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► To cite this version:

Alain Durmus, Eric Moulines, Gersende Fort. New conditions for subgeometric rates of convergence in the Wasserstein distance for Markov chains. 2014. hal-00948661v1

HAL Id: hal-00948661

<https://hal.science/hal-00948661v1>

Preprint submitted on 18 Feb 2014 (v1), last revised 10 Jul 2015 (v3)

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New conditions for subgeometric rates of convergence in the Wasserstein distance for Markov chains

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Abstract: In this paper, we provide sufficient conditions for the existence of the invariant distribution and subgeometric rates of convergence in the Wasserstein distance for general state-space Markov chains which are not phi-irreducible. Our approach is based on a coupling construction adapted to the Wasserstein distance.

Our results are applied to establish the subgeometric ergodicity in Wasserstein distance of non-linear autoregressive models in \mathbb{R}^d and of the preconditioned Crank-Nicolson algorithm MCMC algorithm in a Hilbert space. In particular, for the latter, we show that a simple Hölder condition on the log-density of the target distribution implies the subgeometric ergodicity of the MCMC sampler in a Wasserstein distance.

MSC 2010 subject classifications: 60J10, 60B10, 60J05, 60J22, 65C40.

Keywords and phrases: Markov chains, Wasserstein distance, rate of convergence, Markov Chain Monte Carlo Method in infinite dimension.

1. Introduction

Convergence of general state-space Markov chains in total variation distance (or V -total variation) has been studied by many authors. There is a wealth of contributions establishing explicit rate of convergence under conditions implying geometric ergodicity; see [15, Chapter 16], [16], [1], [5] and the references therein. Subgeometric (or Riemannian) convergence has been more scarcely studied; [18] characterized subgeometric convergence using a sequence of drift conditions, which proved to be difficult to use in practice. [12] have shown that, for polynomial convergence rates, this sequence of drift conditions can be replaced by a single drift conditions, mimicking the classical Foster-Lyapunov approach. This result was later extended by [7] to general subgeometric rate of convergence. Explicit convergence rates were obtained in [19], [9] and [8].

The classical proof of convergence in total variation distance are based either on a regenerative or a coupling construction, which requires the existence of accessible small sets and additional assumptions to control the moments of the

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successive return time to these sets. The existence of an accessible small set implies that the chain is ϕ -irreducible.

In this paper, we establish rate of convergence for general state-space Markov chain which are not ϕ -irreducible. In such case, the Markov chain does not converge in total variation distance, but nevertheless may converge in a weaker sense; see for example [14]. We study in this paper the convergence in Wasserstein distance, which also implies the weak convergence. The use of the Wasserstein distance to obtain explicit rate of convergence has been considered by several authors, most often under conditions implying geometric ergodicity. A significant breakthrough in this domain has been achieved in [10], which has proposed an extension of small set adapted to the Wasserstein distance. The main motivation of [10] was the convergence of the solutions of stochastic delay differential equations (SDDE) to their invariant measure. Nevertheless, the techniques introduced in this work have found several applications. [11] used these techniques to prove the convergence of Markov chain Monte Carlo method to sample in infinite dimensional Hilbert spaces. An application for switched and piecewise deterministic Markov processes can be found in [6].

[4] generalized the results of [10], and established conditions which imply the existence and uniqueness of the invariant distribution, and subgeometric ergodicity of Markov chain (in discrete time) and Markov processes (in continuous time). [4] used this result to establish subgeometric ergodicity of the solutions of SDDE. It is interesting to note that the rates obtained in [4] do not match the rates established in [7] for the V -total variation.

In this paper, we complement and improve the results presented in [4]. The approach developed in this paper is more probabilistic than [4], being extensively based on coupling techniques. We provide a sufficient condition couched in terms of a single drift condition for a coupling kernel outside a appropriately defined coupling set, extending the notion of d -small set of [10]. We then show how this single drift condition implies a sequence of drift inequalities from which we deduce an upper bound of some subgeometric moment of the successive return times to the coupling set. The last step is to show that the Wasserstein distance between the distribution of the chain and the invariant probability measure is controlled by these moments. We apply our result to nonlinear autoregressive model with noise whose distribution can be singular with the Lebesgue measure; we also study the convergence of the preconditionned Crank-Nicolson algorithm for a class of target density having density w.r.t. a Gaussian measure on an Hilbert space, under conditions which are weaker than [11].

The paper is organized as follow: in [section 2](#), the main results on the convergence of Markov chains in Wasserstein distance are presented, under different sets of assumptions. In [section 3](#), the applications of these results to nonlinear algorithm and Crank-Nicolson sampling are considered. The proofs are given in [section 2](#) and [section 5](#).

Notations

Let (E, d) be a Polish space. We denote by $\mathcal{B}(E)$ the associated Borel σ -algebra and $\mathcal{P}(E)$ the set of probability measures on $(E, \mathcal{B}(E))$. Let $\mu, \nu \in \mathcal{P}(E)$; α is a coupling of μ and ν if α is a probability on the product space $(E \times E, \mathcal{B}(E \times E))$, such that $\alpha(A \times E) = \mu(A)$ and $\alpha(E \times A) = \nu(A)$ for all $A \in \mathcal{B}(E)$. The set of couplings of $\mu, \nu \in \mathcal{P}(E)$ is denoted $\mathcal{C}(\mu, \nu)$.

The Wasserstein metric associated with d , between two probability measures $\mu, \nu \in \mathcal{P}(E)$ is defined by:

$$W_d(\mu, \nu) = \inf_{\gamma \in \mathcal{C}(\mu, \nu)} \int_{E \times E} d(x, y) d\gamma(x, y) . \quad (1)$$

When d is the trivial metric $d_0(x, y) = 1_{x \neq y}$, the associated Wasserstein metric is, up to a multiplicative factor, the total variation d_{TV} (see ([20, Chapter 6]) defined by:

$$W_{d_0}(\mu, \nu) = \frac{1}{2} d_{\text{TV}}(\mu, \nu) = \sup_{A \in \mathcal{B}(E)} |\mu(A) - \nu(A)| . \quad (2)$$

When d is bounded, the Monge-Kantorovich duality Theorem implies (see [20, Remark 6.5]) that the lower bound in (1) is reached. In addition, W_d is a metric on $\mathcal{P}(E)$ and $\mathcal{P}(E)$ equipped with W_d is a Polish space; see [20, Theorems 6.8 and 6.16]. Finally, the convergence in W_d implies the weak convergence (see e.g. [20, Corollary 6.11]).

2. Main results

Let (E, d_*) be a Polish space. Our goal is to provide sufficient conditions for the ergodicity of a Markov kernel P on $(E, \mathcal{B}(E))$ at a subgeometric rate in the Wasserstein distance.

Definition 1 (Subgeometric functions). The set of measurable functions $r_0 : \mathbb{R}_+ \rightarrow [2, +\infty)$, such that r_0 is non-decreasing, $x \mapsto \log(r_0(x))/x$ is non-increasing and

$$\frac{\log(r_0(x))}{x} \xrightarrow{x \rightarrow +\infty} 0 \quad (3)$$

is denoted Λ_0 . The set of subgeometric functions Λ is the set of positive functions $r : \mathbb{R}_+ \rightarrow (0, +\infty)$, such that there exists $r_0 \in \Lambda_0$ satisfying:

$$0 < \liminf_{x \rightarrow +\infty} \frac{r(x)}{r_0(x)} \leq \limsup_{x \rightarrow +\infty} \frac{r(x)}{r_0(x)} < +\infty.$$

The set Λ of subgeometric functions contains all the functions on the form $r(x) = (1 + \log(1 + x))^\alpha (1 + x)^\beta \exp(cx^\gamma)$, with $(\alpha, \beta) \in \mathbb{R}^2$ if $c > 0$ and $\gamma \in (0, 1)$, and $(\alpha, \beta) \in (\mathbb{R} \times \mathbb{R}_+^*) \cup (\mathbb{R}_+^* \times \{0\})$ if $\gamma = 0$.

The key ingredient for the derivation of our bounds is the existence for all $(x, y) \in E \times E$ of a coupling kernel $Q((x, y), \cdot)$ of the probability measures $P(x, \cdot)$, $P(y, \cdot)$ such that some iterate Q^ℓ satisfies a strong contraction property when (x, y) belongs to the *coupling set* Δ . This assumption is combined with a condition which implies that in n iterations of the coupling kernel, the number of visits to Δ is large enough so that the Wasserstein distance between $P^n(x, \cdot)$ and $P^n(y, \cdot)$ decreases at a subgeometric rate. Let us give a precise definition of such a coupling set.

Definition 2 (Coupling set). Let $\Delta \in \mathcal{B}(E \times E)$, $\ell \in \mathbb{N}^*$, $\epsilon \in (0, 1)$ and d be a distance on E topologically equivalent to d_* . Δ is a (ℓ, ϵ, d) -coupling set for the Markov kernel P on $(E, \mathcal{B}(E))$ if there exists a kernel Q on $(E \times E, \mathcal{B}(E \times E))$ satisfying the following conditions

- (i) for all $x, y \in E$, $Q((x, y), \cdot)$ is a coupling of $(P(x, \cdot), P(y, \cdot))$.
- (ii) for all $x, y \in E$, $Qd(x, y) \leq d(x, y)$.
- (iii) for all $(x, y) \in \Delta$, $Q^\ell d(x, y) \leq (1 - \epsilon)d(x, y)$.

A simple way to check that $\Delta \in \mathcal{B}(E \times E)$ is a coupling set is the following. Let d be topologically equivalent to d_* , bounded by 1 and let $\epsilon \in (0, 1)$. If for all $(x, y) \in E^2$, $W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y)$, and for all $(x, y) \in \Delta$, $W_d(P(x, \cdot), P(y, \cdot)) \leq (1 - \epsilon)d(x, y)$, then [20, corollary 5.22] implies that there exists a Markov kernel Q on $(E \times E, \mathcal{B}(E \times E))$ which makes Δ a $(1, \epsilon, d)$ -coupling set.

We provide sufficient conditions for the existence of an invariant probability measure π for the Markov kernel P and for subgeometric ergodicity in Wasserstein distance, based on a drift condition on the product space $E \times E$ outside a coupling set. Let us assume

H1. Let $\ell \in \mathbb{N}^*$, $\epsilon \in (0, 1)$ and d be a distance on E topologically equivalent to d_* and bounded by 1. There exist a (ℓ, ϵ, d) -coupling set Δ for P .

H2. There exist

- a concave increasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuously differentiable on $[1, +\infty)$, and satisfying $\phi(0) = 0$, $\lim_{x \rightarrow \infty} \phi(x) = \infty$ and $\lim_{x \rightarrow \infty} \phi'(x) = 0$,
- a constant $b \geq 0$ and a measurable function $V : E \rightarrow [1, +\infty)$ with $\sup_{\Delta} \{V(x) + V(y)\} < +\infty$,

such that for all $x, y \in E$:

$$PV(x) + PV(y) \leq V(x) + V(y) - \phi(V(x) + V(y)) + b1_{\Delta}(x, y). \quad (4)$$

In H2, we can weaken the assumption on $t \mapsto \phi(t)$ by assuming it is concave increasing and continuously differentiable only for large t (say $|t| \geq R_V$). Observe indeed that the function $\tilde{\phi}$ defined by

$$\tilde{\phi}(t) = \begin{cases} (2\phi'(R_V) - \frac{\phi(R_V)}{R_V})t + \frac{2(\phi(R_V) - R_V\phi'(R_V))}{\sqrt{R_V}}\sqrt{t} & \text{for } 0 \leq t < R_V \\ \phi(t) & \text{for } t \geq R_V, \end{cases}$$

is concave increasing and continuously differentiable on $[1, +\infty)$, $\tilde{\phi}(0) = 0$, $\lim_{v \rightarrow \infty} \tilde{\phi}(v) = \infty$ and $\lim_{v \rightarrow \infty} \tilde{\phi}'(v) = 0$. The drift inequality (4) implies that for all $x, y \in E$

$$PV(x) + PV(y) \leq V(x) + V(y) - \tilde{\phi}(V(x) + V(y)) + \tilde{b}1_{\Delta \cup \{V \leq R_V\}^2}(x, y),$$

with $\tilde{b} = b + \sup_{\{(z,t):V(z)+V(t) \leq R_V\}} \left\{ \tilde{\phi}(V(z) + V(t)) - \phi(V(z) + V(t)) \right\}$. Therefore, since $\sup_{(x,y) \in \Delta} V(x) + V(y) < \infty$, the set $\Delta \cup \{V \leq R_V\}^2$ is a coupling set as soon as for any $v > 0$, $\{V \leq v\} \times \{V \leq v\}$ are (ℓ, ϵ, d) -coupling sets; then H2 holds with ϕ replaced with $\tilde{\phi}$.

Examples of functions ϕ satisfying H2 at least for large t are: $t \mapsto t^\gamma, \gamma \in (0, 1)$, $t \mapsto (1 + \log(t))^\alpha, \alpha > 0$, and $t \mapsto t/(1 + \log(t))^\alpha, \alpha > 0$.

Theorem 3 gives sufficient conditions for the existence of a unique invariant probability measure for P .

Theorem 3. Assume H1-H2. Then, P admits a unique invariant probability measure π such that $\pi(\phi \circ V) < \infty$.

Proof: The proof is postponed to subsection 4.3. \square

We now derive expressions of the rate of convergence and the dependence upon the initial condition of the chain. The rate of convergence depends of the concave function ϕ and the integrated subgeometric rate R_ϕ defined as follows (see also [7]). For any nondecreasing concave function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuously differentiable and satisfying $\phi(1) > 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$, set

$$H_\phi(t) = \int_1^t \frac{1}{\phi(s)} ds. \tag{5}$$

Since for $t \geq 1$, $\phi(t) \leq \phi(1) + \phi'(1)(t-1)$, the function H_ϕ is monotone increasing continuously differentiable, and its inverse, denoted H_ϕ^{-1} , is well defined and is continuously differentiable. Define

$$r_\phi(t) = (H_\phi^{-1})'(t) = \phi(H_\phi^{-1}(t)), \tag{6}$$

$$f_{r_\phi}(t) = r_\phi(0) + \int_0^t r_\phi(s) ds. \tag{7}$$

Theorem 4. Assume H1-H2 and there exists C_r such that for all $t_1, t_2 \in \mathbb{R}_+$,

$$f_{r_\phi}(t_1 t_2) \leq C_r f_{r_\phi}(t_1) f_{r_\phi}(t_2). \tag{8}$$

Let π be the invariant probability of P . There exists a constant C such that for all x in E and all $n \geq 1$,

$$W_d(P^n(x, \cdot), \pi) \leq CV(x) / \phi \circ f_{r_\phi} \{n / \log(f_{r_\phi}(n))\}. \tag{9}$$

Proof: The proof is postponed to subsection 4.4 \square

The condition (8) is satisfied for example with $\phi(t) = t^\gamma, \gamma \in (0, 1)$; and with $\phi(t) = (1 + \log(t))^\alpha, \alpha > 0$. In these cases, the rate of convergence $\phi \circ f_{r_\phi}(n/\log(f_{r_\phi}(n)))$ is equivalent when $n \rightarrow \infty$ to resp. $(n/\log(n))^\tau$, for $\tau = \gamma/(1 - \gamma)$; and to $\log(n)^\alpha$. However (8) is not satisfied when $\phi(t) = t/(1 + \log(t))^\alpha, \alpha > 0$. The following result is valid without any restriction on the rate function f_{r_ϕ} ; when applied to rate functions satisfying (8), the rate given by Theorem 5 is smaller than the rate given by Theorem 4.

Theorem 5. *Assume H1 H2. Let π be the invariant probability of P . For all $\delta \in (0, 1)$, there exists a constant C such that for all $x \in E$ and all $n \geq 1$*

$$W_d(P^n(x, \cdot), \pi) \leq C V(x) / \phi\{f_{r_\phi}^\delta(n)\}. \quad (10)$$

Proof: The proof is postponed to subsection 4.5 \square

In the case $\phi(t) = t/(1 + \log(t))^\alpha, \alpha > 0$, which is not covered by Theorem 4, the rate $\phi(f_{r_\phi}^\delta(n))$ is equivalent when $n \rightarrow \infty$ to $n^{-\tau\alpha\delta} \exp(\delta n^\tau)$ with $\tau = 1/(1 + \alpha)$.

We summarize in Table 1 the rates of convergence obtained from Theorem 4 and Theorem 5 for usual concave functions ϕ .

In practice, it is often easier to establish a drift inequality on E instead of a drift inequality on the product space $E \times E$ as in H2. We show in Proposition 7 that H3 implies H1 and H2.

H3. (a) *There exist*

- a concave increasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuously differentiable on $[1, +\infty)$ and satisfying $\phi(0) = 0, \lim_{x \rightarrow \infty} \phi(x) = \infty$ and $\lim_{x \rightarrow \infty} \phi'(x) = 0$,
- a measurable function $V : E \rightarrow [1, +\infty)$ and a constant $b \geq 0$

such that for all $x \in E$,

$$PV(x) \leq V(x) - \phi \circ V(x) + b. \quad (11)$$

(b) *There exists $v > \phi^{-1}(2b)$ such that $\{V \leq v\} \times \{V \leq v\}$ is a (ℓ, ϵ, d) -coupling set, where $\ell \in \mathbb{N}^*$, $\epsilon \in (0, 1)$ and d is a distance on E , bounded by 1, and topologically equivalent to d_* .*

Remark 6. Here again, we can assume without loss of generality that $t \mapsto \phi(t)$ is concave increasing and continuously differentiable only for large t .

Proposition 7. *Assume H3. Set $\mathcal{C} = \{V \leq v\}$ and $c = 1 - 2b/\phi(v)$. Then, H1 holds with $\Delta = \mathcal{C} \times \mathcal{C}$ and H2 holds with the same function V , $\phi \leftarrow c\phi$ and $b \leftarrow 2b$.*

Proof: The proof of Proposition 7 is postponed to subsection 4.6. \square

In many applications (see e.g. section 3), we are able to prove a stronger assumption than H3-(b), namely: for any $u > 0$, there exist $\ell \geq 1, \epsilon \in (0, 1)$ and a distance d bounded by 1 topologically equivalent to d_* such that $\{V \leq u\} \times \{V \leq u\}$

is a (ℓ, ϵ, d) -coupling set. In this case, we can choose ν arbitrary large which yields a constant c arbitrary close to one.

Our framework and results can be compared to [4] who also addresses the convergence in Wasserstein distance at a subgeometric rate under H3-(a) and the assumption

(B) There exists a distance d on E , bounded by 1, such that (E, d) is a Polish space and

- (i) the level set $\Delta = \{(x, y) : V(x) + V(y) \leq \phi^{-1}(2b)\}$ is d -small for P *i.e.* there exists $\epsilon \in (0, 1)$ such that for all $(x, y) \in \Delta$, $W_d(P(x, \cdot), P(y, \cdot)) \leq (1 - \epsilon)d(x, y)$;
- (ii) for all $x, y \in E$, $W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y)$.

Under these conditions, [4, Theorem 2.1] implies the existence and uniqueness of the stationary distribution π and rates of convergence to stationarity; expressions for these rates are displayed in the last row of Table 1 for various functions ϕ . It can be seen that our results always improve on those of [4].

Let us compare our assumption H3-(b) to (B). According to [20, corollary 5.22], (B) implies that there exists $\epsilon \in (0, 1)$ such that Δ is a $(1, \epsilon, d)$ -coupling set. Thus, [4, Theorem 2.1] only covers coupling sets of order 1; this is a serious restriction since in practical examples this order is most likely to be large (see e.g. the examples in Section 3). Checking H3-(b) is easier than checking (B) since allowing the coupling set to be of any order provides far more flexibility.

When some level set $\{V \leq \nu\}$ is (ℓ, ϵ, ν) -small, *i.e.*, there exist $\ell \in \mathbb{N}^*$, $\epsilon \in (0, 1)$ and a probability measure $\nu \in \mathcal{P}(E)$ such that for any $x \in \{V \leq \nu\}$, $P^\ell(x, \cdot) \geq \epsilon\nu$, then H3-(b) is satisfied with $d = d_0$ the trivial distance. In this case, the distance d in Theorem 4 and Theorem 5 is the trivial metric and W_d is the total variation norm (see (2)). Therefore, our results also provide convergence rates in total variation norm and can be compared to the results reported in [7]. In this paper, it is assumed that P is phi-irreducible, aperiodic, that the drift condition H3-(a) hold and that the level sets $\{V \leq u\}$ are (ℓ, ν, ϵ) -small for some $\ell \in \mathbb{N}^*$, $\epsilon \in (0, 1)$ and a probability ν that may depend upon the level set. Under these assumptions, [7, Proposition 2.5.] shows that for any $x \in E$,

$$\lim_n r_\phi(n) d_{TV}(P^n(x, \cdot), \pi) = 0 .$$

Table 1 displays the rate r_ϕ obtained in [7] (see penultimate row) and the rates given by Theorem 4 and Theorem 5 (see rows 2 and 3): our results are nearly the same as in [7]. Nevertheless, we would like to stress that our conditions apply in a much more general context.

Order of the rates of convergence in	$\phi(x) = x^\gamma$ for $\gamma \in (0, 1)$ set $\tau_* = \gamma/(1 - \gamma)$	$\phi(x) = x/(1 + \log(x))^\alpha$ for $\alpha > 0$ set $\tau_* = 1/(1 + \alpha)$	$\phi(x) = (1 + \log(x))^\alpha$ for $\alpha > 0$
Theorem 4	$(\log(n)/n)^{\tau_*}$		$1/\log^\alpha(n)$
Theorem 5 for all $\delta \in (0, 1)$	$1/n^{\tau_*\delta}$	$n^{n\delta\tau_*} \exp(-\delta n^{\tau_*})$	$1/\log^{\alpha\delta}(n)$
[7]	$1/n^{\tau_*}$	$n^{\alpha\tau_*} \exp(-n^{\tau_*})$	$1/\log^\alpha(n)$
[4] for all $\delta \in (0, 1)$	$1/n^{\delta\tau_*}$	$\exists C > 0$ $n^{\delta\alpha\tau_*} \exp(-\delta C n^{\tau_*})$	$1/\log^{\delta\alpha}(n)$

TABLE 1
Comparison of rates of convergence

3. Application

We illustrate our results by establishing the subgeometric ergodicity in Wasserstein distance of a non linear autoregressive model and a MCMC sampler in an infinite dimensional Hilbert space.

3.1. Non linear autoregressive model

For ease of exposition, we assume in this section that $E = \mathbb{R}^p$ for some $p \in \mathbb{N}^*$. We will denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^p . The metric d_* is defined by $d_*(x, y) = 1 \wedge \|x - y\|$, so that (\mathbb{R}^p, d_*) is a Polish space. We consider a Markov chain $\{X_n, n \in \mathbb{N}\}$ on \mathbb{R}^p , defined by the following non linear autoregressive equation of order 1:

$$X_{n+1} = g(X_n) + \epsilon_{n+1},$$

where

AR1. $\{\epsilon_n, n \in \mathbb{N}\}$ is an independent and identically distributed (i.i.d.) zero-mean \mathbb{R}^p -valued sequence, independent of X_0 , and satisfying

$$\mathbb{E}[\exp(z_0 \|\epsilon_0\|^{\gamma_0})] < +\infty$$

for some $z_0 > 0$ and $\gamma_0 \in (0, 1]$.

AR2. $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a measurable function and for all $R > 0$, there exists $k_R \in [0, 1)$ such that g is k_R -Lipschitz on $B(0, R)$ with respect to $\|\cdot\|$. Furthermore, there exist positive constants r, R_0 , and $\rho \in [0, 2)$, such that

$$\|g(x)\| \leq \|x\|(1 - r\|x\|^{-\rho}) \quad \text{if } \|x\| \geq R_0.$$

A simple example of function g satisfying [AR2](#) is $x \mapsto x \cdot \max(1/2, |1 - 1/\|x\|^\rho|)$ with $\rho \in [0, 2)$.

[Proposition 8](#) (combined with [Remark 6](#)) and [Proposition 9](#) establish [H3](#).

Proposition 8. [[7](#), [Theorem 3.3](#)] Assume [AR1](#) and [AR2](#), and let $\rho > \gamma_0$. There exist $R_V, M \geq R_0, z \in (0, z_0)$ and $c > 0$ such that for all $x \in B(0, R)^c$ the drift

inequality (11) holds with

$$\begin{aligned}\phi(x) &:= cx(1 + \log(x))^{1-\rho/(\gamma_0 \wedge (2-\rho))}, \\ V(x) &:= \exp(z\|x\|^{\gamma_0 \wedge (2-\rho)}), \\ \mathcal{C} &:= \{x \in \mathbb{R}^p, \|x\| \leq M\}.\end{aligned}$$

Proof: The proof of Proposition 8 is along the same lines as [7, Theorem 3.3] and is omitted ¹ \square

Consider the basic coupling (X_1, Y_1) between $P(x, \cdot)$ and $P(y, \cdot)$:

$$X_1 = g(x) + \epsilon_1 \quad \text{and} \quad Y_1 = g(y) + \epsilon_1.$$

It defines a Markov kernel Q on $\mathbb{R}^p \times \mathbb{R}^p$ given, for all $x, y \in E$ and $\Delta \in \mathcal{B}(E \times E)$ by

$$Q((x, y), \Delta) = \mu_{\epsilon_1}(\tau_{x,y}^{-1}(\Delta)), \quad (12)$$

where $\tau_{x,y}(z) = (x + z, y + z)$ for all $x, y, z \in E$ and μ_{ϵ_1} is the law of ϵ_1 . We now check H3-b). Proposition 9 implies that $\{V \leq u\}^2$ is a $(1, \epsilon, d)$ -coupling set for the metric \tilde{d} , which depends on the level set, chosen among the family $(d_\eta)_\eta$ indexed by $\eta > 0$ and defined by

$$d_\eta(x, y) := 1 \wedge \eta^{-1}\|x - y\|.$$

For all $\eta_1, \eta_2 > 0$, d_{η_1} and d_{η_2} are Lipschitz equivalent, *i.e.*, there exists two constants C_1 and C_2 such that for all $x, y \in \mathbb{R}^p$, $C_1 d_{\eta_1}(x, y) \leq d_{\eta_2}(x, y) \leq C_2 d_{\eta_1}(x, y)$. Note, in particular, $d_* = d_1$, so that for all $\eta > 0$, d_η is Lipschitz equivalent to d_* , and therefore topologically equivalent. Finally, note that, by the definition of the Wasserstein metric given by (1), W_{d_*} and W_{d_η} are Lipschitz equivalent for all $\eta > 0$.

Proposition 9. *Assume AR1 and AR2. Let \mathcal{C} be a bounded measurable set. Then there exists $\eta > 0$ and $\epsilon \in (0, 1)$ such that $\mathcal{C} \times \mathcal{C}$ is a $(1, \epsilon, d_\eta)$ -coupling set associated with Q defined in (12).*

Proof: Let $R > 0$ be such that $\mathcal{C} \subset \overline{B}(0, R)$. Let $\eta > \text{diam}(\mathcal{C})$. Then $d_\eta(x, y) = \|x - y\|/\eta$ for any $x, y \in \mathcal{C}$.

Then under AR2, there exists $k_R \in [0, 1)$ such that for any $x, y \in \mathcal{C}$,

$$\begin{aligned}\mathbb{E}[d_\eta(g(x) + \epsilon_1, g(y) + \epsilon_1)] &\leq \eta^{-1}\|g(x) - g(y)\| \wedge 1 \\ &\leq k_R \eta^{-1}\|x - y\| = k_R d_\eta(x, y).\end{aligned} \quad (13)$$

Finally, since AR2 implies that g is 1-Lipschitz on \mathbb{R}^p , (13) shows that $\mathbb{E}[d_\eta(g(x) + \epsilon_1, g(y) + \epsilon_1)] \leq d_\eta(x, y)$ for all $x, y \in \mathbb{R}^p$. \square

¹ We point out that in [7], it is additionally required that the distribution of ϵ_0 has a non-trivial absolutely continuous component which is bounded away from zero in a neighborhood of the origin. However, this condition is only required to establish the ϕ -irreducibility of the Markov chain, which is not needed here.

For all η , W_{d_*} and W_{d_η} are Lipschitz equivalent. Therefore, by application of [Theorem 3](#), [Theorem 5](#) and [Proposition 7](#), we deduce from [Proposition 8](#) and [Proposition 9](#), the following rate of ergodicity in d_* -Wasserstein distance.

Theorem 10. *Assume [AR1](#) and [AR2](#) hold, with $\rho > \gamma_0$. Then P admits an unique invariant probability π and there exist two constants C_1 and C_2 such that for all $x \in E$ and $n \in \mathbb{N}^*$*

$$W_{d_*}(P^n(x, \cdot), \pi) \leq C_1 V(x) \exp(-C_2 n^{\tau_*}) ,$$

where $d_*(x, y) = 1 \wedge \|x - y\|$ and $\tau_* = (\gamma_0 \wedge (2 - \rho))/\rho$.

3.2. The preconditioned Crank Nicolson algorithm

In this section, we consider the preconditioned Crank-Nicolson algorithm introduced in [\[2\]](#) and analysed in [\[11\]](#) for sampling a distribution in a separable Hilbert $(\mathcal{H}, \|\cdot\|)$ having a density $\pi \propto \exp(-g)$ with respect to a zero-mean Gaussian measure γ with covariance operator C ; see [\[3\]](#). This algorithm is studied in [\[11\]](#) under conditions which imply the geometric convergence in Wasserstein distance.

Algorithm 1: preconditioned Crank-Nicolson Algorithm

```

Data:  $\rho \in (0, 1]$ 
Result:  $\{X_n, n \in \mathbb{N}\}$ 
begin
  Initialize  $X_0$ 
  for  $n \geq 0$  do
    Generate  $\Xi \sim \gamma$ , and set  $Z = (\rho X_n + \sqrt{1 - \rho^2} \Xi)$ 
    Generate  $U \sim \mathcal{U}([0, 1])$ 
    if  $U \leq \alpha(X_n, Z) = 1 \wedge \exp(g(X_n) - g(Z))$  then
      |  $X_{n+1} = Z$ 
    else
      |  $X_{n+1} = X_n$ 
  end for

```

We consider the convergence of the Crank-Nicolson scheme when the function g satisfies the following conditions:

CN1. *The function $g : \mathcal{H} \rightarrow \mathbb{R}$ is β -Hölder for some $\beta \in (0, 1]$ i.e., there exists C_g , such that for all $x, y \in \mathcal{H}$, $|g(x) - g(y)| \leq C_g \|x - y\|^\beta$.*

Note that under **CN1**, $\exp(-g)$ is γ -integrable (see [Proposition 24](#) in [section 5](#)). Examples of densities satisfying this assumption are $g(x) = -\|x\|^\beta$ with $\beta \in (0, 1]$. The Crank-Nicolson has been shown to be geometrically ergodic by [\[11\]](#) under the assumptions that g is globally Lipschitz and that there exist positive constants C, R_1, R_2 such that for $x \in \mathcal{H}$ with $\|x\| \geq R_1$

$$\inf_{z \in \overline{B}(\rho x, R_2)} \exp(g(x) - g(z)) \geq C ;$$

see [11, Assumption 2.10-2.11]. Such an assumption implies that the acceptance ratio $\alpha(x, \rho x + \sqrt{1 - \rho^2} \xi)$ is bounded from below as $x \rightarrow \infty$ uniformly on the ball $\xi \in \overline{B}(0, R_2/\sqrt{1 - \rho^2})$. In **CN1**, this condition is weakened in order to address situations when the acceptance-rejection ratio vanishes when $\|x\| \rightarrow \infty$: this happens when $\lim_{\|x\| \rightarrow +\infty} g(\rho x) - g(x) = +\infty$.

In the following, we prove that the conditions of **H3** are satisfied.

Proposition 11. *Assume **CN1**, and let $\rho \in [0, 1)$. Set $V(x) = \exp(\|x\|)$. Then there exist $c \in (0, 1)$, $\kappa > 0$, $b, u \in \mathbb{R}_+$ such that for all $x \in \mathcal{H}$*

$$PV(x) \leq V(x) - \phi \circ V(x) + b1_{\{V \leq u\}}(x),$$

where ϕ satisfies the condition **H3-(a)** and $\phi(t) = ct \exp(-\kappa \log(t)^\beta)$ for large enough t .

Proof: The proof is postponed to [subsection 5.1](#). \square

We now deal with showing **H3-(b)**. To that goal, we introduce the distance

$$d_\tau(x, y) = 1 \wedge \tau^{-1} \|x - y\|^\beta, \quad (14)$$

for any $\tau > 0$ and for $x, y \in E$, the basic coupling between $P(x, \cdot)$ and $P(y, \cdot)$: the same Gaussian variable Ξ and the same uniform variable U are generated to build X_1 and Y_1 , with initial conditions x, y . It defines a Markov kernel Q_{pCN} on $E \times E$. Define $\Lambda_{(x,y)}^\rho(z) = (\rho x + \sqrt{1 - \rho^2} z, \rho y + \sqrt{1 - \rho^2} z)$ and $\tilde{\gamma}$ the pushforward of γ by $\Lambda_{(x,y)}^\rho$. Then an explicit form of Q_{pCN} is given by the following expression:

$$\begin{aligned} Q_{\text{pCN}}((x, y), \Delta) &= \int_{\Delta} \alpha(x, z) \wedge \alpha(y, t) d\tilde{\gamma}(z, t) \\ &+ \int_{\mathcal{H} \times \mathcal{H}} (\alpha(y, t) - \alpha(x, z))_+ 1_{\Delta}(x, t) d\tilde{\gamma}(z, t) \\ &+ \int_{\mathcal{H} \times \mathcal{H}} (\alpha(x, z) - \alpha(y, t))_+ 1_{\Delta}(z, y) d\tilde{\gamma}(z, t) \\ &+ \delta_{(x,y)}(\Delta) \int_{\mathcal{H} \times \mathcal{H}} (1 - \alpha(x, z) \vee \alpha(y, t)) d\tilde{\gamma}(z, t) \end{aligned} \quad (15)$$

where for $r \in \mathbb{R}$, $(r)_+ = \max(r, 0)$. In [Proposition 12](#), we prove that there exists $\tau > 0$ such that for any level set $\mathcal{C} = \{V \leq u\}$, $\mathcal{C} \times \mathcal{C}$ is a (ℓ, ϵ, d_τ) coupling set for some $\ell \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$ (the coupling may chosen to be Q_{pCN}), showing **H3-(b)**. Note that for all $\tau > 0$, d_τ is topologically equivalent to $\|\cdot\|$.

Proposition 12. *Assume **CN1**. Set $V(x) = \exp(\|x\|)$. Let $\tau > 0$ be given by [Lemma 25](#). For every $u > 1$, there exist $\ell \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$ such that $\{V \leq u\}^2$ is (ℓ, ϵ, d_τ) -coupling set.*

Proof: See [subsection 5.2](#) \square

As a consequence of [Proposition 11](#), [Proposition 12](#) and [Theorem 5](#), [Proposition 7](#), we have

Theorem 13. *Let P be the kernel of the preconditioned Crank-Nicolson algorithm with target density $d\pi \propto \exp(-g)d\gamma$ and design parameter $\rho \in (0, 1]$. Assume [CN1](#). Then P admits π as an unique invariant probability measure and for $\tau > 0$ sufficiently small and $\delta \in (0, 1)$, there exists C_δ such that for all $n \in \mathbb{N}^*$ and $x \in \mathcal{H}$*

$$W_{d_\tau}(P^n(x, \cdot), \pi) \leq \frac{C_1 \exp(\|x\|)}{\phi(f_{r_\phi}(n)^\delta)}$$

with $\phi(t) = ct \exp(-\kappa \log(t)^\beta)$ for large enough t and $\kappa > 0$, r_ϕ, f_{r_ϕ} are given by [\(6\)](#) and [\(45\)](#) and $d_\tau(x, y) = \tau^{-1} \|x - y\|^\beta \wedge 1$.

We did not find an analytic expression of the rate of convergence in [Theorem 13](#). But it is clear that $t^a \underset{+\infty}{=} o(\phi(t))$ for $a \in (0, 1)$, and $\phi(t) \underset{+\infty}{=} o(t/(1+\log(t))^a)$ for $a \in (0, +\infty)$. Therefore, the rate of convergence given by [Theorem 13](#) is between the polynomial case and the subexponential one ; see [Table 1](#) for details.

4. Proofs of [section 2](#)

Before proceeding to the actual derivation of the proof, we present the roadmap of the proofs. The key step is given by [Lemma 19](#) which provides an explicit expression of $B(n, m)$ such that for any $x, y \in E$

$$W_d(P^n(x, \cdot), P^m(y, \cdot)) \leq B(n, m) (V(x) + V(y)) . \quad (16)$$

First, this inequality will imply that P admits at most one invariant probability. By applying [\(16\)](#) with $n \leftarrow n + m$, and $y \leftarrow x$, we then show that $(P^n(x, \cdot))_{n \geq 0}$ is a Cauchy sequence in $(\mathcal{P}(E), W_d)$ and therefore converges in W_d to some probability measure π_x which may be shown to be invariant for P . Using that P admits one invariant probability measure will imply that π_x does not depend on x , (see [subsection 4.3](#)) .

The proof of [Theorem 4](#) and [Theorem 5](#) also follow from [\(16\)](#) this time taking $n = m$, and integrating this inequality w.r.t. the unique invariant distribution π . The only difficulties to be dealt with stem from the fact that the right hand side of the inequality is not integrable; a truncation is therefore required to conclude the proof.

Let us now explain the computation of the upper bound [\(16\)](#). Let Q be the coupling kernel under which Δ is a (ℓ, ϵ, d) -coupling set. Note that this implies that for any $n \in \mathbb{N}^*$ and $x, y \in E$, $Q^n((x, y), \cdot)$ is a coupling of $(P^n(x, \cdot), P^n(y, \cdot))$. Therefore, by [\(1\)](#),

$$W_d(P^n(x, \cdot), P^n(y, \cdot)) \leq \widetilde{\mathbb{E}}_{x, y} [d(X_n, Y_n)]$$

where $((X_n, Y_n), n \geq 0)$ is a Markov chain on the product space $E \times E$ with Markov kernel Q and $\widetilde{\mathbb{E}}_{x, y}$ is the associated canonical expectation when the initial distribution is the Dirac mass at (x, y) .

The contraction property of Q (see Definition 2)

$$\tilde{\mathbb{E}}_{x,y}[d(X_1, Y_1)] \leq d(x, y) \quad \text{for all } (x, y) \in E \times E, \quad (17)$$

combined with the Markov property of $((X_n, Y_n), n \geq 0)$ imply that $(d(X_n, Y_n), n \geq 0)$ is a supermartingale with respect to the filtration $\tilde{\mathcal{F}}_n = \sigma(X_0, Y_0, \dots, X_n, Y_n)$.

The next step of the proof is to show that this supermartingale property implies that for any $n, m \geq 1$, (see Proposition 17)

$$\tilde{\mathbb{E}}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \tilde{\mathbb{P}}_{x,y}[T_m \geq n]$$

where $(T_m, m \geq 1)$ are the successive return times to Δ . More precisely, set $\tau_\Delta = \inf \{n > 0 | (X_n, Y_n) \in \Delta\}$, $T_0 = \tau_\Delta \circ \theta^{\ell-1} + \ell - 1$ where ℓ is given by H1; and for any $j \geq 1$, define the successive return-times to Δ after $\ell - 1$ steps by

$$T_j = \tau_\Delta \circ \theta^{T_{j-1} + \ell - 1} + T_{j-1} + \ell - 1, \quad (18)$$

where θ is the shift operator.

By the Markov inequality, for any increasing rate function R , it holds

$$\tilde{\mathbb{E}}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \frac{\tilde{\mathbb{E}}_{x,y}[R(T_m)]}{R(n)}. \quad (19)$$

The last step of the proof is to compute an upper bound for the moment $\tilde{\mathbb{E}}_{x,y}[R(T_m)]$. Then m is chosen in order to balance the two terms in the RHS of (19).

To get precise estimate of subgeometric moments of the return times, we introduce, similarly to [18] a sequence of drift conditions; in our setting, it is convenient to formulate this condition on the product space $E \times E$.

H4. *There exist*

- a sequence of measurable functions $(\mathcal{V}_n)_{n \geq 0}$, $\mathcal{V}_n : E \times E \rightarrow \mathbb{R}_+$,
- a set $\Delta \in \mathcal{B}(E \times E)$, a constant $b < \infty$ and a sequence $r \in \Lambda$

such that for all $x, y \in E$ and for every coupling $\alpha \in \mathcal{C}(P(x, \cdot), P(y, \cdot))$:

$$\int_{E \times E} \mathcal{V}_{n+1}(z, t) d\alpha(z, t) \leq \mathcal{V}_n(x, y) - r(n) + br(n)1_\Delta(x, y). \quad (20)$$

Moreover, there exist measurable functions $(V_n)_{n \geq 0}$, $V_n : E \rightarrow \mathbb{R}_+$ such that for all $x, y \in E$ and any $n \geq 0$:

$$\mathcal{V}_n(x, y) \leq V_n(x) + V_n(y) \quad \text{and} \quad PV_{n+1}(x) \leq V_n(x) + br(n). \quad (21)$$

Finally, for all $k \geq 0$,

$$\sup_{(x,y) \in \Delta} \{P^k V_0(x) + P^k V_0(y)\} < +\infty \quad \text{and, for all } x \in E, P^k V_0(x) < +\infty. \quad (22)$$

Under [H4](#), we will get some bounds on the moments $\tilde{\mathbb{E}}_{x,y}[R(T_0)]$ for $x, y \in E$ (see [Proposition 17](#)), where

$$R(n) = \sum_{k=0}^{n-1} r(k) \text{ for } n \geq 1 \quad R(0) = 1. \quad (23)$$

We will then distinguish two cases: these bounds on $R(T_0)$ will provide bounds on the moments $R(T_m/m)$ and $R(T_m)$. To that goal, in the first case R is approximated by a convex function; while in the second case R is approximated by some geometric sequence. This second approach, despite it provides a tight bound when the sequence $(R(n))_n$ is of subexponential order $\exp(cn^\alpha)$, for $c > 0$ and $\alpha \in (0, 1)$, is not appropriate when the sequence is of polynomial or logarithmic order. This is the reason why our convergence results will always be split into two parts (one applicable to polynomial or logarithmic sequences and the other to truly subgeometric sequences). The above discussion is formalized in [Lemma 18](#).

Finally, in [Proposition 22](#), we check that [H4](#) is implied by [H2](#).

4.1. Convergence results under a sequence of drift conditions

Proposition 14. *Assume [H1](#). Then, for all $x, y \in E$, and $n, m \in \mathbb{N}$, $m \geq 1$:*

$$\tilde{\mathbb{E}}_{x,y}[d(X_n, Y_n)] \leq (1 - \epsilon)^{m-1} + \tilde{\mathbb{P}}_{x,y}[T_m \geq n]. \quad (24)$$

We preface the proof by stating the following Lemma, which is a restatement of [[13](#), lemma 3.1].

Lemma 15. *Let $(Z_n)_{n \geq 0}$ be a nonnegative \mathcal{F}_n -supermartingale upper bounded by K . Let $(\tau_n)_n$ be a sequence of increasing stopping times with respect to \mathcal{F}_n , with $\tau_0 = 0$. Assume there exists $\epsilon \in (0, 1)$ such that for every $n \geq 1$ $\mathbb{E}[Z_{\tau_{n+1}} | \mathcal{F}_{\tau_n}] \leq (1 - \epsilon)Z_{\tau_n}$. Then, for all $n, m \in \mathbb{N}$, $m \geq 1$,*

$$\mathbb{E}[Z_n] \leq K((1 - \epsilon)^{m-1} + \mathbb{P}[\tau_m \geq n]).$$

Proof of [Proposition 14](#): Set $Z_n = d(X_n, Y_n)$; under [H1](#), $\{(Z_n, \tilde{\mathcal{F}}_n)\}_{n \geq 0}$ is a bounded non-negative supermartingale and for all $(x, y) \in \Delta$, $\tilde{\mathbb{E}}_{x,y}[Z_\ell] \leq (1 - \epsilon)d(x, y)$. Denote by Z_∞ its $\tilde{\mathbb{P}}_{x,y}$ -a.s limit. By the optional stopping theorem, we have for every $m \geq 0$:

$$\tilde{\mathbb{E}}_{x,y} \left[Z_{T_{m+1}} \Big| \tilde{\mathcal{F}}_{T_m+\ell} \right] \leq Z_{T_m+\ell}. \quad (25)$$

On the other hand, the strong Markov property imply for every $m \geq 0$

$$\tilde{\mathbb{E}}_{x,y} \left[Z_{T_m+\ell} \Big| \tilde{\mathcal{F}}_{T_m} \right] \leq (1 - \epsilon)Z_{T_m}. \quad (26)$$

By (25) and (26), it yields $\tilde{\mathbb{E}}_{x,y} \left[Z_{T_{m+1}} \Big| \tilde{\mathcal{F}}_{T_m} \right] \leq (1 - \epsilon)Z_{T_m}$. Under [H1](#), Z_n is upper bounded by 1 and the proof follows from [Lemma 15](#). \square

To get an estimate of $\tilde{\mathbb{P}}_{x,y}[T_m \geq n]$ for $x, y \in E$ and $n, m \in \mathbb{N}$, we derive from [H4](#) some bound on $\tilde{\mathbb{E}}_{x,y}[R(T_0)]$, where R is given by [\(23\)](#).

Lemma 16. *Assume [H4](#) holds. Then, for all $x, y \in E$ and all $k \geq 0$*

$$\sup_{(x,y) \in \Delta} Q^k \mathcal{V}_0(x, y) < +\infty \quad Q^k \mathcal{V}_0(x, y) < +\infty, \quad .$$

Proof: By [\(21\)](#) and definition of Q , $Q^k \mathcal{V}_0(x, y) \leq P^k V_0(x) + P^k V_0(y)$, for all $k \geq 0$. Eq. [\(22\)](#) concludes the proof. \square

Proposition 17. *Assume [H4](#) holds. Let R be the sequence defined by [\(23\)](#). Then,*

$$\tilde{\mathbb{E}}_{x,y}[R(\tau_\Delta)] \leq \begin{cases} \mathcal{V}_0(x, y), & (x, y) \notin \Delta \\ r(0) + cQ\mathcal{V}_0(x, y), & (x, y) \in E \times E. \end{cases} \quad (27)$$

where $c = \sup_{k \in \mathbb{N}}(r(k+1)/r(k))$ is finite, and

$$\sup_{(x,y) \in \Delta} \tilde{\mathbb{E}}_{x,y}[R(T_0)] < +\infty. \quad (28)$$

In addition, for all $j \geq 0$ and $(x, y) \in E \times E$,

$$\tilde{\mathbb{P}}_{x,y}[T_j < \infty] = 1. \quad (29)$$

Proof: Since $r \in \Lambda$, [Lemma 36](#) shows that the constant c is finite. [\(27\)](#) follows from [[15](#), proposition 11.3.3]. The second statement follows from [\(27\)](#), [Lemma 16](#) and the Markov property. Finally, [\(28\)](#) shows that for any x, y , $\tilde{\mathbb{P}}_{x,y}(T_0 < \infty) = 1$; [\(29\)](#) now follows by a straightforward induction. \square

Lemma 18. *Assume [H1](#) and [H4](#). Let R be the sequence defined by [\(23\)](#). Then,*

- There exists a constant C such that for all $x, y \in E$ and for all $n, m \in \mathbb{N}$,

$$\tilde{\mathbb{P}}_{x,y}[T_m \geq n] \leq \frac{C}{R(\lfloor n/(m+1) \rfloor)} (1 + P^\ell V_0(x) + P^\ell V_0(y)). \quad (30)$$

- For all $\alpha > 0$, there exists a constant C_α satisfying for all $x, y \in E$ and for all $n, m \in \mathbb{N}$,

$$\tilde{\mathbb{P}}_{x,y}[T_m \geq n] \leq \frac{C_\alpha}{R(n)} (1 + P^\ell V_0(x) + P^\ell V_0(y)) (1 + \alpha)^m. \quad (31)$$

Proof: Set $C_\Delta = \sup_{(z,t) \in \Delta} \tilde{\mathbb{E}}_{z,t}[R(T_0)]$, finite by [Proposition 17](#). We first establish [\(30\)](#). Let ψ_r be the increasing convex function given by [Lemma 37](#) such that there exist positive constants C_i , $i \in \{1, 2\}$, for every $n \in \mathbb{N}^*$,

$$C_1 \psi_r(n) \leq R(n) \leq C_2 \psi_r(n). \quad (32)$$

By the Markov inequality, since ψ_r is increasing,

$$\begin{aligned} \tilde{\mathbb{P}}_{x,y}[T_m \geq n] &\leq \psi_r(n/(m+1))^{-1} \tilde{\mathbb{E}}_{x,y}[\psi_r(T_m/(m+1))] \\ &\leq C_2 R(\lfloor n/(m+1) \rfloor)^{-1} \tilde{\mathbb{E}}_{x,y}[\psi_r(T_m/(m+1))] . \end{aligned} \quad (33)$$

By construction,

$$T_m = T_0 + \sum_{k=0}^{m-1} \{ \tau_\Delta \circ \theta^{T_k + \ell - 1} + \ell - 1 \} ,$$

with the convention that $\sum_{k=0}^{-1} = 0$. Since ψ_r is convex it follows from (32), that

$$\begin{aligned} & \tilde{\mathbb{E}}_{x,y} [\psi_r(T_m/(m+1))] \\ & \leq \tilde{\mathbb{E}}_{x,y} \left[\frac{1}{m+1} \left(\psi_r(T_0) + \sum_{k=0}^{m-1} \psi_r(\tau_\Delta \circ \theta^{T_k + \ell - 1} + \ell - 1) \right) \right] \\ & \leq \frac{1}{C_1(m+1)} \tilde{\mathbb{E}}_{x,y} \left[\left(R(T_0) + \sum_{k=0}^{m-1} R(\tau_\Delta \circ \theta^{T_k + \ell - 1} + \ell - 1) \right) \right] . \end{aligned}$$

Using Proposition 17 and the strong Markov property, there exists $C > 0$ such that for any $x, y \in E$ and $m \geq 0$,

$$\tilde{\mathbb{E}}_{x,y} [\psi_r(T_m/(m+1))] \leq \frac{C}{C_1(m+1)} (Q^\ell \mathcal{V}_0(x, y) + mC_\Delta + 1) . \quad (34)$$

It remains to use (21) and plug (34) in (33) to get the first upper bound.

We now consider (31). Again by the Markov inequality, since R is increasing,

$$\tilde{\mathbb{P}}_{x,y} [T_m \geq n] \leq R^{-1}(n) \tilde{\mathbb{E}}_{x,y} [R(T_m)] . \quad (35)$$

If $m = 0$, the result follows from Proposition 17. If $m \geq 1$, using the definitions of T_m and R , given respectively in (18) and (23), and the assertion Lemma 36-(iv)

$$\tilde{\mathbb{E}}_{x,y} [R(T_m)] \leq \tilde{\mathbb{E}}_{x,y} [R(T_{m-1})] + C_1 \tilde{\mathbb{E}}_{x,y} [r(T_{m-1}) R(\tau_\Delta \circ \theta^{T_{m-1} + \ell - 1} + \ell - 1)] ,$$

for a constant $C_1 > 0$. Thus, by the strong Markov property

$$\tilde{\mathbb{E}}_{x,y} [R(T_m)] \leq \tilde{\mathbb{E}}_{x,y} [R(T_{m-1})] + C_2 \tilde{\mathbb{E}}_{x,y} [r(T_{m-1})] , \quad (36)$$

where $C_2 = C_1 C_\Delta$. Let $\alpha > 0$. According to Lemma 35-(iv), there exists N_α such that for any $n \geq N_\alpha$, $r(n) \leq \alpha R(n)$. Then

$$\tilde{\mathbb{E}}_{x,y} [r(T_{m-1})] \leq r(N_\alpha) + \alpha \tilde{\mathbb{E}}_{x,y} [R(T_{m-1})] ,$$

so that (36) becomes

$$\tilde{\mathbb{E}}_{x,y} [R(T_m)] \leq (1 + C_2 \alpha) \tilde{\mathbb{E}}_{x,y} [R(T_{m-1})] + C_2 r(N_\alpha) .$$

By a straightforward induction and definition of N_α , we get,

$$\tilde{\mathbb{E}}_{x,y} [R(T_m)] \leq C_\alpha (1 + C_2 \alpha)^m (\tilde{\mathbb{E}}_{x,y} [R(T_0)] + 1) ,$$

for some constant $C_\alpha > 0$. Plugging this result in (35) and using Proposition 17 concludes the proof. \square

Lemma 19. Assume [H1](#) and [H4](#). Let R be the sequence defined by [\(23\)](#). Then,

- There exists a constant C such that for all $x, y \in E$, all $n, m \in \mathbb{N}$,

$$W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq C \frac{1 + P^\ell V_0(x) + P^{m+\ell} V_0(y)}{R(\lfloor -n \log(1 - \epsilon) / \log(R(n)) \rfloor)}. \quad (37)$$

- For all $\delta \in (0, 1)$, there exists a constant C_δ such that for all $x, y \in E$ and $n, m \in \mathbb{N}$,

$$W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq C_\delta \frac{1 + P^\ell V_0(x) + P^{m+\ell} V_0(y)}{R^\delta(n)}. \quad (38)$$

Proof: We first establish [\(37\)](#). [Lemma 28](#) implies

$$W_d(P^n(x, \cdot), P^{n+m}(y, \cdot)) \leq \inf_{\alpha \in \mathcal{C}(\delta_x, \delta_y, P^m)} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot)) d\alpha(z, t).$$

Since $Q((z, t), \cdot)$ is a coupling of $(P(z, \cdot), P(t, \cdot))$ then for any $n \geq 1$, $Q^n((z, t), \cdot)$ is a coupling of $(P^n(z, \cdot), P^n(t, \cdot))$. Therefore,

$$W_d(P^n(z, \cdot), P^n(t, \cdot)) \leq \tilde{\mathbb{E}}_{z,t} [d(X_n, Y_n)].$$

The next step is to upper bound the RHS. By [Proposition 14](#) and [Lemma 18-\(30\)](#), there exists C such that for all x, y in E and for all $n \geq 0$ and $m \geq 1$

$$\begin{aligned} \tilde{\mathbb{E}}_{x,y} [d(X_n, Y_n)] &\leq (1 - \epsilon)^{m-1} + \tilde{\mathbb{P}}_{x,y} [T_m \geq n] \\ &\leq (1 - \epsilon)^{m-1} + C \frac{1 + P^\ell V_0(x) + P^\ell V_0(y)}{R(\lfloor n/(m+1) \rfloor)}. \end{aligned}$$

Using this inequality with $m = \lfloor -\log(R(n))/\log(1 - \epsilon) \rfloor - 1$ and since R is increasing, there exists a constant C_1 such that for all $z, t \in E$,

$$\tilde{\mathbb{E}}_{z,t} [d(X_n, Y_n)] \leq C_1 \frac{1 + P^\ell V_0(z) + P^\ell V_0(t)}{R(\lfloor -n \log(1 - \epsilon) / \log(R(n)) \rfloor)}. \quad (39)$$

The result now follows easily.

The proof of [\(38\)](#) is along the same lines, using [Lemma 18-\(31\)](#) instead of [Lemma 18-\(30\)](#). In this case, for some fixed $\delta \in (0, 1)$, we choose m such that $(1 - \epsilon)^{m-1} = R^{-\delta}(n)$; and in [Lemma 18-\(31\)](#), we choose $\alpha > 0$ such that

$$\log(1 + \alpha) \leq -\frac{1 - \delta}{\delta} \log(1 - \epsilon).$$

□

Proposition 20. Assume [H1](#) and [H4](#) hold. Then P admits at most one invariant probability measure.

Proof: Under [H1](#), $(\mathcal{P}(E), W_d)$ is a Polish space, and W_d is continuous on $\mathcal{P}(E) \times \mathcal{P}(E)$; see [\[20, Theorem 6.16\]](#). Therefore, $(x, y) \mapsto W_d(P^n(x, \cdot), P^n(y, \cdot))$ is measurable.

Assume that there exist two invariant distributions π and ν , and let α be a coupling of π and ν . According to [Lemma 28](#), we have for every integer n :

$$W_d(\pi, \nu) = W_d(\pi P^n, \nu P^n) \leq \int_{E \times E} W_d(P^n(x, \cdot), P^n(y, \cdot)) \alpha(dx, dy) .$$

By [\(37\)](#), there exists a constant C such that for all $x, y \in E$ and $n \geq 0$,

$$g_n(x, y) \stackrel{\text{def}}{=} W_d(P^n(x, \cdot), P^n(y, \cdot)) \leq C \frac{1 + P^\ell V_0(x) + P^\ell V_0(y)}{R(\lfloor -n \log(1 - \epsilon) / \log(R(n)) \rfloor)} . \quad (40)$$

Since $r \in \Lambda$, [Lemma 36-\(ii\)](#) and [\(v\)](#) shows that

$$\lim_{n \rightarrow +\infty} R(\lfloor -n \log(1 - \epsilon) / \log(R(n)) \rfloor) = +\infty .$$

Eq. [\(40\)](#) shows that the sequence of functions $\{g_n, n \in \mathbb{N}\}$ converges pointwise to 0 and is bounded by 1 since, by assumption, the distance d is bounded by one. Therefore, by the Lebesgue dominated convergence theorem, we have:

$$\int_{E \times E} W_d(P^n(x, \cdot), P^n(y, \cdot)) \alpha(dx, dy) \xrightarrow{n \rightarrow +\infty} 0 ,$$

showing that $W_d(\pi, \nu) = 0$, or equivalently $\nu = \pi$ since W_d is a distance on $\mathcal{P}(E)$.

□

4.2. From the drift condition [H2](#) to the sequence of drifts [H4](#)

Throughout this section, [H2](#) is assumed to hold. Define for $k \geq 0$, $H_k : [1, \infty) \rightarrow \mathbb{R}^+$ and $\mathcal{V}_k : E \times E \rightarrow \mathbb{R}^+$ by

$$H_k(x) = \int_0^{H_\phi(x)} r_\phi(t + k) dt = H_\phi^{-1}(H_\phi(x) + k) - H_\phi^{-1}(k) , \quad (41)$$

$$\mathcal{V}_k(x, y) = H_k(V(x) + V(y)) , \quad (42)$$

where H_ϕ and r_ϕ are respectively given by [\(5\)](#) and [\(6\)](#). Note that $H_0(x) \leq x$ so $\mathcal{V}_0(x, y) \leq V(x) + V(y)$. The proof of the following lemma is adapted from [\[7, Proposition 2.1\]](#).

Lemma 21. *Under the condition [H2](#), for all $x, y \in E$ and every coupling $\alpha \in \mathcal{C}(P(x, \cdot), P(y, \cdot))$, we have:*

$$\int_{E \times E} \mathcal{V}_{k+1}(z, t) d\alpha(z, t) \leq \mathcal{V}_k(x, y) - r_\phi(k) + b \frac{r_\phi(k+1)}{r_\phi(0)} 1_\Delta(x, y) ,$$

where r_ϕ and \mathcal{V}_k are defined in [\(6\)](#) and [\(42\)](#) respectively.

Proof: Set $\mathcal{V}(x, y) = V(x) + V(y)$. First, note H_k that is twice continuously differentiable on $[1, \infty)$ and concave for all $k \geq 0$ (see [Lemma 32](#) and [Proposition 33\(ii\)](#)). This implies that for all $u \geq 1$ and $x \in \mathbb{R}$ such that $x + u \geq 1$, we have

$$H_{k+1}(u+x) - H_{k+1}(u) \leq H'_{k+1}(u)x. \quad (43)$$

In addition, according to the proof of [\[7, Proposition 2.1\]](#): for every $u \geq 1$

$$H_{k+1}(u) - \phi(u)H'_{k+1}(u) \leq H_k(u) - r_\phi(k). \quad (44)$$

Therefore, since H_{k+1} is concave, the Jensen inequality and [\(4\)](#) imply

$$\begin{aligned} \int_{E \times E} \mathcal{V}_{k+1}(z, t) d\alpha(z, t) &\leq H_{k+1} \left(\int_{E \times E} \mathcal{V}(z, t) d\alpha(z, t) \right) \\ &\leq H_{k+1}(\mathcal{V}(x, y) - \phi(\mathcal{V}(x, y)) + b1_\Delta(x, y)). \end{aligned}$$

Using [\(43\)](#), [\(44\)](#) and the inequality $H'_{k+1}(\mathcal{V}(x, y)) \leq H'_{k+1}(1)$ we get that

$$\begin{aligned} \int_{E \times E} \mathcal{V}_{k+1}(z, t) d\alpha(z, t) &\leq H_{k+1}(\mathcal{V}(x, y)) - \phi(\mathcal{V}(x, y))H'_{k+1}(\mathcal{V}(x, y)) + bH'_{k+1}(1)1_\Delta(x, y) \\ &\leq H_k(\mathcal{V}(x, y)) - r_\phi(k) + bH'_{k+1}(1)1_\Delta(x, y). \end{aligned}$$

The proof is concluded upon noting that $H'_{k+1}(1) = r_\phi(k+1)/r_\phi(0)$. \square

Proposition 22. *Assume [H2](#) and let $x_0 \in E$. Then [H4](#) holds with the same set Δ , $r \leftarrow r_\phi$,*

$$b \leftarrow \left(\frac{b + V(x_0)}{r_\phi(0)} + 1 \right) \sup_{k \geq 1} \frac{r_\phi(k+1)}{r_\phi(k)},$$

$\mathcal{V}_n(x, y) \leftarrow H_n(V(x) + V(y))$ and $V_n \leftarrow H_n \circ V + r_\phi(n)$ where H_n is given by [\(41\)](#).

Proof: Since H_k is twice continuously differentiable and V is measurable, \mathcal{V}_n is measurable for all $n \in \mathbb{N}$. By [Lemma 21](#),

$$\int_{E \times E} \mathcal{V}_{k+1}(z, t) d\alpha(z, t) \leq \mathcal{V}_k(x, y) - r_\phi(k) + b \frac{r_\phi(k+1)}{r_\phi(0)} 1_\Delta(x, y).$$

By [Proposition 33\(i\)](#), $r_\phi \in \Lambda$ and [Lemma 36\(iv\)](#) shows that there exists a constant C such that $\sup_k r_\phi(k+1)/r_\phi(k) \leq C$. Therefore

$$\int_{E \times E} \mathcal{V}_{k+1}(z, t) d\alpha(z, t) \leq \mathcal{V}_k(x, y) - r_\phi(k) + \frac{bC}{r_\phi(0)} r_\phi(k) 1_\Delta(x, y).$$

By [Lemma 34\(ii\)](#), for any $k \geq 0$,

$$\mathcal{V}_k(x, y) \leq H_k(V(x)) + H_k(V(y)) + 2r_\phi(k) = V_k(x) + V_k(y).$$

By [Proposition 33-\(iii\)](#), for all $x \in E$ and $k \geq 0$,

$$\begin{aligned} PV_{k+1}(x) &\leq V_k(x) - 2r_\phi(k) + \frac{(b + V(x_0))r_\phi(k+1)}{r_\phi(0)} + r_\phi(k+1) \\ &\leq V_k(x) + r_\phi(k) \left(\frac{b + V(x_0)}{r_\phi(0)} C + C - 2 \right). \end{aligned}$$

Finally $V_0(x) \leq V(x) + r_\phi(0)$, and by [\(4\)](#), $P^k V(x) + P^k V(y) \leq V(x) + V(y) + kb$ for $k \in \mathbb{N}$. Therefore under [H2](#), $\sup_{(x,y) \in \Delta} (P^k V(x) + P^k V(y)) < +\infty$ for all $k \in \mathbb{N}$; and $P^k V_0(x) < \infty$ for any $x \in E$ and $k \in \mathbb{N}$. \square

By [Proposition 22](#), [H2](#) implies [H4](#) with $V_0 \leq V + r_\phi(0)$ and $r \leftarrow r_\phi$, where r_ϕ is given by [\(6\)](#). Thus [Lemma 19](#) and the results of [subsection 4.1](#) apply with $R \leftarrow R_\phi$ where

$$R_\phi(n) = \sum_{k=0}^{n-1} r_\phi(k) \text{ for } n \geq 1 \quad \text{and } R_\phi(0) = 1. \quad (45)$$

Note that by iterating the drift inequality [\(4\)](#) applied with $x = y$, it holds

$$P^\ell V(x) \leq V(x) + \frac{b}{2}\ell, \quad \forall \ell \geq 1, \forall x \in E. \quad (46)$$

4.3. Proof of [Theorem 3](#)

By [Proposition 20](#), if an invariant probability measure exists, it is unique. Let us prove such a measure exists.

Let $x_0 \in E$. We first show there exists $(m_k)_k$ such that $(P^{m_k}(x_0, \cdot))_k$ is a Cauchy sequence for W_d . By [H2](#),

$$PV(x) \leq PV(x) + PV(x_0) \leq V(x) + V(x_0) - \phi \circ V(x_0) + b$$

where we used that $\phi(V(x) + V(x_0)) \geq \phi(V(x))$. This implies, by [Lemma 31](#), that $\lim_n n^{-1} \sum_{k=0}^{n-1} P^k(\phi \circ V)(x_0) \leq b + V(x_0)$. Fix $M_\phi > b + V(x_0)$; there exists an increasing sequence $(n_k)_k$ such that $\lim_k n_k = +\infty$ and

$$P^{n_k}(\phi \circ V)(x_0) \leq M_\phi, \text{ for all } k \in \mathbb{N}. \quad (47)$$

Let $n, k \in \mathbb{N}^*$ and choose $M_V > 0$. By [Lemma 28](#):

$$\begin{aligned} &W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot)) \\ &\leq \inf_{\alpha \in \mathcal{C}(\delta_{x_0}, P^{n_k}(x_0, \cdot))} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \\ &\leq \inf_{\alpha \in \mathcal{C}(\delta_{x_0}, P^{n_k}(x_0, \cdot))} \left\{ \int_{E \times E} \mathbf{1}_{\{V(t) \geq M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \right. \\ &\quad \left. + \int_{E \times E} \mathbf{1}_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \right\}. \end{aligned} \quad (48)$$

We consider the two terms in turn. Let $\alpha \in \mathcal{C}(\delta_{x_0}, P^{n_k}(x_0, \cdot))$. Since W_d is bounded by 1,

$$\begin{aligned} \int_{E \times E} 1_{\{V(t) \geq M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) &\leq P^{n_k}(x_0, \{V \geq M_V\}) \\ &\leq P^{n_k}(x_0, \{\phi \circ V \geq \phi(M_V)\}) \leq \frac{P^{n_k}(\phi \circ V)(x_0)}{\phi(M_V)} \leq \frac{M_\phi}{\phi(M_V)}, \end{aligned} \quad (49)$$

where we used (47) and the Markov inequality. In addition, by Lemma 19-(38) applied with $\delta = 1/2$, there exists $C > 0$ such that:

$$\begin{aligned} \int_{E \times E} 1_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \\ &\leq \frac{C}{\sqrt{R_\phi(n)}} \int_{E \times E} 1_{\{V(t) < M_V\}} (P^\ell V(z) + P^\ell V(t)) \alpha(dz, dt) \\ &\leq \frac{C}{\sqrt{R_\phi(n)}} \int_{E \times E} 1_{\{V(t) < M_V\}} (V(z) + V(t) + \ell b) \alpha(dz, dt), \end{aligned}$$

where we used (46) in the last inequality. Furthermore, $x \mapsto \phi(x)/x$ is non-increasing so that $V(t) \leq M_V \phi(V(t))/\phi(M_V)$ on $\{V \leq M_V\}$. This implies

$$\begin{aligned} \int_{E \times E} 1_{\{V(t) < M_V\}} W_d(P^n(z, \cdot), P^n(t, \cdot)) \alpha(dz, dt) \\ &\leq \frac{C}{\sqrt{R_\phi(n)}} \left(V(x_0) + \ell b + \frac{M_\phi M_V}{\phi(M_V)} \right). \end{aligned} \quad (50)$$

Combining (49) and (50) in (48), we have for every $M_V > 0$, $n, k \in \mathbb{N}^*$

$$W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot)) \leq \frac{M_\phi}{\phi(M_V)} + \frac{C}{\sqrt{R_\phi(n)}} \left(V(x_0) + \ell b + \frac{M_\phi M_V}{\phi(M_V)} \right).$$

Setting $M_V = \sqrt{R_\phi(n)}$, this equation shows there exists a constant C' such that for all $n, k \in \mathbb{N}^*$

$$W_d(P^n(x_0, \cdot), P^{n+n_k}(x_0, \cdot)) \leq C' \frac{V(x_0)}{\phi(\sqrt{R_\phi(n)})}. \quad (51)$$

Let us define the sequence $(m_k)_k$. By Lemma 36-(ii), $\lim_{x \rightarrow +\infty} \phi(x) = +\infty$ and by definition, $\lim_{k \rightarrow +\infty} n_k = +\infty$; hence there exists $(u_k)_k$ such that $u_0 = 1$ and

$$u_{k+1} = \inf \left\{ n_l \mid l \in \mathbb{N}; \phi \left(\sqrt{R_\phi(n_l)} \right)^{-1} \leq 2^{-k-1} \right\}. \quad (52)$$

Set $m_k = \sum_{i=0}^k u_i$. Since for all $k \in \mathbb{N}^*$, $m_{k+1} = m_k + u_{k+1}$, by (51) and (52) $W_d(P^{m_k}(x_0, \cdot), P^{m_{k+1}}(x_0, \cdot)) \leq C' 2^{-k} V(x_0)$, which implies that the series $\sum_k W_d(P^{m_k}(x_0, \cdot), P^{m_{k+1}}(x_0, \cdot))$ converges and $(P^{m_k}(x_0, \cdot))_k$ is Cauchy for W_d .

Since under [H1](#), $(\mathcal{P}(E), W_d)$ is Polish, there exists $\pi \in \mathcal{P}(E)$ such that $\lim_{k \rightarrow +\infty} W_d(P^{m_k}(x_0, \cdot), \pi) = 0$. The second step is to prove that π is invariant. As W_d is continuous on $\mathcal{P}(E) \times \mathcal{P}(E)$, $W_d(\pi, \pi P) = \lim_k W_d(P^{m_k}(x_0, \cdot), \pi P)$. By the triangular inequality, it holds

$$W_d(\pi, \pi P) \leq \lim_{k \rightarrow +\infty} W_d(P^{m_k}(x_0, \cdot), \delta_{x_0} P^{m_k} P) + \lim_{k \rightarrow +\infty} W_d(\delta_{x_0} P^{m_k} P, \pi P) . \quad (53)$$

By [Lemma 19-\(38\)](#) and [Lemma 28](#), there exists C such that for any $k \geq 1$,

$$\begin{aligned} W_d(P^{m_k}(x_0, \cdot), \delta_{x_0} P^{m_k+1}) &\leq \inf_{\alpha \in \mathcal{C}(\delta_{x_0}, \delta_{x_0} P)} \int_{E \times E} W_d(P^{m_k}(z, \cdot), P^{m_k}(t, \cdot)) d\alpha(z, t) \\ &\leq \frac{C}{\sqrt{R_\phi(m_k)}} (P^\ell V(x_0) + P^{\ell+1} V(x_0)) . \end{aligned}$$

By [\(46\)](#), $P^\ell V(x_0) + P^{\ell+1} V(x_0)$ is finite. By definition, $\lim_k m_k = +\infty$ so that by [Lemma 36-\(ii\)](#), the RHS converges to 0 when $k \rightarrow +\infty$. In addition, by [Lemma 30](#), $W_d(\delta_{x_0} P^{m_k} P, \pi P) \leq W_d(P^{m_k}(x_0, \cdot), \pi)$, and this RHS converges to 0 by definition of π . Plugging these results in [\(53\)](#) yields $W_d(\pi, \pi P) = 0$, and therefore $\pi P = \pi$.

Finally, [Lemma 31](#) implies that $\pi(\phi \circ V) < \infty$.

4.4. Proof of [Theorem 4](#)

Fix $M_V > 0$ such that $\pi(V \leq M_V) \geq 1/2$; such a constant exists since $\pi(E) = 1$ and $E = \bigcup_{k \in \mathbb{N}} \{V \leq k\}$. Note that $\pi(\{V \leq M\}) \geq 1/2$ for any $M \geq M_V$. Fix $M > M_V$ and denote by π_M the probability in $\mathcal{P}(E)$ defined by $\pi_M(A) = \pi(A \cap \{V \leq M\}) / \pi(\{V \leq M\})$.

Since π is invariant for P , $W_d(P^n(x, \cdot), \pi) = W_d(P^n(x, \cdot), \pi P^n)$ and the triangular inequality implies for all $n \geq 1$:

$$W_d(P^n(x, \cdot), \pi) \leq W_d(P^n(x, \cdot), \pi_M P^n) + W_d(\pi_M P^n, \pi P^n) . \quad (54)$$

Consider the first term in [\(54\)](#). By [Lemma 28](#), for all $x \in E$ and $n \geq 1$:

$$W_d(P^n(x, \cdot), \pi_M P^n) \leq \inf_{\alpha \in \mathcal{C}(\delta_x, \pi_M)} \int_{E \times E} W_d(P^n(z, \cdot), P^n(t, \cdot)) d\alpha(z, t) .$$

By [Lemma 19-\(37\)](#), there exists $C_1 > 0$ such that for all $x \in E$ and $n \geq 1$

$$\begin{aligned} &W_d(P^n(x, \cdot), \pi_M P^n) \\ &\leq \frac{C_1}{R_\phi \left(\left\lfloor \frac{-n \log(1-\epsilon)}{\log(R_\phi(n))} \right\rfloor \right)} \inf_{\alpha \in \mathcal{C}(\delta_x, \pi_M)} \int_{E \times E} (P^\ell V(z) + P^\ell V(t)) d\alpha(z, t) \\ &\leq \frac{C_1}{R_\phi \left(\left\lfloor \frac{-n \log(1-\epsilon)}{\log(R_\phi(n))} \right\rfloor \right)} (V(x) + \pi_M(V) + b\ell) . \end{aligned} \quad (55)$$

where we used (46) in the last inequality. Finally, since $x \mapsto \phi(x)/x$ is non-increasing and $V(t) \leq M\phi(V(t))/\phi(M)$ on $\{V \leq M\}$. It yields:

$$\pi_M(V) \leq \frac{\pi(\phi \circ V)}{\pi(\{V \leq M\})} \frac{M}{\phi(M)} \leq \frac{2\pi(\phi \circ V)M}{\phi(M)}. \quad (56)$$

We deduce from Proposition 33-(i), Lemma 36-(i) and (8), applied twice, that there exists $C_2 > 0$ such that

$$R_\phi(\lceil -n \log(1 - \epsilon) / \log(R_\phi(n)) \rceil) \geq C_2 f_{r_\phi} \{n / \log(f_{r_\phi}(n))\}. \quad (57)$$

Using (56) and (57), (55) becomes:

$$W_d(P^n(x, \cdot), \pi_M P^n) \leq \frac{C_1(\ell b + V(x) + 2\pi(\phi \circ V)M/\phi(M))}{C_2 f_{r_\phi} \{n / \log(f_{r_\phi}(n))\}}. \quad (58)$$

Consider the second term in (54). Since d is bounded by 1, Lemma 30 and Lemma 26 imply $W_d(\pi_M P^n, \pi P^n) \leq W_d(\pi_M, \pi) \leq d_{\text{TV}}(\pi_M, \pi)$. Since by definition of π_M it holds for any measurable set A

$$\begin{aligned} |\pi_M(A) - \pi(A)| &= |\pi_M(A)(1 - \pi(\{V \leq M\})) + \pi_M(A)\pi(V \leq M) - \pi(A)| \\ &\leq \pi(\{V > M\}) + \pi_M(A)\pi(\{V > M\}) \leq 2\pi(\{V > M\}), \end{aligned}$$

then by (2),

$$W_d(\pi_M P^n, \pi P^n) \leq 2\pi(\{V > M\}) = 2\pi(\{\phi(V) > \phi(M)\}) \leq \frac{2\pi(\phi \circ V)}{\phi(M)}. \quad (59)$$

Combining (59) and (58) in (54), we have for all $M > M_V$ and $n \geq 1$,

$$W_d(P^n(x, \cdot), \pi) \leq \frac{C_1(\ell b + V(x) + 2\pi(\phi \circ V)M/\phi(M))}{C_2 f_{r_\phi} \{n / \log(f_{r_\phi}(n))\}} + \frac{2\pi(\phi \circ V)}{\phi(M)}.$$

For all n large enough, we choose $M = f_{r_\phi} \{n / \log(f_{r_\phi}(n))\}$ (note that by Lemma 36-(i)-(ii) and (v), $\lim_{n \rightarrow +\infty} f_{r_\phi} \{n / \log(f_{r_\phi}(n))\} = +\infty$ so that $M > M_V$ for all n large enough). The proof is concluded upon noting that $\lim_{t \rightarrow \infty} \phi(t)/t = 0$.

4.5. Proof of Theorem 5

The proof is along the same lines as the proof of Theorem 4: the upper bound Lemma 19-(38) is used instead of Lemma 19-(37). Details are omitted.

4.6. Proof of Proposition 7

Proof: Note under H3, $c \in (0, 1)$. It is sufficient to prove that for all $x, y \in E$,

$$PV(x) + PV(y) \leq V(x) + V(y) - c\phi(V(x) + V(y)) + 2b1_{\mathcal{C} \times \mathcal{C}}(x, y). \quad (60)$$

By (11),

$$PV(x) + PV(y) \leq V(x) + V(y) - c\phi(V(x) + V(y)) + 2b1_{\mathcal{C} \times \mathcal{C}} + \Omega(x, y)$$

where

$$\Omega(x, y) = c\phi(V(x) + V(y)) - \phi(V(x)) - \phi(V(y)) + 2b\{1_{E \times \mathcal{C}^c}(x, y) + 1_{\mathcal{C}^c \times E}(x, y)\} .$$

Let us show that for all $x, y \in E$, $\Omega(x, y) \leq 0$. Since ϕ is sub-additive, for all $x, y \in E$

$$\Omega(x, y) \leq -(1 - c)(\phi(V(x)) + \phi(V(y))) + 2b\{1_{E \times \mathcal{C}^c}(x, y) + 1_{\mathcal{C}^c \times E}(x, y)\} .$$

On $(E \times \mathcal{C}^c) \cup (\mathcal{C}^c \times E)$, $\phi(V(x)) + \phi(V(y)) \geq \phi(v)$. The definition of c implies that $\Omega(x, y) \leq 0$. Then, (60) holds and the proof of the proposition follows. \square

5. Proofs of section 3

We will use the following results in the proof.

Lemma 23. Assume CN1 and set $r(x) = (1 - \rho)/2^{1/\beta}\|x\|$. Then, for all $x \in \mathcal{H}$,

$$\inf_{z \in \overline{B}(\rho x, r(x))} \exp(g(x) - g(z)) \geq \exp(-C_g(3/2)(1 - \rho)^\beta \|x\|^\beta) . \quad (61)$$

Proposition 24. Let $(\mathcal{H}, \|\cdot\|)$ be a separable Hilbert space and γ be a Gaussian measure on \mathcal{H} .

1. There exist $\theta \in \mathbb{R}_+$ and a constant C_θ such that

$$\int_{\mathcal{H}} \exp(\theta\|\xi\|^2) d\gamma(\xi) \leq C_\theta .$$

2. There exists a constant C_a such that for all $K > a/(2\theta)$,

$$\int_{\|\xi\| \geq K} \exp(a\|\xi\|) d\gamma(\xi) \leq C_a \exp(-\theta K^2 + aK) .$$

Proof: (1) is Fernique's theorem; see [3, Theorem 2.8.5].

(2) follows from [11, Proposition A.1]. \square

5.1. Proof of Proposition 11

Set $r(x) = (1 - \rho)/2^{1/\beta}\|x\|$. Since $\lim_{x \rightarrow +\infty} r(x) = +\infty$, there exists $R \geq 1$, such that for

$$\frac{r(x)}{\sqrt{1 - \rho^2}} > \frac{\sqrt{1 - \rho^2}}{2\theta} , \quad x \notin B(0, R) , \quad (62)$$

where θ is given by [Proposition 24-\(1\)](#). Using the definition of the Crank-Nicolson kernel,

$$\sup_{x \in B(0,R)} PV(x) \leq \sup_{x \in B(0,R)} \int_{\mathcal{H}} \exp\left(\|x\| + \sqrt{1-\rho^2}\|\xi\|\right) d\gamma(\xi), \quad (63)$$

and [Proposition 24-\(1\)](#) implies that the RHS is finite.

Let $x \in \mathcal{H}$. Define the events $\mathcal{S}(x) = \left\{ \|\Xi\| \leq r(x)/\sqrt{1-\rho^2} \right\}$, $\mathcal{A}(x) = \left\{ \alpha(x, \rho x + \sqrt{1-\rho^2}\Xi) > U \right\}$, and $\mathcal{R}(x) = \left\{ \alpha(x, \rho x + \sqrt{1-\rho^2}\Xi) < U \right\}$, where $U \sim \mathcal{U}([0, 1])$, $\Xi \sim \gamma$, and U and Ξ are independent. It holds

$$PV(x) \leq \mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)^c}] + \mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)}(1_{\mathcal{A}(x)} + 1_{\mathcal{R}(x)})]. \quad (64)$$

For the first term in the RHS, using again $V(X_1) \leq \max(V(x), V(\rho x + \sqrt{1-\rho^2}\Xi))$

$$\mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)^c}] \leq \exp(\|x\|) \int_{\sqrt{1-\rho^2}\|\xi\| \geq r(x)} \exp\left(\sqrt{1-\rho^2}\|\xi\|\right) d\gamma(\xi).$$

By definition of R (see [\(62\)](#)) and by [Proposition 24-\(2\)](#), there exist constants $C_i \in \mathbb{R}_+^*$ such that for any $x \notin B(0, R)$,

$$\mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)^c}] \leq C_1 \exp(-C_2 r(x)^2 + r(x) + \|x\|).$$

The RHS is uniformly bounded on \mathcal{H} since $-C_2 r(x)^2 + r(x) + \|x\| \underset{\|x\| \rightarrow +\infty}{\sim} -C_2 r(x)^2$ when $\|x\|$ tends to infinity. Hence, there exists a constant $b < \infty$ such that

$$\sup_{x \notin B(0,R)} \mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)^c}] \leq b. \quad (65)$$

Consider the second term in the RHS of [\(64\)](#) for $x \notin B(0, R)$. On the event $\mathcal{A}(x) \cap \mathcal{S}(x)$, the move is accepted and $\|X_1 - \rho x\| \leq r(x)$. On $\mathcal{R}(x)$, the move is rejected and $X_1 = x$. Hence,

$$\begin{aligned} & \mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)}(1_{\mathcal{A}(x)} + 1_{\mathcal{R}(x)})] \\ & \leq \mathbb{E}_x \left[\sup_{z \in B(\rho x, r(x))} V(z)1_{\mathcal{S}(x) \cap \mathcal{A}(x)} \right] + \mathbb{E}_x [V(x)1_{\mathcal{S}(x) \cap \mathcal{R}(x)}]. \end{aligned}$$

For $z \in B(\rho x, r(x))$, $V(z) \leq \exp(\rho\|x\| + (1-\rho)/2^{1/\beta}\|x\|) = \exp(C\|x\|)$ where $C \in (0, 1)$ since $\beta \in (0, 1]$ and $\rho \in (0, 1)$. Therefore for any $x \notin B(0, R)$, $\sup_{z \in B(\rho x, r(x))} V(z) \leq lV(x)$, with $l = \exp((C-1)R) < 1$. This yields

$$\begin{aligned} \mathbb{E}_x [V(X_1)1_{\mathcal{S}(x)}(1_{\mathcal{A}(x)} + 1_{\mathcal{R}(x)})] & \leq lV(x)\mathbb{P}[\mathcal{S}(x) \cap \mathcal{A}(x)] + V(x)\mathbb{P}[\mathcal{S}(x) \cap \mathcal{R}(x)] \\ & \leq V(x)\mathbb{P}[\mathcal{S}(x)] - (1-l)V(x)\mathbb{P}[\mathcal{A}(x) \cap \mathcal{S}(x)]. \end{aligned}$$

Since U and Ξ are independent,

$$\mathbb{P}[\mathcal{A}(x) \cap \mathcal{S}(x)] = \mathbb{E} \left[\left(1 \wedge e^{g(x) - g(\rho x + \sqrt{1-\rho^2}\Xi)} \right) 1_{\mathcal{S}(x)} \right].$$

By definition of the set $\mathcal{I}(x)$ and [Lemma 23](#), there exists $\kappa > 0$ such that

$$\mathbb{P}[\mathcal{A}(x) \cap \mathcal{I}(x)] \geq e^{-\kappa \|x\|^\beta} \mathbb{P}[\mathcal{I}(x)] = \exp(-\kappa \log^\beta V(x)) \mathbb{P}[\mathcal{I}(x)] .$$

Hence, for any $x \notin B(0, R)$,

$$\mathbb{E}_x [V(X_1) 1_{\mathcal{I}(x)} (1_{\mathcal{A}(x)} + 1_{\mathcal{R}(x)})] \leq V(x) - (1-l) V(x) \exp(-\kappa \log^\beta V(x)) . \quad (66)$$

Combining [\(65\)](#) and [\(66\)](#) in [\(64\)](#) and using [\(63\)](#), it follows that there exists $\tilde{b} > 0$ such that

$$PV(x) \leq V(x) - (1-l) V(x) \exp(-\kappa \log^\beta V(x)) + \tilde{b} , \quad \forall x \in \mathcal{H} .$$

The proof is then concluded by [Remark 6](#).

5.2. Proof of [Proposition 12](#)

We preface the proof of [Proposition 12](#) by a Lemma establishing a first step to the contracting property of Q_{PCN} . Roughly, the idea of the proof is that the probability the two moves of the basic coupling are accepted can control the probability that only one is.

Lemma 25. *Assume [CN1](#). There exists $\tau > 0$ and for any $L > 0$ there exists $k_L \in (0, 1)$ such that*

- for all $x, y \in B(0, L)$ and $d_\tau(x, y) < 1$,

$$Q_{\text{PCN}} d_\tau(x, y) \leq k_L d_\tau(x, y) . \quad (67)$$

- for all $x, y \in E$,

$$Q_{\text{PCN}} d_\tau(x, y) \leq d_\tau(x, y) . \quad (68)$$

Proof: Let $\tau \in (0, 1)$; for ease of notation, we simply write Q for Q_{PCN} . Let $L > 0$ and $x, y \in B(0, L)$ such that $d_\tau(x, y) < 1$. Let (X_1, Y_1) be the basic coupling between $P(x, \cdot)$ and $P(y, \cdot)$; let Ξ, U be resp. the Gaussian variable and the uniform variable used for the basic coupling. Set

$$\begin{aligned} \mathcal{I} &= \left\{ \sqrt{1 - \rho^2} \|\Xi\| \leq 1 \right\} , \\ \mathcal{A}(x, y) &= \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) \wedge \alpha(y, \rho y + \sqrt{1 - \rho^2} \Xi) > U \right\} , \\ \mathcal{R}(x, y) &= \left\{ \alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) \vee \alpha(y, \rho y + \sqrt{1 - \rho^2} \Xi) < U \right\} . \end{aligned}$$

On the event $\mathcal{A}(x, y)$, the moves are both accepted so that $X_1 = \rho x + \sqrt{1 - \rho^2} \Xi$ and $Y_1 = \rho y + \sqrt{1 - \rho^2} \Xi$; On the event $\mathcal{R}(x, y)$, the moves are both rejected so that $X_1 = x$ and $Y_1 = y$. It holds,

$$\begin{aligned} Q d_\tau(x, y) &\leq \tilde{\mathbb{E}}_{x, y} [d_\tau(X_1, Y_1)] \\ &\leq \tilde{\mathbb{E}}_{x, y} [d_\tau(X_1, Y_1) (1_{\mathcal{A}(x, y) \cup \mathcal{R}(x, y)})] + \mathbb{E} [1_{(\mathcal{A}(x, y) \cup \mathcal{R}(x, y))^c}] , \quad (69) \end{aligned}$$

where we have used d_τ is bounded by 1. Since $d_\tau(X_1, Y_1) = \rho^\beta d_\tau(x, y)$, on $\mathcal{A}(x, y)$, and $d_\tau(X_1, Y_1) = d_\tau(x, y)$, on $\mathcal{B}(x, y)$. Then,

$$\tilde{\mathbb{E}}_{x,y} [d_\tau(X_1, Y_1)(1_{\mathcal{A}(x,y) \cup \mathcal{B}(x,y)})] \leq \rho^\beta d_\tau(x, y) \mathbb{P}[\mathcal{A}(x, y)] + d_\tau(x, y) \mathbb{P}[\mathcal{B}(x, y)] .$$

Since $\mathbb{P}[\mathcal{A}(x, y)] + \mathbb{P}[\mathcal{B}(x, y)] \leq 1$, we have

$$\begin{aligned} & \tilde{\mathbb{E}}_{x,y} [d_\tau(X_1, Y_1)(1_{\mathcal{A}(x,y) \cup \mathcal{B}(x,y)})] \\ & \leq d_\tau(x, y) - (1 - \rho^\beta) d_\tau(x, y) \mathbb{P}[\mathcal{A}(x, y)] \\ & \leq d_\tau(x, y) - (1 - \rho^\beta) d_\tau(x, y) \mathbb{P}[\mathcal{A}(x, y) \cap \mathcal{I}] . \end{aligned} \quad (70)$$

Set

$$\Delta(x, y, \xi) = \left| \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) - \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) \right| . \quad (71)$$

Since d_τ is bounded by 1 and Ξ and U are independent, it follows

$$\mathbb{P}[(\mathcal{A}(x, y) \cup \mathcal{B}(x, y))^c] \leq \int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) . \quad (72)$$

Plugging (70) and (72) in (69) yields

$$\begin{aligned} Qd_\tau(x, y) & \leq d_\tau(x, y) \\ & - (1 - \rho^\beta) d_\tau(x, y) \tilde{\mathbb{P}}_{x,y}[\mathcal{A}(x, y) \cap \mathcal{I}] + \int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) . \end{aligned} \quad (73)$$

Let us now define $h : \mathcal{H} \rightarrow \mathbb{R}$ by

$$h(z) = g(z) - g(\rho z) . \quad (74)$$

We bound from below $\mathbb{P}[\mathcal{A}(x, y) \cap \mathcal{I}]$. Since U is independent of Ξ , it follows

$$\mathbb{P}[\mathcal{A}(x, y) \cap \mathcal{I}] \geq \mathbb{E} \left[\left(\alpha(x, \rho x + \sqrt{1 - \rho^2} \Xi) \wedge \alpha(y, \rho y + \sqrt{1 - \rho^2} \Xi) \right) 1_{\mathcal{I}} \right] .$$

By CN1, for all ξ such that $\sqrt{1 - \rho^2} \|\xi\| \leq 1$, it holds for $z \in \mathcal{H}$

$$g(z) - g(\rho z + \sqrt{1 - \rho^2} \xi) \geq h(z) - C_g .$$

Then,

$$\begin{aligned} & \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) \wedge \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) \\ & \geq 1 \wedge (e^{-C_g} e^{h(x)}) \wedge (e^{-C_g} e^{h(y)}) \geq e^{-C_g} \left[1 \wedge e^{h(x) \wedge h(y)} \right] . \end{aligned}$$

Therefore,

$$\mathbb{P}[\mathcal{A}(x, y) \cap \mathcal{I}] \geq e^{-C_g} \left[1 \wedge e^{h(x) \wedge h(y)} \right] \mathbb{P}[\mathcal{I}] . \quad (75)$$

We now upper bound the integral term in (73). Define the partition of \mathcal{H} ,

$$\begin{aligned}\mathcal{K}_1(x, y) &= \{\xi \in \mathcal{H} : \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) = 1\} \\ \mathcal{K}_2(x, y) &= \{\xi \in \mathcal{H} : \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = 1 > \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi)\} \\ \mathcal{K}_3(x, y) &= \{\xi \in \mathcal{H} : \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) = 1 > \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi)\} \\ \mathcal{K}_4(x, y) &= \{\xi \in \mathcal{H} : \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi) < 1 \text{ and } \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) < 1\}.\end{aligned}$$

Since on $\mathcal{K}_1(x, y)$, $\Delta(x, y, \xi) = 0$,

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) = \sum_{j=2}^4 \int_{\mathcal{K}_j(x, y)} \Delta(x, y, \xi) d\gamma(\xi). \quad (76)$$

For any $a, b > 0$, we have $|a - b| = (a \vee b) [1 - ((a/b) \wedge (b/a))]$. Set

$$S(x, y, \xi) = \alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) \vee \alpha(y, \rho y + \sqrt{1 - \rho^2} \xi).$$

Upon noting that $1 - e^{-t} \leq t$ for any $t \geq 0$, we have

$$\begin{aligned}\Delta(x, y, \xi) &\leq S(x, y, \xi) \left| g(y) - g(x) - g(\rho y + \sqrt{1 - \rho^2} \xi) \right. \\ &\quad \left. + g(\rho x + \sqrt{1 - \rho^2} \xi) \right| 1_{\mathcal{K}_2(x, y) \cup \mathcal{K}_3(x, y) \cup \mathcal{K}_4(x, y)}(\xi).\end{aligned}$$

By **CN1**, this yields

$$\Delta(x, y, \xi) \leq 2C_g \|y - x\|^\beta S(x, y, \xi) \leq 2C_g \tau d_\tau(x, y) S(x, y, \xi).$$

On $\mathcal{K}_2(x, y)$, (74) $h(x) \geq g(\rho x + \sqrt{1 - \rho^2} \xi) - g(\rho x)$, and by **CN1**, $h(x) \geq -C_g(1 - \rho^2)^{\beta/2} \|\xi\|^\beta$. Then,

$$\begin{aligned}\int_{\mathcal{K}_2(x, y)} \Delta(x, y, \xi) d\gamma(\xi) &\leq 2C_g \tau d_\tau(x, y) \int_{\mathcal{K}_2(x, y)} d\gamma(\xi) \\ &\leq 2C_g \tau d_\tau(x, y) \left\{ \left[e^{h(x)} \int_{\mathcal{K}_2(x, y)} e^{C_g(1 - \rho^2)^{\beta/2} \|\xi\|^\beta} d\gamma(\xi) \right] \wedge 1 \right\} \\ &\leq C_I \tau d_\tau(x, y) \left\{ e^{h(x)} \wedge 1 \right\},\end{aligned} \quad (77)$$

for a constant C_I , which is finite according to **Proposition 24-(1)**. By symmetry, on $\mathcal{K}_3(x, y)$,

$$\int_{\mathcal{K}_3(x, y)} \Delta(x, y, \xi) d\gamma(\xi) \leq C_I \tau d_\tau(x, y) \left\{ e^{h(y)} \wedge 1 \right\}. \quad (78)$$

On $\mathcal{K}_4(x, y)$, using **CN1**,

$$\alpha(x, \rho x + \sqrt{1 - \rho^2} \xi) = e^{g(x) - g(\rho x + \sqrt{1 - \rho^2} \xi)} \wedge 1 \leq \left(e^{h(x)} e^{C_g(1 - \rho^2)^{\beta/2} \|\xi\|^\beta} \right) \wedge 1;$$

and by symmetry, we obtain a similar upper bound for $\alpha(y, \rho y \sqrt{1 - \rho^2} \xi)$. It follows

$$S(x, y, \xi) \leq e^{C_g(1-\rho^2)^{\beta/2} \|\xi\|^\beta} (e^{h(x) \vee h(y)} \wedge 1).$$

Hence, using again [Proposition 24-\(1\)](#), there exists $C_I < +\infty$ such that

$$\int_{\mathcal{H}_4(x,y)} \Delta(x, y, \xi) d\gamma(\xi) \leq C_I \tau d_\tau(x, y) \left[e^{h(x) \vee h(y)} \wedge 1 \right]. \quad (79)$$

Plugging [\(77\)](#), [\(78\)](#), [\(79\)](#) in [\(76\)](#), it follows

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) \leq 3C_I \tau d_\tau(x, y) \left[e^{h(x) \vee h(y)} \wedge 1 \right].$$

Finally by [CN1](#) and since $d_\tau(x, y) < 1$, $|h(x) - h(y)| \leq 2C_g \|x - y\|^\beta \leq 2C_g \tau^\beta$. Therefore $e^{h(x) \vee h(y)} \wedge 1 \leq e^{2C_g \tau^\beta} [e^{h(x) \wedge h(y)} \wedge 1]$, and

$$\int_{\mathcal{H}} \Delta(x, y, \xi) d\gamma(\xi) \leq \hat{C}_I \tau d_\tau(x, y) \left[e^{h(x) \wedge h(y)} \wedge 1 \right], \quad (80)$$

for $\hat{C}_I = 3C_I e^{2C_g}$. Plugging [\(75\)](#) and [\(80\)](#) in [\(73\)](#) yields

$$Qd_\tau(x, y) \leq d_\tau(x, y) \left(1 - \left\{ (1 - \rho^\beta) e^{-C_g} \mathbb{P}[\mathcal{I}] - \hat{C}_I \tau \right\} \left[e^{h(x) \wedge h(y)} \wedge 1 \right] \right).$$

Therefore, we can choose τ such that there exists $\delta \in (0, 1)$ and

$$Qd_\tau(x, y) \leq d_\tau(x, y) \left(1 - \delta \left[e^{h(x) \wedge h(y)} \wedge 1 \right] \right). \quad (81)$$

[\(81\)](#) implies [\(67\)](#) upon noting that by definition, $\inf_{B(0,L)} h > -\infty$.

We now consider [\(68\)](#), let $x, y \in \mathcal{H}$. If $d_\tau(x, y) = 1$, $Qd_\tau(x, y) \leq 1$ since d_τ is bounded by 1. If $d_\tau(x, y) < 1$, there exists $L \geq 0$ such that $x, y \in B(0, L)$, and [\(67\)](#) implies $Qd_\tau(x, y) \leq d_\tau(x, y)$. \square

Proof of [Proposition 12](#) Let $\{(X_n, Y_n), n \in \mathbb{N}\}$ be a Markov chain with Markov kernel Q given by [\(15\)](#). We denote for all $n \in \mathbb{N}^*$, Ξ_n and U_n , respectively the common gaussian variable and uniform variable, sampled to build (X_n, Y_n) . Note that by definition the variables $\{\Xi_n, U_n; n \in \mathbb{N}\}$ are independent.

For ease of notation, we simply write d_τ instead of $d_{\tau, Q}$, and Q for Q_{PCN} . Since $\{x : V(x) \leq u\} = \{x : \|x\| \leq \log(u)\}$, for $u \geq 1$, we only prove that for all $L > 0$, there exist $\ell \in \mathbb{N}^*$ and $\epsilon > 0$ such that $\overline{B}(0, L)^2$ is a (ℓ, ϵ, d_τ) -coupling set; see [Definition 2](#) By definition of Q and [Lemma 25](#) the condition [\(i\)](#) and [\(ii\)](#) of [Definition 2](#) are satisfied. Let $L > 0$, and x, y be in $\overline{B}(0, L)$. Assume first $d_\tau(x, y) < 1$. Then by [Lemma 25](#), there exists $k_L \in (0, 1)$, independent of x, y , such that $Qd_\tau(x, y) \leq k_L d_\tau(x, y)$. Then by [Lemma 30](#), for every $n \in \mathbb{N}^*$,

$$Q^n d_\tau(x, y) \leq Q^{n-1} d_\tau(x, y) \leq \dots \leq k_L d_\tau(x, y). \quad (82)$$

Consider now the case $d_\tau(x, y) = 1$. Let $\{(X_n, Y_n), n \in \mathbb{N}\}$ be the Markov chain with Markov kernel Q starting in (x, y) . Let $n \in \mathbb{N}^*$ and denote for all $1 \leq i \leq n$

$$\begin{aligned} \mathcal{A}_i(x, y) &= \{U_i \leq \Psi(X_{i-1}, Y_{i-1}, \Xi_i)\} \\ \widetilde{\mathcal{A}}^i(x, y) &= \bigcap_{1 \leq j \leq i} \left(\{\sqrt{1 - \rho^2} \|\Xi_j\| \leq L/n\} \cap \mathcal{A}_j(x, y) \right), \end{aligned}$$

where $\Psi(X_{i-1}, Y_{i-1}, \Xi_i) = \alpha(X_{i-1}, \rho X_{i-1} + \sqrt{1 - \rho^2} \Xi_i) \wedge \alpha(Y_{i-1}, \rho Y_{i-1} + \sqrt{1 - \rho^2} \Xi_i)$. On the set $\widetilde{\mathcal{A}}^i(x, y)$, $X_j = \rho X_{j-1} + \sqrt{1 - \rho^2} \Xi_j$ and $Y_j = \rho Y_{j-1} + \sqrt{1 - \rho^2} \Xi_j$ for all $1 \leq j \leq i$. Then, since $d_\tau(X_n, Y_n) \leq \tau^{-1} \|X_n - Y_n\|^\beta$, on $\widetilde{\mathcal{A}}^n(x, y)$ it holds $d_\tau(X_n, Y_n) \leq \tau^{-1} \rho^{\beta n} \|x - y\|^\beta$. This inequality and $d_\tau(z, w) \leq 1$ yield

$$\begin{aligned} Q^n d_\tau(x, y) &= \widetilde{\mathbb{E}}_{x, y} \left[d_\tau(X_n, Y_n) (1_{\widetilde{\mathcal{A}}^n(x, y)} + 1_{(\widetilde{\mathcal{A}}^n(x, y))^c}) \right] \\ &\leq \rho^{\beta n} \|x - y\|^\beta \mathbb{P} \left[\widetilde{\mathcal{A}}^n(x, y) \right] + \mathbb{P} \left[(\widetilde{\mathcal{A}}^n(x, y))^c \right] \\ &\leq \rho^{\beta n} (2L)^\beta \mathbb{P} \left[\widetilde{\mathcal{A}}^n(x, y) \right] + \mathbb{P} \left[(\widetilde{\mathcal{A}}^n(x, y))^c \right] \\ &\leq 1 + (\rho^{\beta n} (2L)^\beta - 1) \mathbb{P} \left[\widetilde{\mathcal{A}}^n(x, y) \right]. \end{aligned} \quad (83)$$

As $\rho \in (0, 1)$, there exists ℓ such that, $\rho^{\beta \ell} (2L)^\beta < 1$. It remains to lower bound $\widetilde{\mathbb{P}}_{x, y} \left[\widetilde{\mathcal{A}}^\ell(x, y) \right]$ by a positive constant to conclude, which is done by the following inequalities, where we use the independence of the random variables $\{\Xi_i, U_i; i \in \mathbb{N}^*\}$.

$$\begin{aligned} &\mathbb{P} \left[\widetilde{\mathcal{A}}^\ell(x, y) \right] \\ &= \mathbb{P} \left[\left(\widetilde{\mathcal{A}}^{\ell-1}(x, y) \cap \left(\{\sqrt{1 - \rho^2} \|\Xi_\ell\| \leq L/\ell\} \right) \right) \right] \\ &\quad \times \widetilde{\mathbb{E}}_{x, y} \left[\Psi(X_{\ell-1}, Y_{\ell-1}, \Xi_\ell) \mid \widetilde{\mathcal{A}}^{\ell-1}(x, y) \cap \left(\{\sqrt{1 - \rho^2} \|\Xi_\ell\| \leq L/\ell\} \right) \right]. \end{aligned}$$

For all $1 \leq i \leq \ell$, on the event $\bigcap_{j \leq i} \{\sqrt{1 - \rho^2} \|\Xi_j\| \leq L/\ell\}$, it holds

$$\Psi(X_{i-1}, Y_{i-1}, \Xi_i) \geq \exp \left(- \sup_{z \in B(0, 2L)} g(z) + \inf_{z \in B(0, 2L)} g(z) \right) = \delta,$$

where $\delta \in (0, 1)$. Therefore, since Ξ_i is independent of $\widetilde{\mathcal{A}}^{i-1}(x, y)$, we have

$$\mathbb{P} \left[\widetilde{\mathcal{A}}^\ell(x, y) \right] \geq \delta \mathbb{P} \left[\mathcal{A}_{\ell-1}(x, y) \right] \mathbb{P} \left[\{\sqrt{1 - \rho^2} \|\Xi_\ell\| \leq L/\ell\} \right].$$

An immediate induction leads to

$$\mathbb{P} \left[\widetilde{\mathcal{A}}^\ell(x, y) \right] \geq \left(\mathbb{P} \left[\sqrt{1 - \rho^2} \|\Xi_1\| \leq \frac{L}{\ell} \right] \right)^\ell \delta^\ell.$$

Plugging this result in (83) and (82) implies there exists $s \in (0, 1)$ such that for all $x, y \in \overline{B}(0, L)$, $Q^\ell d_\tau(x, y) \leq s d_\tau(x, y)$. \square

Appendix A: Wasserstein distance: some useful properties

Lemma 26 ([20, Particular Case 6.16]). *Let (E, d) be a Polish space. Then, for all $\mu, \nu \in \mathcal{P}(E)$:*

$$W_d(\mu, \nu) \leq \sup_{x, y \in E} d_*(x, y) d_{\text{TV}}(\mu, \nu) .$$

Hence, when d is bounded, the convergence in total variation distance implies the convergence in the Wasserstein metric.

For any measurable function $l : E \times E \rightarrow \mathbb{R}_+$, we define the optimal transportation for $\mu, \nu \in \mathcal{P}(E)$ by:

$$W_l(\mu, \nu) = \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} l(x, y) d\alpha(x, y) . \quad (84)$$

Note that we may have $W_l(\mu, \nu) = +\infty$, and for all $x, y \in E \times E$, $W_l(\delta_x, \delta_y) = l(x, y)$. We consider the case when the function l is a distance-like function (see also [10])

Definition 27. A function $l : E \times E \rightarrow \mathbb{R}_+$ is said to be a *distance-like* if

1. For all (x, y) in E^2 , $l(x, y) = 0$ if and only if $x = y$.
2. l is lower semicontinuous.
3. For all (x, y) in E^2 , $l(x, y) = l(y, x)$.

The following lemma establishes the convexity of W_l , when l is a distance-like function.

Lemma 28. *Let (E, d) be a Polish space. Let P be a Markov kernel on $(E, \mathcal{B}(E))$ and $l : E \times E \rightarrow \mathbb{R}_+$ be a distance-like function. For any $\mu, \nu \in \mathcal{P}(E)$*

$$W_l(\mu P, \nu P) \leq \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} W_l(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy) .$$

Proof: Let α be a coupling of μ and ν . We get

$$\begin{aligned} \mu P(dz) &= \int_E P(x, dz) \mu(dx) = \int_{E \times E} P(x, dz) \alpha(dx, dy) . \\ \nu P(dz) &= \int_{E \times E} P(y, dz) \alpha(dx, dy) . \end{aligned}$$

Therefore,

$$W_l(\mu P, \nu P) = W_l \left(\int_{E \times E} P(x, \cdot) \alpha(dx, dy), \int_{E \times E} P(y, \cdot) \alpha(dx, dy) \right) .$$

Since l is lower semicontinuous and $l \geq 0$, by [20, Theorem 4.8]

$$W_l(\mu P, \nu P) \leq \int_{E \times E} W_l(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy) .$$

The proof is concluded since this inequality holds for all coupling α . \square

Lemma 29. Let (E, d) be a Polish space and let P be Markov kernel on $(E, \mathcal{B}(E))$. Assume that d is weakly contracting for P , i.e.

$$W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y), \quad \forall (x, y) \in E^2. \quad (85)$$

Then for all $\mu, \nu \in \mathcal{P}(E)$,

$$W_d(\mu P, \nu P) \leq W_d(\mu, \nu). \quad (86)$$

Proof: According to [Lemma 28](#) and using the stated assumptions,

$$\begin{aligned} W_d(\mu P, \nu P) &\leq \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} W_d(P(x, \cdot), P(y, \cdot)) \alpha(dx, dy) \\ &\leq \inf_{\alpha \in \mathcal{C}(\mu, \nu)} \int_{E \times E} d(x, y) \alpha(dx, dy) = W_d(\mu, \nu). \end{aligned}$$

□

Lemma 30. Let (E, d) be a Polish space and let P be a Markov kernel on $(E, \mathcal{B}(E))$. Assume there exists a Markov kernel Q on $(E \times E, \mathcal{B}(E \times E))$ satisfying:

- (i) for all $x, y \in E$, $Q((x, y), \cdot)$ is a coupling of $(P(x, \cdot), P(y, \cdot))$.
- (ii) for all $x, y \in E$, $Qd(x, y) \leq d(x, y)$.

Then for all $x, y \in E$, $W_d(P(x, \cdot), P(y, \cdot)) \leq d(x, y)$ and for all probability measures $\mu, \nu \in \mathcal{P}(E)$,

$$W_d(\mu P, \nu P) \leq W_d(\mu, \nu). \quad (87)$$

Proof: By assumption and the definition of the Wasserstein distance (1), we have for all x, y , $W_d(P(x, \cdot), P(y, \cdot)) \leq Qd(x, y) \leq d(x, y)$. The second statement is a consequence of [Lemma 29](#). □

Appendix B

Lemma 31 ([4, lemma 4.1]). Assume that there exist a measurable function $V : E \rightarrow \mathbb{R}_+$, a nonnegative constant b and a measurable function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$PV + \phi \circ V \leq V + b. \quad (88)$$

Then for every $n \geq 1$,

$$\sum_{i=0}^{n-1} P^i(\phi \circ V) \leq V + nb. \quad (89)$$

If π is an invariant probability measure for P , then $\pi(\phi \circ V) \leq b$.

We remind that for ϕ given by [H2](#), H_ϕ and r_ϕ are respectively given by (5) and (6). Here are some results about H_ϕ .

Lemma 32. Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing, concave, and differentiable function satisfying $\lim_{+\infty} \phi = +\infty$. Then,

- (i) H_ϕ given by (5) is concave, increasing, C^2 on $[1, +\infty)$ and $\lim_{+\infty} H_\phi = +\infty$.
- (ii) $H_\phi^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is C^2 , increasing, convex and $\lim_{+\infty} H_\phi^{-1} = +\infty$.

Proof: (i) is trivial. For (ii), note that $(H_\phi^{-1})' = \phi \circ H_\phi^{-1}$ and since both $y \mapsto H_\phi^{-1}(y)$ and $y \mapsto \phi(y)$ are non decreasing, $y \mapsto (H_\phi^{-1}(y))'$ is non decreasing showing that H_ϕ^{-1} is convex. \square

Let H_k be given by (41) and set

$$\tilde{V}_k = H_k \circ V. \tag{90}$$

Proposition 33 ([7, Lemma 2.3 and Proposition 2.1]). Assume H2 holds.

- (i) $r_\phi \in \Lambda$ and is log-concave.
- (ii) for every $k \geq 0$, H_k is concave.
- (iii) for all $x_0 \in E$ and $k \geq 0$

$$P\tilde{V}_{k+1} \leq \tilde{V}_k - r_\phi(k) + (b + V(x_0)) \frac{r_\phi(k+1)}{r_\phi(0)}.$$

Here are some additional properties on the functions H_k and \tilde{V}_k .

Lemma 34. Assume H2. Let r_ϕ, H_k and \tilde{V}_k be given by (6), (41) and (90).

- (i) There exists some nonnegative constant C such that for every $x \in E$

$$\sup_{k \geq 0} \frac{\tilde{V}_k(x)}{r_\phi(k)} \leq C\tilde{V}_0(x) \leq CV(x).$$

- (ii) For all $x, y \in [1, +\infty[$, and every integer $k \geq 1$

$$H_k(x + y) \leq H_k(x) + H_k(y) + 2r_\phi(k).$$

Proof: (i): By definition for every $x \in \mathbb{R}_+^*$,

$$\frac{H_k(x)}{r_\phi(k)} = \frac{1}{r_\phi(k)} \int_0^{H_\phi(x)} r_\phi(t+k) dt.$$

Since by Proposition 33-(i) $r_\phi \in \Lambda$, Lemma 36-(iv) shows that there exists a constant C such that for any $t, k \geq 0$, $r_\phi(k+t) \leq Cr_\phi(k)r_\phi(t)$. Then

$$\frac{H_k(x)}{r_\phi(k)} \leq C \int_0^{H_\phi(x)} r_\phi(t) dt = CH_0(x) \leq Cx.$$

Applying this inequality with $x \leftarrow V(x)$ concludes the proof.

(ii): by Proposition 33 $z \mapsto H_k(z+1)$ defined on \mathbb{R}_+ , is concave, and $H_k(1) = 0$; thus it is sub-additive. Then, since H_k is nondecreasing, it yields

$$H_k(x + y) \leq H_k(x + y + 1) \leq H_k(x + 1) + H_k(y + 1).$$

Since H_k is concave, for every $z \geq 1$

$$H_k(z+1) - H_k(z) \leq H'_k(z) \leq H'_k(1) = r_\phi(k).$$

These two inequalities imply that for all $x, y \geq 1$,

$$H_k(x+y) \leq H_k(x) + H_k(y) + 2r_\phi(k).$$

□

Appendix C: Subgeometric functions and sequences

For $r \in \Lambda$, we denote by $t \mapsto f_r(t)$ the function

$$f_r(t) = r(0) + \int_0^t r(s) ds. \quad (91)$$

Lemma 35. *Let $r \in \Lambda_0$, R and f_r be resp. given by (23) and (91).*

(i) [17, Lemma 1]: For all $t, u \in \mathbb{R}_+$,

$$r(t+u) \leq r(t)r(u). \quad (92)$$

(ii) f_r is convex, increasing to $+\infty$, and there exists $C > 1$ such that for all $t \geq 0$:

$$f_r(t+1) \leq C f_r(t). \quad (93)$$

(iii) There exists a constant $C \in (0, 1)$ such that

$$\begin{aligned} C f_r(n) &\leq R(n) \leq f_r(n) & \forall n \in \mathbb{N}, \\ C f_r(t) &\leq R(\lfloor t \rfloor) \leq f_r(t) & \forall t \geq 0. \end{aligned} \quad (94)$$

(iv) $\lim_{n \rightarrow \infty} r(n)/R(n) = 0$.

Proof: (ii) By Definition 1 r is non-decreasing, thus is bounded on every compact set; then, f_r is continuous. Moreover, it is differentiable and its derivative is r , which is non-decreasing. Then f_r is convex. In addition $r(0) > 2$, thus f_r is increasing to $+\infty$. Let us show (93). By (91), for all $t \geq 0$ $f_r(t+1) = r(0) + \int_0^1 r(u) du + \int_0^t r(1+u) du$. Then by (92), and since f_r is increasing, for all $t \geq 0$

$$f_r(t+1) \leq f_r(1) + r(1)f_r(t).$$

The proof is concluded since $\lim_{t \rightarrow +\infty} f_r(t) = +\infty$.

(iii) Since r is non-decreasing, by (91) and an integral test we get for all $n \geq 1$,

$$f_r(n-1) \leq R(n) \leq f_r(n).$$

This inequality combined with (93) implies (94). The upper bound in the second inequality is a consequence of the first one and the monotonicity of f_r . For the lower bound, by (94) and the monotonicity of f_r there exists C_1 such that

$$C_1 f_r(t-1) \leq R(\lfloor t \rfloor);$$

by (93) there exists $C_2 > 0$ such that $f_r(t)/C_2 \leq f_r(t-1)$. The monotonicity of f_r concludes the proof.

(iv) Set $u_n := \log(r(n))/n$. By Definition 1 u_n is decreasing, then

$$\log\left(1 + \frac{r(n+1) - r(n)}{r(n)}\right) = \log\left(\frac{r(n+1)}{r(n)}\right) = n(u_{n+1} - u_n) + u_{n+1} \leq u_{n+1}. \tag{95}$$

In addition, by (3) $\lim_{n \rightarrow +\infty} u_n = 0$, so $\lim_{n \rightarrow +\infty} (r(n+1) - r(n))/r(n) = 0$. Therefore, for all $\epsilon > 0$, there exists N such that for all $n \geq N$,

$$(r(n+1) - r(n)) \leq \epsilon r(n).$$

This result implies that for $n \geq N$, $\sum_{k=N}^n (r(k+1) - r(k)) \leq \epsilon \sum_{k=N}^n r(k)$, so that

$$r(n)/R(n) \leq \epsilon + r(N)/R(n) \quad \forall n \geq N. \tag{96}$$

Since $R(n) \geq nr(0)$, $\lim_{n \rightarrow +\infty} R(n) = +\infty$. This concludes the proof. \square

The following lemma is a trivial consequence of Definition 1 and Lemma 35. The proof is omitted.

Lemma 36. *Let $r \in \Lambda$; and let $(R(n))_n$ and f_r be resp. given by (23) and (91).*

(i) *There exist two positive constants C_1, C_2 such that for all $t \geq 0$,*

$$C_1 f_r(t) \leq R(\lfloor t \rfloor) \leq C_2 f_r(t).$$

(ii) $\lim_{n \rightarrow +\infty} R(n) = +\infty$.

(iii) $\lim_{n \rightarrow +\infty} r(n)/R(n) = 0$.

(iv) *There exists a non-negative constant C such that for all $x, y \geq 0$, $r(x+y) \leq Cr(x)r(y)$. In particular, $\limsup_{x \rightarrow \infty} r(x+1)/r(x) < +\infty$.*

(v) $\lim_{n \rightarrow +\infty} \log(R(n))/n = 0$.

Lemma 37. *Let $r \in \Lambda$. There exist a measurable, increasing and convex function ψ_r , and two positive constants C_1, C_2 such that for every integer $n \geq 0$*

$$C_1 \psi_r(n) \leq R(n) \leq C_2 \psi_r(n),$$

where the sequence $(R_n)_n$ is defined by (23).

Proof: Since $r \in \Lambda$, there exist $r_0 \in \Lambda_0$ and positive constants c_1, c_2 such that for any $n \geq 0$, $c_1 r_0(n) \leq r(n) \leq c_2 r_0(n)$. The result now follows from Lemma 35.

\square

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