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DEFORMATIONS OF REDUCIBLE REPRESENTATIONS OF KNOT GROUPS INTO $SL(n, \mathbf{C})$

MICHAEL HEUSENER AND OUARDIA MEDJERAB

ABSTRACT. Let K be a knot in S^3 and X its complement. We study deformations of non-abelian, metabelian, reducible representations of the knot group $\pi_1(X)$ into $SL(n, \mathbf{C})$ which are associated to a simple root of the Alexander polynomial. We prove that some of these metabelian reducible representations are smooth points of the $SL(n, \mathbf{C})$ -representation variety and that they have irreducible deformations.

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1. INTRODUCTION

Let K be a knot in S^3 and $X = \overline{S^3 \setminus V(K)}$ its complement, where $V(K)$ is a tubular neighborhood of K . Moreover, let $\Gamma_K = \pi_1(X)$ denote the fundamental group of X . The aim of this paper is to study deformations of reducible metabelian representations of Γ_K into $SL(n, \mathbf{C})$. The metabelian representations in question were introduced by G. Burde [Bur67] and G. de Rham [dR67]. Let us recall this result: for each nonzero complex number $\lambda \in \mathbf{C}^*$ there exists a diagonal representation $\rho_\lambda: \Gamma_K \rightarrow SL(2, \mathbf{C})$ given by

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

Here $\varphi: \pi_1(X) \rightarrow \mathbf{Z}$ denotes the canonical surjection which maps the meridian μ of K to 1 i.e. $\varphi(\gamma) = \text{lk}(\gamma, K)$. Burde and de Rham proved that there exists a metabelian, non-abelian, reducible representation

$$\rho_\lambda^z: \Gamma_K \rightarrow SL(2, \mathbf{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix} \quad (1.1)$$

if and only if λ^2 is a root of the Alexander polynomial $\Delta_K(t)$. Recall that a representation $\rho: G \rightarrow GL(n, \mathbf{C})$ of a group G is called reducible if the image $\rho(G)$ preserves a proper subspace of \mathbf{C}^n . Otherwise, ρ is called *irreducible*.

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The question whether or not the representation ρ_λ^z is a limit of irreducible representations of Γ_K into $\mathrm{SL}(2, \mathbf{C})$ was studied in [HPSP01]. Theorem 1.1 of [HPSP01] states that a metabelian, non-abelian, reducible representation $\rho_\lambda^z: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C})$ is the limit of irreducible representations if λ^2 is a simple root of $\Delta_K(t)$. Moreover, in this case the representation ρ_λ^z is a smooth point of the representation variety $R(\Gamma_K, \mathrm{SL}(2, \mathbf{C}))$; it is contained in a unique 4-dimensional component $R_\lambda \subset R(\Gamma_K, \mathrm{SL}(2, \mathbf{C}))$.

This article studies the behavior of the representations in question under the composition with the n -dimensional, irreducible, rational representation $r_n: \mathrm{SL}(2, \mathbf{C}) \rightarrow \mathrm{SL}(n, \mathbf{C})$. (for more details see Section 4.1). It is proved in Proposition 3.1 that generically for an irreducible representation $\rho \in R_\lambda$ the representation $\rho_n := r_n \circ \rho \in R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$ is also irreducible. The main result of this article is the following:

Theorem 1.1. *If λ^2 is a simple root of $\Delta_K(t)$ and if $\Delta_K(\lambda^{2k}) \neq 0$ for $2 \leq k \leq n-1$ then the reducible metabelian representation $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z$ is a limit of irreducible representations. More precisely, $\rho_{\lambda,n}^z$ is a smooth point of $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$; it is contained in a unique $(n+2)(n-1)$ -dimensional component $R_{\lambda,n} \subset R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$.*

Remark 1. Let $\rho_{\lambda,n}: \Gamma_K \rightarrow \mathrm{SL}(n, \mathbf{C})$ be the diagonal representation given by $\rho_{\lambda,n} = r_n \circ \rho_\lambda$. The group $\mathrm{SL}(n, \mathbf{C})$ acts on the representation variety $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$ by conjugation, and the orbit $\mathcal{O}(\rho_{\lambda,n})$ is contained in the closure $\overline{\mathcal{O}(\rho_{\lambda,n}^z)}$. Hence $\rho_{\lambda,n}$ and $\rho_{\lambda,n}^z$ project to the same point $\chi_{\lambda,n}$ of the character variety

$$X(\Gamma_K, \mathrm{SL}(n, \mathbf{C})) = R(\Gamma_K, \mathrm{SL}(n, \mathbf{C})) // \mathrm{SL}(n, \mathbf{C}).$$

Here $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C})) // \mathrm{SL}(n, \mathbf{C})$ denotes the GIT quotient of the action (see [New78] for more details). Recall that the GIT quotient parametrizes the closed orbits of the $\mathrm{SL}(n, \mathbf{C})$ action.

It is possible to study the local picture of the character variety at $\chi_{\lambda,n}$ as done in [HPSP01] and [HP05]. Unfortunately, there are additional technical difficulties, and the computations necessary are much more involved. These complications are due to the fact that the diagonal representation $\rho_{\lambda,n}$ is contained in 2^{n-1} components of $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$. Nevertheless, only the component $R_{\lambda,n}$ contains irreducible representations. We will address this topic in a forthcoming paper.

P. Menal-Ferrer and J. Porti [MFP12] showed that the conclusions of the above theorem hold for hyperbolic knots if ρ_λ^z is replaced by a lift of the holonomy, $\widetilde{\mathrm{hol}}: \pi_1(S^3 \setminus K) \rightarrow \mathrm{SL}(2, \mathbf{C})$, of the hyperbolic structure of the complement $S^3 \setminus K$. Note that Theorem 1.1 and Proposition 3.1 do apply to non-hyperbolic knots. Irreducible metabelian representations and their deformations are studied by H. Boden and S. Friedl in a series of articles [BF08, BF11, BF13,

BF14]. In particular the deformations of *irreducible* metabelian representations, which are not considered in this paper, are studied in [BF13].

This article is organized as follows: in Section 2 we will introduce notation and recall some facts which are used. In Section 3 we will prove that the representation variety $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$ contains an irreducible representation if $\Delta_K(t)$ has a simple root (see Proposition 3.1). Moreover we give a streamlined proof of a slightly generalized version of the deformation result used in [MFP12, BAHJ10, BF13] (see Proposition 3.3). The necessary cohomological calculations and the basic facts about the representation theory of $\mathrm{SL}(2, \mathbf{C})$ are presented in Section 4, in order to prove our main result, Theorem 1.1. Finally, in Section 5 some examples are exhibited.

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2. NOTATION AND FACTS

To shorten notation we write $\mathrm{SL}(n)$, $\mathrm{GL}(n)$ and $\mathfrak{sl}(n)$ instead of $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{GL}(n, \mathbf{C})$ and $\mathfrak{sl}(n, \mathbf{C})$.

Let $\varphi: \pi_1(X) \rightarrow \mathbf{Z}$ denote the canonical surjection which maps the meridian μ of K to 1 i.e. $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$. We associate to a nonzero complex number $\alpha \in \mathbf{C}^*$ a homomorphism

$$\alpha^\varphi: \Gamma_K \rightarrow \mathbf{C}^*, \quad \alpha^\varphi: \gamma \mapsto \alpha^{\varphi(\gamma)}.$$

Note that α^φ maps the meridian μ of K to α . We define \mathbf{C}_α to be the Γ_K -module \mathbf{C} with the action induced by α^φ , i.e. $\gamma \cdot x = \alpha^{\varphi(\gamma)}x$ for all $\gamma \in \Gamma_K$ and all $x \in \mathbf{C}$. The trivial Γ_K -module \mathbf{C}_1 is simply denoted \mathbf{C} . With this notation it is easy to see that a map

$$\rho_\lambda^\tilde{z}: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C}), \quad \rho_\lambda^\tilde{z}(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$$

is a homomorphism if and only if the map $z: \Gamma_K \rightarrow \mathbf{C}_{\lambda^2}$ is a 1-cocycle i.e. $z(\gamma_1\gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)}z(\gamma_2)$. Note also that $\rho_\lambda^\tilde{z}$ is abelian if $\lambda = \pm 1$. If $\lambda^2 \neq 1$ then $\rho_\lambda^\tilde{z}$ is abelian if and only if z is a coboundary i.e. there exists an element $x_0 \in \mathbf{C}$ such that $z(\gamma) = (\lambda^{2\varphi(\gamma)} - 1)x_0$. The general reference for group cohomology is Brown's book [Bro82].

In what follows we are mainly interested in the following situation: let X be the complement of a knot $K \subset S^3$ and let A be a $\pi_1(X)$ -module. The spaces X and ∂X are aspherical and hence the natural homomorphisms $H^*(\pi_1(X); A) \rightarrow$

$H^*(X; A)$ and $H^*(\pi_1(\partial X); A) \rightarrow H^*(\partial X; A)$ are isomorphisms. Moreover, the knot complement X has the homotopy type of a 2-dimensional CW-complex which implies that $H^k(\pi_1(X); A) = 0$ and $H_k(\pi_1(X); A) = 0$ for $k \geq 3$. See [Whi78] for more details.

The Laurent polynomial ring $\mathbf{C}[t^{\pm 1}]$ turns into a Γ_K -module via the action $\gamma p(t) = t^{\varphi(\gamma)} p(t)$ for all $\gamma \in \Gamma_K$ and all $p(t) \in \mathbf{C}[t^{\pm 1}]$. Recall that there are isomorphisms of $\mathbf{C}[t^{\pm 1}]$ -modules

$$H_*(\Gamma_K; \mathbf{C}[t^{\pm 1}]) \cong H_*(X; \mathbf{C}[t^{\pm 1}]) \cong H_*(X_\infty; \mathbf{C})$$

where X_∞ denotes the infinite cyclic covering of the knot complement X (see [DK01, Chapter 5]). The module $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$ is a finitely generated torsion module called the *Alexander module* of K . A generator of its order ideal is called the *Alexander polynomial* $\Delta_K(t) \in \mathbf{C}[t^{\pm 1}]$ of K . The Alexander polynomial is unique up to multiplication with a unit in $\mathbf{C}[t^{\pm 1}]$.

For completeness we will state the next lemma which shows that the cohomology groups $H^*(\Gamma_K; \mathbf{C}_\alpha)$ are determined by the Alexander module $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$.

Lemma 2.1. *Let $K \subset S^3$ be a knot and Γ_K its group. Let $\alpha \in \mathbf{C}^*$ be a nonzero complex number and let \mathbf{C}_α denote the Γ_K -module given by the action $\gamma z = \alpha^{\varphi(\gamma)} z$.*

If $\alpha = 1$ then $\mathbf{C}_\alpha = \mathbf{C}$ is a trivial Γ_K -module and $H^k(\Gamma_K, \mathbf{C}) \cong \mathbf{C}$ for $k = 0, 1$ and $H^k(\Gamma_K, \mathbf{C}) = 0$ for $k \geq 2$. If $\alpha \neq 1$ then $H^0(\Gamma_K, \mathbf{C}_\alpha) = 0$ and $\dim H^1(\Gamma_K, \mathbf{C}_\alpha) = \dim H^2(\Gamma_K, \mathbf{C}_\alpha)$. Moreover, $H^1(\Gamma_K, \mathbf{C}_\alpha) \neq 0$ if and only if $\Delta_K(\alpha) = 0$.

Proof. We have $H_0(X_\infty; \mathbf{C}) \cong \mathbf{C} \cong \mathbf{C}[t^{\pm 1}]/(t - 1)$ and $H_k(X_\infty; \mathbf{C}) = 0$ for $k \geq 2$ (see [BZH13, Prop. 8.16]). If $\alpha = 1$ then $H^k(\Gamma_K, \mathbf{C}) \cong \mathbf{C}$ for $k = 0, 1$ and $H^k(\Gamma_K, \mathbf{C}) = 0$ for $k \geq 2$ follows.

Now suppose that $\alpha \in \mathbf{C}^*$, $\alpha \neq 1$, and notice that we have an isomorphism $\mathbf{C}_\alpha \cong \mathbf{C}[t^{\pm 1}]/(t - \alpha)$. The cohomology group $H^0(\Gamma_K, \mathbf{C}_\alpha)$ vanishes, since the Γ_K -module \mathbf{C}_α has no invariants, and $H^k(\Gamma_K, \mathbf{C}_\alpha) = 0$ for $k > 2$ since the knot complement X has the homotopy type of a 2-complex. Recall that the Alexander module $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$ is finitely generated torsion module and hence a sum of non-free cyclic modules since $\mathbf{C}[t^{\pm 1}]$ is a principal ideal domain. The Alexander polynomial is the order ideal of $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$. Since $\alpha \neq 1$, it follows from the universal coefficient theorem that $H^1(\Gamma; \mathbf{C}_\alpha) \cong \text{Hom}(H_1(\Gamma_K; \mathbf{C}); \mathbf{C}_\alpha)$. Hence $H^1(\Gamma_K, \mathbf{C}_\alpha) \neq 0$ if and only if the module $H_1(\Gamma_K; \mathbf{C})$ has $(t - \alpha)$ -torsion which is equivalent to $\Delta_K(\alpha) = 0$. Finally, $\dim H^1(\Gamma_K, \mathbf{C}_\alpha) = \dim H^2(\Gamma_K, \mathbf{C}_\alpha)$ follows since the Euler characteristic of X vanishes. See also [BA00, Proposition 2.1] for more details. \square

In what follows we will also make use of the Poincaré-Lefschetz duality theorem with twisted coefficients: let M^m be a connected, orientable, compact

m -dimensional manifold with boundary ∂M^m and let $\rho: \pi_1(M^m) \rightarrow \mathrm{SL}(n)$ be a representation. Then the cup-product and the Killing form $b: \mathfrak{sl}(n)_\rho \otimes \mathfrak{sl}(n)_\rho \rightarrow \mathbf{C}$ induce a non-degenerate bilinear pairing

$$\begin{aligned} H^k(M^m; \mathfrak{sl}(n)_\rho) \otimes H^{m-k}(M^m, \partial M^m; \mathfrak{sl}(n)_\rho) &\xrightarrow{\sim} \\ H^m(M^m, \partial M^m; \mathfrak{sl}(n)_\rho \otimes \mathfrak{sl}(n)_\rho) &\xrightarrow{b} H^m(M^m, \partial M^m; \mathbf{C}) \cong \mathbf{C} \end{aligned} \quad (2.1)$$

and hence an isomorphism $H^k(M^m; \mathfrak{sl}(n)_\rho) \cong H^{m-k}(M^m, \partial M^m; \mathfrak{sl}(n)_\rho)^*$, for all $0 \leq k \leq m$. See [JM87, Por95] for more details.

2.1. Group cohomology and representation varieties. Let now Γ be a finitely generated group. The set $R_n(\Gamma) := R(\Gamma, \mathrm{SL}(n))$ of homomorphisms of Γ in $\mathrm{SL}(n)$ is called the $\mathrm{SL}(n)$ -representation variety of Γ and has the structure of a (not necessarily irreducible) algebraic set.

Let $\rho: \Gamma \rightarrow \mathrm{SL}(n)$ be a representation. The Lie algebra $\mathfrak{sl}(n)$ turns into a Γ -module via $\mathrm{Ad} \rho$. This module will be simply denoted by $\mathfrak{sl}(n)_\rho$. A cocycle $d \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is a map $d: \Gamma \rightarrow \mathfrak{sl}(n)$ satisfying

$$d(\gamma_1 \gamma_2) = d(\gamma_1) + \rho(\gamma_1) d(\gamma_2) \rho(\gamma_1)^{-1}, \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

It was observed by André Weil [Wei64] that there is a natural inclusion of the Zariski tangent space $T_\rho^{Zar}(R_n(\Gamma)) \hookrightarrow Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Informally speaking, given a smooth curve ρ_ϵ of representations through $\rho_0 = \rho$ one gets a 1-cocycle $d: \Gamma \rightarrow \mathfrak{sl}(n)$ by defining

$$d(\gamma) := \left. \frac{d\rho_\epsilon(\gamma)}{d\epsilon} \right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma.$$

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$. Here, $b: \Gamma \rightarrow \mathfrak{sl}(n)$ is a coboundary if there exists $x \in \mathfrak{sl}(n)$ such that $b(\gamma) = \rho(\gamma) x \rho(\gamma)^{-1} - x$. A detailed account can be found in [LM85].

Let $\dim_\rho R_n(\Gamma)$ be the local dimension of $R_n(\Gamma)$ at ρ (i.e. the maximal dimension of the irreducible components of $R_n(\Gamma)$ containing ρ [Sha77, Ch. II]). So we obtain:

$$\dim_\rho R_n(\Gamma) \leq \dim T_\rho^{Zar}(R_n(\Gamma)) \leq \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho).$$

We will call a representation $\rho \in R_n(\Gamma)$ *regular* if $\dim_\rho R_n(\Gamma) = \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$.

The following lemma follows (for more details see [HPSP01, Lemma 2.6]):

Lemma 2.2. *Let $\rho \in R_n(\Gamma)$ be a representation. If ρ is regular, then ρ is a smooth point of the representation variety $R_n(\Gamma)$ and ρ is contained in a unique component of $R_n(\Gamma)$ of dimension $\dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$.*

Note that there are discrete groups and representations ρ which are smooth points of the representation variety without been regular. (See [LM85, Example 2.10] for more details.)

3. DEFORMING REPRESENTATIONS

The aim of the following sections is to prove that the representation $\rho_{\lambda,n}^z$ from the introduction is a smooth point of the representation variety. We present a more streamlined and slightly generalized version of the deformation result from [HPSP01, HP05, BAHJ10, MFP12, BF14] (see Proposition 3.3). For the convenience of the reader we recall the setup.

First we will prove that the representation $\rho_{\lambda,n}^z \in R_n(\Gamma_K)$ is the limit of irreducible representations if λ^2 is a simple root of of the Alexander polynomial $\Delta_K(t)$. In what follows a property of an irreducible algebraic variety Y is said to be true *generically* if it holds except on a proper Zariski-closed subset of Y , in other words, if it holds on a non-empty Zariski-open subset.

Let $K \subset S^3$ be a knot, $\lambda^2 \in \mathbf{C}$ a simple root of $\Delta_K(t)$ and $z \in Z^1(\Gamma_K, \mathbf{C}_{\lambda^2})$ a cocycle representing a generator of $H^1(\Gamma_K, \mathbf{C}_{\lambda^2})$. Following [HPSP01, Thm 1.1] the representation $\rho_{\lambda}^z \in R_2(\Gamma_K)$ is a smooth point of the representation variety. It is contained in an unique irreducible 4-dimensional component $R_{\lambda} \subset R_2(\Gamma_K)$. Note that generically a representation $\rho \in R_{\lambda}$ is irreducible.

Proposition 3.1. *Let $K \subset S^3$ be a knot, $\lambda^2 \in \mathbf{C}$ a simple root of $\Delta_K(t)$ and let $z \in Z^1(\Gamma_K, \mathbf{C}_{\lambda^2})$ be a cocycle representing a generator of $H^1(\Gamma_K, \mathbf{C}_{\lambda^2})$.*

Then the representation $\rho_{\lambda,n}^z = r_n \circ \rho_{\lambda}^z: \Gamma_K \rightarrow B_n$ is the limit of irreducible representation in $R_n(\Gamma_K)$. More precisely, generically a representation $\rho_n = r_n \circ \rho$, $\rho \in R_{\lambda}$ is irreducible.

Proof. It follows from [HPSP01, Theorem 1.1] that $\rho_{\lambda}^z \in R_2(\Gamma_K)$ is the limit of irreducible representations. Moreover, $\rho_{\lambda}^z \in R_2(\Gamma_K)$ is a smooth point which is contained in a unique 4-dimensional component $R_{\lambda} \subset R_2(\Gamma_K)$.

Let Γ be a discrete group and let $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$ be an irreducible representation. If the image $\rho(\Gamma) \subset \mathrm{SL}(2, \mathbf{C})$ is Zariski-dense then the representation $\rho_n := r_n \circ \rho \in R_n(\Gamma)$ is irreducible. Hence in order to prove the proposition we show that there is a neighborhood $U = U(\rho_{\lambda}^z) \subset R_2(\Gamma_K)$ such that $\rho(\Gamma) \subset \mathrm{SL}(2, \mathbf{C})$ is Zariski-dense for each irreducible $\rho \in U$. Let now $\rho: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C})$ be any irreducible representation and let $G \subset \mathrm{SL}(2)$ denote the Zariski-closure of $\rho(\Gamma_K)$. Suppose that $G \neq \mathrm{SL}(2)$. Since ρ is irreducible it follows that G is, up to conjugation, not a subgroup of upper-triangular matrices of $\mathrm{SL}(2)$. Then by [Kov86, Sec. 1.4] and [Kap57, Theorem 4.12] there are, up to conjugation, only two cases left:

- G is a subgroup of the infinite dihedral group

$$D_{\infty} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in \mathbf{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \mid \alpha \in \mathbf{C}^* \right\}.$$

- G is one of the groups $A_4^{\text{SL}(2)}$ (the tetrahedral group), $S_4^{\text{SL}(2)}$ (the octahedral group) or $A_5^{\text{SL}(2)}$ (the icosahedral group). These groups are the preimages in $\text{SL}(2)$ of the subgroups $A_4, S_4, A_5 \subset \text{PSL}(2, \mathbf{C})$.

In the first case it follows directly from [Nag07] that if ρ is an irreducible metabelian representation then the trace of the image of a meridian $\text{tr}(\rho(\mu)) = 0$ i.e. $\rho(\mu)$ is similar to $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Now, $\text{tr}(\rho_\lambda^z(\mu)) \neq 0$ since $\Delta_K(-1) \neq 0$ and $\Delta_K(\lambda^{\pm 2}) = 0$. For the second case there are up to conjugation only finitely many irreducible representations of Γ_K onto the subgroups $A_4^{\text{SL}(2)}, S_4^{\text{SL}(2)}$ and $A_5^{\text{SL}(2)}$. Note that these finitely many orbits are closed and 3-dimensional. Hence the irreducible $\rho \in R_\lambda$ such that $r_n \circ \rho$ is reducible is contained in a Zariski-closed subset of R_λ . Hence generically $r_n \circ \rho$ is irreducible for $\rho \in R_\lambda$. \square

Remark 2. Recall that a finite group has only finitely many irreducible representations (see [Ser78, FH91]). Hence, the restriction of r_n to the groups $A_4^{\text{SL}(2)}, S_4^{\text{SL}(2)}$ and $A_5^{\text{SL}(2)}$ is reducible, for all but finitely many $n \in \mathbf{N}$.

In order to prove that a certain representation $\rho \in R_n(\Gamma)$ is a smooth point of the representation variety we will prove that every cocycle $u \in Z^1(\Gamma_K; \mathfrak{sl}(n)_\rho)$ is integrable. In order to do this, we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a theorem of Artin [Art68].

The formal deformations of a representation $\rho: \Gamma \rightarrow \text{SL}(n)$ are in general determined by an infinite sequence of obstructions (see [Gol84, BA00, HPSP01]). In what follows we let $C^1(\Gamma; \mathfrak{sl}(n)) := \{c: \Gamma \rightarrow \mathfrak{sl}(n)\}$ denote the 1-cochains of Γ with coefficients in $\mathfrak{sl}(n)$ (see [Bro82, p.59]).

Let $\rho: \Gamma \rightarrow \text{SL}(n)$ be a representation. A formal deformation of ρ is a homomorphism $\rho_\infty: \Gamma \rightarrow \text{SL}(n, \mathbf{C}[[t]])$

$$\rho_\infty(\gamma) = \exp \left(\sum_{i=1}^{\infty} t^i u_i(\gamma) \right) \rho(\gamma), \quad u_i \in C^1(\Gamma; \mathfrak{sl}(n))$$

such that $\text{ev}_0 \circ \rho_\infty = \rho$. Here $\text{ev}_0: \text{SL}(n, \mathbf{C}[[t]]) \rightarrow \text{SL}(n)$ is the evaluation homomorphism at $t = 0$ and $\mathbf{C}[[t]]$ denotes the ring of formal power series.

We will say that ρ_∞ is a formal deformation up to order k of ρ if ρ_∞ is a homomorphism modulo t^{k+1} .

An easy calculation gives that ρ_∞ is a homomorphism up to first order if and only if $u_1 \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is a cocycle. We call a cocycle $u_1 \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ *integrable* if there is a formal deformation of ρ with leading term u_1 .

Lemma 3.2. *Let $u_1, \dots, u_k \in C^1(\Gamma; \mathfrak{sl}(n)_\rho)$ such that*

$$\rho_k(\gamma) = \exp \left(\sum_{i=1}^k t^i u_i(\gamma) \right) \rho(\gamma)$$

is a homomorphism into $\mathrm{SL}(n, \mathbf{C}[[t]]/(t^{k+1}))$. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1, \dots, u_k)} \in H^2(\Gamma, \mathfrak{sl}(n)_\rho)$ with the following properties:

(i) There is a cochain $u_{k+1}: \Gamma \rightarrow \mathfrak{sl}(n)_\rho$ such that

$$\rho_{k+1}(\gamma) = \exp\left(\sum_{i=1}^{k+1} t^i u_i(\gamma)\right) \rho(\gamma)$$

is a homomorphism modulo t^{k+2} if and only if $\zeta_{k+1} = 0$.

(ii) The obstruction ζ_{k+1} is natural, i.e. if $f: \Gamma_1 \rightarrow \Gamma$ is a homomorphism then $f^* \rho_k := \rho_k \circ f$ is also a homomorphism modulo t^{k+1} and $f^*(\zeta_{k+1}^{(u_1, \dots, u_k)}) = \zeta_{k+1}^{(f^* u_1, \dots, f^* u_k)} \in H^2(\Gamma_1; \mathfrak{sl}(n)_{f^* \rho})$.

Proof. The proof is completely analogous to the proof of Proposition 3.1 in [HPSP01]. We replace $\mathrm{SL}(2)$ and $\mathfrak{sl}(2)$ by $\mathrm{SL}(n)$ and $\mathfrak{sl}(n)$ respectively. \square

The following result streamlines the arguments given in [HP05] and [BAHJ10]:

Proposition 3.3. *Let M be a connected, compact, orientable 3-manifold with torus boundary and let $\rho: \pi_1 M \rightarrow \mathrm{SL}(n)$ be a representation.*

If $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$ then ρ is a smooth point of the $\mathrm{SL}(n)$ -representation variety $R_n(\pi_1 M)$. Moreover, ρ is contained in a unique component of dimension $n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho)$.

Proof. First we will show that the map $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$ induced by the inclusion $\partial M \hookrightarrow M$ is injective.

Recall that for any CW-complex X with $\pi_1(X) \cong \pi_1(M)$ and for any $\pi_1 M$ -module A there are natural morphisms $H^i(\pi_1 M; A) \rightarrow H^i(X; A)$ which are isomorphisms for $i = 0, 1$ and an injection for $i = 2$ (see [HP05, Lemma 3.3]). Note also that $\partial M \cong S^1 \times S^1$ is aspherical and hence $H^*(\pi_1 \partial M; A) \rightarrow H^*(\partial M; A)$ is an isomorphism.

First we will prove that for every representation $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$ we have

$$\dim H^0(\mathbf{Z} \oplus \mathbf{Z}; \mathfrak{sl}(n)_\varrho) = \frac{1}{2} \dim H^1(\mathbf{Z} \oplus \mathbf{Z}; \mathfrak{sl}(n)_\varrho) \geq n - 1. \quad (3.1)$$

Moreover, we will prove that $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$ is regular if and only if equality holds in (3.1). It follows from Poincaré duality (2.1) that for every $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$ we have

$$\dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) = \dim H^2(\partial M; \mathfrak{sl}(n)_\varrho),$$

and since the Euler characteristic of M vanishes we obtain the first equality in (3.1):

$$\dim H^1(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim \mathfrak{sl}(n)_{\varrho}^{\mathbf{Z} \oplus \mathbf{Z}}.$$

Now, R.W. Richardson proved in [Ric79, Thm. C] that the representation variety $R_n(\mathbf{Z} \oplus \mathbf{Z})$ is an irreducible algebraic variety of dimension $(n+2)(n-1)$. Hence we obtain for every $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$ that

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_\varrho) \geq (n+2)(n-1) = n^2 + n - 2$$

where the equality holds if and only if ϱ is regular (see Lemma 2.2). At the same time, we have:

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_\varrho) = \dim H^1(\partial M; \mathfrak{sl}(n)_\varrho) + \dim B^1(\partial M; \mathfrak{sl}(n)_\varrho)$$

and the exactness of $0 \rightarrow H^0(\partial M; \mathfrak{sl}(n)_\varrho) \rightarrow \mathfrak{sl}(n) \rightarrow B^1(\partial M; \mathfrak{sl}(n)_\varrho) \rightarrow 0$ gives

$$\dim B^1(\partial M; \mathfrak{sl}(n)_\varrho) = \dim \mathfrak{sl}(n) - \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho).$$

This together with $\dim H^1(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho)$ gives for all $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$:

$$\dim Z^1(\partial M; \mathfrak{sl}(n)_\varrho) = \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) + n^2 - 1 \geq n^2 + n - 2.$$

It follows that

$$\dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) \geq n - 1, \quad \text{for all } \varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z}), \quad (3.2)$$

and $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$ is regular if and only if $\dim H^0(\mathbf{Z} \oplus \mathbf{Z}; \mathfrak{sl}(n)_\varrho) = n - 1$ (see also [Pop08]).

Now, the exact cohomology sequence of the pair $(M, \partial M)$ gives

$$\begin{aligned} &\rightarrow H^1(M, \partial M; \mathfrak{sl}(n)_\rho) \\ &\rightarrow H^1(M; \mathfrak{sl}(n)_\rho) \xrightarrow{\alpha} H^1(\partial M; \mathfrak{sl}(n)_\rho) \xrightarrow{\beta} H^2(M, \partial M; \mathfrak{sl}(n)_\rho) \\ &\rightarrow H^2(M; \mathfrak{sl}(n)_\rho) \xrightarrow{i^*} H^2(\partial M; \mathfrak{sl}(n)_\rho) \rightarrow H^3(M, \partial M; \mathfrak{sl}(n)_\rho) \rightarrow 0. \end{aligned}$$

Poincaré-Lefschetz duality (2.1) implies that α and β are dual to each other. This together with (3.2) gives:

$$\begin{aligned} n - 1 = \dim H^1(M; \mathfrak{sl}(n)_\rho) &\geq \text{rk}(\alpha) = \frac{1}{2} \dim H^1(\partial M; \mathfrak{sl}(n)_\rho) \\ &= \dim H^0(\partial M; \mathfrak{sl}(n)_\rho) \geq n - 1. \end{aligned}$$

Therefore, $\dim H^0(\partial M; \mathfrak{sl}(n)_\rho) = n - 1$ holds in Equation (3.1), and consequently $i^*\rho = \rho \circ i_\# \in R_n(\partial M)$ is regular (here $i: \partial M \rightarrow M$ is the inclusion). Note also that β is surjective, and hence

$$i^*: H^2(M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\partial M; \mathfrak{sl}(n)_\rho)$$

is injective. The following commutative diagram shows that $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$ is also injective:

$$\begin{array}{ccc} H^2(M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\partial M; \mathfrak{sl}(n)_\rho) \\ \uparrow & & \uparrow \cong \\ H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho). \end{array}$$

In order to prove that ρ is a smooth point of $R_n(\pi_1 M)$, we show that all cocycles in $Z^1(\pi_1 M, \mathfrak{sl}(n)_\rho)$ are integrable. In what follows we will prove that all obstructions vanish, by using the fact that the obstructions vanish on the boundary. Let $u_1, \dots, u_k: \pi_1 M \rightarrow \mathfrak{sl}(n)$ be given such that

$$\rho_k(\gamma) = \exp\left(\sum_{i=1}^k t^i u_i(\gamma)\right) \rho(\gamma)$$

is a homomorphism modulo t^{k+1} . Then the restriction $i^* \rho_k: \pi_1(\partial M) \rightarrow \mathrm{SL}(n, \mathbf{C}[[t]])$ is also a formal deformation of order k . Since $i^* \rho$ is a smooth point of the representation variety $R_n(\mathbf{Z} \oplus \mathbf{Z})$, the formal implicit function theorem gives that $i^* \rho_k$ extends to a formal deformation of order $k+1$ (see [HPSP01, Lemma 3.7]). Therefore, we have that

$$0 = \zeta_{k+1}^{(i^* u_1, \dots, i^* u_k)} = i^* \zeta_{k+1}^{(u_1, \dots, u_k)}$$

Now, i^* is injective and the obstruction $\zeta_{k+1}^{(u_1, \dots, u_k)}$ vanishes.

Hence all cocycles in $Z^1(\Gamma, \mathfrak{sl}(n)_\rho)$ are integrable. By applying Artin's theorem [Art68] we obtain from a formal deformation of ρ a convergent deformation (see [HPSP01, Lemma 3.3] or [BA00, § 4.2]).

Thus ρ is a regular point of the representation variety $R_n(\pi_1 M)$. Hence, $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$ and the exactness of

$$0 \rightarrow H^0(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow \mathfrak{sl}(n)_\rho \rightarrow B^1(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow 0$$

implies

$$\dim_\rho R_n(\pi_1 M) = \dim Z^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho).$$

Finally, the proposition follows from Lemma 2.2. \square

Proposition 3.4. *Let $K \subset S^3$ be a knot, $\lambda \in \mathbf{C}^*$ and $n \geq 3$. Suppose that λ^2 is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho_\lambda^z: \Gamma_K \rightarrow \mathrm{SL}(2)$ be a non-abelian representation as in (1.1).*

If $\Delta_K(\lambda^{2i}) \neq 0$ for $2 \leq i \leq n-1$ then for $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z: \Gamma_K \rightarrow \mathrm{SL}(n)$ we have

$$\dim H^1(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \text{ and } H^0(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = 0.$$

Proof. A proof of the cohomological calculation will be given in Section 4. \square

Proof of Theorem 1.1. It follows directly from Propositions 3.3 and 3.4 that $\rho_{\lambda,n}^z$ is a smooth point of $R_n(\Gamma_K)$ which is contained in a unique component $R_{\lambda,n} \subset R_n(\Gamma_K)$, $\dim R_{\lambda,n} = n^2 + n - 2$.

That $\rho_{\lambda,n}^z$ is the limit of irreducible representations which are contained in the component $R_{\lambda,n}$ follows from Proposition 3.1. \square

4. COHOMOLOGICAL CALCULATIONS

For the convenience of the reader we recall some facts from the representation theory of $\mathrm{SL}(2)$. The general reference for this topic is Springer's LNM [Spr77].

4.1. Representation theory of $\mathrm{SL}(2)$. Let V be an n -dimensional complex vector space. In what follows we will call a homomorphism $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$ an *n -dimensional representation* of $\mathrm{SL}(2)$. The vector space V turns into an $\mathrm{SL}(2)$ -module. Two n -dimensional representations $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$ and $r': \mathrm{SL}(2) \rightarrow \mathrm{GL}(V')$ are called *equivalent* if there is an isomorphism $\phi: V \rightarrow V'$ which commutes with the action of $\mathrm{SL}(2)$ i.e. $r'(A)\phi = \phi r(A)$ for all $A \in \mathrm{SL}(2)$. It is clear that equivalent representations give rise to isomorphic $\mathrm{SL}(2)$ -modules.

We let $\mathrm{SL}(2)$ act as a group of automorphisms on the polynomial algebra $R = \mathbf{C}[X, Y]$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$ then there is a unique automorphism $r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of R given by

$$r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(X) = dX - bY \quad \text{and} \quad r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(Y) = -cX + aY.$$

We let $R_{n-1} \subset R$ denote the n -dimensional subspace of homogeneous polynomials of degree $n-1$. The monomials $e_l^{(n-1)} = X^{l-1}Y^{n-l}$, $1 \leq l \leq n$, form a basis of R_{n-1} and $r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ leaves R_{n-1} invariant. In what follows we will identify R_{n-1} and \mathbf{C}^n by fixing the basis $(e_1^{(n-1)}, \dots, e_n^{(n-1)})$ of R_{n-1} . We obtain an n -dimensional representation $r_n: \mathrm{SL}(2) \rightarrow \mathrm{GL}(R_{n-1}) \cong \mathrm{GL}(n)$.

The representation r_n is *rational* i.e. the coefficients of the matrix coordinates of $r_n(A)$ are polynomials in the matrix coordinates of A . We will make use of the following theorem.

Theorem 4.1. (1) *The representation r_n is irreducible i.e. there is no $\mathrm{SL}(2)$ -stable invariant subspace V , $\{0\} \subsetneq V \subsetneq R_{n-1}$ and any irreducible rational representation of $\mathrm{SL}(2)$ is equivalent to some r_n .*

(2) *For an arbitrary rational representation $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$ the $\mathrm{SL}(2)$ -module V is isomorphic to a direct sum of R_n ,*

$$V \cong \bigoplus_{d \geq 0} R_d^{m(d)}.$$

Proof. See Lemma 3.1.3 and Proposition 3.2.1 of [Spr77]. \square

It is easy to see, and it follows also from the general theory, that r_n maps an unipotent matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ onto an unipotent element of $\mathrm{SL}(R_{n-1})$.

Moreover, an explicit calculation shows that the image of a diagonal matrix is the diagonal matrix $r_n(\text{diag}(a, a^{-1})) = \text{diag}(a^{n-1}, a^{n-3}, \dots, a^{-n+3}, a^{-n+1})$. Hence the image of r_n is contained in $\text{SL}(R_{n-1}) \cong \text{SL}(n)$.

Example 1. The representation $r_1: \text{SL}(2) \rightarrow \text{SL}(1) = \{1\}$ is the trivial representation. The representation $r_2: \text{SL}(2) \rightarrow \text{SL}(2)$ is equivalent to the identity:

$$r_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Moreover, it is easy to see that the adjoint representation $\text{Ad}: \text{SL}(2) \rightarrow \text{Aut}(\mathfrak{sl}(2))$ is equivalent to r_3 .

The Lie algebra $\mathfrak{sl}(n)$ of $\text{SL}(n)$ turns into an $\text{SL}(2)$ -module via $\text{Ad} \circ r_n$ where $\text{Ad}: \text{SL}(n) \rightarrow \text{Aut}(\mathfrak{sl}(n))$ denotes the adjoint representation. For this action we have the classical formula of Clebsch–Gordan:

$$\text{Ad} \circ r_n \cong \bigoplus_{i=1}^{n-1} r_{2i+1}. \quad (4.1)$$

Let $B_n \subset \text{SL}(n)$ denote the Borel subgroup of upper triangular matrices. The vector space R_{n-1} turns into a B_2 -module via the restriction of r_n to B_2 . An explicit calculation gives

$$r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_l^{(n-1)} = \lambda^{n-2l+1} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} e_{l-j}^{(n-1)}. \quad (4.2)$$

Hence $r_n(B_2)$ is contained in $B_n \subset \text{SL}(n)$ and the one-dimensional vector space $\langle e_1^{(n-1)} \rangle$ is B_2 invariant: $r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_1^{(n-1)} = \lambda^{n-1} e_1^{(n-1)}$. For a given integer $i \in \mathbf{Z}$ we let $\chi_i: B_2 \rightarrow \mathbf{C}^* = \text{GL}(1, \mathbf{C})$ denote the rational character given by

$$\chi_i \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} = \lambda^i.$$

Now \mathbf{C} turns into a B_2 -module via χ_i i.e. $\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot x = \lambda^i x$ for $x \in \mathbf{C}$. We will denote this B_2 -module by \mathbf{C}_{χ_i} . It follows that the B_2 -module $\langle e_1^{(n-1)} \rangle \in R_{n-1}$ is isomorphic to $\mathbf{C}_{\chi_{n-1}}$ and we obtain a short exact sequence of B_2 -modules

$$0 \rightarrow \mathbf{C}_{\chi_{n-1}} \rightarrow R_{n-1} \rightarrow \bar{R}_{n-1} \rightarrow 1 \quad (4.3)$$

where \bar{R}_{n-1} denotes the quotient $R_{n-1}/\langle e_1^{(n-1)} \rangle$. For a given element $x \in R_{n-1}$ we let $\bar{x} \in \bar{R}_{n-1}$ denote the class represented by x i.e. $\bar{x} = x + \langle e_1^{(n-1)} \rangle$.

For abbreviation, we will drop the representation r_n from the notation and write for $x \in R_{n-1}$

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot x \text{ instead of } r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot x.$$

Lemma 4.2. *The linear map $\phi_{n-3}: R_{n-3} \rightarrow \bar{R}_{n-1}$ defined by*

$$\phi_{n-3}(e_l^{(n-3)}) = \frac{1}{l} \bar{e}_{l+1}^{(n-1)}, \quad l = 1, \dots, n-2,$$

is an injective B_2 -module morphism i.e. for all $x \in R_{n-3}$ we have

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \phi_{n-3}(x) = \phi_{n-3}\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot x\right).$$

Proof. The linear map ϕ_{n-3} is injective since the vectors $\bar{e}_l^{(n-1)}$, $2 \leq l \leq n$, form a basis of \bar{R}_{n-1} . Now

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \phi_{n-3}(e_l^{(n-3)}) = \lambda^{n-2l-1} \frac{1}{l} \sum_{j=0}^l (-b)^j \binom{l}{j} \bar{e}_{l-j+1}^{(n-1)}.$$

Since $\binom{l}{j}(l-j) = l \binom{l-1}{j}$ and $\bar{e}_1^{(n-1)} = 0$ it follows

$$\begin{aligned} \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \phi_{n-3}(e_l^{(n-3)}) &= \lambda^{(n-2)-2l+1} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} \frac{1}{l-j} \bar{e}_{l-j+1}^{(n-1)} \\ &= \phi_{n-3}\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_l^{(n-3)}\right). \end{aligned}$$

Hence ϕ_{n-3} is a B_2 -module morphism. \square

Lemma 4.3. *There is a short exact sequence of B_2 -modules*

$$0 \rightarrow R_{n-3} \xrightarrow{\phi_{n-3}} \bar{R}_{n-1} \rightarrow \mathbf{C}_{\chi_{-n+1}} \rightarrow 0. \quad (4.4)$$

Proof. Again the lemma follows from Equation (4.2):

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_n^{(n-1)} \equiv \lambda^{-n+1} e_n^{(n-1)} \pmod{\langle e_1^{(n-1)}, \dots, e_{n-1}^{(n-1)} \rangle}. \quad \square$$

Let us fix a representation $\rho_\lambda^z: \Gamma_K \rightarrow B_2$. Then R_n turns into a Γ_K -module and the exact sequences (4.3) and (4.4) are exact sequences of Γ_K -modules. Note that $\mathbf{C}_{\chi_k} \cong \mathbf{C}_{\lambda^k}$ since for all $\gamma \in \Gamma_K$ and $k \in \mathbf{Z}$ the equation $\chi_k(\rho_\lambda^z(\gamma)) = \lambda^{k\varphi(\gamma)}$ holds.

Lemma 4.4. *Let $\lambda \in \mathbf{C}^*$, $\lambda \neq 1$, and $n > 3$ be given. If $\Delta_K(\lambda^{n-1}) \neq 0$ and if $\lambda^{n-1} \neq 1$ then*

$$H^*(\Gamma_K; R_{n-1}) \cong H^*(\Gamma_K; R_{n-3}).$$

Proof. The long exact cohomology sequences [Bro82, III.§6] associated to the short exact sequences (4.3) gives:

$$H^k(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) \rightarrow H^k(\Gamma_K; R_{n-1}) \rightarrow H^k(\Gamma_K; \bar{R}_{n-1}) \rightarrow H^{k+1}(\Gamma_K; \mathbf{C}_{\lambda^{n-1}})$$

is exact for $k = 0, 1, 2$. Now $H^0(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) = 0$ since $\lambda^{n-1} \neq 1$ and for $k = 1, 2$ the group $H^k(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) = 0$ since $\Delta_K(\lambda^{n-1}) \neq 0$ (see Lemma 2.1). Hence

$$H^k(\Gamma_K; R_{n-1}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2.$$

Finally, the short exact sequence (4.4), Lemma 2.1 and the assumptions $\Delta_K(\lambda^{n-1}) \neq 0$ with $\lambda^{n-1} \neq 1$ give that

$$H^k(\Gamma_K; R_{n-3}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2$$

are isomorphisms (note that $\Delta_K(t)$ is symmetric). \square

Proposition 4.5. *Let $\lambda \in \mathbf{C}^*$ such that $\Delta_K(\lambda^2) = 0$, $n \geq 3$ and $\rho_\lambda^z: \Gamma_K \rightarrow B_2$ be given as in (1.1). If $\Delta_K(\lambda^{2k}) \neq 0$ and $\lambda^{2k} \neq 1$ for $2 \leq k \leq n-1$ then for $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z: \Gamma_K \rightarrow B_n \subset \text{SL}(n)$ we have*

$$\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).$$

Proof. It follows from (4.1) that we have an isomorphism of Γ_K -modules:

$$\mathfrak{sl}(n)_{\rho_{\lambda,n}^z} \cong \bigoplus_{k=1}^{n-1} R_{2k}.$$

Now Lemma 4.4 implies that $\dim H^*(\Gamma_K, R_{2k}) = \dim H^*(\Gamma_K, R_2)$ since $\Delta_K(\lambda^{2k}) \neq 0$ and $\lambda^{2k} \neq 1$ for $2 \leq k \leq n-1$. Hence the assertion of the proposition follows. \square

Proof of Proposition 3.4. Let $\lambda \in \mathbf{C}^*$ and $n \in \mathbf{Z}$, $n \geq 3$. Suppose that λ^2 is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho_\lambda^z: \Gamma_K \rightarrow B_2$ be a non-abelian representation as in (1.1).

In order to apply Proposition 4.5 we have to show that $\lambda^{2k} \neq 1$ for $2 \leq k \leq n-1$. Suppose that there exists $k \in \mathbf{Z}$, $2 \leq k \leq n-1$, such that $\lambda^{2k} = 1$. Next note that $\lambda^{-2} = \lambda^{2k-2}$ is a root of the Alexander polynomial since $\Delta_K(t)$ is symmetric. Therefore the assumption of the proposition implies that $k = 2$ i.e. $\lambda^4 = 1$ and hence $\lambda^2 = \pm 1$. At the same time, ± 1 is not a root of $\Delta_K(t)$ since $\Delta_K(1) = \pm 1$ and $\Delta_K(-1)$ is an odd integer. This gives a contradiction and hence $\lambda^{2k} \neq 1$ for $2 \leq k \leq n-1$. Therefore, Proposition 4.5 implies that

$$\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).$$

Finally, observe that $\mathfrak{sl}(2)_{\rho_\lambda^z} \cong R_2$ (see Example 1) and $\dim H^1(\Gamma_K; R_2) = 1$ follows from [HP05, Corollary 5.4] or [HPSP01, 4.4]. It is easy to see that $H^0(\Gamma_K; R_2) = 0$ since ρ_λ^z is non-abelian. \square

5. EXAMPLES

Let $K \subset S^3$ be a knot and λ^2 a simple root of $\Delta_K(t)$. Theorem 1.1 implies that if $\Delta_K(\lambda^{2k}) \neq 0$ for all $k \in \mathbf{Z}$, $k \neq \pm 1$, then for all $n \geq 2$, $n \in \mathbf{Z}$, the representation space $R_n(\Gamma_K)$ contains a component $R_{\lambda,n}$ of dimension $n^2 + n - 2$. Moreover, Proposition 3.8 of [New78] shows that if a component contains an irreducible representation, then generic representations on that component are irreducible.

Corollary 5.1. *Let $K \subset S^3$ be a knot with the Alexander polynomial of the figure-eight knot.*

Then the representation variety $R_n(\Gamma_K)$ contains an $(n^2 + n - 2)$ -dimensional component and the irreducible representations form an Zariski-open subset of this component.

Proof. The Alexander polynomial of the figure-eight knot is $\Delta(t) = t^2 - 3t + 1$ and its roots are $\lambda^{\pm 2} = 3/2 \pm \sqrt{5}/2$ and no power $\lambda^{\pm 2k}$, $k \neq \pm 1$, is a root of $\Delta(t)$. \square

The situation for the trefoil knot 3_1 is more complicated since the roots of its Alexander polynomial, $\Delta_{3_1}(t) = t^2 - t + 1$, are the primitive 6-th roots of unity $\lambda^{\pm 2} = e^{\pm i\pi/3}$. Hence $R_n(\Gamma_{3_1})$ contains an $(n^2 + n - 2)$ -dimensional component $R_{\lambda,n}$, for $n \in \{2, 3, 4, 5\}$, since $e^{\pm i\pi/3}$ is a simple root of $\Delta_{3_1}(t)$ and since $\Delta_{3_1}(e^{\pm ik\pi/3}) \neq 0$, for $k \in \{2, 3, 4\}$.

Let us study the case $n = 6$: the group Γ_{3_1} is free product with amalgamation

$$\Gamma_{3_1} = \langle S, T \mid STS = TST \rangle \cong \langle x, y \mid x^2 = y^3 \rangle \cong \langle x \mid - \rangle *_{\langle c \mid - \rangle} \langle y \mid - \rangle$$

where $x = STS$, $y = TS$, and $c = x^2 = y^3$ generates the center of Γ_{3_1} . Note that a meridian μ of 3_1 is represented by the Wirtinger generator $\mu = S = xy^{-1}$. Let $\rho: \Gamma_{3_1} \rightarrow \mathrm{SL}(6)$ be an irreducible representation. It follows from Schur's lemma, that if ρ is irreducible then the generator of the center $x^2 = c = y^3$ has to be mapped into the center

$$C_6 := \left\{ \exp(2\pi \frac{k}{6}) I_6 \mid 1 \leq k \leq 6 \right\} \subset \mathrm{SL}(6)$$

of $\mathrm{SL}(6)$. Notice that for each element of the center C_6 there are only finitely many square and cube roots up to conjugation in $\mathrm{SL}(6)$. This implies that if $R \subset R_6(\Gamma_{3_1})$ is an irreducible component of the representation variety then the conjugacy classes represented by the elements $\rho(c)$, $\rho(x)$, $\rho(y)$ in $\mathrm{SL}(6)$ do not vary with $\rho \in R$. Now let $\lambda = e^{i\pi/6}$ be a primitive 12-th root of unity. A cohomological non-trivial cocycle $z \in Z^1(\Gamma_{3_1}; \mathbf{C}_{\lambda^2})$ is given by $z(S) = 0$ and $z(T) = 1$. Therefore the representation $\rho_\lambda^z: \Gamma_{3_1} \rightarrow \mathrm{SL}(2)$ is given by

$$\rho_\lambda^z(S) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \rho_\lambda^z(T) = \begin{pmatrix} \lambda & \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix} .$$

Hence

$$\rho_{\lambda}^z(x) = \begin{pmatrix} i & \lambda^{-1} \\ 0 & -i \end{pmatrix}, \quad \rho_{\lambda}^z(y) = \begin{pmatrix} \lambda^2 & \lambda^{-2} \\ 0 & \lambda^{-2} \end{pmatrix} \quad \text{and} \quad \rho_{\lambda}^z(c) = -I_2.$$

Proposition 3.1 implies that $\rho_{\lambda,6}^z = r_6 \circ \rho_{\lambda}^z$ is a limit of irreducible representations. Computer supported calculations show that $\dim H^1(\Gamma_{3_1}; R_{10}) = 3$ and Lemma 4.4 implies that $\dim H^1(\Gamma_{3_1}; R_{2k}) = \dim H^1(\Gamma_{3_1}; R_2) = 1$ for $k \in \{2, 3, 4\}$. Hence Formula 4.1 implies that

$$\dim H^1(\Gamma_{3_1}; \mathfrak{sl}(6)_{\rho_{\lambda,6}^z}) = 7 \quad \text{i.e.} \quad Z^1(\Gamma_{3_1}; \mathfrak{sl}(6)_{\rho_{\lambda,6}^z}) = 42.$$

In order to see that $\rho_{\lambda,6}^z$ is contained in a 42-dimensional component of $R_6(\Gamma_{3_1})$ we proceed as follows: let $A = \rho_{\lambda,6}^z(x)$ and $B = \rho_{\lambda,6}^z(y)$ denote the image of x and y respectively. Notice that the matrices A and B are conjugate to $r_6 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $r_6 \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$. Hence

$$A \sim \begin{pmatrix} i & & & & \mathbf{0} \\ & i & & & \\ & & i & & \\ & & & -i & \\ \mathbf{0} & & & & -i \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} -1 & & & & \mathbf{0} \\ & -1 & & & \\ & & \lambda^2 & & \\ & & & \lambda^2 & \\ \mathbf{0} & & & & \lambda^{-2} \\ & & & & & \lambda^{-2} \end{pmatrix}.$$

Further note that a choice of eigenspaces $E_A(i)$, $E_A(-i)$, $E_B(-1)$, $E_B(\lambda^2)$, $E_B(\lambda^{-2})$ such that $E_A(i) \oplus E_A(-i) \cong \mathbf{C}^6$ and $E_B(-1) \oplus E_B(\lambda^2) \oplus E_B(\lambda^{-2}) \cong \mathbf{C}^6$ determines a representation $\rho: \Gamma_{3_1} \rightarrow \text{SL}(6)$ completely.

Let $\text{Gr}(p, n)$ denote the Grassmannian which parametrizes all p -dimensional subspaces of \mathbf{C}^n . Hence the choice of two elements in $\text{Gr}(3, 6)$ in generic position determines A and the choice of three elements in $\text{Gr}(2, 6)$ in generic position determines B . The representation will be irreducible if the eigenspaces of A and B are in general position and reducible if not.

It is well known that $\dim \text{Gr}(p, n) = p(n - p)$ and hence

$$\dim (\text{Gr}(3, 6) \times \text{Gr}(3, 6)) = 18 \quad \text{and} \quad \dim (\text{Gr}(2, 6) \times \text{Gr}(2, 6) \times \text{Gr}(2, 6)) = 24.$$

Therefore, we constructed a 42-dimensional component of representations $C \subset R_6(\Gamma_{3_1})$ which contains $\rho_{\lambda,6}^z = r_6 \circ \rho_{\lambda}^z$ and which also contains irreducible representations. Note that $6^2 + 6 - 2 = 40 < 42$. In conclusion we have:

Corollary 5.2. *The representation variety $R_6(\Gamma_{3_1})$ contains a 42-dimensional component C . The generic representation of C is irreducible and $\rho_{\lambda,6}^z \in C \subset R_6(\Gamma_{3_1})$ is a smooth point.*

Proof. Computer supported calculations give that $\dim Z^1(\Gamma_{3_1}, \mathfrak{sl}(6)_{\rho_{\lambda,6}^z}) = 42$. Additionally, we constructed a 42-dimensional component C containing $\rho_{\lambda,6}^z$. Now, the assertion follows from Lemma 2.2. \square

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