}\text{ar}(l) \cap \mathcal{V}\text{ar}(r)$  occur at depth 0 or 1 in  $l$ . Two rewriting systems  $(\mathcal{R}, \mathcal{F})$  and  $(\mathcal{R}', \mathcal{F})$  are said to be *equivalent* if for all  $n \geq 0$ ,  $\rightarrow_{\mathcal{R}}^n = \rightarrow_{\mathcal{R}'}^n$ . A system  $\mathcal{R}$  is said to *terminate* if there is no infinite derivation in  $\mathcal{R}$ .

### 3. Bounded rewriting

In order to define *bounded rewriting*, we need some marking tools. In the following we assume that  $\mathcal{F}$  is a signature. We shall illustrate many of our definitions with the following system

**Example 3.1.**  $\mathcal{F} = \{a, b, f, g, h, i\}$ ,  $\mathcal{R}_1 = \{f(x) \rightarrow g(x), g(h(x)) \rightarrow i(x), i(x) \rightarrow a, a \rightarrow b\}$ .

#### 3.1. Marking

We mark the symbols of a term using natural integers.

##### 3.1.1. Marked symbols.

**Definition 3.2.** We define the (infinite) *signature of marked symbols*:  $\mathcal{F}^{\mathbb{N}} := \{f^i \mid f \in \mathcal{F}, i \in \mathbb{N}\}$ . For  $j \in \mathbb{N}$ , we denote by  $\mathcal{F}^{\leq j}$  the signature:  $\mathcal{F}^{\leq j} := \{f^i \mid f \in \mathcal{F}, i \leq j\}$ . The mapping  $m : \mathcal{F}^{\mathbb{N}} \rightarrow \mathbb{N}$  maps every marked symbol to its mark:  $m(f^i) = i$ .

##### 3.1.2. Marked terms.

**Definition 3.3.** The terms in  $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$  are called *marked terms*.

The mapping  $m$  is extended to marked terms by: if  $t \in \mathcal{V}$ ,  $m(t) := 0$ , otherwise,  $m(t) := m(\text{root}(t))$ . For every  $f \in \mathcal{F}$ , we identify  $f^0$  and  $f$ ; it follows that  $\mathcal{F} \subset \mathcal{F}^{\mathbb{N}}$ ,  $\mathcal{T}(\mathcal{F}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}})$  and  $\mathcal{T}(\mathcal{F}, \mathcal{V}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ .

We use  $\text{mmax}(t)$  to denote the maximal mark of a marked term  $t$ :  $\text{mmax}(t) := \max\{m(t/u) \mid u \in \mathcal{P}\text{os}(t)\}$ .

**Example 3.4.**  $m(a^1) = 1$ ,  $m(i^0(a^2)) = 0$ ,  $m(h^3(a^0)) = 3$ ,  $m(h^1(x)) = 1$ ,  $m(x) = 0$ ,  $\text{mmax}(i^0(a^1)) = 1$ ,  $\text{mmax}(x) = 0$ .

**Definition 3.5.** Given  $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$  and  $i \in \mathbb{N}$ , we define the marked term  $t^i$  whose marks are all equal to  $i$ :

$$\begin{array}{ll} \text{if } t \text{ is a variable } x & t^i := x \\ \text{if } t \text{ is a constant } c & t^i := c^i \\ \text{otherwise } t = f(t_1, \dots, t_n), \text{ where } n \geq 1 & t^i := f^i(t_1^i, \dots, t_n^i) \end{array}$$

This marking extends to sets of terms  $S$  ( $S^i := \{t^i \mid t \in S\}$ ) and substitutions  $\sigma$  ( $\sigma^i : x \mapsto (x\sigma)^i$ ). *Notation:* in the sequel, given a term  $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ ,  $\bar{t}$  will always refer to a term of  $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$  such that  $\bar{t}^0 = t$ .

**Definition 3.6.** For every marked term  $\bar{t}$ , we denote by  $\widehat{\bar{t}}$  the unique marked term such that:

$$\mathcal{P}\text{os}(\widehat{\bar{t}}) := \mathcal{P}\text{os}(\bar{t}), \quad \forall u \in \mathcal{P}\text{os}_{\overline{\mathcal{V}}}(\bar{t}), \quad m(\widehat{\bar{t}}/u) := \max(m(\bar{t}/u), |u| + 1).$$

We extend this definition to marked substitutions ( $\widehat{\bar{\sigma}} : x \mapsto x\widehat{\bar{\sigma}}$ ) and sets of terms ( $\widehat{\bar{S}} := \{\widehat{\bar{s}} \mid \bar{s} \in \bar{S}\}$ ).

**Example 3.7.** Let  $\bar{t}_1 = f^0(f^1(x))$ , and  $\bar{t}_2 = f^2(f^2(h^2(a^2)))$ . We have:  $\widehat{\bar{t}}_1 = f^1(f^2(x))$ ,  $\widehat{\bar{t}}_2 = f^2(f^2(h^3(a^4)))$ .

### 3.2. Marked rewriting

Let  $\mathcal{R}$  be a left-linear system,  $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ . Let us suppose that  $\bar{s}$  decomposes as

$$\bar{s} = \overline{C}[\bar{l}\bar{\sigma}]_v, \quad \text{with } (l, r) \in \mathcal{R}, \quad (3.1)$$

for some marked context  $\overline{C}[\ ]_v$  and substitution  $\bar{\sigma}$ . We then write  $\bar{s} \circ \rightarrow \bar{t}$  when

$$\bar{s} = \overline{C}[\bar{l}\bar{\sigma}], \quad \bar{t} = \overline{C}[r\widehat{\bar{\sigma}}]. \quad (3.2)$$

More precisely, an ordered pair of marked terms  $(\bar{s}, \bar{t})$  is linked by the relation  $\circ \rightarrow$  iff, there exists  $\overline{C}[\ ]_v, (l, r), \bar{l}, \bar{\sigma}$  fulfilling equations (3.1-3.2). (This is illustrated by Figure 1, where the marks are noted between brackets [...]).

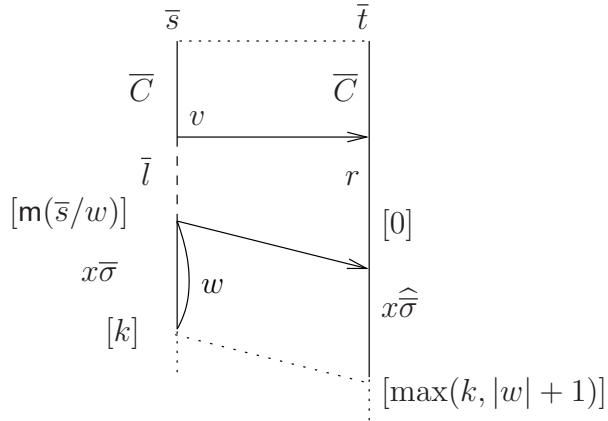


Figure 1: A marked rewriting step

The map  $\bar{s} \mapsto \bar{s}^0$  (from marked terms to unmarked terms) extends into a map from marked derivations to unmarked derivations: every

$$\bar{s}_0 = \overline{C}_0[\bar{l}_0\bar{\sigma}_0]_{v_0} \circ \rightarrow \overline{C}_0[r_0\widehat{\bar{\sigma}}_0]_{v_0} = \bar{s}_1 \circ \rightarrow \dots \circ \rightarrow \overline{C}_{n-1}[r_{n-1}\widehat{\bar{\sigma}}_{n-1}]_{v_{n-1}} = \bar{s}_n \quad (3.3)$$

is mapped to the derivation

$$s_0 = C_0[l_0\sigma_0]_{v_0} \rightarrow C_0[r_0\sigma_0]_{v_0} = s_1 \rightarrow \dots \rightarrow C_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \quad (3.4)$$

The context  $\overline{C}_i[\ ]_{v_i}$ , the rule  $(l_i, r_i)$ , the marked version  $\bar{l}_i$  of  $l_i$  and the substitution  $\bar{\sigma}_i$  completely determine  $\bar{s}_{i+1}$ . Thus, for every fixed pair  $(\bar{s}_0, s_0)$ , this map is a bijection from the set of derivations (3.3), to the set of derivations (3.4).

From now on, each time we deal with a derivation  $s \rightarrow^* t$  between two terms  $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ , we may implicitly decompose it as (3.4) where  $n$  is the length of the derivation,  $s = s_0$  and  $t = s_n$ .

### 3.3. Bounded derivations

**Definition 3.8.** The marked derivation (3.3) is  $k$ -bounded ( $\text{bo}(k)$ ) if for every  $0 \leq i < n$ ,  $\text{mmax}(\bar{l}_i) \leq k$ . The derivation (3.4) is  $\text{bo}(k)$  if the corresponding  $k$ -marked derivation (3.3) is  $\text{bo}(k)$ .

In particular, the marked derivation (3.3) is  $\text{bo}(0)$  iff for every  $0 \leq i < n$ ,  $\bar{l}_i = l_i$ .

**Example 3.9.** Let us consider the following derivations in  $\mathcal{R}_1$ :

- (1)  $f(h(a)) \rightarrow g(h(a)) \rightarrow i(a) \rightarrow a$
- (2)  $f(h(a)) \rightarrow g(h(a)) \rightarrow g(h(b)) \rightarrow i(b) \rightarrow a$

The first derivation is  $\text{bo}(1)$  since the associated marked derivation is  $\text{bo}(1)$ :

$f(h(a)) \circ \rightarrow g(h^1(a^2)) \circ \rightarrow i(a^2) \circ \rightarrow a$ . The second one is  $\text{bo}(2)$ :

$f(h(a)) \circ \rightarrow g(h^1(a^2)) \circ \rightarrow g(h^1(b)) \circ \rightarrow i(b^1) \circ \rightarrow a$ .

Let  $k \in \mathbb{N}$ . It is clear that the composition of two  $\text{bo}(k)$  marked derivations is  $\text{bo}(k)$  too, but the composition of two unmarked  $\text{bo}(k)$  derivations might not be  $\text{bo}(k)$ , as shown in the following example:

**Example 3.10.** The two derivations in  $\mathcal{R}_1$ :  $f(h(a)) \rightarrow g(h(a))$  and  $g(h(a)) \rightarrow i(a) \rightarrow a$  are  $\text{bo}(0)$  while the derivation:  $f(h(a)) \rightarrow g(h(a)) \rightarrow i(a) \rightarrow a$  is not  $\text{bo}(0)$  (but is  $\text{bu}(1)$ ).

In the following we thus (mainly) manipulate *marked*  $\text{bo}(k)$ -derivations. Let us introduce some convenient notations.

**Definition 3.11.** Let  $n, k \in \mathbb{N}$ . The binary relation  $\text{bo}(k) \circ \rightarrow_{\mathcal{R}}^n$  over  $\mathcal{T}(\mathcal{F}^{\mathbb{N}})$  is defined by:  $\bar{s} \text{bo}(k) \circ \rightarrow_{\mathcal{R}}^n \bar{t}$  iff there exists a  $\text{bo}(k)$ -marked derivation from  $\bar{s}$  to  $\bar{t}$  of length  $n$ . The binary relation  $\text{bo}(k) \circ \rightarrow_{\mathcal{R}}^*$  is defined by:  $\bar{s} \text{bo}(k) \circ \rightarrow_{\mathcal{R}}^* \bar{t}$  iff there exists  $m \in \mathbb{N}$  such that  $\bar{s} \text{bo}(k) \circ \rightarrow_{\mathcal{R}}^m \bar{t}$ . The binary relation  $\text{bo}(k) \rightarrow_{\mathcal{R}}^n$  over  $\mathcal{T}(\mathcal{F})$  is defined by:  $s \text{bo}(k) \rightarrow_{\mathcal{R}}^n t$  iff there exists a  $\text{bo}(k)$ -derivation from  $s$  to  $t$  of length  $n$ . The binary relation  $\text{bo}(k) \rightarrow_{\mathcal{R}}^*$  is defined by:  $s \text{bo}(k) \rightarrow_{\mathcal{R}}^* t$  iff there exists  $m \in \mathbb{N}$  such that  $s \text{bo}(k) \rightarrow_{\mathcal{R}}^m t$ .

Next lemma shows that the study of  $\text{bo}(k)$ -derivations can be reduced to the study of  $\text{bo}(0)$ -derivations.

**Lemma 3.12.** Let  $\mathcal{R}$  be a linear rewriting system and let  $k > 0$ . There exists an equivalent linear system  $\mathcal{R}'$  such that: for all  $n \in \mathbb{N}$ ,  $\text{bo}(k) \rightarrow_{\mathcal{R}}^n = \text{bo}(0) \rightarrow_{\mathcal{R}'}^n$ .

*Sketch of proof.* Let  $\mathcal{R}'$  be the rewriting system consisting of the rules:

$$\{l\sigma \rightarrow r\sigma \mid l \rightarrow r \in \mathcal{R}, \sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V}), l\sigma \text{ is standardized, } \text{dpt}(\sigma) \leq k\}.$$

One can easily check that  $\mathcal{R}'$  is finite, equivalent to  $\mathcal{R}$  and that, for all  $n \in \mathbb{N}$ ,  $\text{bo}(k) \rightarrow_{\mathcal{R}}^n = \text{bo}(0) \rightarrow_{\mathcal{R}'}^n$ . ■

**Example 3.13.** Let us consider the  $\text{bo}(1)$ -derivation in example 3.9

$f(h(a)) \rightarrow g(h(a)) \rightarrow i(a) \rightarrow a$  and the system  $\mathcal{R}'$  built for  $\mathcal{R}_1$  and  $k = 1$ . We have:

$$\begin{aligned} \mathcal{R}' = \{ & f(x_1) \rightarrow g(x_1), f(f(x_1)) \rightarrow g(f(x_1)), f(g(x_1)) \rightarrow g(g(x_1)), \\ & f(h(x_1)) \rightarrow g(h(x_1)), f(i(x_1)) \rightarrow g(i(x_1)), f(a) \rightarrow g(a), f(b) \rightarrow g(b), \\ & g(h(x_1)) \rightarrow i(x_1), g(h(f(x_1))) \rightarrow i(f(x_1)), g(h(g(x_1))) \rightarrow i(g(x_1)), \\ & g(h(h(x_1))) \rightarrow i(h(x_1)), g(h(i(x_1))) \rightarrow i(i(x_1)), g(h(a)) \rightarrow i(a), \\ & g(h(b)) \rightarrow i(b), i(x_1) \rightarrow a, i(f(x_1)) \rightarrow a, i(g(x_1)) \rightarrow a, \\ & i(h(x_1)) \rightarrow a, i(i(x_1)) \rightarrow a, i(a) \rightarrow a, i(b) \rightarrow a, a \rightarrow b \} \end{aligned}$$

and the following  $\text{bo}(0)$ -derivation in  $\mathcal{R}'$ :

$$f(h(a)) \circ_{f(h(x_1) \rightarrow g(h(x_1)))} g(h(a^1)) \circ_{h(x_1) \rightarrow i(x_1)} i(a^1) \circ_{i(x_1) \rightarrow a} a.$$

### 3.4. Bounded systems

We introduce here a hierarchy of classes of rewriting systems, based on their ability to meet the bounded restriction over derivations.

**Definition 3.14.** Let  $p$  be some property of derivations. A system  $(\mathcal{R}, \mathcal{F})$  is called  $P$  if  $\forall s, t \in \mathcal{T}(\mathcal{F})$  such that  $s \rightarrow_{\mathcal{R}}^* t$  there exists a  $p$ -derivation from  $s$  to  $t$ .

We denote by  $\text{BO}(k)$  the class of  $\text{BO}(k)$  systems. One can check that, for every  $k > 0$ ,  $\text{BO}(k-1) \subsetneq \text{BO}(k)$ . Finally, the class of *bounded systems*  $\text{BO}$  is defined by:  $\text{BO} = \bigcup_{k \in \mathbb{N}} \text{BO}(k)$ . The membership problem for  $\text{BO}(k)$  is shown to be undecidable in the appendix, using a result of [4].

**Definition 3.15.** We say that the system  $\mathcal{R}$  *bo*( $k$ )-*terminates* iff there is no infinite  $\text{bo}(k)$ -derivation in  $\mathcal{R}$ .

The  $\text{bo}(k)$ -termination problem for a linear system  $\mathcal{R}$  is the following problem:

INSTANCE: A linear system  $\mathcal{R}$ , and an integer  $k$ .

QUESTION: Does  $\mathcal{R}$   $\text{bo}(k)$ -terminate ?

The main result of this paper is the decidability of the  $\text{bo}(k)$ -termination problem.

## 4. Simulation of bounded derivations by a ground rewriting system

In this section, we prove that a  $\text{bo}(0)$ -derivation can be simulated using a ground term rewriting system.

**Definition 4.1.** Let  $\#$  be a constant such that  $\# \notin \mathcal{F}_0$ . Let  $\mathcal{A}$  be the (infinite) rewriting system on  $\mathcal{T}(\mathcal{F} \cup \{\#\})^{\mathbb{N}}$  consisting of the rules:

$$\{f^i(\bar{a}_1, \dots, \bar{a}_n) \rightarrow \#^i \mid i \in \mathbb{N}, f \in \mathcal{F}_n, n > 0, \bar{a}_1, \dots, \bar{a}_n \in (\mathcal{F}_0 \cup \{\#\})^{\mathbb{N}}\}.$$

For  $j \in \mathbb{N}$ , we denote by  $\mathcal{A}^{\leq j}$  the restriction of  $\mathcal{A}$  on  $\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq j})$  consisting of the rules:

$$\{f^i(\bar{a}_1, \dots, \bar{a}_n) \rightarrow \#^i \mid i \leq j, f \in \mathcal{F}_n, n > 0, \bar{a}_1, \dots, \bar{a}_n \in (\mathcal{F}_0 \cup \{\#\})^{\leq j}\}.$$

**Lemma 4.2.** Let  $\bar{s}, \bar{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$ . If  $\bar{s} \rightarrow_{\mathcal{A}}^* \bar{t}$ , then  $\widehat{\bar{s}} \rightarrow_{\mathcal{A}}^* \widehat{\bar{t}}$ . ■



**Definition 4.3.** A marked term  $\bar{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$  is said to be *smoothly-increasing* (s-increasing in short) iff for every branch  $b$ , the sequence of marks on  $b$  has the form:

$$0, 0, \dots, 0, 1, 2, \dots, \ell$$

i.e. more formally: for every  $w \in L_{\mathcal{V}}(t)$ , there exists some  $u \preceq w$  such that,

- $\forall v \preceq u, m(\bar{t}/v) = 0,$
- $\forall v \succeq u, \forall i \in \mathbb{N}, \text{if } v \cdot i \preceq w \text{ then } m(\bar{t}/v \cdot i) = m(\bar{t}/v) + 1.$

A substitution  $\bar{\sigma}$  is said to be s-increasing if for every  $x \in \mathcal{V}$ , the term  $x\bar{\sigma}$  is s-increasing.

Note that by definition of a s-increasing term  $\bar{t}$ , and since the variables are all marked by 0, for all positions  $u \in \mathcal{P}\text{os}_{\mathcal{V}}(t)$ , for all  $v \preceq u$ ,  $m(\bar{t}/v) = 0$ .

**Example 4.4.** The terms  $f^0(h^1(x))$  and  $f^2(h^1(a^2))$  are not s-increasing. The terms  $f^0(f^0(h^1(a^2)))$  and  $f^1(a^2)$  are s-increasing.

**Lemma 4.5.** Let  $\bar{C}[\ ]$  and  $\bar{t}$  be s-increasing. The term  $\bar{C}[\bar{t}]$  is s-increasing.

*Proof.* Since  $\square \in \mathcal{V}$ , it is an immediate consequence of the definition of s-increasing. ■

**Lemma 4.6.** Let  $\bar{s}$  be a s-increasing term and  $\bar{s}_{\text{bo}(0)} \circ \rightarrow_{\mathcal{R}}^* \bar{t}$ . The marked term  $\bar{t}$  is s-increasing.

*Proof.* Let us prove that the result holds by induction on  $n$  the length of the derivation  $\bar{s}_{\text{bo}(0)} \circ \rightarrow_{\mathcal{R}}^* \bar{t}$ . If  $n = 0$ , the result holds. Let  $n > 0$ . There is a term  $\bar{s}_{n-1}$  such that  $\bar{s}_{\text{bo}(0)} \circ \rightarrow_{\mathcal{R}}^{n-1} \bar{s}_{n-1} \text{bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{s}_n$ . By induction hypothesis,  $\bar{s}_{n-1}$  is s-increasing. By definition of a  $\text{bo}(0)$  marked rewriting step, there exists  $\bar{C}[\ ]_v, l \rightarrow r \in \mathcal{R}$  and  $\bar{\sigma}$  such that:  $\bar{s}_{n-1} = \bar{C}[l\bar{\sigma}]_v$ , and  $\bar{t} = \bar{C}[r\widehat{\sigma}]_v$ . As  $\bar{s}_{n-1}$  is s-increasing, since  $m(\bar{s}_{n-1}/v) = 0$ , the context  $\bar{C}[\ ]_v$  is s-increasing. By definition of  $\widehat{\cdot}$ , for every  $x \in \mathcal{V}\text{ar}(r)$ ,  $x\widehat{\sigma}$  is s-increasing. Since the term  $r$  is s-increasing, several applications of lemma 4.5 at each position  $u \in \mathcal{P}\text{os}_{\mathcal{V}}(r)$  show that  $r\widehat{\sigma}$  is s-increasing. We can conclude using the same lemma that  $\bar{C}[r\widehat{\sigma}]_v$  is s-increasing. ■

**Definition 4.7.** Let  $\bar{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$  be a marked term and  $P$  be a subdomain of  $\mathcal{P}\text{os}(t)$  such that  $\mathcal{P}\text{os}_{\mathcal{V}}(t) \subseteq P$ . We define  $\text{Red}(\bar{t}, P)$  as the unique term such that  $\mathcal{P}\text{os}(\bar{t}) = P$  and such that  $\bar{t} \rightarrow_{\mathcal{A}}^* \text{Red}(\bar{t}, P)$ .

The term  $\text{Red}(\bar{t}, P)$  is obtained from  $\bar{t}$  by substituting the subtree  $\bar{t}/u$  by the symbol  $\#^{m(\bar{t}/u)}$ , for every position  $u \in P \setminus L_{\mathcal{V}}(t)$  such that  $\forall i \in \mathbb{N}, u \cdot i \notin P$ .

**Lemma 4.8.** Let  $\bar{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$  and  $P$  be a subdomain of  $\mathcal{P}\text{os}(t)$  such that  $\mathcal{P}\text{os}_{\mathcal{V}}(t) \subseteq P$ . We have  $\text{Red}(\widehat{\bar{t}}, P) = \widehat{\text{Red}(\bar{t}, P)}$ .

*Proof.* Since  $\widehat{\cdot}$  acts only on marks, we have  $\text{Red}(\widehat{\bar{t}}, P)^0 = (\widehat{\text{Red}(\bar{t}, P)})^0$ . Let  $v \in P$ . By definition of  $\widehat{\cdot}$ ,  $m(\widehat{\bar{t}}/v) = \max(m(\bar{t}/v), |v| + 1)$ , and  $m(\text{Red}(\bar{t}, P)/v) = \max(m(\text{Red}(\bar{t}, P)/v), |v| + 1)$ . By definition of  $\mathcal{A}$ ,  $m(\widehat{\bar{t}}/v) = m(\text{Red}(\bar{t}, P)/v)$ . Hence,  $\text{Red}(\widehat{\bar{t}}, P) = \widehat{\text{Red}(\bar{t}, P)}$ . ■

#### 4.0.1. Top of a term.

**Definition 4.9** (Top domain of a term). Let  $\bar{t}$  be a s-increasing term. We define the top domain of  $\bar{t}$ , denoted by  $\text{Topd}(\bar{t})$  as:  $u \in \text{Topd}(\bar{t})$  iff  $u \in \mathcal{P}\text{os}(\bar{t}) \wedge m(\bar{t}/u) \leq 1$ .

Note that by definition of a s-increasing term,  $\text{Topd}(\bar{t})$  is a subdomain of  $t$  and since for every  $u \in \mathcal{P}\text{os}_{\mathcal{V}}(t)$ ,  $m(\bar{t}/u) = 0$ , we have  $\mathcal{P}\text{os}_{\mathcal{V}}(t) \subseteq \text{Topd}(\bar{t})$ .

**Definition 4.10** (Top of a term). Let  $\bar{t}$  be a s-increasing term. We denote by  $\text{Top}(\bar{t})$  the term  $\text{Red}(\bar{t}, \text{Topd}(\bar{t}))$ .

**Example 4.11.** Let  $\bar{t}_1 = f^0(h^1(a^2))$ ,  $\bar{t}_2 = f^0(h^0(a^1))$ . We have:  $\text{Topd}(\bar{t}_1) = \{\epsilon, 0\}$ ,  $\text{Topd}(\bar{t}_2) = \mathcal{P}\text{os}(t_2)$ ,  $\text{Top}(\bar{t}_1) = f(\#^1)$ ,  $\text{Top}(\bar{t}_2) = \bar{t}_2$ .

Intuitively, the top of a term  $\bar{t}$  will be the only part of  $\bar{t}$  which could be used in a  $\text{bo}(0)$ -derivation starting on  $\bar{t}$ . We extend this definition to sets of s-increasing terms ( $\text{Top}(\bar{S}) := \{\text{Top}(\bar{t}) \mid \bar{t} \in \bar{S}\}$ ) and to s-increasing marked substitutions ( $\text{Top}(\bar{\sigma}) : x \mapsto \text{Top}(x\bar{\sigma})$ ).

**Lemma 4.12.** Let  $\bar{C}[\ ]_v, \bar{t}_1$  be s-increasing and let  $\bar{t} = C[\bar{t}_1]$ . We have:

$$\text{Top}(\bar{t}) = \text{Top}(\bar{C}[\ ]_v)[\text{Top}(\bar{t}_1)]_v.$$

*Proof.* By lemma 4.5,  $\bar{t}$  is s-increasing. Since  $\square$  is a variable and since  $\bar{C}[\ ]_v$  is s-increasing, for every  $u \preceq v$ ,  $m(\bar{t}/u) = 0$ . So,  $v \in \text{Topd}(\bar{t})$ , and by definition of  $\text{Top}$ , the result holds. ■

**Lemma 4.13.** Let  $\bar{t}$  and  $\bar{\sigma}$  be s-increasing. We have:  $\text{Top}(\bar{t}\bar{\sigma}) = \text{Top}(\bar{t})\text{Top}(\bar{\sigma})$ .

*Proof.* This lemma is obtained by applying lemma 4.12 several times at each position  $v \in \mathcal{P}\text{os}_{\mathcal{V}}(t)$ . ■

#### 4.0.2. The ground system $\mathcal{S}$ .

**Definition 4.14.** We consider the following ground rewriting system  $\mathcal{S}$  over  $\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$  consisting of all the rules of the form:  $l\bar{\sigma} \rightarrow_{\mathcal{S}} r\widehat{\sigma}$ , where  $l \rightarrow r$  is a rule of  $\mathcal{R}$ , and  $\bar{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ .

Note that since  $\bar{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ , by definition of  $\widehat{\cdot}$ ,  $\widehat{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ . The system  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  will be used to simulate the  $\text{bo}(0)$ -derivations in  $\mathcal{R}$ .

#### 4.0.3. Lifting lemma.

**Lemma 4.15.** Let  $\bar{s}' \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$ ,  $\bar{s}, \bar{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ . Assume that  $\bar{s}' \xrightarrow{*}_{\mathcal{A}} \bar{s} \rightarrow_{\mathcal{S}} \bar{t}$ . There exists a term  $\bar{t}' \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$  such that  $\bar{s}' \xrightarrow{\text{bo}(0)}_{\mathcal{R}} \bar{t}' \xrightarrow{*}_{\mathcal{A}} \bar{t}$ .

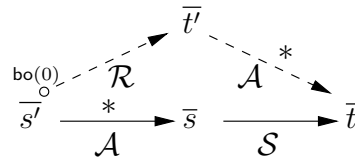


Figure 2: Lemma 4.15

*Proof.* We have  $\bar{s} \rightarrow_{\mathcal{S}} \bar{t}$ . This means that  $\bar{s} = \overline{C}[l\bar{\sigma}]_v$ ,  $\bar{t} = \overline{C}[r\widehat{\sigma}]_v$ , for some rule  $l \rightarrow r \in \mathcal{R}$ , marked context  $\overline{C}[\ ]_v$ , and marked substitution  $\bar{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ . Since  $\bar{s}' \rightarrow_{\mathcal{A}}^* \bar{s}$ , and since  $\mathcal{A}$  goes from bottom to top, there exists a context  $\overline{C}'[\ ]_v$ , a substitution  $\bar{\sigma}'$  such that  $\bar{s}'$  is of the form  $\bar{s}' = \overline{C}'[l\bar{\sigma}']_v$ . with  $\overline{C}'[\ ]_v$  such that  $\overline{C}'[\ ] \rightarrow_{\mathcal{A}}^* \overline{C}[\ ]$ , and  $\bar{\sigma}'$  such that for every  $x \in \mathcal{V}\text{ar}(l)$ ,  $x\bar{\sigma}' \rightarrow_{\mathcal{A}}^* x\bar{\sigma}$ . By definition of  $\text{bo}(0) \circ \rightarrow$ ,  $\bar{s}' = \overline{C}'[l\bar{\sigma}']_v \text{bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{t}' := \overline{C}'[r\widehat{\sigma}]_v$ . By Lemma 4.2, for every  $x \in \mathcal{V}\text{ar}(r)$ ,  $x\widehat{\sigma} \rightarrow_{\mathcal{A}}^* x\widehat{\sigma}$ . Hence,  $\bar{t}' = \overline{C}'[r\widehat{\sigma}]_v \rightarrow_{\mathcal{A}}^* \overline{C}[r\widehat{\sigma}]_v \rightarrow_{\mathcal{A}}^* \overline{C}[r\widehat{\sigma}]_v = \bar{t}$ . We have built a derivation:  $\bar{s}' \text{bo}(0) \circ \rightarrow \bar{t}' \rightarrow_{\mathcal{A}}^* \bar{t}$ . The result holds.  $\blacksquare$

**Example 4.16.** Let us consider the system  $\mathcal{S}$  built from the system  $\mathcal{R}_1$ .

$$\begin{aligned} \mathcal{S} = \{ & f(\#) \rightarrow g(\#^1), f(\#^1) \rightarrow g(\#^1), f(a) \rightarrow g(a^1), f(a^1) \rightarrow g(a^1), \\ & f(b) \rightarrow g(b^1), f(b^1) \rightarrow g(b^1), g(h(\#)) \rightarrow i(\#^1), g(h(\#^1)) \rightarrow i(\#^1), \\ & g(h(a)) \rightarrow i(a^1), g(h(a^1)) \rightarrow i(a^1), g(h(b)) \rightarrow i(b^1), g(h(b^1)) \rightarrow i(b^1), \\ & i(\#) \rightarrow a, i(\#^1) \rightarrow a, i(a) \rightarrow a, i(a^1) \rightarrow a, i(b) \rightarrow a, i(b^1) \rightarrow a, a \rightarrow b\}. \end{aligned}$$

We have the following derivation:

$$g(h(a)) \rightarrow_{\mathcal{A}, a \rightarrow \#} g(h(\#)) \rightarrow_{\mathcal{S}, f(h(\#)) \rightarrow i(\#^1)} i(\#^1).$$

In the proof of lemma 4.15, we build the derivation:

$$g(h(a)) \text{bo}(0) \circ \rightarrow_{\mathcal{R}_1, g(h(x)) \rightarrow i(x)} i(a^1) \rightarrow_{\mathcal{A}, a^1 \rightarrow \#^1} i(\#^1).$$

#### 4.0.4. Projecting lemma.

**Lemma 4.17** (projecting lemma). *Let  $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$  be s-increasing, and  $\bar{s} \text{bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{t}$ . There is a derivation:  $\text{Top}(\bar{s}) \rightarrow_{\mathcal{A}^{\leq 1}}^* \text{Top}(\bar{t})$ .*

$$\begin{array}{ccc} \bar{s} & \xrightarrow{\quad} & \bar{t} \\ \downarrow \text{bo}(0) & \mathcal{R} & \downarrow \\ \text{Top}(\bar{s}) & \xrightarrow[\mathcal{A}^{\leq 1}]{*} & \text{Top}(\bar{t}) \end{array}$$

Figure 3: Projecting lemma

*Proof.* By definition of  $\text{bo}(0) \circ \rightarrow$ , there exists a context  $\overline{C}[\ ]_v$ , a marked substitution  $\bar{\sigma}$ , and a rule  $l \rightarrow r \in \mathcal{R}$  such that  $\bar{s} = \overline{C}[l\bar{\sigma}]_v$  and  $\bar{t} = \overline{C}[r\widehat{\sigma}]_v$ . Since  $\bar{s}$  is s-increasing, by lemma 4.6,  $\bar{t}$  is s-increasing, and  $\text{Top}(\bar{t})$  is well defined. Moreover, the marked context  $\overline{C}[\ ]_v$ , the substitution  $\bar{\sigma}$ , and the terms  $r$  and  $l$  are s-increasing. So, by lemmas 4.12 and 4.13:  $\text{Top}(\bar{s}) = \text{Top}(\overline{C}[\ ]_v)[l\text{Top}(\bar{\sigma})]_v$ , and,  $\text{Top}(\bar{t}) = \text{Top}(\overline{C}[\ ]_v)[r\text{Top}(\widehat{\sigma})]_v$ . By definition of  $\text{Top}$ ,  $\text{Top}(\bar{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ . Let us define the substitution  $\bar{\tau}$  by  $\bar{\tau} : x \mapsto \text{Red}(x\text{Top}(\bar{\sigma}), \text{Topd}(x\text{Top}(\widehat{\sigma})))$ . By definition of  $\text{Red}$ ,  $\text{Top}(\bar{s}) \rightarrow_{\mathcal{A}} \text{Top}(\overline{C}[\ ]_v)[l\bar{\tau}]_v$ . Moreover,  $\text{Top}(\bar{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ . Thus, we have  $\text{Top}(\bar{s}) \rightarrow_{\mathcal{A}^{\leq 1}}^* \text{Top}(\overline{C}[\ ]_v)[l\bar{\tau}]_v$ . Let  $x \in \mathcal{V}\text{ar}(l)$ . Let us prove that  $x\bar{\tau} \in (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ . Let  $u \in \text{Pos}(x\bar{\sigma})$ . If  $|u| \geq 1$ , by definition of  $\widehat{\cdot}$ , we have  $m(x\widehat{\sigma}/u) \geq 2$ , and  $u \notin \text{Topd}(x\text{Top}(\widehat{\sigma}))$ . Thus,  $x\bar{\tau}$  is reduced

to a constant, and since  $\text{Top}(\bar{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ ,  $x\bar{r} \in (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$ . Hence, the rule  $l\bar{r} \rightarrow r\widehat{\bar{r}}$  belongs to  $\mathcal{S}$  and  $\text{Top}(\bar{s}) \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \text{Top}(\bar{C})[l\bar{r}] \rightarrow_{\mathcal{S}} \text{Top}(\bar{C})[r\widehat{\bar{r}}]$ . By lemma 4.8, for all  $x \in \mathcal{V}\text{ar}(r)$ ,

$$x\widehat{\bar{r}} = \text{Red}(x\text{Top}(\bar{\sigma}), \widehat{\text{Topd}(x\text{Top}(\bar{\sigma}))}) = \text{Red}(x\widehat{\text{Top}\bar{\sigma}}, \text{Topd}(x\text{Top}(\bar{\sigma})) = x\text{Top}(\widehat{\bar{\sigma}}).$$

So,  $\widehat{\bar{r}} = \text{Top}(\widehat{\bar{\sigma}})$ , and  $\text{Top}(\bar{t}) = \text{Top}(\bar{C}[\ ]_v)[r\text{Top}(\widehat{\bar{\sigma}})]_v = \text{Top}(\bar{C}[\ ]_v)[r\widehat{\bar{r}}]_v$ . We have built a derivation:  $\text{Top}(\bar{s}) \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \rightarrow_{\mathcal{S}} \text{Top}(\bar{t})$ . The result holds. ■

**Example 4.18.** Let us consider the system  $\mathcal{R}_1$ ,  $\mathcal{S}$  built for this system, and the following  $\text{bo}(0)$  rewriting step:  $\bar{s} = f(f(g^1(a^2))) \circ_{\mathcal{R}_1, f(x) \rightarrow h(x)} \bar{t} = h(f^1(g^2(a^3)))$ .

We have  $\text{Top}(\bar{s}) = f(f(\#^1))$ ,  $\text{Top}(\bar{t}) = h(\#^1)$ , and the following derivation:  $f(f(\#^1)) \rightarrow_{\mathcal{A}^{\leq 1}} f(\#) \rightarrow_{\mathcal{S}, f(\#) \rightarrow h(\#^1)} h(\#^1)$ .

## 5. Decidability of the termination

In this section, we prove that the  $\text{bo}(k)$ -termination problem is decidable.

**Proposition 5.1.** *If the system  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  does not terminate, then  $\mathcal{R}$  does not  $\text{bo}(0)$ -terminate.*

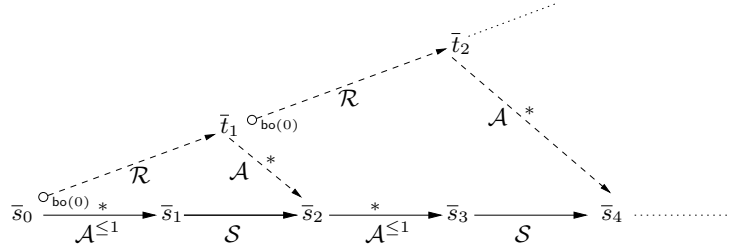


Figure 4: Proposition 5.1

*Proof.* Assume that  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  does not terminate. Then there exists an infinite rewriting sequence of terms in  $\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ . The system  $\mathcal{A}^{\leq 1}$  is obviously terminating. Thus, such an infinite derivation contains an infinite number of steps in  $\mathcal{S}$  and is of the form:

$$s_0 \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \bar{s}_1 \rightarrow_{\mathcal{S}} \bar{s}_2 \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \bar{s}_3 \rightarrow_{\mathcal{S}} \bar{s}_4 \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \dots \rightarrow_{\mathcal{S}} \bar{s}_{2n} \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \dots$$

We now show that repeated applications of lemma 4.15 yields an infinite marked  $\text{bo}(0)$ -derivation in  $\mathcal{R}$ : first, consider  $s_0 \xrightarrow{*}_{\mathcal{A}^{\leq 1}} \bar{s}_1 \rightarrow_{\mathcal{S}} \bar{s}_2$ . By lemma 4.15 there exists  $\bar{t}_1$  such that  $s_0 \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_1 \xrightarrow{*}_{\mathcal{A}} \bar{s}_2$ . Since  $\bar{t}_1 \xrightarrow{*}_{\mathcal{A}} \bar{s}_2$ , we can apply lemma 4.15 to  $\bar{t}_1 \xrightarrow{*}_{\mathcal{A}} \bar{s}_3 \rightarrow_{\mathcal{S}} \bar{s}_4$ . We obtain a term  $\bar{t}_2$  such that  $s_0 \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_1 \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_2 \xrightarrow{*}_{\mathcal{A}} \bar{s}_4$ . Following this process, we obtain an infinite sequence such that  $\bar{s}_0 \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_1 \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_2 \text{bo}(0) \circ_{\mathcal{R}} \dots \text{bo}(0) \circ_{\mathcal{R}} \bar{t}_n \dots$ . We conclude that  $\mathcal{R}$  does not terminate. ■

**Proposition 5.2.** *If  $\mathcal{R}$  does not  $\text{bo}(0)$ -terminate, then  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  does not terminate.*

*Proof.* If  $\mathcal{R}$  does not  $\text{bo}(0)$ -terminate, there is an infinite derivation:

$$s_0 = \bar{s}_0 \text{ bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{s}_1 \text{ bo}(0) \circ \rightarrow_{\mathcal{R}} \dots \bar{s}_n \text{ bo}(0) \circ \rightarrow_{\mathcal{R}} \dots$$

The term  $s_0$  is s-increasing since it has no mark. Moreover, the derivation step  $s_0 \text{ bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{s}_1$  is  $\text{bo}(0)$ . By lemma 4.17,  $s_0 = \text{Top}(s_0) \rightarrow_{\mathcal{A}^{\leq 1}}^* \text{Top}(\bar{s}_1)$ . Another application of lemma 4.17 on  $\bar{s}_1 \text{ bo}(0) \circ \rightarrow_{\mathcal{R}} \bar{s}_2$  leads to a derivation:  $\text{Top}(s_0) \rightarrow_{\mathcal{S} \cup \mathcal{A}^{\leq 1}}^+ \text{Top}(\bar{s}_1) \rightarrow_{\mathcal{S} \cup \mathcal{A}^{\leq 1}}^+ \text{Top}(\bar{s}_2)$ . Following this process, we obtain an infinite derivation:

$$\text{Top}(s_0) \rightarrow_{\mathcal{S} \cup \mathcal{A}^{\leq 1}}^+ \text{Top}(\bar{s}_1) \rightarrow_{\mathcal{S} \cup \mathcal{A}^{\leq 1}}^+ \dots \text{Top}(\bar{s}_n) \rightarrow_{\mathcal{S} \cup \mathcal{A}^{\leq 1}}^+ \dots$$

and  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  does not terminate. ■

**Theorem 5.3.** *The  $\text{bo}(0)$ -termination problem is decidable.*

*Proof.* By propositions 5.1 and 5.2, a linear system  $\mathcal{R}$   $\text{bo}(0)$ -terminates iff the system  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  terminates. It is well known that the termination problem for ground systems is decidable (see e.g.[1, page 99-100]) and we can decide whether the system  $\mathcal{S} \cup \mathcal{A}^{\leq 1}$  terminates. Thus, the  $\text{bo}(0)$ -termination problem is decidable. ■

**Corollary 5.4.** *The  $\text{bo}(k)$ -termination problem is decidable.*

*Proof.* It is just a consequence of theorem 5.3 and lemma 3.12. ■

Note that in general, for a  $\text{BO}(0)$  system, the  $\text{bo}(0)$ -termination property and the termination property are not equivalent.

**Definition 5.5.** Let  $\mathcal{R}$  be a  $\text{BO}(k)$  system. We say that  $\mathcal{R}$  has the  $\text{bo}(k)$  length preservation property if for every  $n \in \mathbb{N}$ :  $\rightarrow_{\mathcal{R}}^n = \text{bo}(k) \rightarrow_{\mathcal{R}}^n$ .

We denote by  $\text{BOLP}(k)$  the class of  $\text{BO}(k)$  systems that have the  $\text{bo}(k)$  length preservation property. Finally, the class of *bounded systems with the length preservation property* is denoted by  $\text{BOLP}$ . One can check that for every  $k > 0$ ,  $\text{BOLP}(k-1) \subsetneq \text{BO}(k)$ .

**Example 5.6.** Let  $\mathcal{R}_2 = \{f(x) \rightarrow g(x), g(a) \rightarrow f(a)\}$ . This system is  $\text{BO}(0)$  but does not have the  $\text{bo}(0)$  length preservation property. There is a derivation of length 2:  $f(a) \rightarrow g(a) \rightarrow f(a)$ , but there is no  $\text{bo}(0)$ -derivation of length 2 from  $f(a)$  to  $f(a)$  (there is one of length 0). Moreover, this system does not terminate but  $\text{bo}(0)$ -terminates.

**Corollary 5.7.** *The termination problem for systems in  $\text{BOLP}(k)$  is decidable.*

*Proof.* Let us prove that, for a system  $\mathcal{R} \in \text{BOLP}(k)$ , the  $\text{bo}(k)$ -termination and termination are equivalent properties. Clearly, if  $\mathcal{R}$  does not  $\text{bo}(k)$  terminate, then the system  $\mathcal{R}$  does not terminate. Conversely, let us suppose that there is an infinite derivation:  $s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n \dots$ . Since  $\mathcal{R}$  has the  $\text{bo}(k)$  length preservation property, there is for each  $m \in \mathbb{N}$  a marked  $\text{bo}(k)$ -derivation  $\overline{D}_m$  such that  $\overline{D}_m = s_0 \text{ bo}(k) \circ \rightarrow_{\mathcal{R}}^m \overline{s}_m$ . The system  $\mathcal{R}$  has a finite number of rules, so there is only a finite number of possible one step rewriting starting on  $s_0$ . Hence, there exists a term  $\overline{s}'_1$  such that the set  $\{m' \mid \overline{D}_{m'} = \overline{s}_0 \text{ bo}(k) \circ \rightarrow_{\mathcal{R}} \overline{s}'_1 \text{ bo}(k) \circ \rightarrow_{\mathcal{R}}^{m'-1} \overline{s}_{m'}\}$  is infinite. Repeating this process, we obtain an infinite derivation:

$$\overline{s}_0 \text{ bo}(k) \circ \rightarrow_{\mathcal{R}} \overline{s}'_1 \text{ bo}(k) \circ \rightarrow_{\mathcal{R}} \dots \text{bo}(k) \circ \rightarrow_{\mathcal{R}} \overline{s}'_n \text{ bo}(k) \circ \rightarrow_{\mathcal{R}} \dots$$

Hence, the system  $\mathcal{R}$  does not  $\text{bo}(k)$ -terminate. We have established that:

$$\mathcal{R} \text{ bo}(k)\text{-terminates} \Leftrightarrow \mathcal{R} \text{ terminates.}$$

By corollary 5.3, the  $\text{bo}(k)$ -termination problem is decidable. Hence, the termination problem for systems in  $\text{BOLP}(k)$  is decidable. ■

We prove in the appendix that several classes of rewriting systems are (strictly) included in the class of BOLP rewriting systems :

- the inverse left-basic semi-Thue systems (viewed as unary term rewriting systems) [12],
- the linear growing term rewriting systems [8],
- the inverse Linear-Finite-Path-Overlapping systems [13],
- the strongly bottom-up linear systems [4].

As a consequence, these systems have a decidable termination problem. Note that, using the same method as the one used to prove the decidability of the termination problem for  $\text{BOLP}(k)$  systems, one can decide for every  $\mathcal{R} \in \text{BOLP}(k)$  and every  $t \in \mathcal{T}(\mathcal{F})$  whether there exists some infinite derivation in  $\mathcal{R}$  starting from  $t$ . It is also shown in the appendix, using a result of [4], that  $\text{BO}(k)$  systems inverse preserve rationality.

## 6. Related works and perspectives

Related works. We borrowed from [4] the idea of simulating derivations according to a special strategy by some ground system. Note however, that the class  $\text{BO}(k)$  itself is new. Its advantages over the class  $\text{BU}(k)$  is that its definition is simpler, it allows a simpler proof of the projecting lemma and it makes lemma 1 true, while this lemma, mutatis mutandis, does not hold for the class  $\text{BU}(k)$ .

The principle of replacing the original rewriting relation over a signature  $\mathcal{F}$  by some other binary relation over a marked-alphabet  $F^{\mathbb{N}}$  was already used in [5] in order to get an algorithm for termination. However, the two marking mechanisms turn out to be different:

- in the case of *word rewriting* systems, the marked derivation used here is not generated by a semi-Thue system while the marked derivation of [5] is generated by an (infinite) semi-Thue system;
- the direct image of a rational set  $R$  by a system which is match-bounded over  $R$  is rational while the direct image of a rational set by a  $\text{BO}(0)$  system needs not be rational; from this point of view our  $\text{BO}(0)$ -semi-Thue systems resemble the inverses of match-bounded systems (though, they are not comparable for inclusion);
- the marking process used here extends naturally to terms while the notion of [5] seems more difficult to extend to terms (although interesting ways of doing such an extension have been studied in [6] and successfully implemented).

Perspectives. Let us mention some natural perspectives of development for this work:

- it is tempting to extend the notion of *bounded* rewriting (resp. system) to left linear systems. This class would extend the class of growing systems studied in [11];
- we think that the direct image of a context-free language through *bounded* rewriting is context-free;
- the whole class of *bounded* systems (at least semi-Thue) should have a decidable termination problem;
- one should try to devise a class of semi-Thue systems that includes both the class of  $\text{BO}(k)$  systems and the class of inverses of match-bounded systems, and still possesses the interesting algorithmic properties of these classes.

Some work in these directions has been undertaken by the authors.

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## Appendix A. Appendix

### A.1. Bottom-up derivations

The class of BO linear systems is closely related to the class of bottom-up systems BU introduced in [4] in the following sense: every BU system is BO, and for every BO system, there is an equivalent system which is BU. The BU systems are also defined using marking tools. The marked derivation used to defined BU system will be denoted by  $\triangleright \rightarrow$ . Let us recall some of the definitions given in [4].

The right-action  $\odot$  of the monoid  $(\mathbb{N}, \max, 0)$  over the set  $\mathcal{F}^{\mathbb{N}}$  consists in applying the operation  $\max$  on every mark: for every  $\bar{t} \in \mathcal{F}^{\mathbb{N}}, n \in \mathbb{N}$ ,

$$\begin{aligned} \mathcal{P}\text{os}(\bar{t} \odot n) &:= \mathcal{P}\text{os}(\bar{t}), \quad \forall u \in \mathcal{P}\text{os}(\bar{t}), \mathfrak{m}((\bar{t} \odot n)/u) := \max(\mathfrak{m}(\bar{t}/u), n), \\ (\bar{t} \odot n)^0 &= \bar{t}^0 \end{aligned}$$

For every linear marked term  $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$  and variable  $x \in \mathcal{V}\text{ar}(\bar{t})$ , we define:

$$M(\bar{t}, x) := \sup\{\mathfrak{m}(\bar{t}/w) \mid w < \text{pos}(\bar{t}, x)\} + 1. \quad (\text{A.1})$$

Let  $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$  and  $t \in \mathcal{T}$ , and let us suppose that  $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$  decomposes as

$$\bar{s} = \bar{C}[\bar{l}\bar{\sigma}]_v, \quad \text{with } (l, r) \in \mathcal{R}, \quad (\text{A.2})$$

for some marked context  $\bar{C}[\ ]_v$  and substitution  $\bar{\sigma}$ . We define a new marked substitution  $\bar{\sigma}$  (such that  $\bar{\sigma}^0 = \bar{\sigma}^0$ ) by: for every  $x \in \mathcal{V}\text{ar}(r)$ ,

$$x\bar{\sigma} := (x\bar{\sigma}) \odot M(\bar{l}, x). \quad (\text{A.3})$$

We then write  $\bar{s} \triangleright \rightarrow \bar{t}$  when

$$\bar{s} = \bar{C}[\bar{l}\bar{\sigma}], \quad \bar{t} = \bar{C}[r\bar{\sigma}]. \quad (\text{A.4})$$

The map  $\bar{s} \mapsto \bar{s}^0$  (from marked terms to unmarked terms) extends into a map from marked derivations to unmarked derivations: every derivation  $d$ :

$$\bar{s}_0 = \bar{C}_0[\bar{l}_0\bar{\sigma}_0]_{v_0} \triangleright \rightarrow \bar{C}_0[r_0\bar{\sigma}_0]_{v_0} = \bar{s}_1 \triangleright \rightarrow \dots \triangleright \rightarrow \bar{C}_{n-1}[r_{n-1}\bar{\sigma}_{n-1}]_{v_{n-1}} = \bar{s}_n \quad (\text{A.5})$$

is mapped to the derivation  $\bar{d}$ :

$$s_0 = C_0[l_0\sigma_0]_{v_0} \rightarrow C_0[r_0\sigma_0]_{v_0} = s_1 \rightarrow \dots \rightarrow C_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \quad (\text{A.6})$$

**Definition A.1** ([4]). The marked derivation (A.5) is *weakly bottom-up* if, for every  $i, 0 \leq i < n$ ,  $\mathfrak{m}(\bar{l}_i) = 0$ .

**Definition A.2** ([4]). The derivation (A.6) is *weakly bottom-up* if the corresponding marked derivation (A.5) starting on the same term  $\bar{s} = s$  is *weakly bottom-up*.

We shall abbreviate “weakly bottom-up” to wbu.

**Definition A.3** ([4]). A derivation is  $\text{bu}(k)$  (resp.  $\text{bu}^-(k)$ ) if it is wbu and, in the corresponding marked derivation  $\forall i, 0 \leq i \leq n$ ,  $\text{mmax}(\bar{s}_i) \leq k$  (resp.  $\forall i, 0 \leq i < n$ ,  $\text{mmax}(\bar{l}_i) < k$ ).

Note that a  $\text{bu}(k)$  derivation is  $\text{bu}^-(k+1)$ .

**Lemma A.4.** *Let  $k \in \mathbb{N}$ . If the derivation (A.6) is  $\text{bu}^-(k)$  then it is  $\text{bu}(k)$ .*



*Sketch of proof.* One can check, by induction on  $n$  the length of the derivation (A.6), that the associated marked derivation (A.5) satisfies:

$$\forall i, 0 \leq i \leq n-1, \text{mmax}(\overline{s_{i+1}}) = \max(\text{mmax}(\overline{s_i}), \text{mmax}(\overline{l_i}) + 1).$$

If the derivation (A.6) is  $\text{bu}^-(k)$ , then the derivation (A.5) satisfies:

$$\forall j, 0 \leq j < n, \text{mmax}(l_j) < k.$$

Hence, for all  $j, 0 \leq j \leq n$ ,  $\text{mmax}(\overline{s_j}) \leq k$  and the derivation (A.6) is  $\text{bu}(k)$ .  $\blacksquare$

Let us introduce a convenient notation.

**Definition A.5.** Let  $k \geq 1$ . The binary relation  $\text{bu}(k) \triangleright_{\mathcal{R}}^* \bar{t}$  over  $\mathcal{T}(\mathcal{F}^{\mathbb{N}})$  is defined by:  $\bar{s} \text{bu}(k) \triangleright_{\mathcal{R}}^* \bar{t}$  if and only if there exists a  $\text{bu}(k)$  marked derivation from  $\bar{s}$  to  $\bar{t}$ . The binary relation  $\text{bu}(k) \rightarrow_{\mathcal{R}}^*$  over  $\mathcal{T}(\mathcal{F})$  is defined by:  $s \text{bu}(k) \rightarrow_{\mathcal{R}}^* t$  if and only if there exists a  $\text{bu}(k)$ -derivation from  $s$  to  $t$ .

**Lemma A.6** ([4]). *Let  $\mathcal{R}$  be a linear system and  $n \in \mathbb{N}$ . If  $s \rightarrow_{\mathcal{R}}^n t$  then there exists a wbu derivation between  $s$  and  $t$  of length  $n$ .*  $\blacksquare$

## A.2. Bottom-up systems

We denote by  $\text{BU}(k)$  the class of  $\text{bu}(k)$  systems, by  $\text{BU}^-(k)$  the class of  $\text{bu}^-(k)$  systems. We define the class of *bottom-up systems*, denoted  $\text{BU}$ , by:

$$\text{BU} = \bigcup_{k \in \mathbb{N}} \text{BU}(k).$$

By definitions of  $\text{bu}$  and  $\text{bu}^-$ , for all  $k > 0$ ,  $\text{BU}(k-1) \subseteq \text{BU}^-(k)$ . By lemma A.4, for all  $k \in \mathbb{N}$ ,  $\text{BU}^-(k) \subseteq \text{BU}(k)$ . Hence, we have:

$$\text{BU} = \bigcup_{k \in \mathbb{N}} \text{BU}^-(k).$$

A system is said to be strongly  $\text{bu}(k)$  iff every wbu derivation is  $\text{bu}(k)$ . The class of strongly  $\text{BU}(k)$  systems is denoted by  $\text{SBU}(k)$ . We define *strongly bottom-up systems*, denoted  $\text{SBU}$  by:

$$\text{SBU} = \bigcup_{k \in \mathbb{N}} \text{SBU}(k).$$

## A.3. Bottom-up systems and Bounded systems

Before proving the equivalence between  $\text{BO}$  and  $\text{BU}$ , we start by giving some technical lemmas. Let us consider  $d$  and  $\bar{d}$ : the derivation (A.6), and its associated marked derivation (A.5) using  $\triangleright_{\mathcal{R}}$ . We denote by  $\tilde{d}$  the associated derivation using  $\circ \rightarrow$ :

$$\tilde{s}_0 = \widetilde{C_0[l_0\tilde{\sigma}_0]}_{v_0} \circ \rightarrow \tilde{s}_1 = \widetilde{C_0[r_0\widehat{\sigma}_0]}_{v_0} \circ \rightarrow \dots \circ \rightarrow \tilde{s}_n = \widetilde{C_{n-1}[r_{n-1}\widehat{\sigma}_{n-1}]}_{v_{n-1}}$$

Let  $e := \max(\{dpt(l) \mid l \in \text{LHS}(\mathcal{R})\})$ . Note that since  $\text{LHS}(\mathcal{R}) \cap \mathcal{V} = \emptyset$ ,  $e \geq 1$ .

**Lemma A.7.** *For all  $i, 0 \leq i \leq n$ , for all  $u \in \text{Pos}(s_i)$ ,*

$$m(\tilde{s}_i/u) = 0 \Leftrightarrow m(\overline{s_i}/u) = 0.$$

*Proof.* We made this proof by induction on  $n$  the length of the derivation  $d$ . If  $n = 0$ , then  $\overline{s}_n = \widetilde{s}_n = s_n$ , and the result holds. Let us suppose that  $n > 0$ . By induction hypothesis on  $n - 1$ , for all  $i, 0 \leq i \leq n - 1$ , for all  $u \in \mathcal{Pos}(s_{n-1})$ ,

$$m(\widetilde{s}_i/u) = 0 \Leftrightarrow \text{mmax}(\overline{s}_i/u) = 0.$$

Let us prove that for all  $u \in \mathcal{Pos}(s_n)$ ,

$$m(\widetilde{s}_n/u) = 0 \Leftrightarrow \text{mmax}(\overline{s}_n/u) = 0.$$

Let  $u \in \mathcal{Pos}(s_n)$ . We distinguish three cases.

- **case 1:**  $u \prec v_{n-1}$  or  $u \perp v_{n-1}$ .

Then  $u \in \mathcal{Pos}_{\overline{v}}(C_{n-1} \parallel_{v_{n-1}})$ , and by induction hypothesis,

$$m(\widetilde{s}_n/u) = m(\widetilde{s}_{n-1}/u) = 0 \Leftrightarrow \text{mmax}(\overline{s}_n/u) = m(\overline{s}_n/u) = 0.$$

The result holds.

- **case 2:**  $\exists w \in \mathcal{Pos}_{\overline{v}}(r_{n-1})$  such that  $u = v_{n-1} \cdot w$ .

Then,  $m(\widetilde{s}_n/u) = m(r_{n-1}/w) = m(\overline{s}_n/u) = 0$ . The result holds.

- **case 3:**  $\exists x \in \mathcal{Var}(r_{n-1})$ ,  $\exists w \in \mathcal{Pos}(x\sigma_{n-1})$  such that  $u = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w$ .

Then, by definition of  $\overline{\sigma}$ ,  $m(\overline{s}_n/u) = m(x\overline{\sigma}/w) > 0$ , and by definition of  $\widehat{\sigma}$ ,

$m(\widetilde{s}_n/u) = m(x\widehat{\sigma}/w) > 0$ . The result holds. ■

**Lemma A.8** ([4]). *If  $d$  is wbu, the following assertion holds:*

$$\forall i, 0 \leq i \leq n, \forall u, v \in \mathcal{Pos}(\overline{s}_i), u \preceq v \Rightarrow m(\overline{s}_i/u) \leq m(\overline{s}_i/v).$$

■

**Lemma A.9.** *If  $d$  is wbu, the following assertion holds: for all  $i, 0 \leq i \leq n$ , for all  $u, v \in \mathcal{Pos}(\overline{s}_i)$ ,  $u \preceq v$ ,*

$$m(\widetilde{s}_i/v) > 0 \Rightarrow m(\widetilde{s}_i/u) - m(\widetilde{s}_i/v) \geq |u| - |v|.$$

*Proof.* We make this proof by induction on  $n$ . If  $n = 0$ , the result holds. Let us suppose that  $n > 0$ , and that the result holds for every  $i, 0 \leq i \leq n - 1$ . Let  $u \in \mathcal{Pos}(s_n)$  and let  $v \preceq u$ . We have to prove that:

$$m(\widetilde{s}_n/v) > 0 \Rightarrow m(\widetilde{s}_n/u) - m(\widetilde{s}_n/v) \geq |u| - |v|.$$

- **case 1:**  $v \prec v_{n-1}$ .

Since  $d$  is wbu,  $m(\overline{l}_{n-1}) = 0$ , and by lemma A.8,  $m(\overline{s}_{n-1}/v) = 0$ . So,

$m(\overline{s}_n/v) = m(\overline{s}_{n-1}/v) = 0$ . By lemma A.7,  $m(\widetilde{s}_n/v) = 0$ . The result holds.

- **case 2:**  $v \perp v_{n-1}$ .

Then, since  $u \perp v$ ,  $m(\widetilde{s}_{n-1}/v) = m(\widetilde{s}_n/v)$ , and  $m(\widetilde{s}_{n-1}/u) = m(\widetilde{s}_n/u)$ . By induction hypothesis on  $n - 1$ ,

$$m(\widetilde{s}_{n-1}/v) > 0 \Rightarrow m(\widetilde{s}_{n-1}/u) - m(\widetilde{s}_{n-1}/v) \geq |u| - |v|.$$

The result holds.

- **case 3:**  $\exists w \in \mathcal{Pos}_{\overline{v}}(r_{n-1})$  such that  $v = v_{n-1} \cdot w$ .

Then,  $m(\widetilde{s}_n/v) = m(r_{n-1}/w) = m(\overline{s}_n/v) = 0$ . The result holds.

- **case 4:**  $\exists x \in \mathcal{Var}(r_{n-1})$ ,  $\exists w \in \mathcal{Pos}(x\sigma_{n-1})$  such that  $v = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w$ .

There exists  $w_1, w \preceq w_1$  such that  $u = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w_1$ .

Let  $v' = v_{n-1} \cdot \text{pos}(l_{n-1}, x) \cdot w$ , and  $u' = v_{n-1} \cdot \text{pos}(l_{n-1}, x) \cdot w_1$ . By definition of  $\sim$ ,

$$m(\widetilde{s}_n/v) = \max(m(\widetilde{s}_{n-1}/v'), |w| + 1),$$

and,

$$\mathfrak{m}(\tilde{s}_n/u) = \max(\mathfrak{m}(\widetilde{s_{n-1}}/u'), |w_1| + 1). \quad (\text{A.7})$$

We distinguish two subcases:

- **case 4.1:**  $\mathfrak{m}(\tilde{s}_n/v) = \mathfrak{m}(\widetilde{s_{n-1}}/v')$ .

Then  $\mathfrak{m}(\widetilde{s_{n-1}}/v') > 0$ , and by induction hypothesis,

$$\mathfrak{m}(\widetilde{s_{n-1}}/u') - \mathfrak{m}(\widetilde{s_{n-1}}/v') \geq |u'| - |v'|.$$

By equation (A.7),  $\mathfrak{m}(\tilde{s}_n/u) \geq \mathfrak{m}(\widetilde{s_{n-1}}/u')$ . Since  $|u'| - |v'| = |u| - |v|$ ,

$$\begin{aligned} \mathfrak{m}(\tilde{s}_n/u) - \mathfrak{m}(\tilde{s}_n/v) &= \mathfrak{m}(\tilde{s}_n/u) - \mathfrak{m}(\widetilde{s_{n-1}}/v') \\ &\geq \mathfrak{m}(\widetilde{s_{n-1}}/u) - \mathfrak{m}(\widetilde{s_{n-1}}/v') \geq |u| - |v|. \end{aligned}$$

The result holds.

- **case 4.2:**  $\mathfrak{m}(\tilde{s}_n/v) = |w| + 1$ .

By equation (A.7),  $\mathfrak{m}(\tilde{s}_n/u) \geq |w_1| + 1$ . Moreover,  $|w_1| = |w| + |u| - |v|$ . Hence,

$$\mathfrak{m}(\tilde{s}_n/u) - \mathfrak{m}(\tilde{s}_n/v) \geq |w_1| + 1 - (|w| + 1) = |u| - |v|.$$

The result holds. ■

**Corollary A.10.** *If  $d$  is wbu, then for all  $n, 0 \leq i \leq n$ , for every  $u, v \in \mathcal{Pos}(\bar{s}_i)$ ,*

$$u \preceq v \Rightarrow \mathfrak{m}(\bar{s}_i/u) \leq \mathfrak{m}(\bar{s}_i/v).$$

*Proof.* It is a immediate consequence of the lemma A.9. ■

**Definition A.11.** Let  $\bar{t}$  be a marked term. We denote by  $N_u(\bar{t})$  the number of positions  $v \prec u$  such that  $\mathfrak{m}(\bar{t}/v) = \mathfrak{m}(\bar{t}/u)$ :

$$N_u(\bar{t}) := |\{v \mid v \prec u, \mathfrak{m}(\bar{t}/v) = \mathfrak{m}(\bar{t}/u)\}|.$$

Note that by lemma A.8, if  $d$  is wbu, for all  $i, 0 \leq i \leq n$ ,  $N_u(\bar{s}_i) = |u| - |v_u|$ , where  $v_u := \inf(\{v \mid v \preceq u, \mathfrak{m}(\bar{s}_i/v) = \mathfrak{m}(\bar{s}_i/u)\})$ .

**Lemma A.12.** *If  $d$  is wbu, then for all  $i, 1 \leq i \leq n$ , for all  $u \in \mathcal{Pos}(s_i)$ ,*

$$\mathfrak{m}(\tilde{s}_i/u) \leq \mathfrak{m}(\bar{s}_i/u) \cdot e + N_u(\bar{s}_i).$$

*Proof.* The proof is made by induction on  $n$  the length of the derivation  $d$ . If  $n = 0$ ,  $\bar{s}_0 = \tilde{s}_0 = s_0$ , and the result holds. Let  $n > 0$ . We have to prove that for every  $u \in \mathcal{Pos}(s_n)$ ,

$$\mathfrak{m}(\tilde{s}_n/u) \leq \mathfrak{m}(\bar{s}_n/u) \cdot e + N_u(\bar{s}_n).$$

We distinguish three cases.

- **case 1:**  $u \prec v_{n-1}$  or  $u \perp v_{n-1}$ .

Then,  $u \in \mathcal{Pos}_{\bar{\mathcal{V}}}(C[\ ]_{v_{n-1}})$ , and  $N_u(\bar{s}_n) = N_u(\bar{s}_{n-1})$ . By induction hypothesis on  $n - 1$ , the result holds.

- **case 2:**  $\exists w \in \mathcal{Pos}_{\bar{\mathcal{V}}}(r_{n-1})$  such that  $u = v_{n-1} \cdot w$ .

Then,  $\mathfrak{m}(\tilde{s}_n/u) = \mathfrak{m}(\bar{s}_n/u) = \mathfrak{m}(r_{n-1}/w) = 0$ , and the result holds.

- **case 3:**  $\exists x \in \mathcal{Var}(r_{n-1}), \exists w \in \mathcal{Pos}(x\sigma_{n-1})$  such that  $u = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w$ :

Let  $u' = v_{n-1} \cdot \text{pos}(l_{n-1}, x) \cdot w$ . By definition,

$$\mathfrak{m}(\tilde{s}_n/u) = \max(|w| + 1, \mathfrak{m}(\widetilde{s_{n-1}}/u')), \quad (\text{A.8})$$

By definition of  $\overline{\sigma_{n-1}}, \overline{x\sigma_{n-1}} = x\overline{\sigma_{n-1}} \odot M(\overline{C}[\overline{l_{n-1}}]_{v_{n-1}}, x)$ . By lemma A.8, and since  $l_{n-1} \notin \mathcal{V}$ ,

$$M(\overline{C}[\overline{l_{n-1}}]_{v_{n-1}}, x) = M(\overline{l_{n-1}}, x) = \mathfrak{m}(\overline{l_{n-1}}/\text{fth}(\text{pos}(l_{n-1}, x))) + 1.$$

Hence, for all  $w_1 \in \mathcal{P}\text{os}(x\sigma_{n-1})$ ,

$$\begin{aligned} \mathfrak{m}(x\overline{\sigma_{n-1}}/w_1) &= \mathfrak{m}(\overline{s_n}/v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w_1) \\ &= \max(\mathfrak{m}(\overline{s_{n-1}}/v_{n-1} \cdot \text{pos}(l_{n-1}, x) \cdot w_1), \mathfrak{m}(\overline{l_{n-1}}/\text{fth}(\text{pos}(l_{n-1}, x)))) + 1). \end{aligned} \quad (\text{A.9})$$

Let  $v := \inf\{w' \mid w' \preceq u', \mathfrak{m}(\overline{s_n}/u) = \mathfrak{m}(\overline{s_n}/w')\}$ , and let  $v' := \inf\{w' \mid w' \preceq u', \mathfrak{m}(\overline{s_{n-1}}/u') = \mathfrak{m}(\overline{s_{n-1}}/w')\}$ . We have:

$$N_u(\overline{s_n}) = |u| - |v| \text{ and } N_{u'}(\overline{s_{n-1}}) = |u'| - |v'|.$$

We distinguish three subcases:

**- case 3.1:**  $v' \preceq v_{n-1}$

Since  $d$  is wbu,  $\mathfrak{m}(\overline{s_{n-1}}/v_{n-1}) = 0$ . By lemma A.8,  $\mathfrak{m}(\overline{s_{n-1}}/v') = 0$ , and by definition of  $v'$ ,  $\mathfrak{m}(\overline{s_{n-1}}/v') = \mathfrak{m}(\overline{s_{n-1}}/u')$ . By lemma A.8, for all  $w_1 \preceq u'$ ,  $\mathfrak{m}(\overline{s_{n-1}}/w_1) = 0$ . By equation (A.9), for all  $w_2, u \preceq w_2 \preceq v_{n-1} \cdot \text{pos}(r_{n-1}, x)$ ,  $\mathfrak{m}(\overline{s_n}/w_2) = 1$ .

Moreover,  $\mathfrak{m}(\overline{s_n}/v_{n-1} \cdot \text{fth}(\text{pos}(r_{n-1}, x))) = \mathfrak{m}(r_{n-1}/\text{fth}(\text{pos}(r_{n-1}, x))) = 0$ , and we have  $v = v_{n-1} \cdot \text{pos}(r_{n-1}, x)$ . So,  $N_u(\overline{s_n}) = |u| - |v_{n-1} \cdot \text{pos}(r_{n-1}, x)| = |w|$ . In addition, since  $\mathfrak{m}(\overline{s_{n-1}}/u') = 0$ , by lemma A.7,  $\mathfrak{m}(\widetilde{s_{n-1}}/u') = 0$ , and by equation (A.8),  $\mathfrak{m}(\widetilde{s_n}/u) = |w| + 1$ . We thus have:

$$\mathfrak{m}(\overline{s_n}/u) = 1, \mathfrak{m}(\widetilde{s_n}/u) = |w| + 1, N_u(\overline{s_n}) = |w|,$$

and since  $e \geq 1$ ,

$$\mathfrak{m}(\widetilde{s_n}/u) = \mathfrak{m}(\overline{s_n}/u) + N_u(\overline{s_n}) \leq \mathfrak{m}(\overline{s_n}/u) \cdot e + N_u(\overline{s_n}).$$

The result holds.

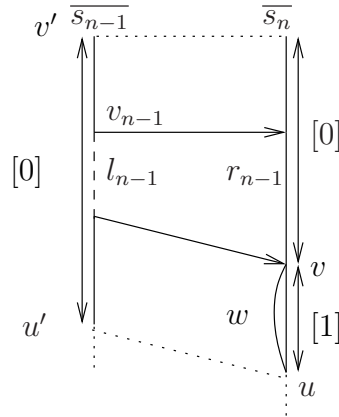


Figure 5: Lemma A.12, case 3.1.

**- case 3.2:**  $\exists w_1 \in \mathcal{P}\text{os}_{\overline{v}}(l_{n-1})$ ,  $|w_1| \geq 1$  such that  $v' = v_{n-1} \cdot w_1$ .

Since  $d$  is wbu,  $\mathfrak{m}(\overline{l_{n-1}}) = \mathfrak{m}(\overline{s_{n-1}}/v_{n-1}) = 0$ . By definition of  $v'$  and by lemma A.8, for all  $w_2, u' \preceq w_2 \preceq v'$ ,  $\mathfrak{m}(\overline{s_{n-1}}/w_2) = \mathfrak{m}(\overline{s_{n-1}}/u')$ . In particular,  $\mathfrak{m}(\overline{s_{n-1}}/v_{n-1} \cdot \text{fth}(\text{pos}(l_{n-1}, x))) = \mathfrak{m}(\overline{s_{n-1}}/u')$ . So, by equation (A.9), for all  $w_2, u \preceq w_2 \preceq v_{n-1} \cdot \text{pos}(r_{n-1}, x)$ ,  $\mathfrak{m}(\overline{s_n}/w_2) = \mathfrak{m}(\overline{s_{n-1}}/u') + 1$ , and since  $\mathfrak{m}(\overline{s_n}/v_{n-1} \cdot \text{fth}(\text{pos}(r_{n-1}, x))) = \mathfrak{m}(r_{n-1}/\text{fth}(\text{pos}(r_{n-1}, x))) = 0$ , we have  $v = v_{n-1} \cdot \text{pos}(r_{n-1}, x)$ . So,  $N_u(\overline{s_n}) = |u| - |v_{n-1} \cdot \text{pos}(r_{n-1}, x)| = |w|$ . In addition, by lemma A.7, since  $\mathfrak{m}(\overline{s_{n-1}}/v') > 0$ , we have  $\mathfrak{m}(\widetilde{s_{n-1}}/v') > 0$ . By lemma A.9,

$$\mathfrak{m}(\widetilde{s_{n-1}}/u') - \mathfrak{m}(\widetilde{s_{n-1}}/v') \geq |u'| - |v'|.$$

Moreover,

$$|u'| - |v'| = |u| - |v| + |\text{pos}(l_{n-1}, x)| - |w_1| > |u| - |v| = |w|.$$

So,  $\mathfrak{m}(\widetilde{s_{n-1}}/u') \geq |w| + 1$ , and by equation (A.8),  $\mathfrak{m}(\widetilde{s_n}/u) = \mathfrak{m}(\widetilde{s_{n-1}}/u')$ . We thus have:

$$\begin{aligned} \mathfrak{m}(\overline{s_n}/u) &= \mathfrak{m}(\overline{s_{n-1}}/u') + 1, \mathfrak{m}(\widetilde{s_n}/u) = \mathfrak{m}(\widetilde{s_{n-1}}/u'), N_u(\overline{s_n}) = |w|, \\ N_{u'}(\overline{s_{n-1}}) &= |u'| - |v'| = N_u(\overline{s_n}) + |\text{pos}(l_{n-1}, x)| - |w_1|. \end{aligned}$$

By induction hypothesis,

$$\mathfrak{m}(\widetilde{s_{n-1}}/u') \leq \mathfrak{m}(\overline{s_{n-1}}/u') \cdot e + N_{u'}(\overline{s_{n-1}}).$$

and since  $|\text{pos}(l_{n-1}, x)| - |w_1| \leq e$ ,

$$\begin{aligned} \mathfrak{m}(\widetilde{s_n}/u) &= \mathfrak{m}(\widetilde{s_{n-1}}/u') \leq (\mathfrak{m}(\overline{s_{n-1}}/u') + 1) \cdot e + N_{u'}(\overline{s_{n-1}}) - e \\ &\leq (\mathfrak{m}(\overline{s_{n-1}}/u') + 1) \cdot e + N_{u'}(\overline{s_{n-1}}) - |\text{pos}(l_{n-1}, x)| + |w_1| \\ &= \mathfrak{m}(\overline{s_n}/u) + N_u(\overline{s_n}). \end{aligned}$$

The result holds.

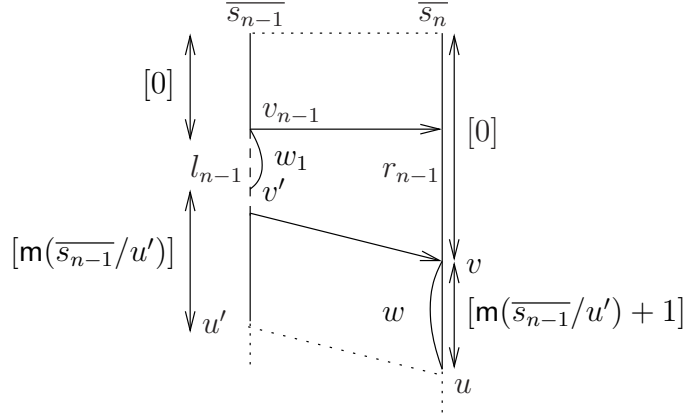


Figure 6: Lemma A.12, case 3.2.

- **case 3.3:**  $\exists w_1 \in \mathcal{P}\text{os}(x\sigma_{n-1})$  such that  $v' = v_{n-1} \cdot \text{pos}(l_{n-1}, x) \cdot w_1$ .

By definition of  $v'$ , and by lemma A.8, for all  $w_2 \prec v'$ ,  $0 \leq \mathfrak{m}(l_{n-1}/w_2) < \mathfrak{m}(\overline{s_{n-1}}/u')$ . Let  $z = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w_1$ . By equation (A.9), for all  $w_2, u \preceq w_2 \preceq \cdot z$ ,

$$\mathfrak{m}(\overline{s_n}/w_2) = \mathfrak{m}(\overline{s_{n-1}}/u'). \quad (\text{A.10})$$

Hence,  $v \preceq v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w_1$ , and there exists  $w'_1 \preceq w_1$  such that  $v = v_{n-1} \cdot \text{pos}(r_{n-1}, x) \cdot w'_1$ . Since  $N_{u'}(\overline{s_{n-1}}) = |u'| - |v'| = |w| - |w_1|$  and since  $N_u(\overline{s_n}) = |u| - |v| = |w| - |w'_1|$ , we have  $N_u(\overline{s_n}) \geq N_{u'}(\overline{s_{n-1}})$ . By lemma A.7, since  $\mathfrak{m}(\overline{s_n}/z) > 0$ , we have  $\mathfrak{m}(\widetilde{s_n}/z) > 0$ , and by lemma A.9,  $\mathfrak{m}(\widetilde{s_n}/u) - \mathfrak{m}(\widetilde{s_n}/z) \geq |u| - |z|$ . By definition,

$$\mathfrak{m}(\widetilde{s_n}/z) = \max(|w_1| + 1, \mathfrak{m}(\widetilde{s_{n-1}}/v')) \geq |w_1| + 1$$

and we have:

$$\mathfrak{m}(\widetilde{s_n}/u) \geq \mathfrak{m}(\widetilde{s_n}/z) + |u| - |z| \geq |w_1| + 1 + |u| - |z|.$$

Since  $|u| - |z| = |u'| - |v'| = |w| - |w_1|$ , we obtain  $\mathfrak{m}(\widetilde{s_n}/u) \geq |w_1| + 1$ , and by equation (A.9),  $\mathfrak{m}(\widetilde{s_n}/u) = \mathfrak{m}(\widetilde{s_{n-1}}/u')$ . We thus have:

$$\mathfrak{m}(\overline{s_n}/u) = \mathfrak{m}(\overline{s_{n-1}}/u'), \mathfrak{m}(\widetilde{s_n}/u) = \mathfrak{m}(\widetilde{s_{n-1}}/u'), N_u(\overline{s_n}) \geq N_{u'}(\overline{s_{n-1}}).$$

By induction hypothesis,

$$m(\widetilde{s_{n-1}/u') \leq m(\overline{s_{n-1}/u') \cdot e + N_{u'}(\overline{s_{n-1}}).$$

Hence,

$$m(\widetilde{s_n/u) = m(\widetilde{s_{n-1}/u') \leq m(\overline{s_{n-1}/u') \cdot e + N_{u'}(\overline{s_{n-1}}) \leq m(\overline{s_n/u) \cdot e + N_u(\overline{s_n}).$$

The result holds.  $\blacksquare$

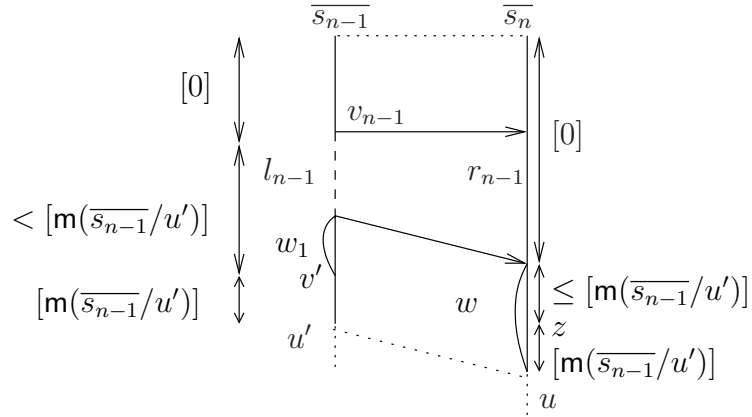


Figure 7: Lemma A.12, case 3.3.

#### A.4. Equivalence between bounded systems and bottom-up systems

We are now ready to prove the equivalence between BO and BU.

**Lemma A.13.** *A derivation is  $\text{bu}^-(1)$  iff it is  $\text{bo}(0)$ .*

*Proof.* The derivation (A.6) is  $\text{bu}^-(1)$  iff the marked derivation using  $\triangleright \rightarrow$  satisfies: for every  $i$ ,  $\bar{l}_i = l_i$ , and this derivation is  $\text{bo}(0)$  iff the marked derivation using  $\circ \rightarrow$  satisfies: for every  $i$ ,  $\tilde{l}_i = l_i$ . We conclude using lemma A.7.  $\blacksquare$

**Corollary A.14.** *Every  $\text{BO}(k)$  system is equivalent to a  $\text{BU}^-(1)$  system.*

*Proof.* It is an immediate consequence of lemmas 3.12, and A.13. Note that the equivalent  $\text{BU}^-(1)$  system can be effectively built.  $\blacksquare$

**Lemma A.15.** *Every  $\text{bu}^-(k)$ -derivation is  $\text{bo}(k \cdot e)$ .*

*Proof.* The derivation  $d$  is  $\text{bu}^-(k)$  iff for all  $i$ ,  $0 \leq i \leq n$ ,  $\text{mmax}(\bar{l}_i) < k$  and  $m(\bar{l}_i) = 0$ . Let  $u \in \text{Pos}_{\overline{v}}(l_i)$  be such that  $m(\bar{l}_i/u) > 0$ . Since  $m(\bar{l}_i) = 0$ , by definition of  $N_{v_i \cdot u}(\bar{s}_i)$  and  $e$ , we have  $N_{v_i \cdot u}(\bar{s}_i) \leq e$ . By lemma A.12,  $m(\tilde{l}_i/u) \leq (k-1) \cdot e + N_{v_i \cdot u}(\bar{s}_i)$ . So,  $m(\tilde{l}_i/u) \leq k \cdot e$ , and  $\text{mmax}(\tilde{l}_i) \leq k \cdot e$ . By definition, the derivation  $d$  is  $\text{bo}(k \cdot e)$ .  $\blacksquare$

**Proposition A.16.** *The classes BO and BU are equivalent in the following sense:*

- (1)  $\text{BU} \subseteq \text{BO}$ .
- (2) For every system  $\mathcal{R} \in \text{BO}$ , there is an equivalent system  $\mathcal{R}'$  in BU.

*Proof.* Since  $\text{BU} = \bigcup_{k \in \mathbb{N}} \text{BU}^-(k)$ , by lemma A.15,  $\text{BU} \subseteq \text{BO}$ . By corollary A.14, for every system  $\mathcal{R} \in \text{BO}$ , there is an equivalent system  $\mathcal{R}' \in \text{BU}^-(1)$ . The result holds.  $\blacksquare$

### A.5. Some other properties of bounded systems

Let us give some properties which directly follow from the equivalence between BU and BO and the results presented in [4].

**Definition A.17.** A system  $(\mathcal{R}, \mathcal{F})$  is said to inverse-preserves rationality if for every recognizable set  $T \subseteq \mathcal{T}(\mathcal{F})$ , the set  $(\rightarrow_{\mathcal{R}}^*)[T] := \{s \in \mathcal{T}(\mathcal{F}) \mid \exists t \in T, s \rightarrow_{\mathcal{R}}^* t\}$  is recognizable too.

**Theorem A.18** ([4]). *Every BU system inverse preserves rationality.* ■

**Corollary A.19.** *Every BO system inverse preserves rationality.*

*Proof.* Using corollary A.14, we can effectively build an equivalent  $\text{BU}^-(1)$  system. We conclude using theorem A.18. ■

**Definition A.20.** The  $\text{BO}(k)$  (respectively  $\text{BU}^-(k)$ ) membership problem is the following:

INSTANCE : An integer  $k$  and a linear rewriting system  $\mathcal{R}$ .

QUESTION : Does  $\mathcal{R}$  belongs to  $\text{BO}(k)$  (resp.  $\text{BU}^-(k)$ ) ?

**Proposition A.21** ([4]). *The  $\text{BU}^-(1)$  membership problem is undecidable.* ■

**Corollary A.22.** *The  $\text{BO}(k)$  membership problem is undecidable.*

*Proof.* Using corollary A.14, and proposition A.21, the result holds. ■

**Lemma A.23.** *If  $\mathcal{R}$  is  $\text{SBU}(k)$ , than  $\mathcal{R}$  is  $\text{BOLP}((k+1) \cdot e)$*

*Proof.* Let  $s \rightarrow_{\mathcal{R}}^n t$ . By lemma A.6, there is a wbu-derivation of length  $n$ . By definition of  $\text{SBU}(k)$ , this derivation is  $\text{bu}(k)$ . Since a  $\text{bu}(k)$  derivation is  $\text{bu}^-(k+1)$ , by lemma A.15, this derivation is  $\text{bo}((k+1) \cdot e)$ . This derivation is of length  $n$ , and the system  $\mathcal{R}$  is  $\text{BOLP}((k+1) \cdot e)$ . ■

**Proposition A.24.** *The termination problem for systems in SBU is decidable.*

*Proof.* This is an immediate consequence of lemma A.23 and of corollary 5.7. ■

It is shown in [4] that classes  $\text{SBU}(k)$  have a decidable membership problem. Note that  $\text{SBU} \subset \text{BOLP}$ , as shown in the following example.

**Example A.25.** Let  $\mathcal{F} = \{f(x), a\}$ .

Let  $\mathcal{R}_3 = \{f(f(x)) \rightarrow f(x)\}$ .

This system is  $\text{bu}(1)$ , and for each  $k \in \mathbb{N}$  we can build a wbu derivation which is not  $\text{bo}(k)$ :

$$f(f(f(\dots(a)) \dots)) \triangleright \rightarrow f(f^1(f^1 \dots (a^1)) \dots) \triangleright \rightarrow f(f^2 \dots (a^2)) \dots \triangleright \rightarrow \dots$$

but for every derivation, there is a  $\text{bo}(0)$ -derivation which has the same length:

$$f(\dots(f(f(f(a)) \dots)) \circ \rightarrow f(\dots(f(f(a^1)) \dots)) \circ \rightarrow f(\dots((f(f^1(a^2)) \dots)).$$

### A.6. Some bounded systems

The class of SBU systems, which is included in BOLP, contains several classes of systems [4]. Among them, there are:

- the inverse left-basic semi-Thue systems (viewed as unary term rewriting systems)
- the linear growing term rewriting systems
- the inverse Linear-Finite-Path-Overlapping systems.

By corollary 5.7, for all these systems, the termination problem is decidable.