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# On the design of full-rate full-diversity space-time block codes for multiple-input–multiple-output systems with a turbo minimum mean square error equaliser at the receiver side

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**Abstract:** Non-orthogonal space-time block (STB) code structures used to be optimised considering a maximum-likelihood detection, but depending on the modulation order and/or the antenna number, the prohibitive complexity of such a receiver makes it infeasible in practice. Suboptimal low-complexity receiver structures like linear detectors can be applied instead, yielding degradation of performance compared with the predictions. When a forward error correction (FEC) code is used upstream of the STB code, an iterative receiver consisting of a FEC decoder and an interference canceller (IC) co-operating according to the turbo equalisation principle, can achieve near-optimal performance. This study aims to define the construction of full-rate full-diversity linear dispersion codes with reduced peak-to-average power ratio (PAPR), taking into account the serial concatenation with a FEC at the transmitter and a minimum mean square error (MMSE) turbo equaliser at the receiver. Optimisation criteria are mutual information maximisation, symbol error rate minimisation at the MMSE–IC output, full-diversity thanks to the threaded algebraic space time concept and PAPR minimisation. Explicit constructions are given and the resulting code efficiency is analysed through comparison with equivalent perfect STB codes.

## 1 Introduction

Space-time block (STB) coding has attracted much attention in wireless multiple-input–multiple-output (MIMO) communications as a solution to increase the data throughput without sacrificing the bandwidth nor increasing the average transmit power. It is known as a coding technique to combat fading by exploiting transmit and receive diversities. Tarokh *et al.* [1] developed criteria [for the high signal-to-noise ratio (SNR) regime] to define full-diversity and maximum coding gain STB codes, based on the pairwise error probability minimisation under the assumption of a maximum-likelihood (ML) decoding at the receiver. Orthogonal STB (OSTB) [2, 3] codes satisfy Tarokh design conditions with a ML linear detection. Their weakness is a sub-optimal transmission rate, which decays as the number of transmit antennas increases. As a consequence, except for the Alamouti code with one receive antenna, OSTB codes do not maximise the capacity. High-rate linear dispersion codes have thus been proposed by Hassibi and Hochwald [4] in order to achieve the ergodic capacity of a MIMO system. A tradeoff between the diversity gain and the multiplexing gain has been established by Zheng and Tse [5]. Many full-diversity full-rate linear dispersion codes among which [6–12] have been developed to achieve the optimum diversity/multiplexing tradeoff under a ML detection, which may not be feasible

in practice. Indeed the complexity of non-orthogonal STB codes ML receiver increases exponentially with the number of transmit antennas and the modulation efficiency. Many works have been done [13, 14] to construct full-rate and full-diversity STB code with low-ML decoding. However, the receiver complexity remains high for large number of transmit antennas and high-order modulation. Relaxing the full-rate constraint, codes allowing reduced decoding complexity have been proposed as single-symbol ML-decodable space-time block codes (STBCs) [15, 16] or asymmetric fast-decodable STBCs as in [17] or [18]. On the other hand, fast ML decoders referred as sphere decoders have been proposed [19] without any constraint on the rate of the STB code, but their non-bounded latency is a major drawback. An alternative is low-complexity linear detection such as minimum mean square error (MMSE) detection. Design of optimal linear dispersion STB codes for MIMO systems equipped with linear MMSE receivers has been studied in [20, 21]. Necessary and sufficient conditions have been derived in order to construct MMSE optimal linear dispersion STB codes.

Furthermore, standards recommend to serially concatenate a forward error correction (FEC) code to the STB code at the transmitter in order to enhance the quality of transmission as proposed in [22, 23]. The optimisation of such schemes assumes an entire ML detection of prohibitive complexity. An alternative to an entire ML receiver achieving

reasonable complexity and near-optimal performance is MMSE turbo equalisation [24, 25]. It combines a MMSE–STB code linear interference canceller (IC) MMSE–IC with a FEC decoder, both involved in an iterative process. In this paper, we propose to optimise the STB code definition considering a MMSE turbo equaliser at the receiver side, in addition to the serial concatenation with a FEC, as design conditions. Assuming such a receiver, we aim to construct a linear dispersion codes (LDC) satisfying three conditions. The first condition is full-diversity and full-rate (optimal diversity/multiplexing tradeoff), achieved by applying the threaded algebraic space time (TAST) principle [7]. The second one is the reduction of the symbol error probability (SEP) at the MMSE–IC output, through signal-to-interference-plus-noise maximisation and correlation reduction between two noise-equivalent components of the MMSE–IC output. The third one is the peak-to-average power ratio (PAPR) minimisation. Indeed LDC involves linear combinations of symbols resulting in power variations, and thus necessary power back-off on the amplifier. We note that in a turbo-like receiver setup, consisting of a MMSE–IC and a FEC decoder, the minimisation of the bit-error-rate (BER) at the output of the entire system can be solved through some convergence analysis based on the extrinsic information transfer chart [25]. One inconvenient of such a method is the definition of STB codes with coefficients depending on the optimisation target SNR value. For this reason, we have chosen the minimisation of the error probability at the MMSE–IC output as a design constraint, assuming the equaliser has converged to its lower bound, which makes the SEP independent of both the FEC code and the SNR value, as will be demonstrated later in the paper.

The major contributions of this paper are

- We provide necessary, and if possible sufficient, conditions to design a full-rate full-diversity LDC, optimised so as to minimise the SEP at the MMSE–IC output and with reduced PAPR.
- We make explicit the construction of such codes for two and four transmit antennas. The proposed construction can be extended to any number of transmit antennas.

*Notations:* Column vectors (resp. matrices) are denoted by boldface lower (resp. capital) case letters. Superscripts  $(\cdot)^T$  and  $(\cdot)^H$  stand for transpose and conjugate transpose, respectively.  $\mathbf{I}_n$  represents the  $n \times n$  identity matrix.  $\mathbb{Z}$ ,  $\mathbb{C}$  and  $\mathbb{Z}[i]$  denote, respectively, the ring of rational integers, the field of complex numbers and the ring of complex integers.  $\mathbf{0}_{n \times m}$  (resp.  $\mathbf{1}_n$ ) denotes the  $n \times m$  matrix having all its elements equal to 0 (resp. 1). Subscripts of matrices indicate their dimensions. The  $\text{vec}(\cdot)$  operator transforms a matrix to a column vector by stacking the associated matrix columns.  $\text{diag}(a_1, \dots, a_n)$  denotes the  $n \times n$  diagonal matrix with entries equal to  $a_k$ ,  $k = 1, \dots, n$ .  $A_{l\cdot}$  (resp.  $A_{\cdot l}$ ) represents the row  $l$  (resp. the column  $l$ ) of the matrix  $A$ .  $\otimes$  denotes the Kronecker product of two matrices  $A$  and  $B$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \dots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nm}B \end{bmatrix}$$

## 2 System model and first design criteria

We consider a MIMO system, with  $n_t$  transmit and  $n_r$  receive antennas, affected by a non-frequency selective Rayleigh block-fading channel. We assume a coherent scenario where the channel is perfectly estimated at the receiver. The received signal is expressed as

$$\mathbf{Y}_{n_r \times T} = \mathbf{H}_{n_r \times n_t} \mathbf{X}_{n_t \times T} + \mathbf{N}_{n_r \times T} \quad (1)$$

where  $T$  is the number of symbol durations to transmit a space-time codeword denoted  $\mathbf{X}_{n_t \times T}$ .  $\mathbf{H}_{n_r \times n_t}$  represents the channel matrix with independent and identically distributed zero-mean unit-variance circularly symmetric Gaussian complex entries.  $\mathbf{N}_{n_r \times T}$  is the noise matrix assumed to be circularly symmetric jointly Gaussian complex with zero mean and covariance matrix  $\sigma^2 \mathbf{I}_{n_r}$ .

A space-time codeword of a linear dispersion STB code can be defined as follows

$$\mathbf{X}_{n_t \times T} = \sum_{l=1}^Q \mathbf{A}_l s_l \quad (2)$$

where  $s_l$ ,  $l = 1, \dots, Q$  are the modulation symbols carved from a constellation. Each modulation symbol  $s_l$  is precoded by a matrix  $\mathbf{A}_l$  of dimension  $n_t \times T$ . To simplify the paper presentation and without loss of generality, we only consider square STB codes where  $T = n_t$ .

Let us define the first constraints considered for the LDC design.

### 2.1 Energy conservation constraint

We impose that the average transmitted power of the space-time coded system be the same as the uncoded system, leading to the first following constraint on the precoding matrices

$$E(\text{tr}(\mathbf{X}\mathbf{X}^H)) = Q \Rightarrow \sum_{l=1}^{n_t} \text{tr}(\mathbf{A}_l \mathbf{A}_l^H) = Q \quad (3)$$

where  $\text{tr}(\cdot)$  denotes the trace of a square matrix,  $E(\cdot)$  denotes the expectation and the average power of a modulation symbol is normalised ( $\sigma_s^2 = E(|s_l|^2) = 1$ ,  $l = 1, \dots, Q$ ).

### 2.2 Information lossless constraint

To avoid any degradation of the channel capacity for the coded system, linear dispersion STB codes must be information lossless codes [6]. In other words, the mutual information of the coded system must be the same as the uncoded spatial multiplexing system. We need to model the space-time transmission scheme (coded system) as an equivalent spatial multiplexing scheme (uncoded system with higher transmit and receive antenna numbers). We thus apply the  $\text{vec}(\cdot)$  operator to the (1). We obtain

$$\mathbf{y}_{n_r T} = \mathbf{H}_{n_r \times n_t} \otimes \mathbf{I}_{n_t} \mathbf{x}_{n_t T} + \mathbf{n}_{n_r T} \quad (4)$$

where  $\mathbf{y}_{n_r T} = \text{vec}(\mathbf{Y})$ ,  $\mathbf{x}_{n_t T} = \text{vec}(\mathbf{X})$ ,  $\mathbf{n}_{n_r T} = \text{vec}(\mathbf{N})$  and  $\otimes$  denotes the Kronecker product. Let  $\tilde{\mathbf{H}} = \mathbf{H} \otimes \mathbf{I}_{n_t}$  and

$\mathbf{a}_l = \text{vec}(\mathbf{A}_l)$ . We can thus write  $\mathbf{x}$  as follows

$$\mathbf{x} = \underbrace{[\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_Q]}_F \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_Q \end{bmatrix} \quad (5)$$

$$\mathbf{X} = \mathbf{F}\mathbf{s} \quad (6)$$

Using (6) in (4), we obtain a new expression for  $\mathbf{y}$

$$\mathbf{y} = \mathbf{H}_{\text{eq}}\mathbf{s} + \mathbf{n}, \quad \text{where } \mathbf{H}_{\text{eq}} = \tilde{\mathbf{H}}\mathbf{F} \quad (7)$$

Thus the LDC-based MIMO system is equivalent to an uncoded spatial multiplexing-based MIMO system with  $n_r n_t$  receive antennas,  $Q$  transmit antennas and  $\mathbf{H}_{\text{eq}}$  as a channel matrix.

To satisfy the information lossless property, we have to solve

$$\begin{aligned} \log \det \left( \mathbf{I}_{n_r n_t} + \frac{\sigma_s^2}{\sigma^2} \mathbf{H}_{\text{eq}} \mathbf{H}_{\text{eq}}^H \right) \\ = \log \det \left( \mathbf{I}_{n_t} + \frac{\sigma_s^2}{\sigma^2} \mathbf{H} \mathbf{H}^H \right) \forall \mathbf{H} \end{aligned} \quad (8)$$

As we consider full-rate linear dispersion STB codes,  $Q = T n_t = n_t^2$ . A necessary and sufficient condition to guarantee the information lossless property [6] is given as follows

$$\mathbf{F}\mathbf{F}^H = \mathbf{I} \text{ and } Q = n_t^2 \quad (9)$$

By developing the above (9), we obtain a second constraint that encompasses the one given by (3)

$$\mathbf{F}^H \mathbf{F} = \begin{bmatrix} \mathbf{a}_1^H \\ \mathbf{a}_2^H \\ \vdots \\ \mathbf{a}_{n_t^2}^H \end{bmatrix} [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_{n_t^2}] = \mathbf{I}_{n_t^2} \quad (10)$$

$$\Leftrightarrow \text{tr}(\mathbf{A}_l^H \mathbf{A}_q) = \delta_{l,q}, \quad l, q = 1, \dots, n_t^2 \quad (11)$$

Linear dispersion STB codes satisfying (11) are referred to as

full-rate trace orthogonal design (TOD) [21, 27]. Equation (11) defines the first design condition.

In [20], the authors proved that a necessary and sufficient condition for a full-rate TOD to achieve minimum BER at the output of a MMSE detector, when the modulation symbols are taken from a quadrature phase shift keying constellation, is given as follows

$$\mathbf{A}_l \mathbf{A}_l^H = \frac{1}{n_t} \mathbf{I}_{n_t} \quad (12)$$

### 3 Optimal STB codes for linear MMSE receivers with a priori information

In this section, we deal with the construction of full-rate and full-diversity linear dispersion STB codes for MIMO-FEC-coded systems (see Fig. 1) equipped at the receiver side with a MMSE turbo equaliser. We thus take into account the FEC code serially concatenated with the STB code and impose the receiver structure. Firstly, we derive the lower bound of the SEP of the MMSE-IC output. Then, we propose design criteria to construct optimised STB codes for such a receiver.

A MMSE turbo equaliser is depicted in Fig. 2. This receiver structure is based on the soft reliable information exchange, according to the turbo principle, between the MMSE-IC and the FEC decoder. Such an iterative receiver structure can fully exploit the available diversity with low complexity compared to the optimal ML decoding strategy [24, 25].

#### 3.1 MMSE-IC lower bound derivation

At each iteration, the MMSE-IC benefits from the channel output and the estimated signal (a-priori information) computed at the previous iteration from the soft FEC decoder output. The MMSE-IC output can be expressed as

$$\hat{s}_l = g_l s_l + \eta_l, \quad l = 1, \dots, n_t^2 \quad (13)$$

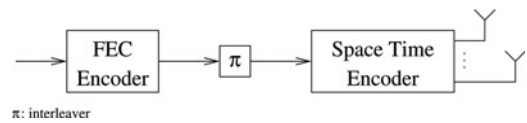


Fig. 1 MIMO-FEC-coded transmitter

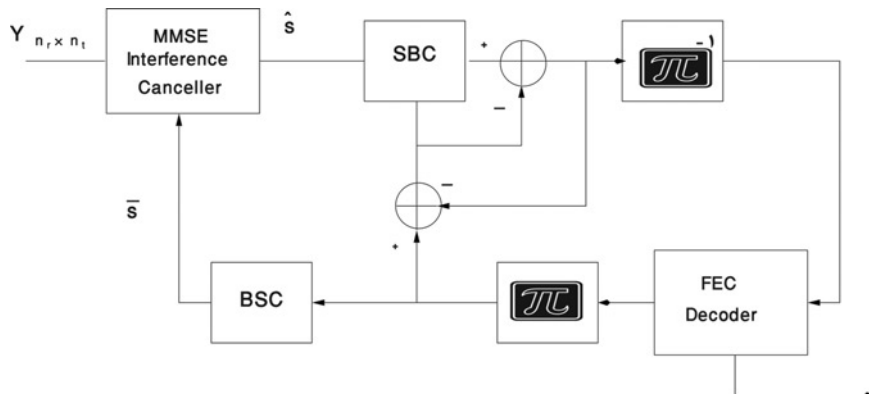


Fig. 2 Turbo MMSE equaliser

where

$$\begin{aligned}
g_l &= \mathbf{W}_{l..} \tilde{\mathbf{H}} \mathbf{F}_{..l} \quad (\text{the gain parameter}) \\
\eta_l &= \sum_{q=1, q \neq l} \mathbf{W}_{l..} \tilde{\mathbf{H}} \mathbf{F}_{..q} (s_q - \bar{s}_q) + \mathbf{W}_{l..} \mathbf{n} \quad (14) \\
&\quad (\text{the interference plus noise term})
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{W}_{l..} &= \sigma_s^2 \mathbf{F}_{..l}^H \tilde{\mathbf{H}}^H \left( (\sigma_s^2 - \sigma_{\bar{s}}^2) \sum_{q=1}^{n_t^2} \tilde{\mathbf{H}} \mathbf{F}_{..q} \mathbf{F}_{..q}^H \tilde{\mathbf{H}}^H \right. \\
&\quad \left. + \sigma_{\bar{s}}^2 \tilde{\mathbf{H}} \mathbf{F}_{..l} \mathbf{F}_{..l}^H \tilde{\mathbf{H}}^H + \sigma^2 \mathbf{I}_{n_r n_t} \right)^{-1} \quad (15)
\end{aligned}$$

and  $\bar{s}_q$  the estimate of  $s_q$  calculated from the soft information provided by the FEC decoder (variance  $\sigma_{\bar{s}}^2$ ).

When the iterative process reaches a steady state and converges to the optimal solution, the interference term in the expression of  $\eta_l$  is equal to zero and the MMSE–IC output is equivalent to the addition of the weighted transmitted symbol vector with a noise-only-dependent vector. The so-called genie bound corresponds to this optimal convergence state of the MMSE turbo equaliser and constitutes a lower bound of the MMSE–IC performance. It is obtained when the interference term is perfectly estimated at the MMSE–IC input and by considering that the symbol to binary converter is feeded with perfect soft a priori information. In such a case,  $\bar{s}_l = s_l$ ,  $l = 1, \dots, n_t^2$  and the variance of estimated symbols  $\sigma_{\bar{s}}^2$  is equal to that of modulation symbols  $\sigma_s^2$ . Thus, in the genie-aided case

$$\begin{aligned}
\mathbf{W}_{l..} &= \sigma_s^2 \mathbf{F}_{..l}^H \tilde{\mathbf{H}}^H (\sigma_s^2 \tilde{\mathbf{H}} \mathbf{F}_{..l} \mathbf{F}_{..l}^H \tilde{\mathbf{H}}^H + \sigma^2 \mathbf{I}_{n_r n_t})^{-1} \\
&= \mathbf{a}_l^H \tilde{\mathbf{H}}^H \left[ \tilde{\mathbf{H}} \mathbf{a}_l \mathbf{a}_l^H \tilde{\mathbf{H}}^H + \frac{1}{\gamma} \mathbf{I}_{n_r n_t} \right]^{-1}, \quad \text{where } \gamma = \frac{\sigma_s^2}{\sigma^2} \quad (16)
\end{aligned}$$

Applying the Woodbury's equality, we obtain

$$\begin{aligned}
\mathbf{W}_{l..} &= \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \left[ \mathbf{I}_{n_r n_t} - \frac{\gamma \tilde{\mathbf{H}} \mathbf{a}_l \mathbf{a}_l^H \tilde{\mathbf{H}}^H}{1 + \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l} \right] \\
&= \frac{\gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H}{1 + \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l} \quad (17)
\end{aligned}$$

Using (17), the gain parameter  $g_l$  and the noise term  $\eta_l$  can be expressed as

$$g_l = \frac{\gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l}{1 + \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l}, \quad l = 1, \dots, n_t^2 \quad (18)$$

$$\eta_l = \mathbf{W}_{l..} \mathbf{n} \quad (19)$$

where  $\eta_l$  is a zero-mean circularly symmetric Gaussian complex variable with variance  $\sigma_{\eta}^2 = \sigma^2 \mathbf{W}_{l..} \mathbf{W}_{l..}^H = \sigma^2 g_l (1 - g_l) \gamma = \sigma_s^2 g_l (1 - g_l)$ . [The inverse of a matrix  $\mathbf{A} = \mathbf{B} + \mathbf{u} \mathbf{u}^H$  is given as  $\mathbf{A}^{-1} = \mathbf{B}^{-1} - ((\mathbf{B}^{-1} \mathbf{u} \mathbf{u}^H \mathbf{B}^{-1}) / (1 + \mathbf{u} \mathbf{B}^{-1} \mathbf{u}^H))$ .]

### 3.2 Additional design criteria

From now on we assume that the information lossless condition given in (11) is satisfied (condition 1).

(1) *Condition 2 – SEP optimisation at the MMSE–IC output*: Considering the optimal convergence state for the MMSE–IC, the SNR for the  $l$ th output is given by

$$\text{SNR}_l = \frac{g_l}{(1 - g_l)} = \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l, \quad l = 1, \dots, n_t^2 \quad (20)$$

By taking the expectation of  $\text{SNR}_l$  with respect to the channel realisation, we obtain

$$\begin{aligned}
\overline{\text{SNR}}_l &= \mathbb{E}_H(\text{SNR}_l) = \gamma \mathbf{a}_l^H (\mathbf{I}_{n_t} \otimes \mathbb{E}(\mathbf{H}^H \mathbf{H})) \mathbf{a}_l = \gamma \mathbf{a}_l^H \mathbf{a}_l \\
&= \gamma \text{tr}(\mathbf{A}_l^H \mathbf{A}_l) = \gamma \quad (21)
\end{aligned}$$

To minimise the SEP at the MMSE–IC output, we must maximise  $\overline{\text{SNR}}_l$  and minimise the correlation between the interference terms affecting two distinct equaliser output components.

One can see that the average SNR is independent of the precoded matrices when the condition given in (11) is satisfied. The FEC decoder is all the more efficient as the soft information provided at its input are less correlated. This can be achieved by reducing the correlation between the noise components relative to two MMSE–IC output components,  $\eta_q$  and  $\eta_l$  whatever  $q \neq l$ . The crosscorrelation between  $\eta_q$  and  $\eta_l$  is denoted  $\rho_{q,l}$  and is equal to

$$\begin{aligned}
\rho_{q,l} &= \mathbb{E}(\eta_q \eta_l^*) = \mathbb{E}(\mathbf{W}_{q..} \mathbf{n} \mathbf{n}^H \mathbf{W}_{l..}^H) = \sigma^2 \mathbf{W}_{q..} \mathbf{W}_{l..}^H \\
&= \frac{\sigma^2 \gamma^2 \mathbf{a}_q^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l}{(1 + \gamma \mathbf{a}_q^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_q)(1 + \gamma \mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l)} \quad (22)
\end{aligned}$$

For high SNR values

$$\begin{aligned}
|\rho_{q,l}|^2 &\simeq \frac{|\sigma^2 \mathbf{a}_q^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l|^2}{|\mathbf{a}_q^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_q|^2 |\mathbf{a}_l^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{a}_l|^2} \\
&= \sigma^4 \frac{|\langle \mathbf{a}_l, \mathbf{a}_q \rangle_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}|^2}{\|\mathbf{a}_q\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^4 \|\mathbf{a}_l\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^4} \quad (23)
\end{aligned}$$

where  $\langle \mathbf{x}, \mathbf{y} \rangle_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}} = \mathbf{y}^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{x}$ .

Designing precoded matrices so as to minimise  $|\rho_{q,l}|^2$  implies that the channel coefficient must be known at the transmitter, as each precoded matrix coefficient is a function of fading parameters in this case. As we assume no channel state information at the transmitter, we have to focus on the  $|\rho_{q,l}|^2$  upper bound minimisation problem. In this case, a necessary condition can be delivered in order to minimise the crosscorrelation.

By applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
|\rho_{q,l}|^2 &\leq \sigma^4 \frac{\|\mathbf{a}_q\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^2 \|\mathbf{a}_l\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^2}{\|\mathbf{a}_q\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^4 \|\mathbf{a}_l\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^4} \\
&= \frac{\sigma^4}{\|\mathbf{a}_q\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^2 \|\mathbf{a}_l\|_{\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}}^2} \\
&= \frac{\sigma^4}{(\mathbf{z}_q^H \mathbf{D} \mathbf{z}_q)(\mathbf{z}_l^H \mathbf{D} \mathbf{z}_l)} \quad (24)
\end{aligned}$$

where  $\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} = \mathbf{U}^H \mathbf{D} \mathbf{U}$  is the eigenvalue decomposition of the Hermitian matrix  $\tilde{\mathbf{H}}^H \tilde{\mathbf{H}}$  with  $\mathbf{U}$  an unitary matrix,  $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_{n_t}^2)$ ,  $\lambda_k \geq 0$ ,  $k = 1, \dots, n_t^2$  and  $\mathbf{z}_l = \mathbf{U} \mathbf{a}_l$ .

A necessary constraint to minimise the upper bound of  $|\rho_{q,l}|^2$  presented in (24) is

$$|z_q(n)| = |z_l(n)|, \quad n = 1, \dots, n_t^2 \quad (25)$$

*Proof:* Let us consider  $\mathbf{z}_q^{\text{opt}}$  and  $\mathbf{z}_l^{\text{opt}}$ ,  $l \neq q$  achieving the minimum of the upper bound given in (24). Minimising the upper bound of  $|\rho_{q,l}|^2$  is equivalent to maximise  $\beta(\mathbf{z}_q, \mathbf{z}_l) = (\mathbf{z}_q^H \mathbf{D} \mathbf{z}_q)(\mathbf{z}_l^H \mathbf{D} \mathbf{z}_l)$ . Using the definition of  $\mathbf{D}$ ,  $\beta(\mathbf{z}_q, \mathbf{z}_l) = \left(\sum_n \lambda_n |z_q(n)|^2\right) \left(\sum_n \lambda_n |z_l(n)|^2\right)$ . Let us take  $\mathbf{z}_l$  and  $\mathbf{z}_q$  such that  $|z_l(n)| = |z_q(n)| = |z_q^{\text{opt}}(n)|$ . We can write

$$\begin{aligned} \beta(\mathbf{z}_q, \mathbf{z}_l) &\leq \beta(\mathbf{z}_q^{\text{opt}}, \mathbf{z}_l^{\text{opt}}) \\ &\Leftrightarrow \left(\sum_n \lambda_n |z_q^{\text{opt}}(n)|^2\right)^2 \leq \left(\sum_n \lambda_n |z_q^{\text{opt}}(n)|^2\right) \left(\sum_n \lambda_n |z_l^{\text{opt}}(n)|^2\right) \\ &\Leftrightarrow \left(\sum_n \lambda_n |z_q^{\text{opt}}(n)|^2\right) \leq \left(\sum_n \lambda_n |z_l^{\text{opt}}(n)|^2\right) \end{aligned} \quad (26)$$

Defining  $\mathbf{z}_l$  and  $\mathbf{z}_q$  such that  $|z_l(n)| = |z_q(n)| = |z_l^{\text{opt}}(n)|$ , with a similar procedure, we obtain

$$\left(\sum_n \lambda_n |z_l^{\text{opt}}(n)|^2\right) \leq \left(\sum_n \lambda_n |z_q^{\text{opt}}(n)|^2\right) \quad (27)$$

Considering (26) and (27) yields

$$\left(\sum_n \lambda_n |z_l^{\text{opt}}(n)|^2\right) = \left(\sum_n \lambda_n |z_q^{\text{opt}}(n)|^2\right) \quad (28)$$

As the condition given in (28) must hold whatever the channel realisation, we deduce that  $|z_l^{\text{opt}}(n)| = |z_q^{\text{opt}}(n)|$ ,  $n = 1, 2, \dots, n_t^2$ .

By developing  $\mathbf{z}_q$ , we obtain

$$\mathbf{z}_q(n) = \mathbf{U}_{n..} \mathbf{a}_q \quad (29)$$

Using (25) and (29), we deduce that for  $l \neq q$  and  $n \in \{1, 2, \dots, n_t^2\}$ , there exists  $\theta_{ql}(n) \in ]0, 2\pi[$  such that

$$\mathbf{U}_{n..} \mathbf{a}_q = \mathbf{U}_{n..} \exp(i\theta_{ql}(n)) \mathbf{a}_l \quad (30)$$

As this condition must hold whatever the channel realisation, and whatever the value of  $n$ , we deduce that there exists  $\theta_{ql} \in ]0, 2\pi[$  such that

$$\mathbf{a}_q = e^{i\theta_{ql}} \mathbf{a}_l \quad (31)$$

Such a constraint is not compatible with  $\mathbf{a}_q^H \mathbf{a}_l = \delta_{ql}$  given by (11). As the priority is given to the information lossless property, we replace the necessary and sufficient condition of (31) by a necessary but not sufficient condition,

compatible with  $\mathbf{a}_q^H \mathbf{a}_l = \delta_{ql}$ . We impose that

$$|a_q(n)| = |a_l(n)|q \neq l, \quad n = 1, \dots, n_t^2 \quad (32)$$

This constraint is a necessary condition to minimise the correlation between noise terms affecting two equaliser output components. Moreover, it guarantees a uniform distribution of the energy transmitted by each antenna and each modulation symbol (we note that a transmitted symbol is a linear combination of  $n_t$  modulation symbols).

By satisfying (11) and (32), we ensure an improvement of the error probability at the MMSE-IC output compared with full-rate full-diversity LDC that do not fulfill these conditions.

(2) *Condition 3 – full-diversity LDC:* We assume that condition 1 (11) and condition 2 (32) are fulfilled. To further ensure a full-diversity LDC, we apply the TAST approach [7]. For the self-completeness of this paper, we recall briefly the threaded algebraic space-time code definition.

A TAST codeword is composed of  $n_t$  layers. Let  $\ell_l$  denotes the layer number  $l$ . It is generated by  $(s_{(l-1)n_t+q})_{1 \leq q \leq n_t}$  and transmitted on time slot  $t$ ,  $1 \leq t \leq n_t$ , by the  $t$ th antenna. Then we associate, to each layer,  $n_t$  precoded matrices  $\mathbf{A}_{(l-1)n_t+q} = \{\phi_l a_{n,p,q}\}_{n,p=1,\dots,n_t}$  where  $a_{n,p,q}$  is defined as follows

$$\begin{aligned} a_{n,p,q} &\neq 0, \quad \text{if } (n,p) \in \{(t+l-2n_t+1, t)\} t = 1, \dots, \\ &n_t = 0, \quad \text{else} \end{aligned} \quad (33)$$

and  $\phi_l$  ( $|\phi_l| = 1$ ) is a diophantine number that ensures an algebraic separation between threads to guarantee the full-transmit diversity. We note that given  $q$ ,  $\mathbf{A}_{(k-1)n_t+q}$  and  $\mathbf{A}_{(l-1)n_t+q}$  associated to layers  $\ell_k$  and  $\ell_l$ , respectively, differ only by the diophantine numbers  $\phi_k$  and  $\phi_l$ . The full-transmit diversity is achieved if each thread ensures a full-transmit diversity when the others are inactive, and by selecting appropriate diophantine numbers that provide an algebraic separation between threads [7].

Coming back to our design, assuming the LDC has the threaded structure described above, the condition given by (32) has to be satisfied inside the group of precoded matrices that belong to a same thread indexed by  $l$ . Let  $\tilde{\mathbf{a}}_{(l-1)n_t+q}$  be the vector associated to the  $n_t$  non-equal to zero elements of  $\mathbf{A}_{(l-1)n_t+q}$  [according to (33),  $\mathbf{A}_{(l-1)n_t+q}$  contains only  $n_t$  non-equal to zero entries]

$$\begin{aligned} \tilde{\mathbf{a}}_{(l-1)n_t+q} &= \phi_l [a_{l,1,q} \quad a_{l+1,2,q} \cdots a_{n_t, n_t-l+1,q} \quad a_{1, n_t-l+2,q} \\ &\quad a_{2, n_t-l+3,q} \cdots a_{l-1, n_t,q}] \end{aligned} \quad (34)$$

Considering (11) and (32), we obtain

$$\begin{aligned} \tilde{\mathbf{a}}_{(l-1)n_t+q}^H \tilde{\mathbf{a}}_{(n-1)n_t+p} &= \delta_{(l,q),(n,p)}, \\ n, l, p, q &= 1, \dots, n_t \end{aligned} \quad (35)$$

$$\begin{aligned} |\tilde{\mathbf{a}}_{(l-1)n_t+q}(n)| &= |\tilde{\mathbf{a}}_{(l-1)n_t+p}(n)|, \\ n, l, p, q &= 1, \dots, n_t \end{aligned} \quad (36)$$

Each thread realises a full-transmit diversity if and only if

$$\forall (s_{(l-1)n_t+1}, \dots, s_{ln_t}) \neq (0, \dots, 0) \in \mathcal{S}^{n_t} \left| \sum_{q=1}^{n_t} \tilde{a}_{(l-1)n_t+q}(n) s_{(l-1)n_t+q} \right| \neq 0 \quad (37)$$

where  $\mathcal{S}$  is a ring that encompasses the modulation alphabet. To have an additional energy efficiency, each antenna must transmit the same average power, thus

$$E(|x_{p,q}|^2) = \sigma_s^2, \quad p, q = 1, \dots, n_t \quad (38)$$

where  $x_{p,q}$  is an entry of a STB codeword. Taking (35), (36) and (38) into account, we can write each entry of a precoded matrix as follows

$$\tilde{a}_k(n) = \frac{1}{\sqrt{n_t}} \exp(i\theta_{k,n}), \quad k = 1, \dots, n_t^2, \quad n = 1, \dots, n_t \quad (39)$$

(3) *Condition 4 – PAPR reduction:* Last construction criterion is the PAPR reduction. Indeed, STB codes introduce a PAPR increase with respect to spatial multiplexing as each transmitted symbol is a linear combination of modulation symbols. High PAPR results in high back-off on the power amplifier to keep it in its linear regime. The PAPR minimisation allows to fix the values of  $\theta_{k,n}$  depending on the number of transmit antennas, by solving this optimisation problem

$$\min_{\{\theta_{k,n}\}} \max_{(s_1, s_2, \dots, s_{n_t}) \in \mathcal{S}^{n_t}} |x_{p,q}|^2 \quad (40)$$

In some transmit antenna configuration, both maximisation of the diversity and minimisation of the PAPR are not compatible. In that case, the priority is given to the diversity and among the set of angles satisfying the diversity constraint, we choose the ones that yield the lower PAPR.

## 4 Explicit construction

As described in the previous section, the TAST concept is applied to the LDC using the same set of precoding vectors for all threads and choosing the appropriate diophantine numbers to separate threads. Let  $a_k$  represent the precoding vector associated to the  $k$ th symbol of the reference thread ( $\ell_1$ ). We select  $\tilde{a}_1 = (1/(\sqrt{n_t})) [1 \ 1 \dots 1]^T$  and  $\phi_1 = 1$  to simplify the construction. We have to determine the  $n_t - 1$  vectors  $\tilde{a}_{k \neq 1}$  such that (35), (36), (39) and (40) are satisfied. To accomplish the construction, diophantine numbers must be chosen according to the following constraints

$$\begin{cases} |\phi_l| = 1 \\ \phi_l \notin \text{the field that encompasses the following ring} \\ \mathcal{S}^t = \left\{ \sum_{q=1}^{n_t} \tilde{a}_q(n) s_q, \quad s_q \in \mathcal{S} \right\}, \quad n, l = 1, \dots, n_t \end{cases} \quad (41)$$

### 4.1 Application of the procedure for $n_t = 2$

We have to determine  $\tilde{a}_2 = (1/\sqrt{2}) [e^{i\theta_1} e^{i\theta_2}]$ . According to (35) and (36), we have  $e^{i\theta_2} = -e^{i\theta_1}$ . Then

$$x_{11} = \frac{1}{\sqrt{2}} (s_1 + e^{i\theta_1} s_2)$$

$$x_{22} = \frac{1}{\sqrt{2}} (s_1 - e^{i\theta_1} s_2)$$

For PAPR minimisation, we must consider the modulation type. Without any loss of generality, we are interested in this paper in a quadrature amplitude modulation (QAM) constellation. For such a modulation, the maximum absolute value of a transmitted symbol  $|x_q|$  is achieved when  $|s_1| = |s_2| = A_{\max}$ . Taking into account the diversity constraint (37), which can be summarised in this case as  $|x_{qq}| \neq 0 \forall (s_1, s_2) \neq \mathbf{0}, q = 1, 2$ , the PAPR constraint yields

$$\theta_1^{\text{optimal}} = \arg \min_{\{\theta_1\}} \max_{\alpha \in \{0, (\pi/2), \pi, (3\pi/2)\}} |1 + e^{i(\theta_1 + \alpha)}|^2 \quad (42)$$

The solution of the optimisation problem defined by (42) is  $\theta_1^{\text{optimal}} = (\pi/4)$  (see the Appendix for the proof) and the proposed  $2 \times 2$  STB code can be written as

$$\mathbf{X}_{2 \times 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} s_1 + e^{i(\pi/4)} s_2 & \phi_2 (s_3 - e^{i(\pi/4)} s_4) \\ \phi_2 (s_3 + e^{i(\pi/4)} s_4) & s_1 - e^{i(\pi/4)} s_2 \end{bmatrix} \quad (43)$$

### 4.2 Application of the procedure for $n_t = 4$

In order to simplify our construction, we consider a canonical base  $\{1, e^{i\theta_k}, e^{2i\theta_k}, e^{3i\theta_k}\}$  for each symbol of a fixed thread. We define  $a_k$  as

$$\tilde{a}_k = \frac{1}{\sqrt{4}} [e^{i(k-1)\theta_1} \quad e^{i(k-1)\theta_2} \quad e^{i(k-1)\theta_3} \quad e^{i(k-1)\theta_4}] \quad (44)$$

By applying (35) and (36), we obtain the following conditions

$$e^{i(k-1)\theta_1} e^{i(k-1)\theta_2} e^{i(k-1)\theta_3} e^{i(k-1)\theta_4} = 0, \quad k = 2, 3, 4 \quad (45)$$

$\theta_q = (i)^{q-1} \theta, q = 1, 2, 3, 4$  is a solution of (45). Thus

$$x_{11} = \frac{1}{2} (s_1 + e^{i\theta} s_2 + e^{i2\theta} s_3 + e^{i3\theta} s_4)$$

$$x_{22} = \frac{1}{2} (s_1 + ie^{i\theta} s_2 - e^{i2\theta} s_3 - ie^{i3\theta} s_4)$$

$$x_{33} = \frac{1}{2} (s_1 - ie^{i\theta} s_2 - e^{i2\theta} s_3 + ie^{i3\theta} s_4)$$

$$x_{44} = \frac{1}{2} (s_1 - e^{i\theta} s_2 + e^{i2\theta} s_3 - e^{i3\theta} s_4)$$

Taking the diversity maximisation and the PAPR minimisation conditions, respectively, given by (37) and (40), we obtain  $\theta = (\pi/8)$  (see the Appendix for the proof). As a result, the  $4 \times 4$  proposed STB code is defined by

$$\mathbf{X}_{4 \times 4} = \frac{1}{2} \begin{bmatrix} x_{11} & x_{42} & x_{33} & x_{24} \\ x_{21} & x_{12} & x_{43} & x_{34} \\ x_{31} & x_{22} & x_{13} & x_{44} \\ x_{41} & x_{32} & x_{23} & x_{14} \end{bmatrix}$$

where

$$\begin{aligned}
 x_{lk} &= \phi_l f_k(s_{4l-3}, s_{4l-2}, s_{4l-1}, s_{4l}) \\
 f_1(a, b, c, d) &= a + e^{i\theta}b + e^{i2\theta}c + e^{i3\theta}d \\
 f_2(a, b, c, d) &= a + ie^{i\theta}b - e^{i2\theta}c - ie^{i3\theta}d \\
 f_3(a, b, c, d) &= a - e^{i\theta}b + e^{i2\theta}c - e^{i3\theta}d \\
 f_4(a, b, c, d) &= a - ie^{i\theta}b - e^{i2\theta}c + ie^{i3\theta}d
 \end{aligned}$$

One can see that the design of the  $4 \times 4$  code is more difficult than the design of the  $2 \times 2$  code. The definition of the precoding matrix satisfying the conditions given in previous sections is obtaining harder as the transmit antenna number increases. The complexity lies in the construction (as it is the case in most other LDC schemes). Once the LDC has been defined according to our proposed procedure, neither the transmitter nor the receiver exhibit any complexity increase compared with other LDC.

## 5 Performance comparisons

This section is dedicated to the simulation of the proposed scheme for  $n_t = 2$  and  $n_t = 4$ . The PSTB code has full-rate, achieves optimal diversity and satisfies the non-vanishing determinant property [10]. Its optimisation assumes a ML detection at the receiver. It is also known in the literature as one of the best performing STB codes in terms of BER. For that reason, we use it as a reference scheme in this section.

### 5.1 PAPR comparison

Compared with the  $2 \times 2$  perfect space-time block code (PSTBC) code (also known as the golden code), and whatever the modulation order, the  $2 \times 2$  proposed LDC code achieves a gain in terms of PAPR equal to (see the

Appendix for the proof)

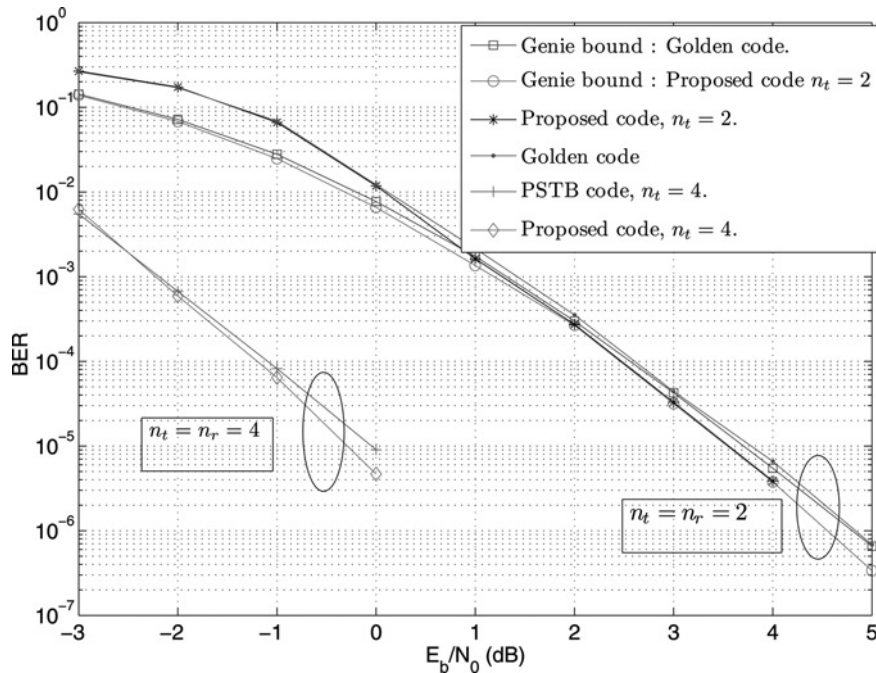
$$\begin{aligned}
 \text{GPAPR}_2 &= 10 \log_{10} \frac{2(1 + \theta_{\text{GC}})^2}{(2 + \sqrt{2})(1 + \theta_{\text{GC}}^2)} = 0.4522 \text{ dB}, \\
 \theta_{\text{GC}} &= \frac{1 + \sqrt{5}}{2}
 \end{aligned} \tag{46}$$

In the  $4 \times 4$  case, the proposed code has a 0.3072 dB PAPR gain with respect to the PSTB code (see the Appendix) using QAM constellation.

### 5.2 BER comparison

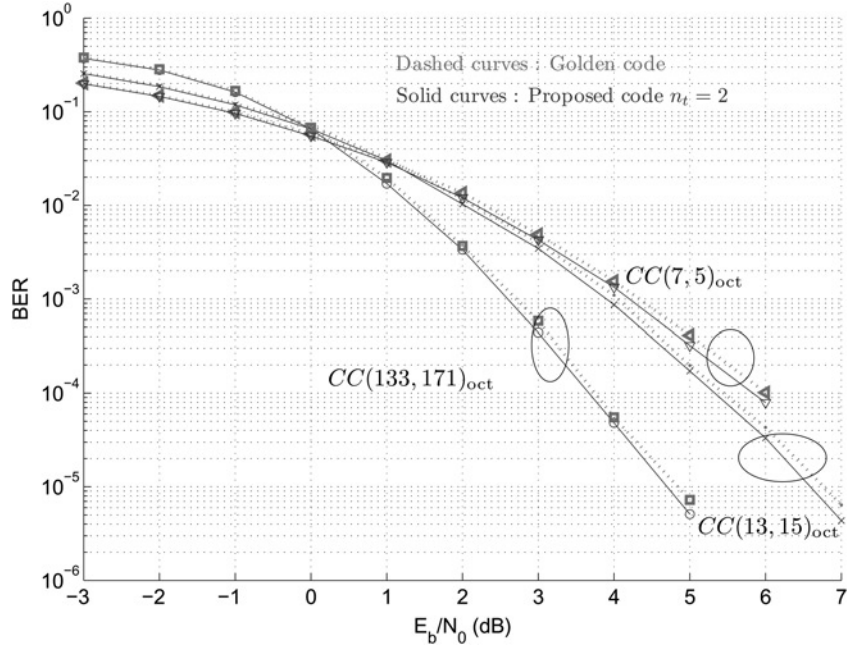
We assume that the proposed code and the PSTBC are serially concatenated to the same outer FEC code (see Fig. 1) and that the receiver consists of a MMSE-IC exchanging information with the FEC decoder in an iterative manner as described in previous sections (see Fig. 2). The BER is measured at the FEC decoder output.

For our simulations, we consider a number of receive antennas equal to the number of transmit antennas. We study QAM constellations with Gray mapping. The channel is a non-frequency selective Rayleigh block fading channel, constant over  $\tau = n_t$  symbol durations. The outer code is a half-rate convolutional code, decoded by applying a soft-input-soft-output Bahl-Cocke-Jelinek-Raviv (BCJR) decoder [26]. We consider the following convolutional codes (CCs)  $\text{CC}(7, 5)_{\text{oct}}$ ,  $\text{CC}(13, 15)_{\text{oct}}$  and  $\text{CC}(133, 171)_{\text{oct}}$  in our simulations. The information frame at the encoder input of the FEC outer code is composed of 4077 information bits in order to guarantee that the BER performance do not change with a higher length frame. We note that plotted curves correspond to the convergence state of the iterative process (i.e. after a given number of iterations, there is no additional performance gain by increasing the iteration number).

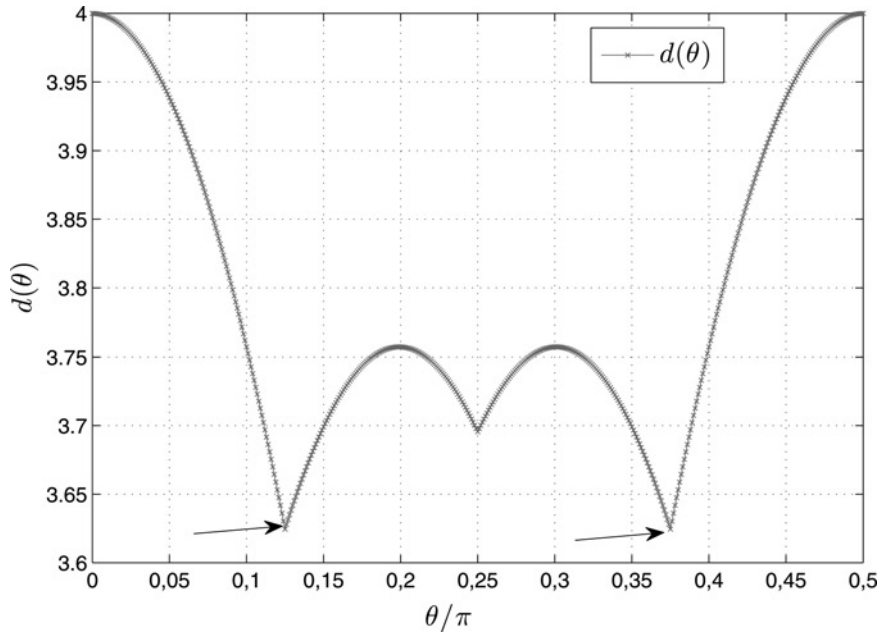


**Fig. 3** Performance comparison between the proposed STB codes and the PSTB codes for  $n_t = 2, 4$  4-QAM,  $\text{CC}(13, 15)_{\text{oct}}$ ,  $\tau = n_t, n_r = n_t$  4077 information bits





**Fig. 4** Genie bound comparison between the proposed STBC code ( $n_t = 2$ ) and the golden code for different outer half-rate CCs 16-QAM,  $\tau = n_t$ ,  $n_r = n_t$ . 4077 information bits



**Fig. 5**  $d(\theta)$  function

Fig. 3 shows that, for  $n_t = 2$ , both schemes reach their respective genie bound after four iterations. Moreover, the proposed code outperforms by 0.25 dB the golden code at a BER =  $10^{-6}$ . For  $n_t = 4$ , similar conclusions can be drawn. A gain of 0.2 dB at a BER =  $10^{-5}$  is achieved by the proposed code with respect to the PSTB code.

Fig. 4 compares the genie bounds of both schemes for  $n_t = 2$ , using a 16-QAM constellation, for different half-rate CCs. One can see that the scheme using the proposed optimised code outperforms the scheme based on the golden code, whatever the correction capability of the outer convolutional code. This results confirm that the lower bound performance of the proposed codes is better than the existing codes independently of the error correction capability of the outer code.

For all simulated configurations, both the genie bound and the convergence state of the turbo equaliser are better for the proposed code than for the PSTBC. This outperformance comes from the SEP reduction criterion, used in the construction of the proposed code. It allows the MMSE turbo equaliser to fully exploit the available diversity.

## 6 Conclusion

In this paper, we have proposed a new approach to construct full-rate full-diversity STB codes taking into account the serial concatenation with an outer FEC code and assuming an iterative receiver consisting of a MMSE-IC exchanging information with the FEC decoder. We have considered full-rate LDCs. The precoding matrices have been optimised so

as to satisfy four constraints. First, we have imposed that the full-rate LDC be information lossless. This constraint guarantee no capacity degradation compared to the equivalent uncoded spatial multiplexing structure. Second, necessary conditions have been established to reduce the SEP at the MMSE–IC output, through noise-equivalent component de-correlation. Third, full-diversity has been achieved applying the threaded layering concept (TAST). The two latest constraints ensure an improvement of the performance in terms of BER. Fourth, the PAPR minimisation has been imposed to end the code definition. Explicit construction of such codes has been presented for  $n_t = 2$  and 4. The simulations have shown that the resulting codes, when concatenated to an outer FEC code and using a MMSE–IC turbo equaliser, outperform the PSTB codes both in terms of BER lower bound and achieved BER, without any complexity penalty neither at the transmitter nor at the receiver. The theoretical PAPR computation has exhibited a gain for  $n_t = 2$  and 4 whatever the modulation order.

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## 9 Appendix

Assuming a QAM constellation, we compute in this Appendix optimal values of  $\theta_{k,n}$ ,  $k = 1, \dots, n_t^2$ ,  $n = 1, \dots, n_t$  defined in (40). We also calculate the PAPR values for the golden code, the PSTB code ( $n_t = 4$ ) and the proposed STB codes ( $n_t = 2, 4$ ).

Let us denote  $\mathcal{K}$  the set defined by  $\mathcal{K} = \{\pm 1, \pm 3, \dots, \pm(\sqrt{2^{m_b}} - 1)\}$ . The  $2^{m_b}$ -QAM alphabet is defined as  $\mathcal{S} = \mathcal{K}[i] = \{a + ib | a, b \in \mathcal{K}\}$ . Let  $s$  be a modulation symbol taking values with equal probability in  $\mathcal{S}$ . One can prove that  $E(|s|^2) = 2/3(2^{m_b} - 1)$ .

### 9.1 PAPR minimisation for the $2 \times 2$ proposed STB code

For  $n_t = 2$ , the optimal value of  $\theta$  is defined as

$$\theta_{\text{opt}} = \arg \min_{\{\theta_1\}} \max_{\alpha \in \{0, \pi/2, \pi, 3\pi/2\}} |1 + e^{i(\theta_1 + \alpha)}|^2 \quad (47)$$

To solve the above optimisation problem, four cases can be considered

$$\begin{cases} \alpha = 0 \Rightarrow d_1(\theta) = |1 + e^{i(\theta + \alpha)}|^2 = |1 + e^{i\theta}|^2 = 2 + 2 \cos(\theta) \\ \alpha = \frac{\pi}{2} \Rightarrow d_2(\theta) = |1 + e^{i(\theta + \alpha)}|^2 = |1 + ie^{i\theta}|^2 = 2 - 2 \sin(\theta) \\ \alpha = \pi \Rightarrow d_3(\theta) = |1 + e^{i(\theta + \alpha)}|^2 = |1 - e^{i\theta}|^2 = 2 - 2 \cos(\theta) \\ \alpha = \frac{3\pi}{2} \Rightarrow d_4(\theta) = |1 + e^{i(\theta + \alpha)}|^2 = |1 - ie^{i\theta}|^2 = 2 + 2 \sin(\theta) \end{cases} \quad (48)$$

Thus

$$\theta_{\text{opt}} = \min_{0 \leq \theta \leq \pi/2} \max(d_1(\theta), d_4(\theta)) \quad (49)$$

For  $0 \leq \theta \leq \pi/2$ , the function  $d_1(\theta)$  decreases from  $d_1(0) = 4$  to  $d_1(\pi/2) = 2$ , and the function  $d_4(\theta)$  increases from  $d_4(0) = 2$  to  $d_4(\pi/2) = 4$ . Then, the solution of (49) corresponds to the intersection point between  $d_1(0)$  and

$d_4(\theta)$ , and it can be determined by solving

$$\begin{aligned} d_1(\theta_{\text{opt}}) = d_4(\theta_{\text{opt}}) &\Rightarrow \cos(\theta_{\text{opt}}) = \sin(\theta_{\text{opt}}), \\ 0 \leq \theta &\leq \frac{\pi}{2} \end{aligned} \quad (50)$$

We obtain  $\theta_{\text{opt}} = \pi/4$ . Calculating the PAPR for the proposed code ( $n_t = 2$ ) taking into account that the maximum modulus of any STB code matrix entry corresponds to a linear combination of modulation symbols with maximum modulus, we have

$$\begin{aligned} \text{PAPR} &= \frac{\max_{(s_1, s_2) \in \mathcal{S}^2} |s_1 + e^{i\pi/4} s_2|^2}{\text{E}(|s_1 + e^{i\pi/4} s_2|^2)} \\ &\text{such that } |s_1|^2 = |s_2|^2 = 2(\sqrt{2^{m_b}} - 1)^2 \end{aligned} \quad (51)$$

$$= \frac{(4 + 2\sqrt{2})(\sqrt{2^{m_b}} - 1)^2}{4/3(2^{m_b} - 1)} \quad (52)$$

$$= \frac{3(2 + \sqrt{2})(\sqrt{2^{m_b}} - 1)^2}{2(\sqrt{2^{m_b}} + 1)} \quad (53)$$

## 9.2 PAPR for the golden code

We recall that a STB codeword for the golden code can be written as follows

$$\mathbf{X}_{2 \times 2} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(s_1 + \theta s_2) & \alpha(s_3 + \theta s_4) \\ \bar{\alpha}(s_1 + \bar{\theta} s_2) & \bar{\alpha}(s_3 + \bar{\theta} s_4) \end{bmatrix} \quad (54)$$

where  $s_k \in \mathcal{S}$ ,  $k = 1, \dots, 4$ ,  $\theta = 1 + \sqrt{5}/2$ ,  $\bar{\theta} = 1 - \theta$ ,  $\alpha = 1 + i\bar{\theta}$  and  $\bar{\alpha} = 1 + i\theta$ . Thanks to the symmetry in the entries of a STB code matrix, the PAPR can be obtained as

$$\text{PAPR}_{\text{Golden}} = \frac{\max_{(s_1, s_2) \in \mathcal{S}^2} |\alpha s_1 + \theta s_2|^2}{\text{E}(|\alpha s_1 + \theta s_2|^2)} \quad (55)$$

$$= \frac{\max_{\delta \in \{0, \pm\pi/2, \pi\}} 2(\sqrt{2^{m_b}} - 1)^2 |1 + \theta e^{i\delta}|^2}{2/3(2^{m_b} - 1)(1 + \theta^2)} \quad (56)$$

$$= \frac{(1 + \theta)^2}{(1 + \theta^2)} \frac{3(\sqrt{2^{m_b}} - 1)}{(\sqrt{2^{m_b}} + 1)} \quad (57)$$

$$= \frac{(1 - \bar{\theta})^2}{(1 + \bar{\theta}^2)} \frac{3(\sqrt{2^{m_b}} - 1)}{(\sqrt{2^{m_b}} + 1)} \quad (57)$$

## 9.3 PAPR for the $4 \times 4$ proposed STB codes

For  $n_t = 4$ , each layer can be expressed as

$$x = \frac{1}{2} \phi(s_1 + \alpha e^{i\theta} s_2 + \beta e^{i2\theta} s_3 + \gamma e^{i3\theta} s_4) \quad (58)$$

where  $(s_1, s_2, s_3, s_4) \in \mathcal{S}^4$ ,  $|\phi| = 1$  and  $\alpha, \beta, \gamma \in \{1, -1, i, -i\}$ . Taking into account that the modulus of any STB code matrix entry is maximum when all modulation symbol moduli are maximum, the minimisation of the PAPR imposes that  $\theta$  must be chosen as follows

$$\theta_{\text{opt}} = \arg \max_{0 \leq \theta \leq \pi/2} \max_{(s_1, s_2, s_3, s_4) \in \mathcal{S}^4} |x|^2 \quad (59)$$

$$= \arg \min_{0 \leq \theta \leq \pi/2} \max_{(a, b, c, d) \in \{1, -1, i, -i\}^4} |(a + e^{i\theta} b + \beta e^{i2\theta} c + e^{i3\theta} d)|^2 \quad (60)$$

By plotting the function  $d(\theta) = \max_{(a, b, c, d) \in \{1, -1, i, -i\}^4} |(a + e^{i\theta} b + e^{i2\theta} c + e^{i3\theta} d)|$  (see Fig. 5), two solutions of equation (59) can be determined  $\theta_{\text{opt}} = \pi/8$  or  $\theta_{\text{opt}} = 3\pi/8$ . Since, we deal with  $e^{i\theta}$ ,  $e^{i2\theta}$  and  $e^{i3\theta}$  terms, we obtain two equivalent STB codes. According to Fig. 5, the minimal value of  $d(\theta)$  is equal to  $d(\theta_{\text{opt}}) = 3.6245$ , which corresponds to  $(a, b, c, d) = (1, 1, 1, 1)$ . Therefore the PAPR of the proposed STB code is equal to

$$\text{PAPR} = 3.6245^2 \frac{2(\sqrt{2^{m_b}} - 1)^2}{2/3(2^{m_b} - 1)} \quad (61)$$

$$= 13.1371 \frac{3(\sqrt{2^{m_b}} - 1)}{(\sqrt{2^{m_b}} + 1)} \quad (62)$$

## 9.4 PAPR for the $4 \times 4$ PSTB code

Using exhaustive computer search and focusing on maximum modulus modulation symbols as they maximize the modulus of any PSTB code matrix entry, we obtain

$$\text{PAPR} = 3.755^2 \frac{2(\sqrt{2^{m_b}} - 1)^2}{2/3(2^{m_b} - 1)} \quad (63)$$

$$= 14.1 \frac{3(\sqrt{2^{m_b}} - 1)}{(\sqrt{2^{m_b}} + 1)} \quad (64)$$