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CONTROL OF BLOW-UP SINGULARITIES FOR NONLINEAR WAVE EQUATIONS

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ABSTRACT. While the global boundary control of nonlinear wave equations that exhibit blow-up is generally impossible, we show on a typical example, motivated by laser breakdown, that it is possible to control solutions with small data so that they blow up on a prescribed compact set bounded away from the boundary of the domain. This is achieved using the representation of singular solutions with prescribed blow-up surface given by Fuchsian reduction. We outline on this example simple methods that may be of wider applicability.

1. Introduction.

1.1. **Objectives.** It is well-known that the boundary control of solutions of nonlinear Klein-Gordon equations that exhibit blow-up in finite time is, in general, impossible. Indeed, assume that the given initial and boundary data lead to blow-up for $t = t_0$ and $x = x_0$, where the distance of x_0 from the boundary is greater than ct_0 , c being the speed of propagation; then, the boundary data do not have time to influence the solution at x_0 before the blow-up time. The solution near the blow-up point is entirely determined by the initial conditions, and no choice of boundary conditions can modify this blow-up behavior. In other words, boundary conditions, whatever their type, cannot, in general, arrest blow-up. Nevertheless, it is often possible to steer small data to zero (“local controllability”).¹ The purpose of this paper is to show that it is also possible, for cubic wave equations, to steer small data in order to achieve blow-up on a prescribed compact set in the interior of the domain.

This possibility is suggested by the method of Fuchsian reduction [10, 11, 9] that yields solutions that blow up on a given set in spacetime, for wide classes of equations. More precisely, given a sufficiently smooth graph $\Sigma = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t = \psi(x)\}$, with $|\nabla\psi| < 1$ on the entire space, the method yields a solution $u(x, t)$ that becomes singular precisely as $t \rightarrow \psi(x)-$, and that is defined and regular in

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¹General methods and results on the control of hyperbolic problems, together with further references, may be found in the following papers: [17, 12, 13, 14, 15, 16, 1, 23, 24]. Among works more particularly relevant to the present paper, we may mention [4, 5, 20, 8, 21, 22], where further references may be found. We do not aim at completeness. The considerations of the present paper also apply to other nonlinearities, such as those considered in [21]. For recent results on interior control, see e.g. [6].

an open set limited by Σ . Walter Littman observed to me, many years ago, that this type of result suggests the possibility of a control of blow-up singularities, since it furnishes an explicit construction of boundary data that steer a particular set of Cauchy data so that they blow up on a specified set, while remaining smooth elsewhere. Indeed, taking ψ to be positive, and a smooth bounded domain $\Omega \subset \mathbb{R}^n$ that contains a compact set K on which ψ reaches its (positive) minimum ψ_{\min} , the restriction of u and u_t to the set where $t = 0$ and $x \in \Omega$ furnishes a pair of Cauchy data, and its trace on the boundary of Ω provides Dirichlet boundary data, such that the solution of the initial-boundary value problem with these data first blows up precisely for $t = \psi_{\min}$ and $x \in K$. In other words,

It may not be possible to arrest blow-up, but it may be possible to force blow-up to occur at a specified time and place.

This explicit construction has the same advantages as the classical restriction argument for the linear wave equation [18, 16]. However, it requires the initial data, as well as the boundary data, to be chosen in a special way to ensure blow-up occurs only on Σ . We show in this paper, on a typical example that may be of some interest in applications, that if the problem is locally controllable, it is possible to steer the solution, starting from arbitrary small data, so that it blows up on a prescribed set inside the domain. In a nutshell, the above construction will be modified so as to ensure that u has not only smooth, but also *small* data, that may, in turn, be steered to zero by local controllability.

1.2. The model. Our model is

$$\square u = 2u^3, \tag{1}$$

in three space dimensions, to fix ideas. Similar considerations apply to complex solutions of $\square u + \alpha u_z = \beta u|u|^2$, where α and β are constants. This latter problem is a model for the envelope of the electric field of an ultra-short optical pulse, taking normal dispersion and paraxiality corrections into account. In the language of laser breakdown, our statement may be translated as follows:

While it is impossible to arrest laser self-focusing, it is possible to arrange boundary data so that breakdown occurs at a place and time specified in advance.

Before outlining the proof of this statement, we introduce some notation. We shall have to work with two sets of variables: the original space and time variables $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and variables adapted to Σ : $T = \psi(x) - t$; $X = x$. We write ∂_i for $\partial/\partial X_i$, where i runs from 1 to n ; we have $n = 3$ in the example from nonlinear optics, but this will not be used in the sequel. Also, ψ may be viewed as a function of x or X . The smoothness of Σ is measured in Sobolev spaces: we take $\psi \in H^\sigma$, where σ will be taken sufficiently large. Throughout, we assume $\sup |\nabla \psi| < 1$ and $\sup |\psi| < 1$ everywhere. Therefore, $|t - T| < 1$. The regularity of the solution $u(X, T)$ will be estimated in $H^s \times H^{s-1}$, with $s \geq \sigma - 4$, this choice being dictated by the regularity of the coefficients of the Fuchsian system introduced in Sect. 3. We also let $S = (1 - \Delta)^{s/2}$. Finally, $\| \cdot \|_s$ stands for the norm in H^s . Recall that multiplication is a continuous bilinear map from $H^s \times H^s$ to H^s .

1.3. Outline of the argument. The argument for proving this controllability of blow-up singularities for the model at hand is as follows. We are given the compact set K within Ω . We are also given s_0 and $\varepsilon > 0$ so that one has local controllability in the smooth bounded domain Ω for Cauchy data of norm less than

ε in $H^{s_0} \times H^{s_0-1}(\Omega)$ [20, 4]. We may assume $s_0 > n/2 + 1$ without loss of generality. We first of all choose $\alpha > 2$ so that the Cauchy data for the exact solution $1/t$, for $t = \alpha$, have norm less than $\varepsilon/4$ in $H^{s_0} \times H^{s_0-1}$. The objective is to show that, if the constants α and σ are taken large enough, there is a constant μ such that $\|\psi\|_\sigma < \mu$ ensures that there are solutions u of (1) that blow-up for $t = \psi(x)$ and have data on the hyperplane ($t = \alpha$) that are less than ε in $H^{s_0} \times H^{s_0-1}(\Omega)$. It is always possible to choose ψ so that it vanishes precisely on K , and is negative elsewhere; one may also assume its H^σ norm to be as small as we wish—consider $\lambda\psi$, with λ positive and small if necessary. By time reversal (considering $u(\alpha - t)$), we obtain a solution with Cauchy data on ($t = 0$) that are less than ε in norm, and that first blows up on K . Taking the trace of this solution on $\partial\Omega$, we obtain the result

There are small Cauchy data, and boundary controls on $\partial\Omega$ that yield a solution that blows up for $t = \alpha$ and $x \in K$, and remains finite for $t = \alpha$ and $x \in \Omega \setminus K$.

Combining this with the local controllability result gives the desired boundary control of blow-up singularities.

The rest of the paper is devoted to showing that one can choose α , σ and μ with the above properties. This is achieved by constructing a solution u of (1) consisting of three parts:

$$u = \frac{1}{t} + \Phi + T^3w, \quad (2)$$

where $T = t - \psi(x)$, and Φ is an explicit expression involving ψ and its derivatives, and that vanishes when ψ is identically zero. In fact, $1/t$ is an exact solution of (1). In Sect. 2, Φ is obtained by truncating a formal solution of (1); it is completely determined by ψ , and has small Cauchy data on ($t = \alpha$) if ψ is small. In Sect. 3, w is found as the solution of a degenerate initial-value problem of Fuchsian type. The restriction w_0 of w to Σ , must be specified in order to determine w . Sect. 4 deals with the estimation of w . Since $|t - T| < 1$, in order to estimate the Cauchy data of T^3w in $H^{s_0} \times H^{s_0-1}$ on the hyperplane $\{t = \alpha\}$, it suffices to estimate the *space-time* Sobolev norm of index s of w on some slab of the form $(\alpha - 1 \leq T \leq b)$, where $b > \alpha + 1$, with $s > s_0 + 1/2$, and then take the traces of T^3w and $(T^3w)_t$ on the hyperplane ($t = \alpha$). It is such an estimate that we obtain in Sect. 5: these traces are small if $\|\psi\|_\sigma + \|w_0\|_s$ is, provided σ is large enough. The estimation of u is then easily completed: α has been chosen at the outset to make the Cauchy data of $1/t$ less than $\varepsilon/4$; the Cauchy data of Φ have the same property if $\|\psi\|_\sigma$ is less than some M_1 , and those of T^3w are less than $\varepsilon/2$ if $\|\psi\|_\sigma + \|w_0\|_s$ does not exceed some M_2 . Therefore, if $\|\psi\|_\sigma < \min(M_1, M_2)$, and ψ is nonpositive and vanishes only on K , we may take w_0 so that the resulting solution u has Cauchy data less than ε in norm, and blows up precisely on the compact K , as desired.

2. Step 1: Introduction of the formal expansion of u . In the variables (X, T) , the wave equation (1) takes the form

$$\gamma u_{TT} - \Delta_X u + 2\nabla\psi \cdot \nabla u_T + u_T \Delta\psi = 2u^3, \quad (3)$$

where $\gamma = 1 - |\nabla\psi|^2$. The solvability of the standard Cauchy problem for (3) with Cauchy data (f, g) on Σ means that, if f and g in suitable function spaces, there

is, near Σ , a unique function v such that $u = f + Tg + T^2v$ solves (3).² Singular solutions may be obtained by a closely related Ansatz: seek solutions in the form

$$u = \frac{1}{T} \{u_0 + u_1T + \dots\}.$$

This may be viewed as a perturbation of the exact solution $1/t$ of (1).³ In the present situation, (1) is formally solved by an expression of the form

$$u = \frac{1}{T} \{u_0 + u_1T + u_2T^2 + u_3T^3 + u_{4,1}T^4 \ln T + T^4w\},$$

where w is a series in T and $T \ln T$, with coefficients depending on X .⁴ The coefficients u_0, u_1, u_2, u_3 and $u_{4,1}$ are entirely determined by ψ and its derivatives up to order four; one finds $u_0 = \sqrt{\gamma}$. They may be found recursively, and have a geometric interpretation [9, pp. 271–273], [2]. All we need here is that they are obtained by dividing polynomials in derivatives of ψ by powers of γ and that, apart from u_0 , they vanish when ψ is identically zero. The rest of the series is entirely determined by the value $w_0(X)$ of w for $T = 0$, and the coefficients of the expansion of w may be found recursively, as long as ψ possesses sufficiently many derivatives. Thus, the Cauchy data are replaced by the pair of *singularity data* (ψ, w_0) .

There are two essential differences with the Cauchy problem: the expansion must include logarithmic terms, even if the solution is infinitely smooth off Σ , and the singularity data are not the first two terms in the expansion.⁵ For our purposes, the exact expression for the coefficients of the expansion is not needed. It suffices to write

$$u = \frac{1}{t} + \Phi + T^3w,$$

where

$$\Phi = \left(\frac{u_0}{T} - \frac{1}{t} \right) + u_1 + u_2T + u_3T^2 + u_{4,1}T^3 \ln T.$$

Since $\alpha > 2$, this expression has no singularity on $(t = \alpha)$; it also vanishes with ψ (and its derivatives). Because smooth functions act on Sobolev spaces of index higher than $n/2$, it follows that

$$\|\Phi\|_s + \|\Phi_t\|_{s-1} \leq C(\alpha)\|\psi\|_\sigma(1 + \|\psi\|_\sigma^q)$$

for a suitable integer q , and the Sobolev norms are taken on the hyperplane $t = \alpha$. By the above estimate on Φ , there is a constant M_1 , that also depends on α , hence on ε , such that the Cauchy data of Φ have the same property if $\|\psi\|_\sigma < M_1$.

²The local solvability of this problem is a very special case of standard results on symmetric-hyperbolic systems, since any strictly hyperbolic operator admits of symmetrization, see e.g. [19, §5.2–5.3]. An explicit reduction is given below, see (5).

³The existence of an exact solution simplifies matters, but is not essential: it is possible to construct singular solutions by a similar Ansatz even if there is no exact solution independent of space variables. Also, since $-u$ is a solution if u is, there are also solutions that blow up to $-\infty$. Both solutions are very easy to produce numerically, by using a local explicit scheme [2].

⁴It is convenient to treat T and $T \ln T$ as if they were independent variables for bookkeeping purposes, when computing formal series solutions.

⁵There are general rules to determine the form of the expansion and the nature of the data (see [9]), that generalize the usual rules for the form of series solutions of ODEs of Fuchsian type, such as the Bessel or hypergeometric equations, hence the name of the method; however, the present solutions are not necessarily analytic. More general examples require even more complicated series solutions that have no counterpart in the ODE case.

3. Step 2: Reduced equation for w . Let us examine the equation satisfied by w : this is the reduced equation (RE). Writing $D = T\partial_T$, its form is

$$(\gamma D(D+5) - T^2 \Delta_X + T^2 \nabla \psi \nabla \partial_T)w = \text{lower-order terms},$$

where $\gamma = 1 - |\nabla \psi(x)|^2$. Observe that the singular set ($T = 0$) is characteristic for the operator on the left-hand side, but not for the wave operator. To obtain w , we solve the initial-value problem for the RE with only one initial condition: $w(T = 0) = w_0$. This may be achieved by casting the problem in the form of a first-order reduced system (RS), that is symmetric hyperbolic for $T \neq 0$ [9, Th. 10.10, pp. 186-8]:

Theorem 3.1. *For s and m large enough, there are symmetric matrices Q and A^j , $1 \leq j \leq n$, a constant matrix A and functions f_0 and f_1 such that the solution $\mathbf{w} = (w, w_{(0)}, w_{(i)})$ of the reduced system (RS)*

$$Q(D+A)\mathbf{w} = TA^j \partial_j \mathbf{w} + Tf_0(T, T \ln T, X, \mathbf{w}) + T \ln T f_1(T, T \ln T, X, \mathbf{w}), \quad (4)$$

exists for small T , and generates a solution $u = 1/t + \Phi + T^3 w$ of the wave equation, provided that $\mathbf{w}(T = 0)$ is small in H^s and belongs to the null-space of A .

Proof. The RS is derived from the usual symmetric system associated with the wave equation (1) in the new variables X and T ; letting $\mathbf{u} := (u, u_{(0)}, u_{(i)})$, where $u_{(0)}$ and the $u_{(i)}$ correspond to the time and space derivatives of u , this system reads:

$$\begin{cases} \partial_T u = u_{(0)}, \\ (1 - |\nabla \psi|^2) \partial_T u_{(0)} = \sum_i (\partial_i u_{(i)} - 2\psi_i \partial_i u_{(0)}) - (\Delta \psi) u_{(0)} + 2u^3, \\ \partial_T u_{(i)} = \partial_i u_{(0)}. \end{cases} \quad (5)$$

Define the unknown $\mathbf{w} = (w, w_{(0)}, w_{(i)})$ through

$$\begin{cases} u = \frac{u_0}{t_0} + u_1 + u_2 t_0 + u_3 t_0^2 + u_{4,1} t_0^2 t_1 + w(t_0, t_1, X) t_0^3, \\ u_{(0)} = -\frac{u_0}{t_0^2} + u_2 + 2u_3 t_0 + (t_0^2 + 3t_0 t_1) u_{4,1} + w_{(0)}(t_0, t_1, X) t_0^2, \\ u_{(i)} = \frac{u_{0i}}{t_0} + u_{1i} + u_{2i} t_0 + u_{3i} t_0^2 + w_{(i)}(t_0, t_1, X) t_0^2. \end{cases} \quad (6)$$

About the derivation of this expression, see Remark 1 below. After substitution, the symmetric system for $(u, u_{(0)}, u_{(i)})$ goes into the desired system, where Q , A^j , A are given by

$$Q = \begin{bmatrix} 1 & & \\ & \gamma & \\ & & I_n \end{bmatrix}, \quad \text{and } A^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\psi^j & e^j \\ 0 & t e^j & 0 \end{bmatrix},$$

where $e^j = (0, \dots, 1, \dots, 0)$ is the j th vector of the standard basis of n -space, and

$$A = \left[\begin{array}{cc|cc} 3 & -1 & 0 & \cdots & 0 \\ -6 & 2 & 0 & \cdots & 0 \\ \hline 0 & 0 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 2 \end{array} \right].$$

The matrix A is constant, with eigenvalues 0 and 5; the former is simple. For $T = 0$, (4) forces $Aw(T = 0) = 0$, which is why the solution of the RS is determined

by only one initial value, namely, the first component of \mathbf{w} . The exact form of $f = Tf_0 + T \ln T f_1$ is again not essential:⁶ all we need is that

$$f = g_0 + g_1 w + g_2 w^2 + g_3 w^3,$$

where the coefficients g_k are polynomials in T and $T \ln T$ without constant term, with coefficients that are products of the coefficients of Φ and of γ . The coefficient g_0 vanishes with ψ . \square

Remark 1. The RS was derived by the following argument: to obtain the reduced first-order system corresponding to a nonlinear wave equation, first reduce it to a symmetric-hyperbolic system for the unknown u and its first derivatives u_k ($0 \leq k \leq n$). Determine the formal expansion a_h of u up to some given order h inclusive, and the expansion $a_{h-1,k}$ of u_k up to order $h-1$. Then, let $v = (u - a_h)/T^h$, $v_k = (u_k - a_{h-1,k})/T^{h-1}$. The resulting system for v and the v_k is the desired reduced system if h is sufficiently large.

Remark 2. It may be shown that the reduced system has a unique local solution that may be viewed as a continuous function of T and $T \ln T$ with values in a Sobolev space, the existence proof being carried out by performing the same computations on a regularized system obtained by Yosida regularization, or using Friedrichs mollifiers, as in the symmetric-hyperbolic case, see e.g. [19, 9].

Remark 3. Even though Q is positive definite, and A has no eigenvalues with negative real parts, A is not positive definite. For this reason, we shall introduce in the next section a weighted the L^2 scalar product. This annoyance could have been avoided at the expense of further expansion of the solution by introducing a new unknown $\mathbf{z} = [\mathbf{w} - \mathbf{w}_0 - T\mathbf{w}_1(T, X) - \dots - \mathbf{w}_h(T, X)]/T^h$, where $\mathbf{w}_0 + T\mathbf{w}_1(T, X) + \dots$ is the formal expansion of \mathbf{w} , and the \mathbf{w}_k ($k = 0, 1, \dots$) are polynomials in $\ln T$. By substitution, one checks that \mathbf{z} solves a system of the same form as the first reduced system, but with A replaced by $A + h$. Taking h large enough, one may always assume that $A + h$ is positive definite.

4. Step 3: Estimating w . The local solution of the reduced system is obtained by a modification of the method of solution of symmetric hyperbolic systems. The net result, for our purposes, is that there is a $T_0 > 0$ (possibly smaller than b or α), and a unique solution w for every choice of w_0 small in H^s , that is continuous with values in H^s . Let us therefore fix some constant $M > \varepsilon$ such that, for $\|\psi\|_\sigma + \|w_0\|_s \leq M$, we have $\|\mathbf{w}(T)\|_s \leq 2M$ for $T \leq T_0$. By the continuity of w , this is certainly true for small T_0 . Since $s > n/2$, this implies an L^∞ bound as well. We may also assume that ψ is small enough in H^σ to ensure that $\gamma (= 1 - |\nabla\psi|^2)$ remains bounded away from zero. Since the RS is a standard symmetric hyperbolic system for $T > 0$, its solutions persist as long as they do not blow up in C^1 (this means that u has no singularity other than the one for $T = 0$). Since $s > n/2 + 1$, this follows from a bound in $H^s \times H^{s-1}$. We proceed to show that \mathbf{w} actually satisfies stronger estimates that will enable us to show that it actually extends to all $T \in [0, b]$, and remains small there.

For this, we must first estimate $f = Tf_0 + T \ln T f_1$. Recall that, by Moser-type estimates, the H^s Sobolev norms of the powers of \mathbf{w} are estimated linearly in terms

⁶See [9, p. 187] for the complete expressions.

of the Sobolev norms of \mathbf{w} if \mathbf{w} is known to be bounded. By consideration of the expression for f , one obtains an estimate of the form

$$\|f\|_s \leq K_1(M)T|\ln T|(1+T^8)(\|\psi\|_\sigma + \|\mathbf{w}(T)\|_s).$$

Similarly, we also have an H^0 (i.e., L^2) estimate

$$\|f\|_0 \leq K_2(M)T|\ln T|(1+T^8)(\|\psi\|_\sigma + \|\mathbf{w}(T)\|_0).$$

The derivation of the L^2 estimates on \mathbf{w} requires a slight modification of the standard scalar product. Let us write (f, g) for the real L^2 scalar product on functions of the space variables X . Introduce the matrix $V = \text{diag}(6\gamma, 1, \dots, 1)$; the matrix VQA is then nonnegative, and satisfies $VA^i = A^i$. Multiplying (4) by V and taking the scalar product with \mathbf{w} , we obtain, since VQ is independent of T , $D(\mathbf{w}, VQ\mathbf{w}) = 2(\mathbf{w}, VQD\mathbf{w})$, hence

$$\frac{1}{2}D(\mathbf{w}, VQ\mathbf{w}) + (\mathbf{w}, VQA\mathbf{w}) = \sum_i T(\mathbf{w}, A^i \partial_i \mathbf{w}) + (\mathbf{w}, Vf).$$

Now, $(\mathbf{w}, VQA\mathbf{w}) \geq 0$ and $(\mathbf{w}, A^i \partial_i \mathbf{w}) = (-\partial_i[A^i \mathbf{w}], \mathbf{w})$ since A^i is symmetric. It follows, by expanding $\partial_i[A^i \mathbf{w}]$, that $2(\mathbf{w}, A^i \partial_i \mathbf{w}) = (-[\partial_i A^i] \mathbf{w}, \mathbf{w})$. This quantity may be estimated by $C\|\partial_i A^i\|_{L^\infty}(\mathbf{w}, V\mathbf{w}) \leq C(M)(\mathbf{w}, V\mathbf{w})$, since A^i involves the first derivatives of ψ . Consider now $e_0(T) = (\mathbf{w}, VQ\mathbf{w})(T)$, a quantity equivalent to the L^2 norm since γ is bounded away from zero. We obtain

$$De_0 \leq T|\ln T|(K_3(M)\|\psi\|_\sigma + K_4(M)e_0)(1+T^8),$$

hence (remembering that $D = T\partial_T$),

$$\partial_T \{\ln(K_3\|\psi\|_\sigma + K_4e_0(T))\} \leq |\ln T|(1+T^8).$$

Integrating, we obtain that for $0 \leq T \leq b$, one has

$$K_3\|\psi\|_\sigma + K_4e_0(T) \leq (K_3\|\psi\|_\sigma + K_4e_0(0)) \exp\left\{\int_0^b |\ln \tau|(1+\tau^8)d\tau\right\}.$$

Since $e_0(T)$ is estimated by a multiple of $\|w_0\|_s$, we obtain an inequality of the form

$$e_0(T) \leq K_5\|\psi\|_\sigma + K_6\|w_0\|_s.$$

To obtain spatial derivative norms, one performs the same work on the system solved by $\mathbf{v} = S\mathbf{w}$, estimating $e_s(T) = (S\mathbf{w}, VS\mathbf{w})$ (this is equivalent to the norm of $\mathbf{w}(T)$ in H^s). This system satisfies the same assumptions as the original one because of commutator estimates. Time derivatives are then estimated using the reduced system itself.

Take now s to be an integer. If σ is large enough at the outset, there is a positive δ such that the inequality $\|\psi\|_\sigma + \|w_0\|_s < \delta$ ensures that w remains less than ε , hence less than M up to time $T = b$ at least, and therefore is well-defined on the slab $\alpha - 1 \leq T \leq b$. Furthermore, by induction, $\partial_T^k w$ is, for $s - k > n/2$, bounded in H^{s-k} in this slab, so that w belongs to the Sobolev class H^m in space and time with respect to the (X, T) variables if the integer m satisfies $2m > s$ and $s - m > n/2$. For σ , hence s large enough, we may take m greater than both $n/2 + 1$ and $s_0 + 1$. Since the mapping $(t, x) \mapsto (t - \psi(x), x)$ is a local diffeomorphism of class H^s for any $s > n/2 + 1$,⁷ w is also of class H^m with respect to the original (x, t) variables. This is also true of $T^3 w$. The traces of this function and its t -derivative on $(t = \alpha)$

⁷The composition of Sobolev maps with $s > n/2 + 1$ is discussed, for instance, in [7, p. 108].

therefore belong in particular to H^{s_0} and H^{s_0-1} respectively, and are small in these spaces if $\|\psi\|_\sigma + \|w_0\|_s$ is small enough.

5. Step 5: Estimating the Cauchy data of u . At this stage, we know that the Cauchy data of $1/t$ and Φ on $\{(x, t) : t = \alpha \text{ and } x \in \Omega\}$ are both less than $\varepsilon/4$ if ψ is small enough. We also know that the Cauchy data of T^3w on $(t = \alpha)$ may be made less than $\varepsilon/2$ in $H^{s_0} \times H^{s_0-1}$ by choosing σ and s large enough, and the singularity data ψ and w_0 small enough in their respective spaces. Therefore, the Cauchy data of $u = 1/t + \Phi + T^3w$ are less than ε in $H^{s_0} \times H^{s_0-1}$ if the singularity data are small enough, QED.

6. Concluding remarks. We have shown that the reachable set in the cubic nonlinear wave equation contains solutions that blow up on any prescribed compact set. The control time is here a priori very large if ε is (that is, if the local controllability set is very small). However, the solution is far from unique, since all solutions having the same ψ but different w_0 all have the same blow-up set. This raises the question whether one may optimize the control time by proper choice of w_0 .⁸ Also, since there are many different functions ψ that have the same zero set, the choice of ψ could also be of some interest, given that the curvature of the blow-up set is related to the rate of concentration of the so-called “energy” [3].⁹

The possibility of control of singularities seems to be a further illustration of Russell’s suggestion [17, pp. 640–641] that the development of control theory was slowed down by the “historically dominant emphasis on well-posedness and regularity” in the study of PDEs, putting to the fore the search for conditions under which the influence of the data on the solution is “not too great.” By contrast, in control theory “we want to know that the influence of the control functions on [the solution] is ‘not too little’—” and the present paper shows that this influence may be largest possible, and force the solution to become infinite. The main technical point is that the solution admits a stable parameterization of solutions by singularity data—and not only by Cauchy data: blow-up is a stable phenomenon, amenable to control.

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⁸From a practical standpoint, one suggestion would be to determine a formal solution to high order, taking $w_0 = \theta Z(X)$, with Z a bump function and θ small, and to plot the values of the norm of the Cauchy data of the solution on some hyperplane ($t = \alpha'$), with α' not too large, as a function of θ . It is conceivable that a value of θ yielding a minimum of this norm would be a good candidate.

⁹Thus, for blow-up at a single point, a sharply peaked ψ could be preferable in some applications, and in others a shallow extremum would on the contrary be desirable.

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