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# DEFORMATIONS OF REDUCIBLE REPRESENTATIONS OF KNOT GROUPS INTO $\mathrm{SL}(n, \mathbf{C})$

MICHAEL HEUSENER AND OUARDIA MEDJERAB

ABSTRACT. Let  $K$  be a knot in  $S^3$  and  $X$  its complement. We study deformations of non-abelian, metabelian, reducible representations of the knot group  $\pi_1(X)$  into  $\mathrm{SL}(n, \mathbf{C})$  which are associated to a simple root of the Alexander polynomial. We prove that certain of these metabelian reducible representations are smooth points of the  $\mathrm{SL}(n, \mathbf{C})$ -representation variety and that they have irreducible deformations.

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## 1. INTRODUCTION

Let  $K$  be a knot in  $S^3$  and  $X = \overline{S^3 \setminus V(K)}$  its complement, where  $V(K)$  is a tubular neighborhood of  $K$ . Moreover, let  $\Gamma_K = \pi_1(X)$  denote the fundamental group of  $X$ . The aim of this paper is to study deformations of reducible metabelian representations of  $\Gamma_K$  into  $\mathrm{SL}(n, \mathbf{C})$ . The metabelian representations in question were introduced by G. Burde [Bur67] and G. de Rham [dR67]. Let us recall this result: for each nonzero complex number  $\lambda \in \mathbf{C}^*$  there exists a diagonal representation  $\rho_\lambda: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C})$  given by

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

Here  $\varphi: \pi_1(X) \rightarrow \mathbf{Z}$  denotes the canonical surjection which maps the meridian  $\mu$  of  $K$  to 1 i.e.  $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$ . Burde and de Rham proved that there exists a metabelian, non-abelian representation

$$\rho_\lambda^z: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$$

if and only if  $\lambda^2$  is a root of the Alexander polynomial  $\Delta_K(t)$ .

The question whether or not the representation  $\rho_\lambda^z$  is a limit of irreducible representations of  $\Gamma_K$  into  $\mathrm{SL}(2, \mathbf{C})$  was studied in [HPSP01]. Theorem 1.1

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of [HPSP01] states that a metabelian non-abelian representation  $\rho_\lambda^z: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C})$  is the limit of irreducible representations if  $\lambda^2$  is a simple root of  $\Delta_K(t)$ . Moreover, in this case the representation  $\rho_\lambda^z$  is a smooth point of the representation variety  $R(\Gamma_K, \mathrm{SL}(2, \mathbf{C}))$ ; it is contained in a unique 4-dimensional component  $R_\lambda \subset R(\Gamma_K, \mathrm{SL}(2, \mathbf{C}))$ .

This article studies the behavior of the representations in question under the composition with the  $n$ -dimensional, irreducible, rational representation  $r_n: \mathrm{SL}(2, \mathbf{C}) \rightarrow \mathrm{SL}(n, \mathbf{C})$  (for more details see Section 4.1). It is proved in Proposition 3.1 that generically for an irreducible representation  $\rho \in R_\lambda$  the representation  $\rho_n := r_n \circ \rho \in R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$  is also irreducible. The main result of this article is the following:

**Theorem 1.1.** *If  $\lambda^2$  is a simple root of  $\Delta_K(t)$  and if  $\Delta_K(\lambda^{2k}) \neq 0$  for  $2 \leq k \leq n-1$  then the reducible metabelian representation  $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z$  is a limit of irreducible representations. More precisely,  $\rho_{\lambda,n}^z$  is a smooth point of  $R_n(\Gamma_K)$ ; it is contained in a unique  $(n+2)(n-1)$ -dimensional component  $R_{\lambda,n} \subset R_n(\Gamma_K)$ .*

This theorem is somehow complementary to a result of P. Menal-Ferrer and J. Porti [MFP12] where the authors showed that the conclusions of the above theorem hold for hyperbolic knots if  $\rho_\lambda^z$  is replaced by a lift of the holonomy,  $\widetilde{\mathrm{hol}}: \pi_1(S^3 \setminus K) \rightarrow \mathrm{SL}(2, \mathbf{C})$ , of the hyperbolic structure of the complement  $S^3 \setminus K$ . Note that Theorem 1.1 and Proposition 3.1 do apply to non-hyperbolic knots. Irreducible metabelian representations and their deformations are studied by Boden and Friedl in [BF08, BF11, BF12].

This article is organized as follows: in Section 2 we will introduce notations and recall some facts which are used. In Section 3 we will prove that the representation variety  $R(\Gamma_K, \mathrm{SL}(n, \mathbf{C}))$  contains irreducible representation if  $\Delta_K(t)$  has a simple root (see Proposition 3.1). Moreover we give a streamlined proof of a slightly generalized version of the deformation result used in [MFP12] and [BAHJ10] (see Proposition 3.3). In order to prove our main result Theorem 1.1 the necessary cohomological calculations and the basic facts about the representation theory of  $\mathrm{SL}(2, \mathbf{C})$  are presented in Section 4. Finally, in Section 5 some examples are exhibited.

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## 2. NOTATIONS AND FACTS

To shorten notation we write  $\mathrm{SL}(n)$ ,  $\mathrm{GL}(n)$  and  $\mathfrak{sl}(n)$  instead of  $\mathrm{SL}(n, \mathbf{C})$ ,  $\mathrm{GL}(n, \mathbf{C})$  and  $\mathfrak{sl}(n, \mathbf{C})$ .

Let  $\varphi: \pi_1(X) \rightarrow \mathbf{Z}$  denote the canonical surjection which maps the meridian  $\mu$  of  $K$  to 1 i.e.  $\varphi(\gamma) = \text{lk}(\gamma, K)$ . We associate to a nonzero complex number  $\alpha \in \mathbf{C}^*$  a homomorphism

$$\alpha^\varphi: \Gamma \rightarrow \mathbf{C}^*, \quad \alpha^\varphi: \gamma \mapsto \alpha^{\varphi(\gamma)}.$$

Note that  $\alpha^\varphi$  maps the meridian  $\mu$  of  $K$  to  $\alpha$ . We define  $\mathbf{C}_\alpha$  to be the  $\Gamma$ -module  $\mathbf{C}$  with the action induced by  $\alpha^\varphi$ , i.e.  $\gamma \cdot x = \alpha^{\varphi(\gamma)}x$  for all  $\gamma \in \Gamma$  and all  $x \in \mathbf{C}$ . The trivial  $\Gamma$ -module  $\mathbf{C}_1$  is simply denoted  $\mathbf{C}$ . With this notation it is easy to see that a map

$$\rho_\lambda^z: \Gamma \rightarrow \text{SL}(2, \mathbf{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$$

is a homomorphism if and only if the map  $z: \Gamma \rightarrow \mathbf{C}_{\lambda^2}$  is a 1-cocycle i.e.  $z(\gamma_1\gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)}z(\gamma_2)$ . Note also that  $\rho_\lambda^z$  is abelian if  $\lambda = \pm 1$ . If  $\lambda^2 \neq 1$  then  $\rho_\lambda^z$  is abelian if and only if  $z$  is a coboundary i.e. there exists an element  $x_0 \in \mathbf{C}$  such that  $z(\gamma) = (\lambda^{2\varphi(\gamma)} - 1)x_0$ . The general reference for group cohomology is Brown's book [Bro82].

In what follows we are mainly interested in the following situation: let  $X$  be the complement of a knot  $K \subset S^3$  and let  $A$  be a  $\pi_1(X)$ -module. The spaces  $X$  and  $\partial X$  are aspherical and hence the natural homomorphisms  $H^*(\pi_1(X); A) \rightarrow H^*(X; A)$  and  $H^*(\pi_1(\partial X); A) \rightarrow H^*(\partial X; A)$  are isomorphisms. Moreover, the knot complement  $X$  has the homotopy type of a 2-dimensional CW-complex which implies that  $H^k(\pi_1(X); A) = 0$  and  $H_k(\pi_1(X); A) = 0$  for  $k \geq 3$ . See [Whi78] for more details.

The Laurent polynomial ring  $\mathbf{C}[t^{\pm 1}]$  turns into a  $\Gamma_K$ -module via the action  $\gamma p(t) = t^{\varphi(\gamma)}p(t)$  for all  $\gamma \in \Gamma_K$  and all  $p(t) \in \mathbf{C}[t^{\pm 1}]$ . Recall that there are isomorphism of  $\mathbf{C}[t^{\pm 1}]$ -modules

$$H_*(\Gamma_K; \mathbf{C}[t^{\pm 1}]) \cong H_*(X; \mathbf{C}[t^{\pm 1}]) \cong H_*(X_\infty; \mathbf{C})$$

where  $X_\infty$  denotes the infinite cyclic covering of the knot complement  $X$  (see [DK01, Chapter 5]). The module  $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$  is a finitely generated torsion module it is called the *Alexander module* of  $K$ . A generator of its order ideal is called the *Alexander polynomial*  $\Delta_K(t) \in \mathbf{C}[t^{\pm 1}]$  of  $K$ . The Alexander polynomial is unique up to multiplication with a unit in  $\mathbf{C}[t^{\pm 1}]$ .

For completeness we will state the next lemma which shows that the cohomology groups  $H^*(\Gamma_K; \mathbf{C}_\alpha)$  are determined by the Alexander module  $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$ .

**Lemma 2.1.** *Let  $K \subset S^3$  be a knot and  $\Gamma_K$  its group. Let  $\alpha \in \mathbf{C}^*$  be a nonzero complex number and let  $\mathbf{C}_\alpha$  denote the  $\Gamma_K$ -module given by the action  $\gamma z = \alpha^{\varphi(\gamma)}z$ .*

*If  $\alpha = 1$  then  $\mathbf{C}_\alpha = \mathbf{C}$  is a trivial  $\Gamma_K$ -module and  $H^k(\Gamma_K, \mathbf{C}) \cong \mathbf{C}$  for  $k = 0, 1$  and  $H^k(\Gamma_K, \mathbf{C}) = 0$  for  $k \geq 2$ . If  $\alpha \neq 1$  then  $H^0(\Gamma_K, \mathbf{C}_\alpha) = 0$  and*

$\dim H^1(\Gamma_K, \mathbf{C}_\alpha) = \dim H^2(\Gamma_K, \mathbf{C}_\alpha)$ . Moreover,  $H^1(\Gamma_K, \mathbf{C}_\alpha) \neq 0$  if and only if  $\Delta_K(\alpha) = 0$ .

*Proof.* We have  $H_0(X_\infty; \mathbf{C}) \cong \mathbf{C} \cong \mathbf{C}[t^{\pm 1}]/(t-1)$  and  $H_k(X_\infty; \mathbf{C}) = 0$  for  $k \geq 2$  (see [BZH13, Prop. 8.16]).

First note that for all  $\alpha \in \mathbf{C}^*$  we have an isomorphism  $\mathbf{C}_\alpha \cong \mathbf{C}[t^{\pm 1}]/(t-\alpha)$ . Recall that the Alexander module  $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$  is finitely generated torsion module and hence a sum of non-free cyclic modules since  $\mathbf{C}[t^{\pm 1}]$  is a principal ideal domain. The Alexander polynomial is the order ideal of  $H_1(\Gamma_K; \mathbf{C}[t^{\pm 1}])$ . The lemma follows from the universal coefficient theorem and the fact that  $\Delta_K(1) \neq 0$ . See [BA00, Proposition 2.1] for more details.  $\square$

**2.1. Group cohomology and representation varieties.** Let now  $\Gamma$  be a finitely generated group. The set  $R_n(\Gamma) := R(\Gamma, \mathrm{SL}(n))$  of homomorphisms of  $\Gamma$  in  $\mathrm{SL}(n)$  is called the  $\mathrm{SL}(n)$ -representation variety of  $\Gamma$  and has the structure of a (not necessarily irreducible) algebraic set.

Let  $\rho: \Gamma \rightarrow \mathrm{SL}(n)$  be a representation. The Lie algebra  $\mathfrak{sl}(n)$  turns into a  $\Gamma$ -module via  $\mathrm{Ad} \rho$ . This module will be simply denoted by  $\mathfrak{sl}(n)_\rho$ . A cocycle  $d \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$  is a map  $d: \Gamma \rightarrow \mathfrak{sl}(n)$  satisfying

$$d(\gamma_1 \gamma_2) = d(\gamma_1) + \rho(\gamma_1) d(\gamma_2) \rho(\gamma_1)^{-1}, \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

It was observed by André Weil [Wei64] that there is a natural inclusion of the Zariski tangent space  $T_\rho^{Zar}(R_n(\Gamma)) \hookrightarrow Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ . Informally speaking, given a smooth curve  $\rho_\epsilon$  of representations through  $\rho_0 = \rho$  one gets a 1-cocycle  $d: \Gamma \rightarrow \mathfrak{sl}(n)$  by defining

$$d(\gamma) := \left. \frac{d\rho_\epsilon(\gamma)}{d\epsilon} \right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma.$$

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries  $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$ . Here,  $b: \Gamma \rightarrow \mathfrak{sl}(n)$  is a coboundary if there exists  $x \in \mathfrak{sl}(n)$  such that  $b(\gamma) = \rho(\gamma) x \rho(\gamma)^{-1} - x$ . A detailed account can be found in [LM85].

Let  $\dim_\rho R_n(\Gamma)$  be the local dimension of  $R_n(\Gamma)$  at  $\rho$  (i.e. the maximal dimension of the irreducible components of  $R_n(\Gamma)$  containing  $\rho$  [Sha77, Ch. II]). So we obtain:

$$\dim_\rho R_n(\Gamma) \leq \dim T_\rho^{Zar}(R_n(\Gamma)) \leq \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$$

and the following lemma follows (for more details see [HPSP01, Lemma 2.6]):

**Lemma 2.2.** *Let  $\rho \in R_n(\Gamma)$  be given. If  $\dim_\rho R_n(\Gamma) = \dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$  then  $\rho$  is a smooth point of the representation variety  $R_n(\Gamma)$  and  $\rho$  is contained in a unique component of  $R_n(\Gamma)$  of dimension  $\dim Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ .*

## 3. DEFORMING REPRESENTATIONS

The aim of the following sections is to prove that the representation  $\rho_{\lambda,n}^z$  from the introduction is a smooth point of the representation variety. We present a more streamlined and slightly generalized version of the deformation result from [HPSP01, HP05, BAHJ10, MFP12] (see Proposition 3.3). For the convenience of the reader we recall the setup.

First we will prove that the representation  $\rho_{\lambda,n}^z \in R_n(\Gamma_K)$  is the limit of irreducible representations if  $\lambda^2$  is a simple root of the Alexander polynomial  $\Delta_K(t)$ . In what follows a property of an irreducible algebraic variety  $Y$  is said to be true *generically* if it holds except on a proper Zariski-closed subset of  $Y$ , in other words, if it holds on a non-empty Zariski-open subset.

Let  $K \subset S^3$  be a knot,  $\lambda^2 \in \mathbf{C}$  a simple root of  $\Delta_K(t)$  and  $z \in Z^1(\Gamma_K, \mathbf{C}_{\lambda^2})$  a cocycle representing a generator of  $H^1(\Gamma_K, \mathbf{C}_{\lambda^2})$ . Following [HPSP01, Thm 1.1] the representation  $\rho_\lambda^z \in R_2(\Gamma_K)$  is a smooth point of the representation variety. It is contained in a unique irreducible 4-dimensional component  $R_\lambda \subset R_2(\Gamma_K)$ . Note that generically a representation  $\rho \in R_\lambda$  is irreducible.

**Proposition 3.1.** *Let  $K \subset S^3$  be a knot,  $\lambda^2 \in \mathbf{C}$  a simple root of  $\Delta_K(t)$  and let  $z \in Z^1(\Gamma_K, \mathbf{C}_{\lambda^2})$  be a cocycle representing a generator of  $H^1(\Gamma_K, \mathbf{C}_{\lambda^2})$ .*

*Then the representation  $\rho_{\lambda,n}^z = r_n \circ \rho_\lambda^z: \Gamma_K \rightarrow B_n$  is the limit of irreducible representation in  $R_n(\Gamma_K)$ . More precisely, generically a representation  $\rho_n = r_n \circ \rho$ ,  $\rho \in R_\lambda$  is irreducible.*

*Proof.* It follows from [HPSP01, Theorem 1.1] that  $\rho_\lambda^z \in R_2(\Gamma_K)$  is the limit of irreducible representations. Moreover,  $\rho_\lambda^z \in R_2(\Gamma_K)$  is a smooth point which is contained in a unique 4-dimensional component  $R_\lambda \subset R_2(\Gamma_K)$ .

Let  $\Gamma$  be a discrete group and let  $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$  be an irreducible representation. If the image  $\rho(\Gamma) \subset \mathrm{SL}(2, \mathbf{C})$  is Zariski-dense then the representation  $\rho_n := r_n \circ \rho \in R_n(\Gamma)$  is irreducible. Hence in order to prove the proposition we show that there is a neighborhood  $U = U(\rho_\lambda^z) \subset R_2(\Gamma_K)$  such that  $\rho(\Gamma) \subset \mathrm{SL}(2, \mathbf{C})$  is Zariski-dense for each irreducible  $\rho \in U$ . Let now  $\rho: \Gamma_K \rightarrow \mathrm{SL}(2, \mathbf{C})$  be any irreducible representation and let  $G \subset \mathrm{SL}(2)$  denote the Zariski-closure of  $\rho(\Gamma_K)$ . Suppose that  $G \neq \mathrm{SL}(2)$ . Since  $\rho$  is irreducible it follows that  $G$  is, up to conjugation, not a subgroup of upper-triangular matrices of  $\mathrm{SL}(2)$ . Then by [Kov86, Sec. 1.4] and [Kap57, Theorem 4.12] there are, up to conjugation, only two cases left:

- $G$  is a subgroup of the infinite dihedral group

$$D_\infty = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in \mathbf{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \mid \alpha \in \mathbf{C}^* \right\}.$$

- $G$  is one of the groups  $A_4^{\text{SL}(2)}$  (the tetrahedral group),  $S_4^{\text{SL}(2)}$  (the octahedral group) or  $A_5^{\text{SL}(2)}$  (the icosahedral group). These groups are the preimages in  $\text{SL}(2)$  of the subgroups  $A_4, S_4, A_5 \subset \text{PSL}(2, \mathbf{C})$ .

In the first case it follows directly from [Nag07] that if  $\rho$  is an irreducible metabelian representation then the trace of the image of a meridian  $\text{tr}(\rho(\mu)) = 0$  i.e.  $\rho(\mu)$  is similar to  $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ . Now,  $\text{tr}(\rho_\lambda^z(\mu)) \neq 0$  since  $\Delta_K(-1) \neq 0$  and  $\Delta_K(\lambda^{\pm 2}) = 0$ . For the second case there are up to conjugation only finitely many irreducible representations of  $\Gamma_K$  onto the subgroups  $A_4^{\text{SL}(2)}, S_4^{\text{SL}(2)}$  and  $A_5^{\text{SL}(2)}$ . Note that these finitely many orbits are closed and 3-dimensional. Hence the irreducible  $\rho \in R_\lambda$  such that  $r_n \circ \rho$  is reducible is contained in a Zariski-closed subset of  $R_\lambda$ . Hence generically  $r_n \circ \rho$  is irreducible for  $\rho \in R_\lambda$ .  $\square$

In order to proof that a certain representation  $\rho \in R_n(\Gamma)$  is a smooth point of the representation variety we will prove that every cocycle  $u \in Z^1(\Gamma_K; \mathfrak{sl}(n)_\rho)$  is integrable. In order to do this, we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a theorem of Artin [Art68].

The formal deformations of a representation  $\rho: \Gamma \rightarrow \text{SL}(n)$  are in general determined by an infinite sequence of obstructions (see [Gol84, BA00, HPSP01]). In what follows we let  $C^1(\Gamma; \mathfrak{sl}(n)_\rho) := \{c: \Gamma \rightarrow \mathfrak{sl}(n)_\rho\}$  denote the 1-cochains of  $\Gamma$  with coefficients in  $\mathfrak{sl}(n)_\rho$  (see [Bro82, p.59]).

Let  $\rho: \Gamma \rightarrow \text{SL}(n)$  be a representation. A formal deformation of  $\rho$  is a homomorphism  $\rho_\infty: \Gamma \rightarrow \text{SL}(n, \mathbf{C}[[t]])$

$$\rho_\infty(\gamma) = \exp \left( \sum_{i=1}^{\infty} t^i u_i(\gamma) \right) \rho(\gamma), \quad u_i \in C^1(\Gamma; \mathfrak{sl}(n)_\rho)$$

such that  $\text{ev}_0 \circ \rho_\infty = \rho$ . Here  $\text{ev}_0: \text{SL}(n, \mathbf{C}[[t]]) \rightarrow \text{SL}(n)$  is the evaluation homomorphism at  $t = 0$  and  $\mathbf{C}[[t]]$  denotes the ring of formal power series.

We will say that  $\rho_\infty$  is a formal deformation up to order  $k$  of  $\rho$  if  $\rho_\infty$  is a homomorphism modulo  $t^{k+1}$ .

An easy calculation gives that  $\rho_\infty$  is a homomorphism up to first order if and only if  $u_1 \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$  is a cocycle. We call a cocycle  $u_1 \in Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$  *integrable* if there is a formal deformation of  $\rho$  with leading term  $u_1$ .

**Lemma 3.2.** *Let  $u_1, \dots, u_k \in C^1(\Gamma; \mathfrak{sl}(n)_\rho)$  such that*

$$\rho_k(\gamma) = \exp \left( \sum_{i=1}^k t^i u_i(\gamma) \right) \rho(\gamma)$$

*is a homomorphism into  $\text{SL}(n, \mathbf{C}[[t]])$  modulo  $t^{k+1}$ . Then there exists an obstruction class  $\zeta_{k+1} := \zeta_{k+1}^{(u_1, \dots, u_k)} \in H^2(\Gamma, \mathfrak{sl}(n)_\rho)$  with the following properties:*

(i) There is a cochain  $u_{k+1}: \Gamma \rightarrow \mathfrak{sl}(n)_\rho$  such that

$$\rho_{k+1}(\gamma) = \exp \left( \sum_{i=1}^{k+1} t^i u_i(\gamma) \right) \rho(\gamma)$$

is a homomorphism modulo  $t^{k+2}$  if and only if  $\zeta_{k+1} = 0$ .

(ii) The obstruction  $\zeta_{k+1}$  is natural, i.e. if  $f: \Gamma_1 \rightarrow \Gamma$  is a homomorphism then  $f^* \rho_k := \rho_k \circ f$  is also a homomorphism modulo  $t^{k+1}$  and  $f^*(\zeta_{k+1}^{(u_1, \dots, u_k)}) = \zeta_{k+1}^{(f^* u_1, \dots, f^* u_k)} \in H^2(\Gamma_1; \mathfrak{sl}(n)_{f^* \rho})$ .

*Proof.* The proof is completely analogous to the proof of Proposition 3.1 in [HPSP01]. We replace  $\mathrm{SL}(2)$  and  $\mathfrak{sl}(2)$  by  $\mathrm{SL}(n)$  and  $\mathfrak{sl}(n)$  respectively.  $\square$

The following result streamlines the arguments given in [HP05] and [BAHJ10]:

**Proposition 3.3.** *Let  $M$  be a connected, compact, orientable 3-manifold with torus boundary and let  $\rho: \pi_1 M \rightarrow \mathrm{SL}(n)$  be a representation.*

*If  $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$  then  $\rho$  is a smooth point of the  $\mathrm{SL}(n)$ -representation variety  $R_n(\pi_1 M)$ . Moreover,  $\rho$  is contained in a unique component of dimension  $n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho)$ .*

*Proof.* First we will show that the map  $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$  induced by the inclusion  $\partial M \hookrightarrow M$  is injective.

Recall that for any CW-complex  $X$  with  $\pi_1(X) \cong \pi_1(M)$  and for any  $\pi_1 M$ -module  $A$  there are natural morphisms  $H^i(\pi_1 M; A) \rightarrow H^i(X; A)$  which are isomorphisms for  $i = 0, 1$  and an injection for  $i = 2$  (see [HP05, Lemma 3.3]). Note also that  $\partial M \cong S^1 \times S^1$  is aspherical and hence  $H^*(\pi_1 \partial M; A) \rightarrow H^*(\partial M; A)$  is an isomorphism.

We will make use of a result of Richardson [Ric79, Thm. C]. Richardson proved that the representation variety  $R_n(\mathbf{Z} \oplus \mathbf{Z})$  is an irreducible algebraic variety of dimension  $(n + 2)(n - 1)$ . It follows from Poincaré duality and the Euler characteristic that for every  $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$

$$\dim H^1(\partial M; \mathfrak{sl}(n)_\varrho) = 2 \dim H^0(\partial M; \mathfrak{sl}(n)_\varrho) \geq 2(n - 1). \quad (3.1)$$

Hence  $\varrho \in R_n(\mathbf{Z} \oplus \mathbf{Z})$  is a smooth point if and only if the dimension of  $\mathfrak{sl}(n)_\varrho^{\mathbf{Z} \oplus \mathbf{Z}} = H^0(\mathbf{Z} \oplus \mathbf{Z}; \mathfrak{sl}(n)_\varrho)$  is minimal i.e.  $\dim H^0(\mathbf{Z} \oplus \mathbf{Z}; \mathfrak{sl}(n)_\varrho) = n - 1$  (see also [Pop08]).

The exact cohomology sequence of the pair  $(M, \partial M)$  gives

$$\begin{aligned} & \rightarrow H^1(M, \partial M; \mathfrak{sl}(n)_\rho) \\ & \rightarrow H^1(M; \mathfrak{sl}(n)_\rho) \xrightarrow{\alpha} H^1(\partial M; \mathfrak{sl}(n)_\rho) \xrightarrow{\beta} H^2(M, \partial M; \mathfrak{sl}(n)_\rho) \\ & \rightarrow H^2(M; \mathfrak{sl}(n)_\rho) \xrightarrow{i^*} H^2(\partial M; \mathfrak{sl}(n)_\rho) \rightarrow H^3(M, \partial M; \mathfrak{sl}(n)_\rho) \rightarrow 0. \end{aligned}$$



Poincaré–Lefschetz duality implies that  $\alpha$  and  $\beta$  are dual to each other. Therefore we have

$$n - 1 = \dim H^1(M; \mathfrak{sl}(n)_\rho) \geq \text{rk}(\alpha) = \frac{1}{2} \dim H^1(\partial M; \mathfrak{sl}(n)_\rho) \geq n - 1$$

and Equation (3.1) together with Richardson’s result imply that  $i^*\rho = \rho \circ i_\# \in R_n(\partial M)$  is a smooth point of the representation variety (here  $i: \partial M \rightarrow M$  is the inclusion). Note also that  $\beta$  is surjective and hence

$$i^*: H^2(M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\partial M; \mathfrak{sl}(n)_\rho)$$

is injective. The following commutative diagram shows that  $i^*: H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho)$  is injective:

$$\begin{array}{ccc} H^2(M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\partial M; \mathfrak{sl}(n)_\rho) \\ \uparrow & & \uparrow \cong \\ H^2(\pi_1 M; \mathfrak{sl}(n)_\rho) & \xrightarrow{i^*} & H^2(\pi_1 \partial M; \mathfrak{sl}(n)_\rho). \end{array}$$

In order to prove that  $\rho$  is a smooth point of  $R_n(\pi_1 M)$ , we show that all cocycles in  $Z^1(\pi_1 M, \mathfrak{sl}(n)_\rho)$  are integrable. In what follows we will prove that all obstructions vanish, by using the fact that the obstructions vanish on the boundary. Let  $u_1, \dots, u_k: \pi_1 M \rightarrow \mathfrak{sl}(n)$  be given such that

$$\rho_k(\gamma) = \exp \left( \sum_{i=1}^k t^i u_i(\gamma) \right) \rho(\gamma)$$

is a homomorphism modulo  $t^{k+1}$ . Then the restriction  $i^*\rho_k: \pi_1(\partial M) \rightarrow \text{SL}(n, \mathbf{C}[[t]])$  is also a formal deformation of order  $k$ . Since  $i^*\rho$  is a smooth point of the representation variety  $R_n(\mathbf{Z} \oplus \mathbf{Z})$ , the formal implicit function theorem gives that  $i^*\rho_k$  extends to a formal deformation of order  $k+1$  (see [HPSP01, Lemma 3.7]). Therefore, we have that

$$0 = \zeta_{k+1}^{(i^*u_1, \dots, i^*u_k)} = i^* \zeta_{k+1}^{(u_1, \dots, u_k)}$$

Now,  $i^*$  is injective and the obstruction  $\zeta_{k+1}^{(u_1, \dots, u_k)}$  vanishes.

Hence all cocycles in  $Z^1(\Gamma, \mathfrak{sl}(n)_\rho)$  are integrable. By applying Artin’s theorem [Art68] we obtain from a formal deformation of  $\rho$  a convergent deformation (see [HPSP01, Lemma 3.3] or [BA00, § 4.2]).

Thus  $\rho$  is a smooth point of the representation variety  $R_n(\pi_1 M)$ . Hence,  $\dim H^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n - 1$  and the exactness of

$$0 \rightarrow H^0(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow \mathfrak{sl}(n)_\rho \rightarrow B^1(\pi_1 M; \mathfrak{sl}(n)_\rho) \rightarrow 0$$

implies

$$\dim Z^1(\pi_1 M; \mathfrak{sl}(n)_\rho) = n^2 + n - 2 - \dim H^0(\pi_1 M; \mathfrak{sl}(n)_\rho).$$

□

**Proposition 3.4.** *Let  $K \subset S^3$  be a knot,  $\lambda \in \mathbf{C}^*$  and  $n \geq 3$ . Suppose that  $\lambda^2$  is a simple root of the Alexander polynomial  $\Delta_K(t)$  and let  $\rho_\lambda^z: \Gamma \rightarrow B_2$  be a non-abelian representation as in the introduction.*

*If  $\Delta_K(\lambda^{2i}) \neq 0$  for  $2 \leq i \leq n-1$  then for  $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z: \Gamma \rightarrow B_n \subset \mathrm{SL}(n)$  we have*

$$\dim H^1(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \text{ and } H^0(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = 0.$$

*Proof.* A proof of the cohomological calculation will be given in Section 4. □

*Proof of Theorem 1.1.* It follows directly from Propositions 3.3 and 3.4 that  $\rho_{\lambda,n}^z$  is a smooth point of  $R_n(\Gamma_K)$  which is contained in a unique component  $R_{\lambda,n} \subset R_n(\Gamma_K)$ ,  $\dim R_{\lambda,n} = n^2 + n - 2$ .

That  $\rho_{\lambda,n}^z$  is the limit of irreducible representation which are contained in the component  $R_{\lambda,n}$  follows from Proposition 3.1. □

#### 4. COHOMOLOGICAL CALCULATIONS

For the convenience of the reader we recall some facts from the representation theory of  $\mathrm{SL}(2)$ . The general reference for this topic is Springer's LNM [Spr77].

**4.1. Representation theory of  $\mathrm{SL}(2)$ .** Let  $V$  be an  $n$ -dimensional complex vector space. In what follows we will call a homomorphism  $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$  an  $n$ -dimensional representation of  $\mathrm{SL}(2)$ . The vector space  $V$  turns into a  $\mathrm{SL}(2)$ -module. Two  $n$ -dimensional representations  $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$  and  $r': \mathrm{SL}(2) \rightarrow \mathrm{GL}(V')$  are called *equivalent* if there is an isomorphism  $\phi: V \rightarrow V'$  which commutes with the action of  $\mathrm{SL}(2)$  i.e.  $r'(A)\phi = \phi r(A)$  for all  $A \in \mathrm{SL}(2)$ . It is clear that equivalent representations give rise to isomorphic  $\mathrm{SL}(2)$ -modules.

We let  $\mathrm{SL}(2)$  act as a group of automorphism on the polynomial algebra  $R = \mathbf{C}[X, Y]$ . If  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$  then there is a unique automorphism  $r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$  of  $R$  given by

$$r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X) = dX - bY \quad \text{and} \quad r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(Y) = -cX + aY.$$

We let  $R_{n-1} \subset R$  denote the  $n$ -dimensional subspace of homogeneous polynomials of degree  $n-1$ . The monomials  $e_l^{(n-1)} = X^{l-1}Y^{n-l}$ ,  $1 \leq l \leq n$ , form a basis of  $R_{n-1}$  and  $r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$  leaves  $R_{n-1}$  invariant. In what follows we will identify  $R_{n-1}$  and  $\mathbf{C}^n$  by fixing the basis  $(e_1^{(n-1)}, \dots, e_n^{(n-1)})$  of  $R_{n-1}$ . We obtain an  $n$ -dimensional representation  $r_n: \mathrm{SL}(2) \rightarrow \mathrm{GL}(R_{n-1}) \cong \mathrm{GL}(n)$ .

The representation  $r_n$  is *rational* i.e. the coefficients of the matrix coordinates of  $r_n(A)$  are polynomials in the matrix coordinates of  $A$ . We will make use of the following theorem.

**Theorem 4.1.** (1) *The representation  $r_n$  is irreducible i.e. there is no  $\mathrm{SL}(2)$ -stable invariant subspaces  $V$ ,  $\{0\} \subsetneq V \subsetneq R_{n-1}$  and any irreducible rational representation of  $\mathrm{SL}(2)$  is equivalent to some  $r_n$ .*  
 (2) *For an arbitrary rational representation  $r: \mathrm{SL}(2) \rightarrow \mathrm{GL}(V)$  the  $\mathrm{SL}(2)$ -module  $V$  is isomorphic to a direct sum of  $R_n$ ,*

$$V \cong \bigoplus_{d \geq 0} R_d^{m(k)}.$$

*Proof.* For (1) see Lemma 3.1.3 and Proposition 3.2.1 of [Spr77].  $\square$

It is easy to see, and it follows also from the general theory, that  $r_n$  maps an unipotent matrix  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$  onto an unipotent element of  $\mathrm{SL}(R_{n-1})$ . Moreover, an explicit calculations shows that the image of a diagonal matrix is the diagonal matrix  $r_n(\mathrm{diag}(a, a^{-1})) = \mathrm{diag}(a^{n-1}, a^{n-3}, \dots, a^{-n+3}, a^{-n+1})$ . Hence the image of  $r_n$  is contained in  $\mathrm{SL}(R_{n-1}) \cong \mathrm{SL}(n)$ .

*Example 1.* The representation  $r_1: \mathrm{SL}(2) \rightarrow \mathrm{SL}(1) = \{1\}$  is the trivial representation. The representation  $r_2: \mathrm{SL}(2) \rightarrow \mathrm{SL}(2)$  is equivalent to the identity:

$$r_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Moreover, it is easy to see that the adjoint representation  $\mathrm{Ad}: \mathrm{SL}(2) \rightarrow \mathrm{Aut}(\mathfrak{sl}(2))$  is equivalent to  $r_3$ .

The Lie algebra  $\mathfrak{sl}(n)$  of  $\mathrm{SL}(n)$  turns into an  $\mathrm{SL}(2)$ -module via  $\mathrm{Ad} \circ r_n$  where  $\mathrm{Ad}: \mathrm{SL}(n) \rightarrow \mathrm{Aut}(\mathfrak{sl}(n))$  denotes the adjoint representation. For this action we have the classical formula of Clebsch–Gordan:

$$\mathrm{Ad} \circ r_n \cong \bigoplus_{i=1}^{n-1} r_{2i+1}. \quad (4.1)$$

Let  $B_n \subset \mathfrak{sl}(n)$  denote the Borel subgroup of upper triangular matrices. The vector space  $R_{n-1}$  turns into a  $B_2$ -module via restriction of  $r_n$  to  $B_2$ . An explicit calculation gives

$$r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_l^{(n-1)} = \lambda^{n-2l+1} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} e_{l-j}^{(n-1)}. \quad (4.2)$$

Hence  $r_n(B_2)$  is contained in  $B_n \subset \mathrm{SL}(n)$  and the one-dimensional vector space  $\langle e_1^{(n-1)} \rangle$  is  $B_2$  invariant:  $r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_1^{(n-1)} = \lambda^{n-1} e_1^{(n-1)}$ . For a given integer  $i \in \mathbf{Z}$  we let  $\chi_i: B_2 \rightarrow \mathbf{C}^* = \mathrm{GL}(1, \mathbf{C})$  denote the rational character given by

$$\chi_i \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} = \lambda^i.$$

Now  $\mathbf{C}$  turns into a  $B_2$ -module via  $\chi_i$  i.e.  $\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.x = \lambda^i x$  for  $x \in \mathbf{C}$ . We will denote this  $B_2$ -module by  $\mathbf{C}_{\chi_i}$ . It follows that the  $B_2$ -module  $\langle e_1^{(n-1)} \rangle \in R_{n-1}$  is isomorphic to  $\mathbf{C}_{\chi_{n-1}}$  and we obtain a short exact sequence of  $B_2$ -modules

$$0 \rightarrow \mathbf{C}_{\chi_{n-1}} \rightarrow R_{n-1} \rightarrow \bar{R}_{n-1} \rightarrow 1 \quad (4.3)$$

where  $\bar{R}_{n-1}$  denotes the quotient  $R_{n-1}/\langle e_1^{(n-1)} \rangle$ . For a given element  $x \in R_{n-1}$  we let  $\bar{x} \in \bar{R}_{n-1}$  denote the class represented by  $x$  i.e.  $\bar{x} = x + \langle e_1^{(n-1)} \rangle$ .

For abbreviation, we will drop the representation  $r_n$  from the notation and write for  $x \in R_{n-1}$

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.x \text{ instead of } r_n \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.x.$$

**Lemma 4.2.** *The linear map  $\phi_{n-3}: R_{n-3} \rightarrow \bar{R}_{n-1}$  defined by*

$$\phi_{n-3}(e_l^{(n-3)}) = \frac{1}{l} \bar{e}_{l+1}^{(n-1)}, \quad l = 1, \dots, n-2,$$

*is an injective  $B_2$ -module morphism i.e. for all  $x \in R_{n-3}$  we have*

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.\phi_{n-3}(x) = \phi_{n-3}\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.x\right).$$

*Proof.* The linear map  $\phi_{n-3}$  is injective since the vectors  $\bar{e}_l^{(n-1)}$ ,  $2 \leq l \leq n$ , form a basis of  $\bar{R}_{n-1}$ . Now

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.\phi_{n-3}(e_l^{(n-3)}) = \lambda^{n-2l-1} \frac{1}{l} \sum_{j=0}^l (-b)^j \binom{l}{j} \bar{e}_{l-j+1}^{(n-1)}.$$

Since  $\binom{l}{j}(l-j) = l \binom{l-1}{j}$  and  $\bar{e}_1^{(n-1)} = 0$  it follows

$$\begin{aligned} \begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.\phi_{n-3}(e_l^{(n-3)}) &= \lambda^{(n-2)-2l+1} \sum_{j=0}^{l-1} (-b)^j \binom{l-1}{j} \frac{1}{l-j} \bar{e}_{l-j+1}^{(n-1)} \\ &= \phi_{n-3}\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.e_l^{(n-3)}\right). \end{aligned}$$

Hence  $\phi_{n-3}$  is a  $B_2$ -module morphism.  $\square$

**Lemma 4.3.** *There is a short exact sequence of  $B_2$ -modules*

$$0 \rightarrow R_{n-3} \xrightarrow{\phi_{n-3}} \bar{R}_{n-1} \rightarrow \mathbf{C}_{\chi_{-n+1}} \rightarrow 0. \quad (4.4)$$

*Proof.* Again the lemma follows from Equation (4.2):

$$\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}.e_n^{(n-1)} \equiv \lambda^{-n+1} e_n^{(n-1)} \pmod{\langle e_1^{(n-1)}, \dots, e_{n-1}^{(n-1)} \rangle}. \quad \square$$

Let us fix a representation  $\rho_\lambda^\gamma: \Gamma \rightarrow B_2$ . Then  $R_n$  turns into a  $\Gamma$ -module and the exact sequences (4.3) and (4.4) are exact sequences of  $\Gamma$ -modules. Note that  $\mathbf{C}_{\chi_k} \cong \mathbf{C}_{\lambda^k}$  since for all  $\gamma \in \Gamma$  and  $k \in \mathbf{Z}$  the equation  $\chi_k(\rho_\lambda^\gamma(\gamma)) = \lambda^{k\varphi(\gamma)}$  holds.

**Lemma 4.4.** *Let  $\lambda \in \mathbf{C}^*$ ,  $\lambda \neq 1$ , and  $n > 3$  be given. If  $\Delta_K(\lambda^{n-1}) \neq 0$  and if  $\lambda^{n-1} \neq 1$  then*

$$H^*(\Gamma_K; R_{n-1}) \cong H^*(\Gamma_K; R_{n-3}).$$

*Proof.* The long exact cohomology sequences [Bro82, III.§6] associated to the short exact sequences (4.3) gives:

$$H^k(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) \rightarrow H^k(\Gamma_K; R_{n-1}) \rightarrow H^k(\Gamma_K; \bar{R}_{n-1}) \rightarrow H^{k+1}(\Gamma_K; \mathbf{C}_{\lambda^{n-1}})$$

is exact for  $k = 0, 1, 2$ . Now  $H^0(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) = 0$  since  $\lambda^{n-1} \neq 1$  and for  $k = 1, 2$  the group  $H^k(\Gamma_K; \mathbf{C}_{\lambda^{n-1}}) = 0$  since  $\Delta_K(\lambda^{n-1}) \neq 0$  (see Lemma 2.1). Hence

$$H^k(\Gamma_K; R_{n-1}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2.$$

Finally, the short exact Sequence (4.4), Lemma 2.1 and the assumptions  $\Delta_K(\lambda^{n-1}) \neq 0$  and if  $\lambda^{n-1} \neq 1$  give that

$$H^k(\Gamma_K; R_{n-3}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{n-1}) \quad \text{for } k = 0, 1, 2.$$

are isomorphisms (note that  $\Delta_K(t)$  is symmetric).  $\square$

**Proposition 4.5.** *Let  $\lambda \in \mathbf{C}^*$ ,  $\Delta_K(\lambda^2) = 0$ ,  $n \geq 3$  and  $\rho_\lambda^z: \Gamma \rightarrow B_2$  be given as in the introduction. If  $\Delta_K(\lambda^{2k}) \neq 0$  and  $\lambda^{2k} \neq 1$  for  $2 \leq k \leq n-1$  then for  $\rho_{\lambda,n}^z := r_n \circ \rho_\lambda^z: \Gamma \rightarrow B_n \subset \text{SL}(n)$  we have*

$$\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).$$

*Proof.* It follows from (4.1) that we have an isomorphism of  $\Gamma$ -modules:

$$\mathfrak{sl}(n)_{\rho_n} \cong \bigoplus_{k=1}^{n-1} R_{2k}.$$

Now Lemma 4.4 implies that  $\dim H^*(\Gamma, R_{2k}) = \dim H^*(\Gamma, R_2)$  since  $\Delta_K(\lambda^{2k}) \neq 0$  and  $\lambda^{2k} \neq 1$  for  $2 \leq k \leq n-1$ . Hence the assertion of the proposition follows.  $\square$

*Proof of Proposition 3.4.* Let  $\lambda \in \mathbf{C}^*$  and  $n \in \mathbf{Z}$ ,  $n \geq 3$ . Suppose that  $\lambda^2$  is a simple root of the Alexander polynomial  $\Delta_K(t)$  and let  $\rho_\lambda^z: \Gamma \rightarrow B_2$  be a non-abelian representation as in the introduction.

In order to apply Proposition 4.5 we have to show that  $\lambda^{2k} \neq 1$  for  $2 \leq k \leq n-1$ . Suppose that there exists  $k \in \mathbf{Z}$ ,  $2 \leq k \leq n-1$ , such that  $\lambda^{2k} = 1$ . Next note that  $\lambda^{-2} = \lambda^{2k-2}$  is a root of the Alexander polynomial since  $\Delta_K(t)$  is symmetric. Therefore the assumption of the proposition implies that  $k = 2$  i.e.  $\lambda^4 = 1$  and hence  $\lambda^2 = \pm 1$ . On the other hand,  $\pm 1$  is not a root of  $\Delta_K(t)$  since  $\Delta_K(1) = \pm 1$  and  $\Delta_K(-1)$  is an odd integer. This gives a contradiction and hence  $\lambda^{2k} \neq 1$  for  $2 \leq k \leq n-1$ . Therefore, Proposition 4.5 implies that

$$\dim H^*(\Gamma_K; \mathfrak{sl}(n)_{\rho_{\lambda,n}^z}) = (n-1) \dim H^*(\Gamma_K; R_2).$$

Finally, observe that  $\mathfrak{sl}(2)_{\rho_{\tilde{\lambda}}} \cong R_2$  (see Example 1) and  $\dim H^1(\Gamma_K; R_2) = 1$  follows from [HP05, Corollary 5.4] or [HPSP01, 4.4]. It is easy to see that  $H^0(\Gamma_K; R_2) = 0$  since  $\rho_{\tilde{\lambda}}$  is non-abelian.  $\square$

## 5. EXAMPLES

Let  $K \subset S^3$  be a knot and  $\lambda^2$  a simple root of  $\Delta_K(t)$ . Theorem 1.1 implies that if  $\Delta_K(\lambda^{2k}) \neq 0$  for all  $k \in \mathbf{Z}$ ,  $k \neq \pm 1$ , then for all  $n \geq 2$ ,  $n \in \mathbf{Z}$ , the representation space  $R_n(\Gamma_K)$  contains a component  $R_{\lambda,n}$  of dimension  $n^2 + n - 2$ . Moreover,  $R_{\lambda,n}$  contains irreducible representations and generically a representation  $\rho \in R_{\lambda,n}$  is irreducible (this follows for example from [New78, Prop. 3.8]).

**Corollary 5.1.** *Let  $K \subset S^3$  be a knot with the Alexander polynomial of the figure-eight knot.*

*Then the representation variety  $R_n(\Gamma_K)$  contains an  $(n^2 + n - 2)$ -dimensional component and the irreducible representations form a Zariski-open subset of this component.*

*Proof.* The Alexander polynomial of the figure-eight knot is  $\Delta(t) = t^2 - 3t + 1$  and its roots are  $\lambda^{\pm 2} = 3/2 \pm \sqrt{5}/2$  and no power  $\lambda^{\pm 2k}$ ,  $k \neq \pm 1$ , is a root of  $\Delta(t)$ .  $\square$

The situation for the trefoil knot  $3_1$  is more complicate since the roots of its Alexander polynomial  $\Delta_{3_1}(t) = t^2 - t + 1$  are the 6-th roots of unity  $\lambda^{\pm 2} = e^{\pm i\pi/3}$ . Hence  $R_n(\Gamma_{3_1})$  contains an  $(n^2 + n - 2)$ -dimensional component  $R_{\lambda,n}$  for  $n \in \{2, 3, 4, 5\}$  since  $e^{\pm i\pi/3}$  is a simple root of  $\Delta_{3_1}(t)$  and since  $\Delta_{3_1}(e^{\pm ik\pi/3}) \neq 0$  for  $k \in \{2, 3, 4\}$ . Let us study the case  $n = 6$ : the group  $\Gamma_{3_1}$  is free product with amalgamation

$$\Gamma_{3_1} = \langle S, T \mid STS = TST \rangle \cong \langle x, y \mid x^2 = y^3 \rangle \cong \langle x \mid - \rangle *_{\langle c \mid - \rangle} \langle y \mid - \rangle$$

where  $x = STS$ ,  $y = TS$ , and  $c = x^2 = y^3$  generates the center of  $\Gamma_{3_1}$ . Note that a meridian  $\mu$  of  $3_1$  is represented by the Wirtinger generator  $\mu = S = xy^{-1}$ . Let  $\rho: \Gamma_{3_1} \rightarrow \mathrm{SL}(n)$  be an irreducible representation. Since  $\rho$  is irreducible the generator of the center  $x^2 = c = y^3$  has to be mapped into the center

$$C_n := \{ \exp(2\pi \frac{k}{n}) I_n \mid 1 \leq k \leq n \} \subset \mathrm{SL}(n)$$

of  $\mathrm{SL}(n)$ . It follows that if  $R \subset R_n(\Gamma_{3_1})$  is an irreducible component of the representation variety then the conjugacy classes represented by the elements  $\rho(c)$ ,  $\rho(x)$ ,  $\rho(y)$  in  $\mathrm{SL}(n)$  do not vary with  $\rho \in R$ . Now let  $\lambda = e^{i\pi/6}$  be a primitive 12-th root of unity. A cohomological non-trivial cocycle  $z \in Z^1(\Gamma_{3_1}; \mathbf{C}_{\lambda^2})$  is given by  $z(S) = 0$  and  $z(T) = 1$ . Therefore the representation  $\rho_{\tilde{\lambda}}: \Gamma_{3_1} \rightarrow \mathrm{SL}(2)$  is

given by

$$\rho_\lambda^z(S) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \rho_\lambda^z(T) = \begin{pmatrix} \lambda & \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Hence

$$\rho_\lambda^z(x) = \begin{pmatrix} i & \lambda^{-1} \\ 0 & -i \end{pmatrix}, \quad \rho_\lambda^z(y) = \begin{pmatrix} \lambda^2 & \lambda^{-2} \\ 0 & \lambda^{-2} \end{pmatrix} \quad \text{and} \quad \rho_\lambda^z(c) = -I_2.$$

Proposition 3.1 implies that  $\rho_{\lambda,n}^z = r_n \circ \rho_\lambda^z$  is a limit of irreducible representations. Computer supported calculations show that  $\dim H^1(\Gamma_K; R_{10}) = 3$  and Lemma 4.4 implies that  $\dim H^1(\Gamma_K; R_{2k}) = \dim H^1(\Gamma_K; R_2) = 1$  for  $k \in \{2, 3, 4\}$ . Hence Formula 4.1 implies that

$$\dim H^1(\Gamma_K; \mathfrak{sl}(6)) = 7 \quad \text{i.e.} \quad Z^1(\Gamma_K; \mathfrak{sl}(6)) = 42.$$

In order to see that  $\rho_{\lambda,n}^z$  is contained in a 42-dimensional component of  $R_6(\Gamma_K)$  we proceed as follows: let  $A = \rho_{\lambda,n}^z(x)$  and  $B = \rho_{\lambda,n}^z(y)$  denote the image of  $x$  and  $y$  respectively. Notice that the matrices  $A$  and  $B$  are conjugate to  $r_6\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  and  $r_6\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$ . Hence

$$A \sim \begin{pmatrix} i & & & & \mathbf{0} \\ & i & & & \\ & & i & & \\ & & & -i & \\ \mathbf{0} & & & & -i \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} -1 & & & & \mathbf{0} \\ & -1 & & & \\ & & \lambda^2 & & \\ & & & \lambda^2 & \\ \mathbf{0} & & & & \lambda^{-2} \end{pmatrix}.$$

Further note that a choice of eigenspaces  $E_A(i)$ ,  $E_A(-i)$ ,  $E_B(-1)$ ,  $E_B(\lambda^2)$ ,  $E_B(\lambda^{-2})$  such that  $E_A(i) \oplus E_A(-i) \cong \mathbf{C}^6$  and  $E_B(-1) \oplus E_B(\lambda^2) \oplus E_B(\lambda^{-2}) \cong \mathbf{C}^6$  determines a representation  $\rho: \Gamma_{3_1} \rightarrow \mathrm{SL}(6)$  completely.

Let  $\mathrm{Gr}(p, n)$  denote the Grassmannian which parametrizes all  $p$ -dimensional subspaces of  $\mathbf{C}^n$ . Hence the choice of two elements in  $\mathrm{Gr}(3, 6)$  in generic position determines  $A$  and the choice of three elements in  $\mathrm{Gr}(2, 6)$  in generic position determines  $B$ . The representation will be irreducible if the eigenspaces of  $A$  and  $B$  are mutual in general position and reducible if not.

It is well known that  $\dim \mathrm{Gr}(p, n) = p(n - p)$  and hence

$$\dim (\mathrm{Gr}(3, 6) \times \mathrm{Gr}(3, 6)) = 18 \quad \text{and} \quad \dim (\mathrm{Gr}(2, 6) \times \mathrm{Gr}(2, 6) \times \mathrm{Gr}(2, 6)) = 24.$$

Therefore, we constructed a 42-dimensional component of representations  $C \subset R_6(\Gamma_{3_1})$  which contains  $\rho_{\lambda,6}^z = r_6 \circ \rho_\lambda^z$  and which contains irreducible representations. Note that  $6+6-2 = 40 < 42$ . In conclusion we have:

**Corollary 5.2.** *The representation variety  $R_6(\Gamma_{3_1})$  contains a 42-dimensional component  $C$ . The generic representation of  $C$  is irreducible and  $\rho_{\lambda,6}^z \in C \subset R_6(\Gamma_{3_1})$  is a smooth point.*

*Proof.* Computer supported calculations give that  $\dim Z^1(\Gamma_K, \mathfrak{sl}(6)) = 42$ . On the other hand, we constructed a 42-dimensional component  $C$  containing  $\rho_{\lambda,6}^z$ . Now, the assertion follows from Lemma 2.2.  $\square$

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