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# Computation of leaky modes in three-dimensional open elastic waveguides

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## Abstract

Elastic guided waves are of interest for inspecting structures due to their ability to propagate over long distances. When the guiding structure is embedded into a solid matrix, waveguides are open and waves can be trapped or leaky. With numerical methods, one of the difficulty is that leaky modes attenuate along the axis (complex wavenumber) and exponentially grow along the transverse direction. The goal of this work is to propose a numerical approach for computing modes in open elastic waveguides combining the so-called semi-analytical finite element method (SAFE) and a perfectly matched layer (PML) technique.

## Introduction

The simulation of open waveguides can be done through a simple numerical method which consists in using absorbing layers of artificially growing viscoelasticity [1]. To circumvent the transverse exponential growth of leaky modes, an alternative approach is to use a PML method instead of absorbing layers. Such a technique has already been applied to the scalar wave equation [2], [3]. In the present work, a SAFE-PML approach is applied to the equation of elastodynamics (non-scalar) in order to compute leaky modes in three-dimensional waveguides of arbitrary cross-section.

## 1 SAFE-PML formulations

### 1.1 Cartesian PML

One assumes a linearly elastic material in a domain  $\Omega = \{S, z\}$ . The time harmonic dependence is chosen as  $e^{-i\omega t}$ .  $z$  is the waveguide axis,  $S$  is the transverse section of the waveguide.  $S = (x, y)$  in Cartesian coordinates or  $(r, \theta)$  in cylindrical coordinates. Acoustic sources and external forces are suppressed for the purpose of studying propagation modes.

The 3D variational formulation governing elastodynamics is given by:

$$\int_{\Omega} \delta \tilde{\epsilon}^T \tilde{\sigma} d\tilde{\Omega} - \omega^2 \int_{\Omega} \tilde{\rho} \delta \tilde{\mathbf{u}}^T \tilde{\mathbf{u}} d\tilde{\Omega} = 0 \quad (1)$$

where  $d\tilde{\Omega} = d\tilde{x}d\tilde{y}d\tilde{z} = \tilde{r}d\tilde{r}d\tilde{\theta}d\tilde{z}$  (the tilde notation will be explained later). The variational formulation holds for any kinematically admissible trial displacement field  $\delta \mathbf{u}$ .  $\delta \tilde{\epsilon}$  denotes the virtual strain vector and  $\tilde{\sigma}$  is the stress vector. The superscript  $T$  denotes the matrix transpose.  $\tilde{\rho}$  is the material density.

The stress-strain relationship is  $\tilde{\sigma} = \tilde{\mathbf{C}}\tilde{\epsilon}$ , where  $\tilde{\mathbf{C}}$  is the matrix of material properties.

With a Cartesian PML in the  $x, y$  direction, the formulation (1) can be interpreted as the analytical continuation of the equilibrium equations into the complex spatial coordinate  $\tilde{x}, \tilde{y}$ , with:

$$\tilde{x} = \int_0^x \gamma_x(\xi) d\xi, \quad \tilde{y} = \int_0^y \gamma_y(\xi) d\xi \quad (2)$$

$\gamma_x$  is a complex-valued function of  $x$ , satisfying  $\gamma_x(x)=1$  for  $|x| \leq d_x$ ;  $\text{Im}\{\gamma_x(x)\} > 0$  for  $|x| > d_x$ . The definition of  $\gamma_y$  is analogous.  $d_x, d_y$  are the interfaces between the PML and physical domains.

From Eq. (2), the change of variables  $\tilde{x} \mapsto x$  yields for any function  $\tilde{f}$ :

$$\frac{\partial \tilde{f}}{\partial \tilde{x}} = \frac{1}{\gamma_x} \frac{\partial f}{\partial x}, \quad \frac{\partial \tilde{f}}{\partial \tilde{y}} = \frac{1}{\gamma_y} \frac{\partial f}{\partial y}, \quad d\tilde{x} = \gamma_x dx, \quad d\tilde{y} = \gamma_y dy \quad (3)$$

where  $\tilde{f}(\tilde{x}(x), \tilde{y}(y)) = f(x, y)$ .

In addition to the PML technique, the SAFE method is applied, which consists in assuming an  $e^{ikz}$  dependence, where  $k$  is the axial wavenumber. Combining SAFE and PML approach, the strain-displacement relation can be written as follows by separating transverse from axial derivatives:

$$\boldsymbol{\epsilon} = (\mathbf{L}_{\tilde{x}\tilde{y}} + ik\mathbf{L}_z) \mathbf{u} \quad (4)$$

where  $\mathbf{L}_{\tilde{x}\tilde{y}}$  is the operator containing derivatives with respect to transverse direction  $(\tilde{x}, \tilde{y})$ .

$$\mathbf{L}_{\tilde{x}\tilde{y}} = \begin{bmatrix} \frac{1}{\gamma_x} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{1}{\gamma_y} \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \frac{1}{\gamma_y} \frac{\partial}{\partial y} & \frac{1}{\gamma_x} \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{1}{\gamma_x} \frac{\partial}{\partial x} \\ 0 & 0 & \frac{1}{\gamma_y} \frac{\partial}{\partial y} \end{bmatrix}, \quad \mathbf{L}_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (5)$$

Finally, the FE discretization of the variation formulation (1) along the transverse directions  $x, y$  yields:

$$\{\mathbf{K}_1 - \omega^2 \mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2 \mathbf{K}_3\} \mathbf{U} = \mathbf{0} \quad (6)$$

with the following elementary matrices:

$$\begin{aligned} \mathbf{K}_1^e &= \int_{S^e} \mathbf{N}^{eT} \mathbf{L}_{\tilde{x}\tilde{y}}^T \mathbf{C} \mathbf{L}_{\tilde{x}\tilde{y}} \mathbf{N}^e \gamma_x \gamma_y dx dy \\ \mathbf{K}_2^e &= \int_{S^e} \mathbf{N}^{eT} \mathbf{L}_{\tilde{x}\tilde{y}}^T \mathbf{C} \mathbf{L}_z \mathbf{N}^e \gamma_x \gamma_y dx dy \\ \mathbf{K}_3^e &= \int_{S^e} \mathbf{N}^{eT} \mathbf{L}_z^T \mathbf{C} \mathbf{L}_z \mathbf{N}^e \gamma_x \gamma_y dx dy \\ \mathbf{M}^e &= \int_{S^e} \rho \mathbf{N}^{eT} \mathbf{N}^e \gamma_x \gamma_y dx dy \end{aligned}$$

where the column vector  $\mathbf{U}$  contains nodal displacements and  $\mathbf{N}^e$  is a matrix of nodal interpolating functions of displacement on the element.

Given  $\omega$  and finding  $k$ , the eigenproblem (6) is quadratic. The linearization of this eigensystem is detailed in [4] for instance.

### 1.2 Cylindrical PML

The cylindrical PML defines the complex radial coordinate  $\tilde{r}$  :

$$\tilde{r} = \int_0^r \gamma(\xi) d\xi \quad (7)$$

where  $\gamma(r)=1$  for  $r \leq d_r$ ;  $Im\{\gamma(r)\} > 0$  for  $r > d_r$ . By using the change of variable  $\tilde{r} \rightarrow r$  and a SAFE method, the strain-displacement relation becomes:

$$\boldsymbol{\epsilon} = (\mathbf{L}_{\tilde{r}\theta} + ik\mathbf{L}_z) \mathbf{u} \quad (8)$$

Before FE discretization, the formulation (1) and Eq. (8) are rewritten in Cartesian coordinates. For paper conciseness, the operator matrix  $\mathbf{L}_{\tilde{r}\theta}$  in Cartesian coordinates is not presented here.

Finally, it can be shown that the FE discretization of the variation formulation (1) along the transverse section yields the same equation as Eq.(6) with the elementary matrices obtained by replacing  $\mathbf{L}_{\tilde{x}\tilde{y}}$  with  $\mathbf{L}_{\tilde{r}\theta}$  and  $\gamma_x \gamma_y$  with  $\tilde{r} \gamma / r$ .

## 2 Results

A numerical test is taken from the literature [1], consisting of a steel bar buried in a concrete infinite domain. The continuity of displacements and stresses is enforced at each interface, i.e. between the core and semi-infinite layers. Following the suggestion of [2], [3], the PML layer is close to the core in order to reduce the effects of the exponential growth of leaky modes on the numerical results. A Dirichlet condition is chosen at the exterior boundary of truncated domain. Finite elements are triangles with six

nodes.  $\gamma_x, \gamma_y, \gamma$  in Eqs. (2) and (7) should be chosen as smooth as possible to minimize numerical reflection [5]. They are parabolic functions in this work.

A difficulty is that the method will not only provide trapped and leaky modes but also non-intrinsic modes corresponding to continua of radiation modes which are mainly resonating in the artificial layers and depend on the characteristics of these layers. A modal filtering step consists in identifying and separating physical modes from unwanted modes. The filtering criterion used for our tests is the ratio of kinetic energy in the PML region over the kinetic energy in the whole domain. Physical modes are then identified if this criterion is smaller than a user-defined value.

Numerical results will be shown during the conference to validate the SAFE-PML methods.

Further work will consist in extending the proposed approach to twisted waveguides in order to simulate helical structures buried in infinite solid media.

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