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DEFORMATIONS OF REDUCIBLE REPRESENTATIONS OF
KNOT GROUPS INTO $\text{SL}(n, \mathbb{C})$

MICHAEL HEUSENER AND OUARDIA MEDJERAB

Abstract. Let $K$ be a knot in $S^3$ and $X$ its complement. We study deformations of non-abelian, metabelian, reducible representations of the knot group $\pi_1(X)$ into $\text{SL}(n, \mathbb{C})$ which are associated to a simple root of the Alexander polynomial. We prove that some of these metabelian reducible representations are smooth points of the $\text{SL}(n, \mathbb{C})$-representation variety and that they have irreducible deformations.

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1. Introduction

Let $K$ be a knot in $S^3$ and $X = S^3 \setminus V(K)$ its complement, where $V(K)$ is a tubular neighborhood of $K$. Moreover, let $\Gamma_K = \pi_1(X)$ denote the fundamental group of $X$. The aim of this paper is to study deformations of reducible metabelian representations of $\Gamma_K$ into $\text{SL}(n, \mathbb{C})$. The metabelian representations in question were introduced by G. Burde [Bur67] and G. de Rham [dR67]. Let us recall this result: for each nonzero complex number $\lambda \in \mathbb{C}^*$ there exists a diagonal representation $\rho_\lambda: \Gamma_K \rightarrow \text{SL}(2, \mathbb{C})$ given by

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$ 

Here $\varphi: \pi_1(X) \rightarrow \mathbb{Z}$ denotes the canonical surjection which maps the meridian $\mu$ of $K$ to 1 i.e. $\varphi(\gamma) = \text{lk}(\gamma, K)$. Burde and de Rham proved that there exists a metabelian, non-abelian, reducible representation

$$\rho_\lambda^2: \Gamma_K \rightarrow \text{SL}(2, \mathbb{C}), \quad \rho_\lambda^2(\gamma) = \begin{pmatrix} 1 & \varphi(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$$

(1.1)

if and only if $\lambda^2$ is a root of the Alexander polynomial $\Delta_K(t)$. Recall that a representation $\rho: G \rightarrow \text{GL}(n, \mathbb{C})$ of a group $G$ is called reducible if the image $\rho(G)$ preserves a proper subspace of $\mathbb{C}^n$. Otherwise, $\rho$ is called irreducible.

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The question whether or not the representation $\rho^z_\lambda$ is a limit of irreducible representations of $\Gamma_K$ into $SL(2, \mathbb{C})$ was studied in [HPSP01]. Theorem 1.1 of [HPSP01] states that a metabelian, non-abelian, reducible representation $\rho^z_\lambda: \Gamma_K \to SL(2, \mathbb{C})$ is the limit of irreducible representations if $\lambda^2$ is a simple root of $\Delta_K(t)$. Moreover, in this case the representation $\rho^z_\lambda$ is a smooth point of the representation variety $R(\Gamma_K, SL(2, \mathbb{C}))$; it is contained in a unique 4-dimensional component $R_\lambda \subset R(\Gamma_K, SL(2, \mathbb{C}))$.

This article studies the behavior of the representations in question under the composition with the $n$-dimensional, irreducible, rational representation $r_n: SL(2, \mathbb{C}) \to SL(n, \mathbb{C})$. (for more details see Section 4.1). It is proved in Proposition 3.1 that generically for an irreducible representation $\rho \in R^\lambda_\lambda$ the representation $\rho^n := r_n \circ \rho \in R(\Gamma_K, SL(n, \mathbb{C}))$ is also irreducible. The main result of this article is the following:

**Theorem 1.1.** If $\lambda^2$ is a simple root of $\Delta_K(t)$ and if $\Delta_K(\lambda^{2k}) \neq 0$ for $2 \leq k \leq n-1$ then the reducible metabelian representation $\rho^z_{\lambda,n} := r_n \circ \rho^z_\lambda$ is a limit of irreducible representations. More precisely, $\rho^z_{\lambda,n}$ is a smooth point of $R(\Gamma_K, SL(n, \mathbb{C}))$; it is contained in a unique $(n+2)(n-1)$-dimensional component $R_{\lambda,n} \subset R(\Gamma_K, SL(n, \mathbb{C}))$.

**Remark 1.** Let $\rho_{\lambda,n}: \Gamma_K \to SL(n, \mathbb{C})$ be the diagonal representation given by $\rho_{\lambda,n} = r_n \circ \rho^z_\lambda$. The group $SL(n, \mathbb{C})$ acts on the representation variety $R(\Gamma_K, SL(n, \mathbb{C}))$ by conjugation, and the orbit $\mathcal{O}(\rho_{\lambda,n})$ is contained in the closure $\overline{\mathcal{O}(\rho^z_{\lambda,n})}$. Hence $\rho_{\lambda,n}$ and $\rho^z_{\lambda,n}$ project to the same point $\chi_{\lambda,n}$ of the character variety $X(\Gamma_K, SL(n, \mathbb{C})) = R(\Gamma_K, SL(n, \mathbb{C})) / SL(n, \mathbb{C})$. Here $R(\Gamma_K, SL(n, \mathbb{C})) / SL(n, \mathbb{C})$ denotes the GIT quotient of the action (see [New78] for more details). Recall that the GIT quotient parametrizes the closed orbits of the $SL(n, \mathbb{C})$ action.

It is possible to study the local picture of the character variety at $\chi_{\lambda,n}$ as done in [HPSP01] and [HP05]. Unfortunately, there are additional technical difficulties, and the computations necessary are much more involved. These complications are due to the fact that the diagonal representation $\rho_{\lambda,n}$ is contained in $2^{n-1}$ components of $R(\Gamma_K, SL(n, \mathbb{C}))$. Nevertheless, only the component $R_{\lambda,n}$ contains irreducible representations. We will address this topic in a forthcoming paper.

P. Menal-Ferrer and J. Porti [MFP12] showed that the conclusions of the above theorem hold for hyperbolic knots if $\rho^z_\lambda$ is replaced by a lift of the holonomy, $\tilde{\text{hol}}: \pi_1(S^3 \setminus K) \to SL(2, \mathbb{C})$, of the hyperbolic structure of the complement $S^3 \setminus K$. Note that Theorem 1.1 and Proposition 3.1 do apply to non-hyperbolic knots. Irreducible metabelian representations and their deformations are studied by H. Boden and S. Friedl in a series of articles [BF08, BF11, BF13].
In particular the deformations of irreducible metabelian representations, which are not considered in this paper, are studied in [BF13].

This article is organized as follows: in Section 2 we will introduce notation and recall some facts which are used. In Section 3 we will prove that the representation variety \( R(\Gamma_K, \text{SL}(n, \mathbb{C})) \) contains an irreducible representation if \( \Delta_K(t) \) has a simple root (see Proposition 3.1). Moreover we give a streamlined proof of a slightly generalized version of the deformation result used in [MFP12, BAHJ10, BF13] (see Proposition 3.3). The necessary cohomological calculations and the basic facts about the representation theory of \( \text{SL}(2, \mathbb{C}) \) are presented in Section 4, in order to prove our main result, Theorem 1.1. Finally, in Section 5 some examples are exhibited.

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2. Notation and Facts

To shorten notation we write \( \text{SL}(n), \text{GL}(n) \) and \( \mathfrak{sl}(n) \) instead of \( \text{SL}(n, \mathbb{C}), \text{GL}(n, \mathbb{C}) \) and \( \mathfrak{sl}(n, \mathbb{C}) \).

Let \( \varphi : \pi_1(X) \to \mathbb{Z} \) denote the canonical surjection which maps the meridian \( \mu \) of \( K \) to 1 i.e. \( \varphi(\gamma) = \text{lk}(\gamma, K) \). We associate to a nonzero complex number \( \alpha \in \mathbb{C}^\ast \) a homomorphism \( \alpha^\varphi : \Gamma_K \to \mathbb{C}^\ast, \quad \alpha^\varphi : \gamma \mapsto \alpha^{\varphi(\gamma)}. \)

Note that \( \alpha^\varphi \) maps the meridian \( \mu \) of \( K \) to \( \alpha \). We define \( C_{\alpha} \) to be the \( \Gamma_K \)-module \( C \) with the action induced by \( \alpha^\varphi \), i.e. \( \gamma : x = \alpha^\varphi(\gamma)x \) for all \( \gamma \in \Gamma_K \) and all \( x \in C \). The trivial \( \Gamma_K \)-module \( C_1 \) is simply denoted \( C \). With this notation it is easy to see that a map

\[
\rho^\chi_\lambda : \Gamma_K \to \text{SL}(2, \mathbb{C}), \quad \rho^\chi_\lambda(\gamma) = \begin{pmatrix}
1 & z(\gamma) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\lambda^{\varphi(\gamma)} & 0 \\
0 & \lambda^{-\varphi(\gamma)}
\end{pmatrix}
\]

is a homomorphism if and only if the map \( z : \Gamma_K \to C_{\lambda^2} \) is a 1-cocycle i.e. \( z(\gamma_1 \gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)}z(\gamma_2) \). Note also that \( \rho^\chi_\lambda \) is abelian if \( \lambda = \pm 1 \). If \( \lambda^2 \neq 1 \) then \( \rho^\chi_\lambda \) is abelian and if only if \( z \) is a coboundary i.e. there exists an element \( x_0 \in C \) such that \( z(\gamma) = (\lambda^{2\varphi(\gamma)} - 1)x_0 \). The general reference for group cohomology is Brown’s book [Bro82].

In what follows we are mainly interested in the following situation: let \( X \) be the complement of a knot \( K \subset S^3 \) and let \( A \) be a \( \pi_1(X) \)-module. The spaces \( X \) and \( \partial X \) are aspherical and hence the natural homomorphisms \( H^*(\pi_1(X); A) \to \)
$H^*(X;A)$ and $H^*(\pi_1(\partial X);A) \to H^*(\partial X;A)$ are isomorphisms. Moreover, the knot complement $X$ has the homotopy type of a 2-dimensional CW-complex which implies that $H^k(\pi_1(X);A) = 0$ and $H_k(\pi_1(X);A) = 0$ for $k \geq 3$. See [Whi78] for more details.

The Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$ turns into a $\Gamma_K$-module via the action $\gamma p(t) = t^{\gamma(\gamma)} p(t)$ for all $\gamma \in \Gamma_K$ and all $p(t) \in \mathbb{C}[t^{\pm 1}]$. Recall that there are isomorphisms of $\mathbb{C}[t^{\pm 1}]$-modules

$$H_\ast(\Gamma_K;\mathbb{C}[t^{\pm 1}]) \cong H_\ast(X;\mathbb{C}[t^{\pm 1}]) \cong H_\ast(X_\infty;\mathbb{C})$$

where $X_\infty$ denotes the infinite cyclic covering of the knot complement $X$ (see [DK01] Chapter 5)). The module $H_1(\Gamma_K;\mathbb{C}[t^{\pm 1}])$ is a finitely generated torsion module called the Alexander module of $K$. A generator of its order ideal is called the Alexander polynomial $\Delta_K(t) \in \mathbb{C}[t^{\pm 1}]$ of $K$. The Alexander polynomial is unique up to multiplication with a unit in $\mathbb{C}[t^{\pm 1}]$.

For completeness we will state the next lemma which shows that the cohomology groups $H^\ast(\Gamma_K;\mathbb{C}_\alpha)$ are determined by the Alexander module $H_1(\Gamma_K;\mathbb{C}[t^{\pm 1}])$.

**Lemma 2.1.** Let $K \subset S^3$ be a knot and $\Gamma_K$ its group. Let $\alpha \in \mathbb{C}^*$ be a nonzero complex number and let $\mathbb{C}_\alpha$ denote the $\Gamma_K$-module given by the action $\alpha z = \alpha^{\phi(\gamma)} z$.

If $\alpha = 1$ then $\mathbb{C}_\alpha = \mathbb{C}$ is a trivial $\Gamma_K$-module and $H^k(\Gamma_K,\mathbb{C}) \cong \mathbb{C}$ for $k = 0,1$ and $H^k(\Gamma_K,\mathbb{C}) = 0$ for $k \geq 2$. If $\alpha \neq 1$ then $H^0(\Gamma_K,\mathbb{C}_\alpha) = 0$ and $\dim H^1(\Gamma_K,\mathbb{C}_\alpha) = \dim H^2(\Gamma_K,\mathbb{C}_\alpha)$.

Moreover, $H^1(\Gamma_K,\mathbb{C}_\alpha) \neq 0$ if and only if $\Delta_K(\alpha) = 0$.

**Proof.** We have $H_0(X_\infty;\mathbb{C}) \cong \mathbb{C} \cong \mathbb{C}[t^{\pm 1}] / (t - 1)$ and $H_k(X_\infty;\mathbb{C}) = 0$ for $k \geq 2$ (see [BZH13] Prop. 8.16)). If $\alpha = 1$ then $H^k(\Gamma_K,\mathbb{C}) \cong \mathbb{C}$ for $k = 0,1$ and $H^k(\Gamma_K,\mathbb{C}) = 0$ for $k \geq 2$ follows.

Now suppose that $\alpha \in \mathbb{C}^*$, $\alpha \neq 1$, and notice that we have an isomorphism $\mathbb{C}_\alpha \cong \mathbb{C}[t^{\pm 1}] / (t - \alpha)$. The cohomology group $H^0(\Gamma_K,\mathbb{C}_\alpha)$ vanishes, since the $\Gamma_K$-module $\mathbb{C}_\alpha$ has no invariants, and $H^k(\Gamma_K,\mathbb{C}_\alpha) = 0$ for $k > 2$ since the knot complement $X$ has the homotopy type of a 2-complex. Recall that the Alexander module $H_1(\Gamma_K;\mathbb{C}[t^{\pm 1}])$ is finitely generated torsion module and hence a sum of non-free cyclic modules since $\mathbb{C}[t^{\pm 1}]$ is a principal ideal domain. The Alexander polynomial is the order ideal of $H_1(\Gamma_K;\mathbb{C}[t^{\pm 1}])$. Since $\alpha \neq 1$, it follows from the universal coefficient theorem that $H^1(\Gamma;\mathbb{C}_\alpha) \cong \text{Hom}(H_1(\Gamma_K;\mathbb{C});\mathbb{C}_\alpha)$. Hence $H^1(\Gamma_K,\mathbb{C}_\alpha) = 0$ if and only if the module $H_1(\Gamma_K;\mathbb{C})$ has $(t - \alpha)$-torsion which is equivalent to $\Delta_K(\alpha) = 0$. Finally, $\dim H^1(\Gamma_K,\mathbb{C}_\alpha) = \dim H^2(\Gamma_K,\mathbb{C}_\alpha)$ follows since the Euler characteristic of $X$ vanishes. See also [BA00] Proposition 2.1 for more details. \qed

In what follows we will also make use of the Poincaré-Lefschetz duality theorem with twisted coefficients: let $M^m$ be a connected, orientable, compact
m-dimensional manifold with boundary $\partial M^m$ and let $\rho: \pi_1(M^m) \to \text{SL}(n)$ be a representation. Then the cup-product and the Killing form $b: \mathfrak{sl}(n)_\rho \otimes \mathfrak{sl}(n)_\rho \to \mathbb{C}$ induce a non-degenerate bilinear pairing

$$H^k(M^m; \mathfrak{sl}(n)_\rho) \otimes H^{m-k}(M^m, \partial M^m; \mathfrak{sl}(n)_\rho) \xrightarrow{\cup}$$

$$H^m(M^m, \partial M^m; \mathfrak{sl}(n)_\rho \otimes \mathfrak{sl}(n)_\rho) \xrightarrow{b} H^m(M^m, \partial M^m; \mathbb{C}) \cong \mathbb{C} \quad (2.1)$$

and hence an isomorphism $H^k(M^m; \mathfrak{sl}(n)_\rho) \cong H^{m-k}(M^m, \partial M^m; \mathfrak{sl}(n)_\rho)^*$, for all $0 \leq k \leq m$. See [LM87] for more details.

2.1. **Group cohomology and representation varieties.** Let now $\Gamma$ be a finitely generated group. The set $R_n(\Gamma) := R(\Gamma, \text{SL}(n))$ of homomorphisms of $\Gamma$ in $\text{SL}(n)$ is called the $\text{SL}(n)$-representation variety of $\Gamma$ and has the structure of a (not necessarily irreducible) algebraic set.

Let $\rho: \Gamma \to \text{SL}(n)$ be a representation. The Lie algebra $\mathfrak{sl}(n)$ turns into a $\Gamma$-module via $\text{Ad} \rho$. This module will be simply denoted by $\mathfrak{sl}(n)_\rho$. A cocycle $d \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is a map $d: \Gamma \to \mathfrak{sl}(n)$ satisfying

$$d(\gamma_1 \gamma_2) = d(\gamma_1) + \rho(\gamma_1) d(\gamma_2) \rho(\gamma_1)^{-1}, \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

It was observed by André Weil [Wei64] that there is a natural inclusion of the Zariski tangent space $T^Z(\Gamma; \mathfrak{sl}(n)_\rho) \hookrightarrow Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Informally speaking, given a smooth curve $\rho_\epsilon$ of representations through $\rho_0 = \rho$ one gets a 1-cocycle $d: \Gamma \to \mathfrak{sl}(n)$ by defining

$$d(\gamma) := \left. \frac{d \rho_\epsilon(\gamma)}{d \epsilon} \right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma.$$

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Here, $b: \Gamma \to \mathfrak{sl}(n)$ is a coboundary if there exists $x \in \mathfrak{sl}(n)$ such that $b(\gamma) = \rho(\gamma) x \rho(\gamma)^{-1} - x$. A detailed account can be found in [LM87].

Let $\dim_\rho R_n(\Gamma)$ be the local dimension of $R_n(\Gamma)$ at $\rho$ (i.e. the maximal dimension of the irreducible components of $R_n(\Gamma)$ containing $\rho$ [Sha77, Ch. II]). So we obtain:

$$\dim_\rho R_n(\Gamma) \leq \dim T^Z(\Gamma; \mathfrak{sl}(n)_\rho) \leq \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho).$$

We will call a representation $\rho \in R_n(\Gamma)$ **regular** if $\dim_\rho R_n(\Gamma) = \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$.

The following lemma follows (for more details see [HPSP01, Lemma 2.6]):

**Lemma 2.2.** Let $\rho \in R_n(\Gamma)$ be a representation. If $\rho$ is regular, then $\rho$ is a smooth point of the representation variety $R_n(\Gamma)$ and $\rho$ is contained in a unique component of $R_n(\Gamma)$ of dimension $\dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$. 
Note that there are discrete groups and representations \( \rho \) which are smooth points of the representation variety without being regular. (See [LMS85] Example 2.10 for more details.)

3. Deforming representations

The aim of the following sections is to prove that the representation \( \rho_{\lambda,n} \) from the introduction is a smooth point of the representation variety. We present a more streamlined and slightly generalized version of the deformation result from [HPSP01, HP05, BAHJ10, MFPI12, BF14] (see Proposition 3.3). For the convenience of the reader we recall the setup.

First we will prove that the representation \( \rho_{\lambda,n} \in R_n(\Gamma_K) \) is the limit of irreducible representations if \( \lambda^2 \) is a simple root of the Alexander polynomial \( \Delta_K(t) \). In what follows a property of an irreducible algebraic variety \( Y \) is said to be true generically if it holds except on a proper Zariski-closed subset of \( Y \), in other words, if it holds on a non-empty Zariski-open subset.

Let \( K \subset S^3 \) be a knot, \( \lambda^2 \in \mathbb{C} \) a simple root of \( \Delta_K(t) \) and \( z \in Z^1(\Gamma_K, \mathbb{C}_{\lambda^2}) \) a cocycle representing a generator of \( H^1(\Gamma_K, \mathbb{C}_{\lambda^2}) \). Following [HPSP01, Thm 1.1] the representation \( \rho_{\lambda} \in R_2(\Gamma_K) \) is a smooth point of the representation variety. Note that generically a representation \( \rho \in R_\lambda \) is irreducible.

**Proposition 3.1.** Let \( K \subset S^3 \) be a knot, \( \lambda^2 \in \mathbb{C} \) a simple root of \( \Delta_K(t) \) and let \( z \in Z^1(\Gamma_K, \mathbb{C}_{\lambda^2}) \) be a cocycle representing a generator of \( H^1(\Gamma_K, \mathbb{C}_{\lambda^2}) \).

Then the representation \( \rho_{\lambda,n} = r_n \circ \rho_{\lambda} : \Gamma_K \to B_n \) is the limit of irreducible representation in \( R_n(\Gamma_K) \). More precisely, generically a representation \( \rho_n = r_n \circ \rho, \rho \in R_\lambda \) is irreducible.

**Proof.** It follows from [HPSP01] Theorem 1.1] that \( \rho_{\lambda} \in R_2(\Gamma_K) \) is the limit of irreducible representations. Moreover, \( \rho_{\lambda} \in R_2(\Gamma_K) \) is a smooth point which is contained in a unique 4-dimensional component \( R_\lambda \subset R_2(\Gamma_K) \).

Let \( \Gamma \) be a discrete group and let \( \rho : \Gamma \to SL(2, \mathbb{C}) \) be an irreducible representation. If the image \( \rho(\Gamma) \subset SL(2, \mathbb{C}) \) is Zariski-dense then the representation \( \rho_n := r_n \circ \rho \in R_n(\Gamma) \) is irreducible. Hence in order to prove the proposition we show that there is a neighborhood \( U = U(\rho_{\lambda}) \subset R_2(\Gamma_K) \) such that \( \rho(\Gamma) \subset SL(2, \mathbb{C}) \) is Zariski-dense for each irreducible \( \rho \in U \). Let now \( \rho : \Gamma_K \to SL(2, \mathbb{C}) \) be any irreducible representation and let \( G \subset SL(2) \) denote the Zariski-closure of \( \rho(\Gamma_K) \). Suppose that \( G \neq SL(2) \). Since \( \rho \) is irreducible it follows that \( G \) is, up to conjugation, a subgroup of upper-triangular matrices of \( SL(2) \). Then by [Kov86, Sec. 1.4] and [Kap57, Theorem 4.12] there are, up to conjugation, only two cases left:

- \( G \) is a subgroup of the infinite dihedral group

\[
D_\infty = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \mid \alpha \in \mathbb{C}^* \right\}.
\]
• $G$ is one of the groups $A_4^{SL(2)}$ (the tetrahedral group), $S_4^{SL(2)}$ (the octahedral group) or $A_5^{SL(2)}$ (the icosahedral group). These groups are the preimages in $SL(2)$ of the subgroups $A_4$, $S_4$, $A_5 \subset PSL(2, \mathbb{C})$.

In the first case it follows directly from [Nag07] that if $\rho$ is an irreducible metabelian representation then the trace of the image of a meridian $\text{tr}(\rho(\mu)) = 0$ i.e. $\rho(\mu)$ is similar to $\pm(i00-1)$. Now, $\text{tr}(\rho^*(\mu)) \neq 0$ since $\Delta^K(-1) \neq 0$ and $\Delta(K(\lambda \pm 2)) = 0$. For the second case there are up to conjugation only finitely many irreducible representations of $\Gamma_K$ onto the subgroups $A_4^{SL(2)}$, $S_4^{SL(2)}$ and $A_5^{SL(2)}$. Note that these finitely many orbits are closed and 3-dimensional. Hence the irreducible $\rho \in R_\lambda$ such that $r_n \circ \rho$ is reducible is contained in a Zariski-closed subset of $R_\lambda$. Hence generically $r_n \circ \rho$ is irreducible for $\rho \in R_\lambda$. □

Remark 2. Recall that a finite group has only finitely many irreducible representations (see [Ser78, FH91]). Hence, the restriction of $r_n$ to the groups $A_4^{SL(2)}$, $S_4^{SL(2)}$ and $A_5^{SL(2)}$ is reducible, for all but finitely many $n \in \mathbb{N}$.

In order to prove that a certain representation $\rho : \Gamma \to SL(n)$ is a smooth point of the representation variety we will prove that every cocycle $u \in Z^1(\Gamma_K; sl(n)_\rho)$ is integrable. In order to do this, we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a theorem of Artin [Art68].

The formal deformations of a representation $\rho : \Gamma \to SL(n)$ are in general determined by an infinite sequence of obstructions (see [Gol84, BA 00, HPSP01]). In what follows we let $C^1(\Gamma; sl(n)) := \{c : \Gamma \to sl(n)\}$ denote the 1-cochains of $\Gamma$ with coefficients in $sl(n)$ (see [Bro82] p.59).

Let $\rho : \Gamma \to SL(n)$ be a representation. A formal deformation of $\rho$ is a homomorphism $\rho_\infty : \Gamma \to SL(n, \mathbb{C}[t])$

$$\rho_\infty(\gamma) = \exp \left( \sum_{i=1}^{\infty} t^iu_i(\gamma) \right) \rho(\gamma), \quad u_i \in C^1(\Gamma; sl(n))$$

such that $ev_0 \circ \rho_\infty = \rho$. Here $ev_0 : SL(n, \mathbb{C}[t]) \to SL(n)$ is the evaluation homomorphism at $t = 0$ and $\mathbb{C}[t]$ denotes the ring of formal power series.

We will say that $\rho_\infty$ is a formal deformation up to order $k$ of $\rho$ if $\rho_\infty$ is a homomorphism modulo $t^{k+1}$.

An easy calculation gives that $\rho_\infty$ is a homomorphism up to first order if and only if $u_1 \in Z^1(\Gamma; sl(n)_\rho)$ is a cocycle. We call a cocycle $u_1 \in Z^1(\Gamma; sl(n)_\rho)$ integrable if there is a formal deformation of $\rho$ with leading term $u_1$.

Lemma 3.2. Let $u_1, \ldots, u_k \in C^1(\Gamma; sl(n)_\rho)$ such that

$$\rho_k(\gamma) = \exp \left( \sum_{i=1}^{k} t^iu_i(\gamma) \right) \rho(\gamma)$$
is a homomorphism into $\text{SL}(n, \mathbb{C}[t]/(t^{k+1}))$. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1, \ldots, u_k)} \in H^2(\Gamma; \mathfrak{sl}(n)_\rho)$ with the following properties:

(i) There is a cochain $u_{k+1}: \Gamma \to \mathfrak{sl}(n)_\rho$ such that

$$
\rho_{k+1}(\gamma) = \exp \left( \sum_{i=1}^{k+1} t^i u_i(\gamma) \right) \rho(\gamma)
$$

is a homomorphism modulo $t^{k+2}$ if and only if $\zeta_{k+1} = 0$.

(ii) The obstruction $\zeta_{k+1}$ is natural, i.e. if $f: \Gamma \to \tilde{\Gamma}$ is a homomorphism then $f^* \rho_k := \rho_k \circ f$ is also a homomorphism modulo $t^{k+1}$ and $f^*(\zeta_{k+1}^{(u_1, \ldots, u_k)}) = \zeta_{k+1}^{(f^* u_1, \ldots, f^* u_k)} \in H^2(\Gamma_1; \mathfrak{sl}(n)_{f^* \rho})$.

Proof. The proof is completely analogous to the proof of Proposition 3.1 in [HP05]. We replace $\text{SL}(2)$ and $\mathfrak{sl}(2)$ by $\text{SL}(n)$ and $\mathfrak{sl}(n)$ respectively. \qed

The following result streamlines the arguments given in [HP05] and [BHT10]:

**Proposition 3.3.** Let $M$ be a connected, compact, orientable 3-manifold with torus boundary and let $\rho: \pi_1 M \to \text{SL}(n)$ be a representation.

If $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$ then $\rho$ is a smooth point of the $\text{SL}(n)$-representation variety $R_n(\pi_1 M)$. Moreover, $\rho$ is contained in a unique component of dimension $n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho)$.

Proof. First we will show that the map $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \to H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$ induced by the inclusion $\partial M \hookrightarrow M$ is injective.

Recall that for any CW-complex $X$ with $\pi_1(X) \cong \pi_1(M)$ and for any $\pi_1 M$-module $A$ there are natural morphisms $H^i(\pi_1 M; A) \to H^i(X; A)$ which are isomorphisms for $i = 0$ and 1 and an injection for $i = 2$ (see [HP05 Lemma 3.3]). Note also that $\partial M \cong S^1 \times S^1$ is aspherical and hence $H^*(\pi_1 \partial M; A) \to H^*(\partial M; A)$ is an isomorphism.

First we will prove that for every representation $\varrho \in R_n(\mathbb{Z} \oplus \mathbb{Z})$ we have

$$
\dim H^0(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}(n)_\varrho) = \frac{1}{2} \dim H^1(\mathbb{Z} \oplus \mathbb{Z}; \mathfrak{sl}(n)_\varrho) \geq n - 1. \quad (3.1)
$$

Moreover, we will prove that $\varrho \in R_n(\mathbb{Z} \oplus \mathbb{Z})$ is regular if and only if equality holds in (3.1). It follows from Poincaré duality (2.1) that for every $\varrho \in R_n(\mathbb{Z} \oplus \mathbb{Z})$ we have

$$
\dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) = \dim H^2(\partial M; \mathfrak{sl}(n)_\varrho),
$$

and since the Euler characteristic of $M$ vanishes we obtain the first equality in (3.1):

$$
\dim H^1(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim \mathfrak{sl}(n)_{\varrho}^{\mathbb{Z} \oplus \mathbb{Z}}.
$$
Now, R.W. Richardson proved in [Ric79, Thm. C] that the representation variety $R_n(Z \oplus Z)$ is an irreducible algebraic variety of dimension $(n + 2)(n - 1)$. Hence we obtain for every $\rho \in R_n(Z \oplus Z)$ that

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_{\rho}) \geq (n + 2)(n - 1) = n^2 + n - 2$$

where the equality holds if and only if $\rho$ is regular (see Lemma 22). At the same time, we have:

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_{\rho}) = \dim H^1(\partial M; \mathfrak{sl}(n)_{\rho}) + \dim B^1(\partial M; \mathfrak{sl}(n)_{\rho})$$

and the exactness of $0 \to H^0(\partial M; \mathfrak{sl}(n)_{\rho}) \to \mathfrak{sl}(n) \to B^1(\partial M; \mathfrak{sl}(n)_{\rho}) \to 0$ gives

$$\dim B^1(\partial M; \mathfrak{sl}(n)_{\rho}) = \dim \mathfrak{sl}(n) - \dim H^0(\partial M; \mathfrak{sl}(n)_{\rho}).$$

This together with $\dim H^1(\partial M; \mathfrak{sl}(n)_{\rho}) = 2 \dim H^0(\partial M; \mathfrak{sl}(n)_{\rho})$ gives for all $\rho \in R_n(Z \oplus Z)$:

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_{\rho}) = \dim H^0(\partial M; \mathfrak{sl}(n)_{\rho}) + n^2 - 1 \geq n^2 + n - 2.$$ 

It follows that

$$\dim H^0(\partial M; \mathfrak{sl}(n)_{\rho}) \geq n - 1, \text{ for all } \rho \in R_n(Z \oplus Z), \quad (3.2)$$

and $\rho \in R_n(Z \oplus Z)$ is regular if and only if $\dim H^0(Z \oplus Z; \mathfrak{sl}(n)_{\rho}) = n - 1$ (see also [Pop08]).

Now, the exact cohomology sequence of the pair $(M, \partial M)$ gives

$$0 \to H^1(M, \partial M; \mathfrak{sl}(n)_{\rho}) \to H^1(\partial M; \mathfrak{sl}(n)_{\rho}) \stackrel{\alpha}{\to} H^2(\partial M, \partial M; \mathfrak{sl}(n)_{\rho}) \to H^2(M, \partial M; \mathfrak{sl}(n)_{\rho}) \to 0 \to H^3(M, \partial M; \mathfrak{sl}(n)_{\rho}) \to 0.$$

Poincaré-Lefschetz duality (24) implies that $\alpha$ and $\beta$ are dual to each other. This together with (3.2) gives:

$$n - 1 = \dim H^1(M; \mathfrak{sl}(n)_{\rho}) \geq \text{rk}(\alpha) = \frac{1}{2} \dim H^1(\partial M; \mathfrak{sl}(n)_{\rho}) = \dim H^0(\partial M; \mathfrak{sl}(n)_{\rho}) \geq n - 1.$$

Therefore, $\dim H^0(\partial M; \mathfrak{sl}(n)_{\rho}) = n - 1$ holds in Equation (3.1), and consequently $i^\ast \rho = \rho \circ i_{\#} \in R_n(\partial M)$ is regular (here $i: \partial M \to M$ is the inclusion). Note also that $\beta$ is surjective, and hence

$$i^\ast: H^2(M; \mathfrak{sl}(n)_{\rho}) \to H^2(\partial M; \mathfrak{sl}(n)_{\rho}).$$
is injective. The following commutative diagram shows that $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \to H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$ is also injective:

\[
\begin{array}{ccc}
H^2(M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\partial M; \mathfrak{sl}(n)_\rho) \\
\uparrow & & \uparrow \\
H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho).
\end{array}
\]

In order to prove that $\rho$ is a smooth point of $R_n(\pi_1 M)$, we show that all obstructions vanish, by using the fact that the obstructions vanish on the boundary. Let $u_1, \ldots, u_k: \pi_1 M \to \mathfrak{sl}(n)$ be given such that

\[\rho_k(\gamma) = \exp \left( \sum_{i=1}^k t^i u_i(\gamma) \right) \rho(\gamma)\]

is a homomorphism modulo $t^{k+1}$. Then the restriction $i^* \rho_k: \pi_1(\partial M) \to \text{SL}(n, \mathbb{C}[t])$ is also a formal deformation of order $k$. Since $i^* \rho$ is a smooth point of the representation variety $R_n(\mathbb{Z} \oplus \mathbb{Z})$, the formal implicit function theorem gives that $i^* \rho_k$ extends to a formal deformation of order $k + 1$ (see [HPSP01, Lemma 3.3]). Therefore, we have that

\[0 = \zeta^{(u_1, \ldots, u_k)}_{k+1} = i^* \zeta^{(u_1, \ldots, u_k)}_{k+1}\]

Now, $i^*$ is injective and the obstruction $\zeta^{(u_1, \ldots, u_k)}_{k+1}$ vanishes.

Hence all cocycles in $Z^1(\Gamma, \mathfrak{sl}(n)_\rho)$ are integrable. By applying Artin’s theorem [Art68] we obtain from a formal deformation of $\rho$ a convergent deformation (see [HPSP01] Lemma 3.3) or [BA00, § 4.2]).

Thus $\rho$ is a regular point of the representation variety $R_n(\pi_1 M)$. Hence, $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$ and the exactness of

\[0 \to H^0(\pi_1 M; \mathfrak{sl}(n)_\rho) \to \mathfrak{sl}(n)_\rho \to B^1(\pi_1 M; \mathfrak{sl}(n)_\rho) \to 0\]

implies

\[\dim R_n(\pi_1 M) = \dim Z^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho).
\]

Finally, the proposition follows from Lemma 2.2 \(\square\)

**Proposition 3.4.** Let $K \subset S^3$ be a knot, $\lambda \in \mathbb{C}^*$ and $n \geq 3$. Suppose that $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho^\lambda_\lambda: \Gamma_K \to \text{SL}(2)$ be a non-abelian representation as in (1).

If $\Delta_K(\lambda^2) \neq 0$ for $2 \leq i \leq n - 1$ then for $\rho^\lambda_{\lambda,n} := r_n \circ \rho^\lambda_{\lambda,1}: \Gamma_K \to \text{SL}(n)$ we have

\[\dim H^1(\Gamma_K; \mathfrak{sl}(n)_{\rho^\lambda_{\lambda,n}}) = (n - 1) \text{ and } H^0(\Gamma_K; \mathfrak{sl}(n)_{\rho^\lambda_{\lambda,n}}) = 0.
\]

**Proof.** A proof of the cohomological calculation will be given in Section 4. \(\square\)
Proof of Theorem 1.1. It follows directly from Propositions 3.3 and 3.4 that \( \rho_{\lambda,n}^z \) is a smooth point of \( R_n(\Gamma_K) \) which is contained in a unique component \( R_{\lambda,n} \subset R_n(\Gamma_K) \), \( \dim R_{\lambda,n} = n^2 + n - 2 \).

That \( \rho_{\lambda,n}^z \) is the limit of irreducible representations which are contained in the component \( R_{\lambda,n} \) follows from Proposition 3.1. □

4. Cohomological calculations

For the convenience of the reader we recall some facts from the representation theory of SL(2). The general reference for this topic is Springer’s LNM [Spr77].

4.1. Representation theory of SL(2). Let \( V \) be an \( n \)-dimensional complex vector space. In what follows we will call a homomorphism \( r : SL(2) \rightarrow GL(V) \) an \( n \)-dimensional representation of \( SL(2) \). The vector space \( V \) turns into an \( SL(2) \)-module. Two \( n \)-dimensional representations \( r \) and \( r' : SL(2) \rightarrow GL(V') \) are called equivalent if there is an isomorphism \( \phi : V \rightarrow V' \) which commutes with the action of \( SL(2) \) i.e. \( r'(A) \phi = \phi r(A) \) for all \( A \in SL(2) \).

It is clear that equivalent representations give rise to isomorphic \( SL(2) \)-modules.

We let \( SL(2) \) act as a group of automorphisms on the polynomial algebra \( R = \mathbb{C}[X,Y] \). If \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2) \) then there is a unique automorphism \( r \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) of \( R \) given by

\[
\begin{align*}
    r \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (X) &= dX - bY  \quad \text{and} \quad r \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (Y) = -cX + aY.
\end{align*}
\]

We let \( R_{n-1} \subset R \) denote the \( n \)-dimensional subspace of homogeneous polynomials of degree \( n - 1 \). The monomials \( e_l^{(n-1)} = X^{i-1}Y^{n-l} \), \( 1 \leq l \leq n \), form a basis of \( R_{n-1} \) and \( r \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) leaves \( R_{n-1} \) invariant. In what follows we will identify \( R_{n-1} \) and \( \mathbb{C}^n \) by fixing the basis \((e_1^{(n-1)}, \ldots, e_n^{(n-1)})\) of \( R_{n-1} \). We obtain an \( n \)-dimensional representation \( r_n : SL(2) \rightarrow GL(R_{n-1}) \cong GL(n) \).

The representation \( r_n \) is rational i.e. the coefficients of the matrix coordinates of \( r_n(A) \) are polynomials in the matrix coordinates of \( A \). We will make use of the following theorem.

Theorem 4.1. (1) The representation \( r_n \) is irreducible i.e. there is no \( SL(2) \)-stable invariant subspace \( V \), \( \{0\} \subsetneq V \subsetneq R_{n-1} \) and any irreducible rational representation of \( SL(2) \) is equivalent to some \( r_n \).

(2) For an arbitrary rational representation \( r : SL(2) \rightarrow GL(V) \) the \( SL(2) \)-module \( V \) is isomorphic to a direct sum of \( R_n \),

\[
V \cong \bigoplus_{d \geq 0} R_{d}^{m(k)}.
\]

Proof. See Lemma 3.1.3 and Proposition 3.2.1 of [Spr77]. □

It is easy to see, and it follows also from the general theory, that \( r_n \) maps an unipotent matrix \( \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \) and \( \left( \begin{array}{cc} 1 & 0 \\ c & 1 \end{array} \right) \) onto an unipotent element of \( SL(R_{n-1}) \).
Moreover, an explicit calculation shows that the image of a diagonal matrix is the diagonal matrix \( r_n(\text{diag}(a, a^{-1})) = \text{diag}(a^{n-1}, a^{n-3}, \ldots, a^{-n+3}, a^{-n+1}) \). Hence the image of \( r_n \) is contained in \( SL(R_{n-1}) \cong SL(n) \).

**Example 1.** The representation \( r_1 : SL(2) \to SL(1) = \{1\} \) is the trivial representation. The representation \( r_2 : SL(2) \to SL(2) \) is equivalent to the identity:

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix}.
\]

Moreover, it is easy to see that the adjoint representation \( \text{Ad} : SL(2) \to \text{Aut}(sl(2)) \) is equivalent to \( r_3 \).

The Lie algebra \( sl(n) \) of \( SL(n) \) turns into an \( SL(2) \)-module via \( \text{Ad} \circ r_n \) where \( \text{Ad} : SL(n) \to \text{Aut}(sl(n)) \) denotes the adjoint representation. For this action we have the classical formula of Clebsch–Gordan:

\[
\text{Ad} \circ r_n \cong \bigoplus_{i=1}^{n-1} r_{2i+1}.
\] (4.1)

Let \( B_n \subset SL(n) \) denote the Borel subgroup of upper triangular matrices. The vector space \( R_{n-1} \) turns into a \( B_2 \)-module via the restriction of \( r_n \) to \( B_2 \). An explicit calculation gives

\[
r_n(\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix}), e_1^{(n-1)} = \lambda^{n-2l+1} \sum_{j=0}^{l-1} (-b)^j \begin{pmatrix} l-1 \\ j \end{pmatrix} e_{l-j}^{(n-1)}.
\] (4.2)

Hence \( r_n(B_2) \) is contained in \( B_n \subset SL(n) \) and the one-dimensional vector space \( \langle e_1^{(n-1)} \rangle \) is \( B_2 \)-invariant: \( r_n(\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix}), e_1^{(n-1)} = \lambda^{n-1} e_1^{(n-1)} \). For a given integer \( i \in \mathbb{Z} \) we let \( \chi_i : B_2 \to \mathbb{C}^* = GL(1, \mathbb{C}) \) denote the rational character given by

\[
\chi_i(\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix}) = \lambda^i.
\]

Now \( \mathbb{C} \) turns into a \( B_2 \)-module via \( \chi_i \) i.e. \( (\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix})x = \lambda^i x \) for \( x \in \mathbb{C} \). We will denote this \( B_2 \)-module by \( \mathbb{C}_{\chi_i} \). It follows that the \( B_2 \)-module \( \langle e_1^{(n-1)} \rangle \) is isomorphic to \( \mathbb{C}_{\chi_n-1} \) and we obtain a short exact sequence of \( B_2 \)-modules

\[
0 \to \mathbb{C}_{\chi_{n-1}} \to R_{n-1} \to \tilde{R}_{n-1} \to 1
\] (4.3)

where \( \tilde{R}_{n-1} \) denotes the quotient \( R_{n-1}/\langle e_1^{(n-1)} \rangle \). For a given element \( x \in R_{n-1} \) we let \( \bar{x} \in \tilde{R}_{n-1} \) denote the class represented by \( x \) i.e. \( \bar{x} = x + \langle e_1^{(n-1)} \rangle \).

For abbreviation, we will drop the representation \( r_n \) from the notation and write for \( x \in R_{n-1} \)

\[
(\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix})x.
\] instead of \( r_n(\begin{pmatrix}
\lambda & \lambda^{-1}b \\
0 & \lambda^{-1}
\end{pmatrix}), x \).
Lemma 4.2. The linear map $\phi_{n-3} : R_{n-3} \to \bar{R}_{n-1}$ defined by

$$\phi_{n-3}(e_l^{(n-3)}) = \frac{1}{l} e_l^{(n-1)}, \quad l = 1, \ldots, n-2,$$

is an injective $B_2$-module morphism i.e. for all $x \in R_{n-3}$ we have

$$(\lambda^i \lambda^{-1}b, \phi_{n-3}(x)) = \phi_{n-3}(\left(\begin{array}{c} \lambda^i \lambda^{-1}b \\ 0 \end{array}\right)_l x).$$

Proof. The linear map $\phi_{n-3}$ is injective since the vectors $e_l^{(n-1)}, 2 \leq l \leq n,$ form a basis of $R_{n-1}$. Now

$$(\lambda^i \lambda^{-1}b, \phi_{n-3}(e_l^{(n-3)}) = \lambda^{n-2l+1} \frac{1}{l} \sum_{j=0}^{l} (-b)^j \binom{l}{j} e_{l-j+1}^{(n-1)}.$$

Since $\binom{j}{l-j} (l-j) = l \binom{l-1}{j-1}$ and $e_l^{(n-1)} = 0$ it follows

$$(\lambda^i \lambda^{-1}b, \phi_{n-3}(e_l^{(n-3)}) = \lambda^{n-2l+1} \frac{1}{l} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} \frac{1}{l-j-1} e_{l-j+1}^{(n-1)} = \phi_{n-3}(\left(\begin{array}{c} \lambda^i \lambda^{-1}b \\ 0 \end{array}\right)_l e_l^{(n-3)}).$$

Hence $\phi_{n-3}$ is a $B_2$-module morphism. \qed

Lemma 4.3. There is a short exact sequence of $B_2$-modules

$$0 \to R_{n-3} \xrightarrow{\phi_{n-3}} \bar{R}_{n-1} \to C_{\chi_{n+1}} \to 0.$$ \hfill (4.4)

Proof. Again the lemma follows from Equation (4.2):

$$(\lambda^i \lambda^{-1}b, e_n^{(n-1)}) = \lambda^{-n} e_n^{(n-1)} \mod (e_1^{(n-1)}, \ldots, e_{n-1}^{(n-1)}).$$ \qed

Let us fix a representation $\rho_{\gamma}^k : \Gamma_K \to B_2$. Then $R_n$ turns into a $\Gamma_K$-module and the exact sequences (1.3) and (1.4) are exact sequences of $\Gamma_K$-modules. Note that $C_{\chi_k} \cong C_{\lambda_k}$ since for all $\gamma \in \Gamma_K$ and $k \in \mathbb{Z}$ the equation $\chi_k(\rho_{\gamma}^k(\gamma)) = \lambda_k e(\gamma)$ holds.

Lemma 4.4. Let $\lambda \in C^*, \lambda \neq 1$, and $n > 3$ be given. If $\Delta_K(\lambda^{-n}) \neq 0$ and if $\lambda^{n-1} \neq 1$ then

$$H^*(\Gamma_K; R_{n-1}) \cong H^*(\Gamma_K; R_{n-3}).$$

Proof. The long exact cohomology sequences [Bros2] III.$[6]$ associated to the short exact sequences (1.3) gives:

$$H^k(\Gamma_K; C_{\lambda^{n-1}}) \to H^k(\Gamma_K; R_{n-1}) \to H^k(\Gamma_K; \bar{R}_{n-1}) \to H^{k+1}(\Gamma_K; C_{\lambda^{n-1}})$$
Let Proposition 4.5. Let \( \lambda \) and hence \( H \) be given as in (1.1) (see Lemma 2.1). Hence
\[
H^k(\Gamma_K; R_{n-1}) \cong H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2.
\]

Finally, the short exact sequence (4.4), Lemma 2.1 and the assumptions \( \Delta_K(\lambda^{n-1}) \neq 0 \) give that
\[
H^k(\Gamma_K; R_{n-3}) \cong H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2
\]
are isomorphisms (note that \( \Delta_K(t) \) is symmetric).

**Proposition 4.5.** Let \( \lambda \in \mathbb{C}^* \) such that \( \Delta_K(\lambda^2) = 0 \), \( n \geq 3 \) and \( \rho_\lambda^z : \Gamma_K \to B_2 \) be given as in (1.1). If \( \Delta_K(\lambda^{2k}) \neq 0 \) and \( \lambda^{2k} \neq 1 \) for \( 2 \leq k \leq n-1 \) then for \( \rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z : \Gamma_K \to B_n \subset \text{SL}(n) \) we have
\[
\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).
\]

**Proof.** It follows from (4.1) that we have an isomorphism of \( \Gamma_K \)-modules:
\[
\mathfrak{sl}(n)_{\rho_{\lambda,n}^z} \cong \bigoplus_{k=1}^{n-1} R_{2k}.
\]

Now Lemma 1.4 implies that \( \dim H^*(\Gamma_K, R_{2k}) = \dim H^*(\Gamma_K, R_2) \) since \( \Delta_K(\lambda^{2k}) \neq 0 \) and \( \lambda^{2k} \neq 1 \) for \( 2 \leq k \leq n-1 \). Hence the assertion of the proposition follows.

**Proof of Proposition 4.5.** Let \( \lambda \in \mathbb{C}^* \) and \( n \in \mathbb{Z}, n \geq 3 \). Suppose that \( \lambda^2 \) is a simple root of the Alexander polynomial \( \Delta_K(t) \) and let \( \rho_\lambda^z : \Gamma_K \to B_2 \) be a non-abelian representation as in (1.1).

In order to apply Proposition 4.5 we have to show that \( \lambda^{2k} \neq 1 \) for \( 2 \leq k \leq n-1 \). Suppose that there exists \( k \in \mathbb{Z}, 2 \leq k \leq n-1 \), such that \( \lambda^{2k} = 1 \). Next note that \( \lambda^2 = \lambda^{2k-2} \) is a root of the Alexander polynomial since \( \Delta_K(t) \) is symmetric. Therefore the assumption of the proposition implies that \( k = 2 \) i.e. \( \lambda^2 = 1 \) and hence \( \lambda^2 = \pm1 \). At the same time, \( \pm1 \) is not a root of \( \Delta_K(t) \) since \( \Delta_K(1) = \pm1 \) and \( \Delta_K(-1) \) is an odd integer. This gives a contradiction and hence \( \lambda^{2k} \neq 1 \) for \( 2 \leq k \leq n-1 \). Therefore, Proposition 4.5 implies that
\[
\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).
\]

Finally, observe that \( \mathfrak{sl}(2)_{\rho_\lambda^z} \cong R_2 \) (see Example 1) and \( \dim H^1(\Gamma_K; R_2) = 1 \) follows from [HP05, Corollary 5.4] or [HPSP01, 4.4]. It is easy to see that \( H^0(\Gamma_K; R_2) = 0 \) since \( \rho_\lambda^z \) is non-abelian.
5. Examples

Let $K \subset S^3$ be a knot and $\lambda^2$ a simple root of $\Delta_K(t)$. Theorem 1.1 implies that if $\Delta_K(\lambda^{2k}) \neq 0$ for all $k \in \mathbb{Z}$, $k \neq \pm 1$, then for all $n \geq 2$, $n \in \mathbb{Z}$, the representation space $R_n(\Gamma_K)$ contains a component $R_{\lambda,n}$ of dimension $n^2 + n - 2$. Moreover, Proposition 3.8 of [New78] shows that if a component contains an irreducible representation, then generic representations on that component are irreducible.

**Corollary 5.1.** Let $K \subset S^3$ be a knot with the Alexander polynomial of the figure-eight knot.

Then the representation variety $R_n(\Gamma_K)$ contains an $(n^2 + n - 2)$-dimensional component and the irreducible representations form a Zariski-open subset of this component.

**Proof.** The Alexander polynomial of the figure-eight knot is $\Delta(t) = t^2 - 3t + 1$ and its roots are $\lambda^{\pm 2} = 3/2 \pm \sqrt{5}/2$ and no power $\lambda^{2k}$, $k \neq \pm 1$, is a root of $\Delta(t)$. □

The situation for the trefoil knot $3_1$ is more complicated since the roots of its Alexander polynomial, $\Delta_{3_1}(t) = t^2 - t + 1$, are the primitive 6-th roots of unity $\lambda^{\pm 2} = e^{\pm i\pi/3}$. Hence $R_n(\Gamma_{3_1})$ contains an $(n^2 + n - 2)$-dimensional component $R_{\lambda,n}$, for $n \in \{2, 3, 4, 5\}$, since $e^{\pm i\pi/3}$ is a simple root of $\Delta_{3_1}(t)$ and since $\Delta_{3_1}(e^{\pm ik\pi/3}) \neq 0$, for $k \in \{2, 3, 4\}$.

Let us study the case $n = 6$: the group $\Gamma_{3_1}$ is free product with amalgamation

$$\Gamma_{3_1} = \langle S, T \mid STS = TST \rangle \cong \langle x, y \mid x^2 = y^3 \rangle \cong \langle x \mid x^2 = y^3 \rangle \ast \langle y \mid y^3 \rangle$$

where $x = STS$, $y = TS$, and $c = x^2 = y^3$ generates the center of $\Gamma_{3_1}$. Note that a meridian $\mu$ of $3_1$ is represented by the Wirtinger generator $\mu = S = x y^{-1}$. Let $\rho: \Gamma_{3_1} \to \text{SL}(6)$ be an irreducible representation. It follows from Schur’s lemma, that if $\rho$ is irreducible then the generator of the center $x^2 = c = y^3$ has to be mapped into the center $C_6 := \{ \exp(2\pi i k/6) | 1 \leq k \leq 6 \} \subset \text{SL}(6)$ of $\text{SL}(6)$. Notice that for each element of the center $C_6$ there are only finitely many square and cube roots up to conjugation in $\text{SL}(6)$. This implies that if $R \subset R_6(\Gamma_{3_1})$ is an irreducible component of the representation variety then the conjugacy classes represented by the elements $\rho(c)$, $\rho(x)$, $\rho(y)$ in $\text{SL}(6)$ do not vary with $\rho \in R$. Now let $\lambda = e^{i\pi/6}$ be a primitive 12-th root of unity. A cohomological non-trivial cocycle $z \in Z^1(\Gamma_{3_1}; C_{\mathbb{Z}^2})$ is given by $z(S) = 0$ and $z(T) = 1$. Therefore the representation $\rho_{\lambda}^z: \Gamma_{3_1} \to \text{SL}(2)$ is given by

$$\rho_{\lambda}^z(S) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \rho_{\lambda}(T) = \begin{pmatrix} \lambda & \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}.$$
Hence
\[
\rho_A^i(x) = \begin{pmatrix} i & \lambda^{-1} \\ 0 & -i \end{pmatrix}, \quad \rho_A^i(y) = \begin{pmatrix} \lambda^2 & \lambda^{-2} \\ 0 & \lambda^{-2} \end{pmatrix} \quad \text{and} \quad \rho_A^i(c) = -I_2.
\]

Proposition 3.1 implies that \( \rho_{3,6}^A = r_6 \circ \rho_A^i \) is a limit of irreducible representations. Computer supported calculations show that \( \dim H^1(\Gamma_{3}; R_{10}) = 3 \) and Lemma 4.4 implies that \( \dim H^1(\Gamma_{3}; R_{2k}) = \dim H^1(\Gamma_{3}; R_2) = 1 \) for \( k \in \{2, 3, 4\} \). Hence Formula 4.1 implies that
\[
\dim H^1(\Gamma_{31}; \mathfrak{s}(6)_{\rho_{3,6}^A}) = 7 \quad \text{i.e.} \quad Z^1(\Gamma_{31}; \mathfrak{s}(6)_{\rho_{3,6}^A}) = 42.
\]

In order to see that \( \rho_{3,6}^A \) is contained in a 42-dimensional component of \( R_6(\Gamma_{31}) \) we proceed as follows: let \( A = \rho_{3,6}^A(x) \) and \( B = \rho_{3,6}^A(y) \) denote the image of \( x \) and \( y \) respectively. Notice that the matrices \( A \) and \( B \) are conjugate to \( r_6 \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right) \) and \( r_6 \left( \begin{smallmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{smallmatrix} \right) \). Hence
\[
A \sim \begin{pmatrix} i & 0 \\ i & -i \\ 0 & -i \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} -1 & 0 \\ -1 & \lambda^2 \\ 0 & \lambda^{-2} \end{pmatrix}.
\]

Further note that a choice of eigenspaces \( E_A(i), E_A(-i), E_B(-1), E_B(\lambda^2), E_B(\lambda^{-2}) \) such that \( E_A(i) \oplus E_A(-i) \cong \mathbb{C}^6 \) and \( E_B(-1) \oplus E_B(\lambda^2) \oplus E_B(\lambda^{-2}) \cong \mathbb{C}^6 \) determines a representation \( \rho: \Gamma_{31} \to \text{SL}(6) \) completely.

Let \( \text{Gr}(p, n) \) denote the Grassmannian which parametrizes all \( p \)-dimensional subspaces of \( \mathbb{C}^n \). Hence the choice of two elements in \( \text{Gr}(3, 6) \) in generic position determines \( A \) and the choice of three elements in \( \text{Gr}(2, 6) \) in generic position determines \( B \). The representation will be irreducible if the eigenspaces of \( A \) and \( B \) are in general position and reducible if not.

It is well known that \( \dim \text{Gr}(p, n) = p(n - p) \) and hence
\[
\dim (\text{Gr}(3,6) \times \text{Gr}(3,6)) = 18 \quad \text{and} \quad \dim (\text{Gr}(2,6) \times \text{Gr}(2,6) \times \text{Gr}(2,6)) = 24.
\]

Therefore, we constructed a 42-dimensional component of representations \( C \subset R_6(\Gamma_{31}) \) which contains \( \rho_{3,6}^A = r_6 \circ \rho_A^i \) and which also contains irreducible representations. Note that \( 6^2 + 6 - 2 = 40 < 42 \). In conclusion we have:

**Corollary 5.2.** The representation variety \( R_6(\Gamma_{31}) \) contains a 42-dimensional component \( C \). The generic representation of \( C \) is irreducible and \( \rho_{3,6}^C \in C \subset R_6(\Gamma_{31}) \) is a smooth point.

**Proof.** Computer supported calculations give that \( \dim Z^1(\Gamma_{31}; \mathfrak{s}(6)_{\rho_{3,6}^C}) = 42 \). Additionally, we constructed a 42-dimensional component \( C \) containing \( \rho_{3,6}^C \). Now, the assertion follows from Lemma 2.2. \( \square \)
References


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