Density Bounds for some Degenerate Stable Driven SDEs

Lorick Huang, Stephane Menozzi

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Lorick Huang
Université Paris 7 - Diderot *

Stéphane Menozzi
Université d'Evry Val-d'Essonne †

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Abstract

We consider a stable driven degenerate stochastic differential equation, whose coefficients satisfy a kind of weak Hörmander condition. Under mild smoothness assumptions we prove the uniqueness of the martingale problem for the associated generator. Also, in the scalar case we establish density bounds reflecting the multi-scale behavior of the process.

1 Introduction

The aim of this paper is to study degenerate stable driven stochastic differential equations of the following form:

$$\begin{aligned} dX_t^1 &= \left(a_t^{1,1}X_t^1 + \cdots + a_t^{1,n}X_t^n\right)dt + \sigma(t, X_{t^-})dZ_t \\ dX_t^2 &= \left(a_t^{2,1}X_t^1 + \cdots + a_t^{2,n}X_t^n\right)dt \\ dX_t^3 &= \left(a_t^{3,2}X_t^2 + \cdots + a_t^{3,n}X_t^n\right)dt \\ &\vdots \\ dX_t^n &= \left(a_t^{n,n-1}X_t^{n-1} + a_t^{n,n}X_t^n\right)dt,\ X_0 = x \in \mathbb{R}^{nd}, \end{aligned} \tag{1.1}$$

where Z is an $\mathbb{R}^d$ valued symmetric α stable process ($\alpha \in (0,2)$), $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$, $a^{i,j} : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^d$, $i \in [\![1,n]\!]$, $j \in [\![(i-1)\vee 1, n]\!]$. Observe that X is $\mathbb{R}^{nd}$ valued. We will often use the shortened form:

$$dX_t = A_t X_t dt + B\sigma(t, X_{t^-})dZ_t,\ X_0 = x, \tag{1.2}$$

where $B = \left(I_{d\times d}\ 0_{(n-1)d\times d}\right)^*$ denotes the injection matrix from $\mathbb{R}^d$ into $\mathbb{R}^{nd}$ and A_t is the matrix :

$$A_t = \begin{pmatrix} a_t^{1,1} & \dots & \dots & \dots & a_t^{1,n} \\ a_t^{2,1} & \ddots & & & a_t^{2,n} \\ 0 & a_t^{3,2} & \ddots & & a_t^{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_t^{n,n-1} & a_t^{n,n} \end{pmatrix}.$$

The previous system appears in many applicative fields. It is for instance related for $n = 2$ to the pricing of Asian options in jump diffusions models (see e.g. Jeanblanc *et al* [13] or Barucci *et al* [2] in the Brownian case). The Hamiltonian formulation in mechanics can lead to systems corresponding to the drift part of (1.1) (still with $n = 2$). The associated Brownian perturbation has been thoroughly studied, see e.g. Talay [32] or Stuart *et al.* [22] for the convergence of approximation schemes to equilibrium, but to the best of our knowledge other perturbations, like the current stable one, have not yet been considered. For a general n, equation (1.1) can be seen as the linear dynamics of n coupled oscillators in dimension d perturbed by a stable anisotropic noise. Observe also that in the diffusive case these oscillator chains naturally appear in statistical mechanics, see e.g Eckman *et al.* [9].

*Laboratoire de Probabilités et Modèles Aléatoires, Bâtiment Sophie Germain, 5 rue Thomas Mann, 75205 Paris CEDEX 13, France. huang@math.univ-paris-diderot.fr

†Laboratoire d'Analyse et Probabilités, 23 Boulevard de France, 91037 Evry, France. stephane.menozzi@univ-evry.fr

Equation (1.1) is degenerate in the sense that the noise only acts on the first component of the system. Additionally to the non-degeneracy of the *volatility* σ, we will assume a kind of weak Hörmander condition on the drift component in order to allow the noise propagation into the system.

A huge literature exists on degenerate Brownian diffusions under the *strong* Hörmander condition, i.e. when the underlying space is spanned by the diffusive vector fields and their iterated Lie brackets. The major works in that framework have been obtained in a series of papers by Kusuoka ans Stroock, [17], [18], [19], using a Malliavin calculus approach.

For the weak Hörmander case, many questions are still open even in the Brownian setting. Let us mention in this framework the papers [8], [24] and [16] dealing respectively with density estimates, martingale problems and random walk approximations for systems of type (1.1) or that can be linearized around such systems. In those works a global multi-scale Gaussian regime holds. For highly non-linear first order vector fields, Franchi [10] and Cinti *et al.* [7] address issues for which there is not a single regime anymore. A specificity of the weak Hörmander condition is the unbounded first order term which does not lead to a time-space separation in the off-diagonal bounds for the density estimates as in the sub-Riemannian setting, see e.g. [19], Ben Arous and Léandre [4] and references therein. The energy of the associated deterministic control problem has to be considered instead, see e.g. [8]. We have a similar feature in our current stable setting.

In this work we are first interested in proving the uniqueness of the martingale problem associated to the generator $(L_t)_{t\geq}$ of (1.1), i.e.

$$\forall \varphi \in C_0^1(\mathbb{R}^{nd},\mathbb{R}),\ \forall x\in\mathbb{R}^{nd},\ L_t\varphi(x)=\langle A_t x,\nabla\varphi(x)\rangle+\int_{\mathbb{R}^d}\left(\varphi(x+B\sigma(t,x)z)-\varphi(x)-\frac{\langle\nabla\varphi(x),B\sigma(t,x)z\rangle}{1+|z|^2}\right)\nu(dz), \tag{1.3}$$

under some mild assumptions on the volatility σ and the Lévy measure ν of Z. To this end, the key tool consists in exploiting some properties of the joint densities of stable processes and their iterated integrals, corresponding to the proxy model in a *parametrix* continuity technique. Following the strategies developed in [3], [24] we then derive uniqueness exploiting the smoothing properties of the parametrix kernel. Let us emphasize that the above mentioned densities actually behave as the density of an α stable process in dimension nd with a modified Lévy measure, where $n-1$ is the numbers of iterated integrals considered and d is the initial dimension of the process. They also exhibit different time-scales. Roughly speaking, the typical time scale of the initial stable process is $t^{1/\alpha}$ and $t^{(i-1)+1/\alpha}$ for the associated $(i-1)^{\text{th}}$ integral. Also, the process will deviate from the transport of the initial condition by the deterministic system, i.e. setting $\frac{d}{dt}R_t = A_t R_t,\ R_0 = I$, the mean of the process is $R_t x$, accordingly to the associated component wise time scales. We establish two sided estimates for those densities, see Proposition 3.3.

When turning to density estimates, one of the dramatic differences with the Gaussian case is the lack of integrability of the driving process. For non-degenerate stable driven SDEs, this difficulty can be bypassed to derive two-sided pointwise bounds for the SDE that are homogeneous to the density of the driving process, see e.g. Kolokoltsov [14], establishing in the stable case the analogue of the *Aronson* bounds for diffusions, see e.g. Sheu [28] or [1]. For approximation schemes of non-degenerate stable-driven SDEs we also mention [15]. In our current degenerate framework, working under somehow minimal assumptions to derive pointwise density bounds, that is Hölder continuity of the coefficients, we did not succeed to get rid of those integrability problems. For technical reasons that will appear later on, we obtain when $d=1, n=2$ (scalar non-degenerate diffusion and associated non-degenerate integral) the expected upper-bound up to an additional logarithmic contribution and the expected diagonal lower bound, see Theorem 2.2. To this end we use a parametrix approach similar to the one of Mc Kean and Singer [23]. Working with smoother coefficients would have allowed to consider Malliavin calculus type techniques. In the jump case, this approach has been investigated to establish existence/smoothness of the density for SDEs by Bichteler *et al.* in the non-degenerate case [5], and Léandre in the degenerate one, see [20],[21]. Let us eventually mention some related works. Priola and Zabczyk establish in [26] existence of the density for processes of type (1.1), under the same kind of weak Hörmander assumption and when σ is constant, for a general driving Lévy process Z provided its Lévy measure is infinite and has itself a density on compact sets. Also, Picard, [25] investigates similar problems for singular Lévy measures. Other results concerning the smoothness of the density of Lévy driven SDEs have been obtained by Ishikawa and Kunita [11] in the non-degenerate case but with mild conditions on the Lévy measure and by Cass [6] who gets smoothness in the weak Hörmander framework under technical restrictions.

The article is organized as follows. We state our main results in Section 2. In Section 3, we explain the procedure to derive those results and also state the density estimates on the process in (1.1) when $\sigma(t,x)=\sigma(t)$ (*frozen process*). We then prove the uniqueness of the martingale problem in Section 4. Sections 5 and 6 are the technical core of the paper. In particular, we prove there the existence of the density and the associated estimates for the frozen process and establish the smoothing properties of the parametrix kernel. Appendices

A and B are dedicated to the derivation of stable density bounds and kernels following the procedure of [14] in our current degenerate setting.

2 Assumptions, and Main Result.

Let $(Z_t)_{t\geq 0}$ be an α stable symmetric process, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, that is a Lévy process with Fourier exponent:

$$\mathbb{E}e^{i\langle p, Z_t\rangle} = \exp\left(-t\int_{S^{d-1}} |\langle p, \varsigma\rangle|^\alpha \mu(d\varsigma)\right),\ \forall p \in \mathbb{R}^d.$$

In the above expression, we denote by S^{d-1} the unit sphere in $\mathbb{R}^d$, and by μ the spectral measure of Z. This measure is related to the Lévy measure of Z as follows. If ν is the Lévy measure of Z, its decomposition in polar coordinates writes: $\nu(dz) = \frac{d\rho}{\rho^{1+\alpha}}\tilde{\mu}(d\varsigma)$, where $z = \rho\varsigma,\ (\rho, \varsigma) \in \mathbb{R}_+ \times S^{d-1}$. Then, $\mu = C_{\alpha,d}\tilde{\mu}$ (see Sato [27] for the exact value of $C_{\alpha,d}$).

We will make the following assumptions:

[H-1]: (**Hölder regularity**) $\exists H > 0,\ \eta \in (0, 1],\ \forall x, y \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$||\sigma(t, x) - \sigma(t, y)|| \leq H|x - y|^\eta.$$

[H-2]: (**Non degeneracy of the spectral measure**) $\exists \Lambda_1, \Lambda_2 \in \mathbb{R}_+^*$, $\forall u \in \mathbb{R}^d$,

$$\Lambda_1|u|^\alpha \leq \int_{S^{d-1}} |\langle u, \varsigma\rangle|^\alpha \mu(d\varsigma) \leq \Lambda_2|u|^\alpha. \tag{2.1}$$

[H-3]: (**Ellipticity**) $\exists\ \overline{c},\ \underline{c} > 0,\ \forall \xi \in \mathbb{R}^d,\ \forall z \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$\underline{c}|\xi|^2 \leq \langle \xi, \sigma\sigma^*(t, z)\xi\rangle \leq \overline{c}|\xi|^2. \tag{2.2}$$

[H-4]: (**Hörmander-like condition for** $(A_t)_{t\geq 0}$) $\exists \overline{\alpha},\ \underline{\alpha} \in \mathbb{R}_+^*,\ \forall \xi \in \mathbb{R}^{nd}$ and $\forall t \geq 0,\ \underline{\alpha}|\xi|^2 \leq \langle a_t^{i,i-1}\xi, \xi\rangle \leq \overline{\alpha}|\xi|^2,\ \forall i \in [\![2, n-1]\!]$. Also, for all $(i, j) \in [\![1, n]\!]^2,\ \|a_t^{i,j}\| \leq \overline{\alpha}$.

We say that **[H]** holds if conditions **[H-1]** to **[H-4]** are fulfilled.

Our main results are the following.

Theorem 2.1 (**Weak Uniqueness**). *Under **[H]**, the martingale problem associated with the generator $(L_t)_{t\geq 0}$, defined in* (1.3)*, of the degenerate equation* (1.1)*:*

$$dX_t = A_tX_tdt + B\sigma(t, X_{t\_})dZ_t,$$

admits a unique solution. That is, for every $x \in \mathbb{R}^{nd}$, there exists a unique probability measure $\mathbb{P}$ on $\Omega = \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^{nd}, \mathbb{R})$ the space of càdlàg functions, such that for all $f \in C_0^{1,1}(\mathbb{R}_+ \times \mathbb{R}^{nd}, \mathbb{R})$, denoting by $(X_t)_{t\geq 0}$ the canonical process, we have:

$$\mathbb{P}(X_0 = x) = 1 \quad \textit{and} \quad f(t, X_t) - \int_0^t (\partial_u + L_u)f(u, X_u)du \quad \textit{is a } \mathbb{P}\textit{- martingale.}$$

Hence, weak uniqueness holds for (1.1)*.*

Also, when $d = 1$ and $n = 2$ in (1.1) we are able to prove the following density estimates.

Theorem 2.2 (**Density Estimates**). *Assume that $d = 1,\ n = 2$. Under **[H]**, the unique weak solution of* (1.1) *has for every $s > 0$ a density with respect to the Lebesgue measure. Precisely, for all $0 \leq t < s$ and $x \in \mathbb{R}^2$,*

$$\mathbb{P}(X_s \in dy | X_t = x) = p(t, s, x, y)dy. \tag{2.3}$$

Also, for a deterministic time horizon $T > 0$, and fixed threshold $K > 0$, the following upper bound holds:
$\exists C_{2.2} := C_{2.2}(\textbf{[H]}, T, K) \geq 1,\ s.t.\ \forall 0 \leq t < s \leq T,\ \forall (x, y) \in (\mathbb{R}^2)^2$,

$$p(t, s, x, y) \leq C_{2.2}\bar{p}_\alpha(t, s, x, y)\left(1 + \log(K \vee |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))|\right), \tag{2.4}$$

where

$$\bar{p}_\alpha(t,s,x,y) = C_\alpha \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{\{K \vee |(\mathbb{T}^\alpha_{s-t})^{-1}(y - R_{s,t}(x))|\}^{2+\alpha}}, \text{ and } \forall u \in \mathbb{R}_+,\ \mathbb{T}^\alpha_u := \mathrm{Diag}\big((u^{1/\alpha}, u^{1+1/\alpha})\big).$$

Here, $R_{s,t}$ stands for the resolvent associated to the deterministic part of (1.1)*, i.e. $\frac{d}{ds}R_{s,t} = A_s R_{s,t},\ R_{t,t} = I_2$, and C_α is s.t. $\int_{\mathbb{R}^2} \bar{p}_\alpha(t,s,x,y)dy = 1$.*

Eventually for $0 < T \le T_0 := T_0(\textbf{[H]}, K)$ small enough, the following diagonal lower bound holds:

$$\forall 0 \le t < s \le T,\ \forall (x,y) \in (\mathbb{R}^2)^2 \text{ s.t. } |(\mathbb{T}^\alpha_{s-t})^{-1}(y - R_{s,t}(x))| \le K,\ p(t,s,x,y) \ge C^{-1}_{2.2}\det(\mathbb{T}^\alpha_{s-t})^{-1}. \tag{2.5}$$

Under the current assumptions, Theorem 2.1 is proved following the lines of [3] and [24]. In the Gaussian framework, those assumptions are sufficient to derive homogeneous two-sided multi-scale Gaussian bounds, see [8]. However, in the current context, we only managed to obtain the expected upper bound up to a logarithmic factor and a diagonal lower bound for $d = 1$ and $n = 2$. This is mainly due to a lack of integrability of the stable process which becomes really delicate to handle in the degenerate case. Indeed, in the non-degenerate context, Kolokoltsov [14] successfully gives two sided bounds for the density of the SDE. The technical reasons leading to the restriction of Theorem 2.2 will be discussed thoroughly in the dedicated sections (see Sections 3.3 and 6). Let us mention that the above results could be extended to the case of a d-dimensional non-degenerate stable-driven SDE and the integral of **one** of its components. We emphasize as well, that our estimates still hold if we had a non-linear bounded drift in the dynamics of X^1 if $\alpha > 1$ (see Remark 5.5). We conclude this paragraph saying that the uniqueness of the martingale problem and the estimates of Section 6 allow to extend in the non-degenerate case, the stable two-sided *Aronson like* estimates of [14] for Hölder coefficients.

Constants and usual notations:

- The capital letter C will denote a constant whose value may change from line to line, and can depend on the hypotheses **[H]**. Other dependencies (in particular in time), will be specified, using explicit under scripts.
- We will often use the notation $\asymp$ to express equivalence between functions. If f and g are two real valued nonnegative functions, we denote $f(x) \asymp g(x),\ x \in I \subset \mathbb{R}^p,\ p \in \mathbb{N}$, when there exists a constant $C \ge 1$, possibly depending on **[H]**, I s.t. $C^{-1}f(x) \le g(x) \le Cf(x),\ \forall x \in I$.
- For $x = (x_1, \cdots, x_{nd}) \in \mathbb{R}^{nd}$ and for all $k \in [\![1, n]\!]$, we define $x^k := (x_{(k-1)d+1}, \cdots, x_{kd}) \in \mathbb{R}^d$. Accordingly, $x = (x^1, \cdots, x^n)$.

From now on, we assume **[H]** to be in force.

3 Continuity techniques : the Frozen equation and the parametrix series.

For density estimates, a continuity technique consists in considering a simpler equation as *proxy* model for the initial equation. The *proxy* will be significant if it achieves two properties:

- It admits an explicit density or a density that is well estimated.
- The difference between the density of the initial SDE and the one of the proxy can be well controlled.

For the last point a usual strategy consists in expressing the difference of the densities through the difference of the generators of the two SDEs, using Kolmogorov's equations. This approach is known as the *parametrix* method. In the current work, we will use the procedure developed by Mc Kean and Singer [23], which turns out to be well-suited to handle coefficients with mild smoothness properties.

We first introduce the *proxy* model in Section 3.1, and give some associated density bounds. We then analyze in Sections 3.2, 3.3 how this choice can formally lead through a parametrix expansion to a density estimate, exploiting some suitable regularization properties in time. These arguments can be made rigorous provided that the initial SDE admits a Feller transition function. The uniqueness of the martingale problem will actually give this property.

3.1 The Frozen Process.

In this section, we give results that hold in any dimension d, and for any fixed number of oscillators n. Let $T > 0$ (arbitrary deterministic time) and $y \in \mathbb{R}^{nd}$ (final freezing point) be given. Heuristically, y is the point where we want to estimate the density of (1.1) at time T provided it exists. We introduce the *frozen* process as follows:

$$d\tilde{X}_s^{T,y} = A_s \tilde{X}_s^{T,y} ds + B\sigma(s, R_{s,T}(y))dZ_s. \tag{3.1}$$

In this equation, $R_{s,T}(y)$ is the resolvent of the deterministic equation associated, i.e. it satisfies $\frac{d}{ds}R_{s,T} = A_s R_{s,T}$, with $R_{T,T} = I_{nd}$ in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$. Let us emphasize that the previous choice can seem awkward at first sight. Indeed, a very natural approach for a proxy model would consist in freezing the diffusion coefficient at the terminal point, see e.g. Kolokoltsov [14]. In our current weak Hörmander setting we need to take into account the backward transport of the final point by the deterministic differential system. This particular choice is actually imposed by the natural metric appearing in the density of the frozen process, see Proposition 3.3. This allows the comparison of the singular parts of the generators of (1.1) and (3.1) applied to the frozen density, see Proposition 3.5 and Lemma 3.9.

Proposition 3.1. *Fix $(t,x) \in [0,T] \times \mathbb{R}^{nd}$. The unique solution of (3.1) starting from x at time t writes:*

$$\tilde{X}_s^{t,x,T,y} = R_{s,t}x + \int_t^s R_{s,u}B\sigma(u, R_{u,T}(y))dZ_u. \tag{3.2}$$

Proof. Equation (3.1) is a linear SDE, with constant diffusion coefficient. As such, it admits a unique strong solution. The representation (3.2) follows from Itô's formula. □

Introduce for all $u \in \mathbb{R}^+$, the diagonal time scale matrix:

$$\mathbb{T}_u^\alpha = \begin{pmatrix} u^{\frac{1}{\alpha}} I_d & & 0 \\ & \ddots & \\ 0 & & u^{n-1+\frac{1}{\alpha}} I_d \end{pmatrix}. \tag{3.3}$$

This extends the definition of Theorem 2.2 for $n = 2$. The entries of this matrix correspond to the *intrinsic* time scales of the iterated integrals of a stable process with index α observed at time u. They reflect the multi-scale behavior of our system. We first give an expression of the density of $\tilde{X}_s^{t,x,T,y}$ in terms of its inverse Fourier transform. We refer to Section 5.2 for the proof of this result.

Proposition 3.2. *The frozen process $(\tilde{X}_s^{t,x,T,y})_{s \geq t}$ has for all $s > t$ a density w.r.t. the Lebesgue measure, that is:*

$$\mathbb{P}(\tilde{X}_s^{T,y} \in dz | \tilde{X}_t^{T,y} = x) = \tilde{p}_\alpha^{T,y}(t,s,x,z)dz.$$

*For $0 < T - t \leq T_0 := T_0(\textbf{\textit{[H]}}) \leq 1$ there exists a symmetric measure $\mu_S^* = \mu_S^*(t,T,s,y)$ on S^{nd-1} satisfying **[H-2]** in $\mathbb{R}^{nd}$ (non degeneracy of the spectral measure), uniformly in the parameters t, T, s, y, such that:*

$$\tilde{p}_\alpha^{T,y}(t,s,x,z) = \frac{\det\left(\mathbb{T}_{s-t}^\alpha\right)^{-1}}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} e^{-i\langle q, (\mathbb{T}_{s-t}^\alpha)^{-1}(z - R_{s,t}(x))\rangle} \exp\left(-\int_{S^{nd-1}} |\langle q, \eta\rangle|^\alpha \mu_S^*(d\eta)\right) dq. \tag{3.4}$$

Remark 3.1. The above proposition is important in that it shows why the density of a d-dimensional stable process with index $\alpha \in (0,2)$ and its n iterated integrals actually behaves as the density of an nd-dimensional *multi-scale* stable process, where the various scales are read through the matrix $\mathbb{T}^\alpha$.

From the previous remark, following the computations in [14] which are recalled in the appendix, we derive the following estimates.

Proposition 3.3. *Fix $T > 0$, a threshold $K > 0$ and $y \in \mathbb{R}^{nd}$. For all $(t,x) \in [0,T) \times \mathbb{R}^{nd}$, the density $\tilde{p}_\alpha^{T,y}(t,s,x,z)$ of the frozen process $(\tilde{X}_s^{t,x,T,y})_{s \in (t,T]}$ in (3.2) satisfies the following estimates. There exists $C_{3.3} := C_{3.3}$ (**H-2,H-3,H-4**,K)≥ 1, s.t. for all $0 \leq t < s \leq T$, $(x,z) \in (\mathbb{R}^{nd})^2$:*

$$C_{3.3}^{-1} \bar{p}_\alpha(t,s,x,z) \leq \tilde{p}_\alpha^{T,y}(t,s,x,z) \leq C_{3.3} \bar{p}_\alpha(t,s,x,z), \tag{3.5}$$

where extending the definition of Theorem 2.2, we write:

$$\bar{p}_\alpha(t,s,x,y) = C_\alpha \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{\{K \vee |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))|\}^{nd+\alpha}}. \tag{3.6}$$

We refer to Section 5.2 and Appendix A for the proof of this result. As a corollary, we have the following important property.

Corollary 3.4 ("Semigroup" property). *There exists $C_{3.4} := C_{3.4}$ (**H-2,H-3,H-4**,K) ≥ 1 s.t. for all $0 \leq t < \tau < s,\ (x,y) \in (\mathbb{R}^{nd})^2$:*

$$\int_{\mathbb{R}^{nd}} \bar{p}_\alpha(t,\tau,x,z)\bar{p}_\alpha(\tau,s,z,y)dz \leq C_{3.4}\bar{p}_\alpha(t,s,x,y).$$

Proof. From the lower bound in (3.5) we get for all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^{nd}$:

$$\int_{\mathbb{R}^{nd}} \bar{p}_\alpha(t,\tau,x,z)\bar{p}_\alpha(\tau,s,z,y)dz \leq C_{3.3}^2 \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha^{s,y}(t,\tau,x,z)\tilde{p}_\alpha^{s,y}(\tau,s,z,y)dz = C_{3.3}^2 \tilde{p}_\alpha^{s,y}(t,s,x,y),$$

because $\tilde{p}_\alpha^{s,y}$ enjoys the semigroup property. The upper-bound of (3.5) finally yields:

$$\int_{\mathbb{R}^{nd}} \bar{p}_\alpha(t,\tau,x,z)\bar{p}_\alpha(\tau,s,z,y)dz \leq C_{3.3}^3 \bar{p}_\alpha(t,s,x,y).$$

□

Remark 3.2. As an easy corollary of this semigroup property, we derive that for all $T_1,T_2,T_3 > 0$, $y_1,y_2,y_3 \in \mathbb{R}^{nd}$, for all $t < \tau < s$ and $x,y \in \mathbb{R}^{nd}$:

$$\int_{\mathbb{R}^{nd}} \tilde{p}_\alpha^{T_1,y_1}(t,\tau,x,z)\tilde{p}_\alpha^{T_2,y_2}(\tau,s,z,y)dz \leq C_{3.3}^3 \tilde{p}_\alpha^{T_3,y_3}(t,s,x,y). \tag{3.7}$$

The above control reads as a semigroup property on the frozen densities with possible different freezing points.

3.2 The Parametrix Series.

We assume here that the generator $(L_t)_{t\geq 0}$ of (1.1) generates a Feller inhomogeneous semigroup $(P_{t,s})_{0\leq t\leq s}$. Using the Chapman-Kolmogorov equations satisfied by the semigroup and the pointwise Kolmogorov equations for the *proxy* model, we derive a formal representation of the semigroup in terms of a series, involving the difference of the generators of the initial and frozen processes. Let L_t (already defined in (1.3)) and $\tilde{L}_t^{T,y}$ denote the generators of $X^{t,x}$ and $\tilde{X}^{t,x,T,y}$ at time t respectively. For $\phi \in C_0^1(\mathbb{R}^{nd},\mathbb{R})$, denoting by ν the Lévy measure of Z, we have for all $x \in \mathbb{R}^{nd}$:

$$L_t\phi(x) = \langle \nabla\phi(x), A_t x\rangle + \int_{\mathbb{R}^d} \left(\phi(x + B\sigma(t,x)z) - \phi(x) - \frac{\langle \nabla\phi(x), B\sigma(t,x)z\rangle}{1+|z|^2}\right)\nu(dz),$$

$$\tilde{L}_t^{T,y}\phi(x) = \langle \nabla\phi(x), A_t x\rangle + \int_{\mathbb{R}^d} \left(\phi(x + B\sigma(t,R_{t,T}(y))z) - \phi(x) - \frac{\langle \nabla\phi(x), B\sigma(t,R_{t,T}(y))z\rangle}{1+|z|^2}\right)\nu(dz). \tag{3.8}$$

Observe that for $\tilde{X}_s^{t,x,T,y}$ defined in (3.2), its density $\tilde{p}_\alpha^{T,y}(t,s,x,\cdot)$ exists and is smooth under **[H]** for $s > t$ (see Proposition 3.2 above).

Proposition 3.5. *Suppose that there exists a unique weak solution $(X_s^{t,x})_{0\leq t\leq s}$ to (1.1) which has a Feller semigroup $(P_{t,s})_{0\leq t\leq s}$. We have the following formal representation. For all $0 \leq t < T,\ (x,y) \in (\mathbb{R}^{nd})^2$ and any bounded measurable $f : \mathbb{R}^{nd} \to \mathbb{R}$:*

$$P_{t,T}f(x) = \mathbb{E}[f(X_T)|X_t = x] = \int_{\mathbb{R}^{nd}} \left(\sum_{r=0}^{+\infty} (\tilde{p}_\alpha \otimes H^{(r)})(t,T,x,y)\right) f(y), \tag{3.9}$$

where H is the parametrix kernel:

$$\forall 0 \leq t < T,\ (x,y) \in (\mathbb{R}^{nd})^2,\ H(t,T,x,y) := (L_t - \tilde{L}_t^{T,y})\tilde{p}_\alpha^{T,y}(t,T,x,y). \tag{3.10}$$

In equation (3.9), we denote for all $0 \leq t < u \leq T,\ (x,z) \in (\mathbb{R}^{nd})^2,\ \tilde{p}_\alpha(t,u,x,z) := \tilde{p}_\alpha^{u,z}(t,u,x,z)$. Also, the notation $\otimes$ stands for the time space convolution:

$$f \otimes g(t,T,x,y) = \int_t^T du \int_{\mathbb{R}^{nd}} dz f(t,u,x,z)g(u,T,z,y).$$

Besides, $H^{(0)} = I$ *and* $\forall r \in \mathbb{N},\ H^{(r)}(t,T,x,y) = H^{(r-1)} \otimes H(t,T,x,y)$.

Furthermore, when the above representation can be justified, it yields the existence as well as a representation for the density of the initial process. Namely $\mathbb{P}[X_T \in dy | X_t = x] = p(t,T,x,y)dy$ *where :*

$$\forall 0 \leq t < T,\ (x,y) \in (\mathbb{R}^{nd})^2,\ p(t,T,x,y) = \sum_{r=0}^{+\infty} (\tilde{p}_\alpha \otimes H^{(r)})(t,T,x,y). \tag{3.11}$$

Proof. Let us first emphasize that the density $\tilde{p}_\alpha^{T,y}(t,s,x,z)$ of $\tilde{X}_s^{t,x,T,y}$ at point z solves the Kolmogorov *backward* equation:

$$\frac{\partial \tilde{p}_\alpha^{T,y}}{\partial t}(t,s,x,z) = -\tilde{L}_t^{T,y} \tilde{p}_\alpha^{T,y}(t,s,x,z),\ \text{for all } t < s,\ (x,z) \in \mathbb{R}^{nd} \times \mathbb{R}^{nd},\ \lim_{t \uparrow s} \tilde{p}_\alpha^{T,y}(t,s,\cdot,z) = \delta_z(\cdot). \tag{3.12}$$

Here, $\tilde{L}_t^{T,y}$ acts on the variable x. Let us now introduce the family of operators $(\tilde{P}_{t,s})_{0 \leq t \leq s}$. For $0 \leq t \leq s$ and any bounded measurable function $f : \mathbb{R}^{nd} \to \mathbb{R}$:

$$\tilde{P}_{t,s} f(x) := \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha^{T,y}(t,T,x,y) f(y) dy := \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha(t,T,x,y) f(y) dy. \tag{3.13}$$

Observe that the family $(\tilde{P}_{t,s})_{0 \leq t \leq s}$ is not a semigroup. Anyhow, we can still establish, see Lemma 4.1, that for a continuous f:

$$\lim_{s \to t} \tilde{P}_{s,t} f(x) = f(x). \tag{3.14}$$

This convergence is not a direct consequence of the bounded convergence theorem since the freezing parameter is also the integration variable.

The boundary condition (3.14) and the Feller property yield:

$$(P_{t,T} - \tilde{P}_{t,T}) f(x) = \int_t^T du \frac{\partial}{\partial u} \Big\{ P_{t,u}(\tilde{P}_{u,T} f(x)) \Big\}.$$

Computing the derivative under the integral leads to:

$$(P_{t,T} - \tilde{P}_{t,T}) f(x) = \int_t^T du \Big\{ \partial_u P_{t,u}(\tilde{P}_{u,T} f(x)) + P_{t,u}(\partial_u (\tilde{P}_{u,T} f(x))) \Big\}.$$

Using the Kolmogorov equation (3.12) and the Chapman-Kolmogorov relation $\partial_u P_{t,u} \varphi(x) = P_{t,u}(L_u \varphi(x))$, $\forall \varphi \in C_b^1(\mathbb{R}^{nd}, \mathbb{R})$ we get:

$$(P_{t,T} - \tilde{P}_{t,T}) f(x) = \int_t^T du P_{t,u} \Big(L_u \tilde{P}_{u,T} f \Big)(x) - P_{t,u} \left(\int_{\mathbb{R}^{nd}} f(y) \tilde{L}_u^{T,y} \tilde{p}_\alpha(u,T,\cdot,y) dy \right)(x).$$

Define now the operator:

$$\mathcal{H}_{u,T} \varphi(z) := \int_{\mathbb{R}^{nd}} \varphi(y)(L_u - \tilde{L}_u^{T,y}) \tilde{p}_\alpha(u,T,z,y) dy = \int_{\mathbb{R}^{nd}} \varphi(y) H(u,T,z,y). \tag{3.15}$$

We can thus rewrite:

$$P_{t,T} f(x) = \tilde{P}_{t,T} f(x) + \int_t^T P_{t,u} \big(\mathcal{H}_{u,T}(f) \big)(x) du.$$

The idea is now to reproduce this procedure for $P_{t,u}$ applied to $\mathcal{H}_{u,T}(f)$. This recursively yields the formal representation:

$$P_{t,T} f(x) = \tilde{P}_{t,T} f(x) + \sum_{r \geq 1} \int_t^T du_1 \int_t^{u_1} du_2 \ldots \int_t^{u_{r-1}} du_r \tilde{P}_{t,u_r} \big(\mathcal{H}_{u_r,u_{r-1}} \circ \cdots \circ \mathcal{H}_{u_1,T} \big)(f)(x).$$

Equation (3.9) then formally follows from the following identification. For all $r \in \mathbb{N}^*$:

$$\int_t^T du_1 \int_t^{u_1} du_2 \ldots \int_t^{u_{r-1}} du_r \tilde{P}_{t,u_r} \big(\mathcal{H}_{u_r,u_{r-1}} \circ \cdots \circ \mathcal{H}_{u_1,T} \big)(f)(x) du = \int_{\mathbb{R}^{nd}} dy f(y) \tilde{p}_\alpha \otimes H^{(r)}(t,T,x,y) dy.$$

We can proceed by immediate induction:

$$\begin{aligned}
&\int_t^T du_1 \int_t^{u_1} du_2 \dots \int_t^{u_{r-1}} du_r \tilde{P}_{t,u_r}\big(\mathcal{H}_{u_r,u_{r-1}} \circ \cdots \circ \mathcal{H}_{u_1,T}\big)(f)(x) du \\
= &\int_t^T du_1 \int_t^{u_1} du_2 \dots \int_t^{u_{r-1}} du_r \int_{\mathbb{R}^{nd}} dz \mathcal{H}_{u_r,u_{r-1}} \circ \cdots \circ \mathcal{H}_{u_1,T}(f)(z) \tilde{p}_\alpha(t,u_r,x,z) \qquad (3.16) \\
\overset{(3.15)}{=} &\int_t^T du_1 \int_t^{u_1} du_2 \dots \int_t^{u_{r-1}} du_r \int_{\mathbb{R}^{nd}} dz \int_{\mathbb{R}^{nd}} dy \mathcal{H}_{u_{r-1},u_{r-2}} \circ \cdots \circ \mathcal{H}_{u_1,T}(f)(y) H(u_r,u_{r-1},z,y) \tilde{p}_\alpha(t,u_r,x,z) \\
= &\int_t^T du_1 \int_t^{u_1} du_2 \dots \int_t^{u_{r-2}} du_{r-1} \int_{\mathbb{R}^{nd}} dy \mathcal{H}_{u_{r-1},u_{r-2}} \circ \cdots \circ \mathcal{H}_{u_1,T}(f)(y) \tilde{p}_\alpha \otimes H(t,u_{r-1},x,y).
\end{aligned}$$

Thus, we can iterate the procedure from (3.16) with $\tilde{p}_\alpha \otimes H$ instead of $\tilde{p}_\alpha$.

□

Observe that in order to make the identification above, we have exchanged various integrals. Hence, so far the representation (3.11) is *formal*. It will become rigorous provided that we manage to show the convergence of the series and get integrable bounds on the sum of the series. To answer these queries, one needs to give precise bounds on the iterated time-space convolutions appearing in the series. Such controls are stated in Section 3.3 and proved in Section 6 below.

3.3 Controls on the iterated kernels

From now on, we assume w.l.o.g. that $0 < T \leq T_0 := T_0(\textbf{[H]}) \leq 1$. The choice of T_0 depends on the constants appearing in **[H]** and will be clear from the proof of Lemma 5.1. Theorems 2.1 and 2.2 can anyhow be obtained for an arbitrary fixed finite $T > 0$, from the results for T sufficiently small. Indeed, the uniqueness of the martingale problem simply follows from the Markov property whereas the upper density estimate stems from the *semigroup* property of $\bar{p}_\alpha$ (see Corollary 3.4 and Lemma 3.11 for the convolutions involving the logarithmic correction). From now on, we consider that the threshold $K > 0$ appearing in Lemma 3.3 is fixed.

We first give pointwise results on the convolution kernel, that hold in any dimension d, and for any number of oscillators n.

Lemma 3.6 (Control of the kernel). *There exists constants $C_{3.6} := C_{3.6}(\textbf{[H]}, K)$, $\delta := \delta(\textbf{[H]}) > 0$ s.t. for all $T \in (0, T_0]$ and $(t,x,y) \in [0,T) \times (\mathbb{R}^{nd})^2$:*

$$|H(t,T,x,y)| \leq C_{3.6} \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha \wedge 1)}}{T-t} \tilde{p}_\alpha^{T,y}(t,T,x,y). \qquad (3.17)$$

Once integrated in space, this pointwise estimate yields the following *smoothing* property in time.

Lemma 3.7. *There exists $C_{3.7} := C_{3.7}(\textbf{[H]}, K)$ s.t. for all $T \in (0, T_0], (x,y) \in (\mathbb{R}^{nd})^2$, $\tau \in [t,T)$, we have the estimate*

$$\begin{aligned}
\int_{\mathbb{R}^{nd}} \delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha^{T,y}(\tau,T,z,y) dz &\leq C_{3.7}(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)}, \qquad (3.18) \\
\int_{\mathbb{R}^{nd}} \delta \wedge |z - R_{\tau,t,}(x)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha^{T,y}(t,\tau,x,z) dz &\leq C_{3.7}(\tau-t)^{\eta(\frac{1}{\alpha}\wedge 1)}. \qquad (3.19)
\end{aligned}$$

The proof of these results will be given in Section 5.3 and Appendix B.

Remark 3.3. We can now justify from this Lemma our previous choice for the *proxy* model. Indeed, the contributions $|z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}$, $|z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)}$ come from the difference of the generators and turn out to be compatible, up to using the Lipschitz property of the flow, with the bounds appearing in Proposition 3.3 for the frozen density. This is what gives this smoothing property and thus allows to get rid-off the diagonal singularities coming from the bound (3.17).

The technical computations in Section 6, based on the previous controls on the kernel H, yield the following bound for the first step of the parametrix procedure.

Lemma 3.8. *There exist $C_{3.8} := C_{3.8}(\textbf{[H]}, K)$, $\omega := \omega(\textbf{[H]}) \in (0,1]$ s.t. for all $T \in (0, T_0]$ and $(t,x,y) \in [0,T) \times (\mathbb{R}^{nd})^2$:*

$$\begin{aligned}
|\tilde{p}_\alpha \otimes H(t,T,x,y)| \leq C_{3.8} \tilde{p}_\alpha(t,T,x,y) \Big(&(T-t)^\omega + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \\
&+ \log(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|) \Big).
\end{aligned}$$

Remark 3.4. We point out that this control is not sufficient to establish the convergence of the series (3.11). The additional logarithmic contribution prevents indeed from getting a smoothing effect in time at each iteration. The first term in the above r.h.s. regularizes, the second one also, but at the next iteration step thanks to Lemma 3.7. The last one would appear as such at each iteration. Actually, this contribution comes from the fact that, when the frozen density is in the off-diagonal regime through one of its first $n_0 := n_0(\alpha, d, n)$ components, i.e. $\exists i \in [\![1, n_0]\!]$, *s.t.* $\frac{|(R_{T,t}x-y)_i|}{(T-t)^{(i-1)+1/\alpha}} \asymp |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}x - y)| \geq K$, we are led to integrate in time the contribution $(T-\tau)^{-1}$ in (3.17) up to a time $\tau_0 = T - |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}x-y)|^\gamma$ for $\gamma > 0$ to be specified later on (see the proof of Lemma 6.2). This is a specific feature of our multi-scale framework. When a *slow* component dominates there are cases for which the smoothing effect in time coming from the intrinsic time scales of the components is not enough to compensate the faster diagonal decay. If a *fast* component dominates in $|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}x-y)|$, the analysis is similar to the non-degenerate case of Kolokoltsov [14]. We mention that this problem occurs when considering the *tails* of the density and would be very likely to disappear adding some integrability at infinity considering for instance tempered stable processes.

Up to the end of section we restrict to the case $d = 1$ and $n = 2$, for which we have been able to refine the above results and to derive the convergence of (3.11). This restriction will be discussed thoroughly in Section 6.

Lemma 3.9 (**Control of the iterated kernels**). *There exist $C_{3.9} := C_{3.9}(\textbf{\textit{[H]}}, K)$, $\omega := \omega(\textbf{\textit{[H]}}) \in (0,1]$ s.t. for all $T \leq T_0$ and $(t,x,y) \in [0,T) \times (\mathbb{R}^2)^2$:*

$$\begin{aligned}
|\tilde{p}_\alpha \otimes H(t,T,x,y)| &\leq C_{3.9}\Big((T-t)^\omega \tilde{p}_\alpha(t,T,x,y) + \bar{q}_\alpha(t,T,x,y)\Big),\\
|\bar{q}_\alpha \otimes H(t,T,x,y)| &\leq C_{3.9}(T-t)^\omega\Big(\tilde{p}_\alpha(t,T,x,y) + \bar{q}_\alpha(t,T,x,y)\Big),
\end{aligned}$$

where we denoted

$$\bar{q}_\alpha(t,T,x,y) = \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}\tilde{p}_\alpha(t,T,x,y)\Big(1 + \log(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|)\ \Big).$$

Now for all $k \geq 1$,

$$\begin{aligned}
|\tilde{p}_\alpha \otimes H^{(2k)}(t,T,x,y)| &\leq (4C_{3.9})^{2k}(T-t)^{k\omega}\Big((T-t)^{k\omega}\tilde{p}_\alpha(t,T,x,y) + (\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\Big),\\
|\tilde{p}_\alpha \otimes H^{(2k+1)}(t,T,x,y)| &\leq (4C_{3.9})^{2k+1}(T-t)^{k\omega}\Big((T-t)^{(k+1)\omega}\tilde{p}_\alpha + (T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha) + \bar{q}_\alpha\Big)(t,T,x,y).
\end{aligned}$$

The above controls allow to derive under the sole assumption **[H]** an upper bound for the sum of the parametrix series (3.11) in small time.

Proposition 3.10 (**Sum of the parametrix series**). *Assume **[H]** is in force. There exists $\bar{T}_0 := \bar{T}_0(\textbf{\textit{[H]}}) \leq T_0$, $C_{3.10} := C_{3.10}(\textbf{\textit{[H]}}, K, \bar{T}_0)$ s.t. for all $T \in (0, \bar{T}_0]$ and $(t,x,y) \in [0,T) \times (\mathbb{R}^2)^2$:*

$$\begin{aligned}
\sum_{r\geq 0} |\tilde{p}_\alpha \otimes H^{(r)}(t,T,x,y)| &\leq C_{3.10}\Big(\tilde{p}_\alpha(t,T,x,y) + \bar{q}_\alpha(t,T,x,y)\Big),\\
C_{3.10}\det(\mathbb{T}^\alpha_{T-t})^{-1} &\leq \sum_{r\geq 0}\tilde{p}_\alpha \otimes H^{(r)}(t,T,x,y), \text{ for } |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x) - y)| \leq K.
\end{aligned}$$

The proofs of Lemmas 3.6 and 3.9 are postponed to Section 6.4. Using those controls on the iterated convolutions, we can prove Proposition 3.10.

Proof. Let us write:

$$\sum_{r\geq 0} |\tilde{p}_\alpha \otimes H^{(r)}|(t,T,x,y) = \sum_{k\geq 0} |\tilde{p}_\alpha \otimes H^{(2k)}(t,T,x,y)| + \sum_{k\geq 0} |\tilde{p}_\alpha \otimes H^{(2k+1)}(t,T,x,y)|.$$

Plugging for each sum the corresponding estimate of Lemma 3.9 yields:

$$\begin{aligned}
\sum_{k\geq 0} |\tilde{p}_\alpha \otimes H^{(2k)}(t,T,x,y)| &\leq \sum_{k\geq 0} C_{3.9}^{2k}(T-t)^{k\omega}\Big((T-t)^{k\omega}\tilde{p}_\alpha(t,T,x,y) + (\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\Big)\\
&\leq \tilde{p}_\alpha(t,T,x,y)\frac{1}{1 - C_{3.9}^2(T-t)^{2\omega}} + (\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\frac{1}{1 - C_{3.9}^2(T-t)^\omega},
\end{aligned}$$

$$\begin{aligned}\sum_{k\geq 0}|\tilde{p}_\alpha\otimes H^{(2k+1)}(t,T,x,y)| &\leq \sum_{k\geq 0} C_{3.9}^{2k+1}(T-t)^{k\omega}\Big((T-t)^{(k+1)\omega}\tilde{p}_\alpha+(T-t)^{\omega}(\tilde{p}_\alpha+\bar{q}_\alpha)+\bar{q}_\alpha\Big)(t,T,x,y)\\ &\leq \tilde{p}_\alpha(t,T,x,y)\frac{C_{3.9}(T-t)^{\omega}}{1-C_{3.9}^2(T-t)^{2\omega}}+(\tilde{p}_\alpha+\bar{q}_\alpha(t,T,x,y)\frac{C_{3.9}(T-t)^{\omega}}{1-C_{3.9}^2(T-t)^{\omega}}\\ &\quad +\bar{q}_\alpha(t,T,x,y)\frac{C_{3.9}}{1-C_{3.9}^2(T-t)^{\omega}},\end{aligned}$$

using that $C_{3.9}^2(\bar{T}_0)^\omega<1$. To get the diagonal lower bound, we first write:

$$\sum_{k\geq 0}\tilde{p}_\alpha\otimes H^{(k)}(t,T,x,y)=\tilde{p}_\alpha(t,T,x,y)+\left(\sum_{k\geq 0}\tilde{p}_\alpha\otimes H^{(k)}\right)\otimes H(t,T,x,y).$$

Now, since

$$\sum_{k\geq 0}|\tilde{p}_\alpha\otimes H^{(k)}(t,T,x,y)|\leq C(\tilde{p}_\alpha+\bar{q}_\alpha)(t,T,x,y),$$

we derive:

$$\left|\left(\sum_{k\geq 0}\tilde{p}_\alpha\otimes H^{(k)}\right)\otimes H(t,T,x,y)\right|\leq C|(\tilde{p}_\alpha+\bar{q}_\alpha)\otimes H(t,T,x,y)|.$$

Using once again the first part of Lemma 3.9, we thus get that

$$\left|\left(\sum_{k\geq 0}\tilde{p}_\alpha\otimes H^{(k)}\right)\otimes H(t,T,x,y)\right|\leq C\left\{(T-t)^\omega\tilde{p}_\alpha(t,T,x,y)+\bar{q}_\alpha(t,T,x,y)+(T-t)^\omega(\tilde{p}_\alpha+\bar{q}_\alpha)(t,T,x,y)\right\}.$$

Now, if the global regime is diagonal, i.e. $|(\mathbb{T}^\alpha_{T-t})^{-1}(y-R_{T,t}(x))|\leq K$, the logarithm contribution vanishes in $\bar{q}_\alpha$. Observe also that

$$\begin{aligned}\delta\wedge|x-R_{t,T}(y)|^{\eta(\alpha\wedge 1)} &\leq C^{\eta(\alpha\wedge 1)}|R_{T,t}(x)-y|^{\eta(\alpha\wedge 1)}\leq C^{\eta(\alpha\wedge 1)}(T-t)^{\eta(1/\alpha\wedge 1)}|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x)-y)|^{\eta(\alpha\wedge 1)}\\ &\leq (CK)^{\eta(\alpha\wedge 1)}(T-t)^{\eta(1/\alpha\wedge 1)}.\end{aligned}$$

Hence $\left|\left(\sum_{k\geq 0}\tilde{p}_\alpha\otimes H^{(k)}\right)\otimes H(t,T,x,y)\right|\leq C(T-t)^\omega\det(\mathbb{T}^\alpha_{T-t})^{-1}$. Taking $T-t$ small enough yields the announced bound. □

The additional logarithmic contribution prevents us from deriving two-sided global bounds as in the non-degenerate case of [14]. We conclude anyhow the section stating a Lemma that allows to extend the upper bound in Theorem 2.2 to an arbitrary given fixed time. The arguments for its proof would be similar to those of Lemma 6.3.

Lemma 3.11 (**Semigroup property for $\bar{q}_\alpha$**). *With the notations of Proposition 3.10, for any $T\in[0,\bar{T}_0)$, we have that there exists $C_{3.11}:=C_{3.11}([\mathbf{H}],\bar{T}_0)\geq 1$ s.t.:*

$$\forall(x,y)\in\mathbb{R}^{nd},\ \forall n\in\mathbb{N},\ \int_{\mathbb{R}^{nd}}\bar{q}_\alpha(0,nT,x,z)\bar{q}_\alpha(nT,(n+1)T,z,y)dz\leq C_{3.11}^{n+2}\bar{q}(0,(n+1)T,x,y).$$

Observe now that Theorem 2.1 yields that $(X_t)_{t\geq 0}$, the canonical process of $\mathbb{P}$, admits a Feller transition function. On the other hand, when $d=1,n=2$ we have from Proposition 3.10 that the series appearing in equation (3.9) of Proposition 3.5 is absolutely convergent. This allows to derive that the Feller transition is absolutely continuous, which in particular means that the process $(X_t)_{t\geq 0}$ admits for all $t>0$ a density, satisfying the bounds of Proposition 3.10.

4 Proof of the uniqueness of the Martingale Problem associated with (1.1).

In this section, d and n are arbitrary integers. As a corollary to the bounds of Section 3.3, specifically Lemmas 3.6 and 3.7 (controls on the kernel and associated smoothing effect), we prove here Theorem 2.1. The existence of a solution to the martingale problem can be derived by compactness arguments adapting the proof of Theorem 2.2 from [30], even though our coefficients are not bounded.

Uniqueness of the Martingale Problem associated with (1.3). Suppose we are given two solutions $\mathbb{P}^1$ and $\mathbb{P}^2$ of the martingale problem associated to $(L_s)_{s\in[t,T]}$, starting in x at time t. We can assume w.l.o.g. that $T \leq T_0 := T_0(\mathbf{[H]})$. Define for a bounded Borel function $f : [0,T] \times \mathbb{R}^{nd} \to \mathbb{R}$,

$$S^i f = \mathbb{E}^i \left(\int_t^T f(s, X_s) ds \right), \ i \in \{1,2\},$$

where $(X_s)_{s\in[t,T]}$ stands for the canonical process associated with $(\mathbb{P}^i)_{i\in\{1,2\}}$. Let us specify that $S^i f$ is *a priori* only a linear functional and not a function since $\mathbb{P}^i$ does not need to come from a Markov process. We denote:

$$S^\Delta f = S^1 f - S^2 f.$$

If $f \in C_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$, since $(\mathbb{P}^i)_{i\in\{1,2\}}$ both solve the martingale problem, we have:

$$f(t,x) + \mathbb{E}^i \left(\int_t^T (\partial_s + L_s) f(s, X_s) ds \right) = 0, \ i \in \{1,2\}. \tag{4.1}$$

For a fixed point $y \in \mathbb{R}^{nd}$ and a given $\varepsilon \geq 0$, introduce now for all $f \in C_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$ the Green function:

$$\forall (t,x) \in [0,T) \times \mathbb{R}^{nd}, G^{\varepsilon,y} f(t,x) = \int_t^T ds \int_{\mathbb{R}^{nd}} dz \tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,z) f(s,z).$$

We recall here that $\tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,z)$ stands for the density at time s and point z of the process $\tilde{X}^{s+\varepsilon,y}$ defined in (3.2) starting from x at time t. In particular, ε can be equal to zero in the previous definition. One now easily checks that:

$$\forall (t,x,z) \in [0,s) \times (\mathbb{R}^{nd})^2, \left(\partial_t + \tilde{L}_t^{s+\varepsilon,y}\right) \tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,z) = 0, \lim_{s\downarrow t} \tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,\cdot) = \delta_x(.). \tag{4.2}$$

Introducing for all $f \in C_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$ the quantity:

$$M_{t,x}^{\varepsilon,y} f(t,x) = \int_t^T ds \int_{\mathbb{R}^{nd}} dz \tilde{L}_t^{s+\varepsilon,y} \tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,z) f(s,z), \tag{4.3}$$

we derive from (4.2) and the definition of $G^{\varepsilon,y}$ that the following equality holds:

$$\partial_t G^{\varepsilon,y} f(t,x) + M_{t,x}^{\varepsilon,y} f(t,x) = -f(t,x), \ \forall (t,x) \in [0,T) \times \mathbb{R}^{nd}. \tag{4.4}$$

Now, let $h \in C_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$ be an arbitrary function and define for all $(t,x) \in [0,T) \times \mathbb{R}^{nd}$:

$$\phi^{\varepsilon,y}(t,x) := \tilde{p}_\alpha^{t+\varepsilon,y}(t,t+\varepsilon,x,y) h(s,y), \Psi_\varepsilon(t,x) := \int_{\mathbb{R}^{nd}} dy G^{\varepsilon,y}(\phi^{\varepsilon,y})(t,x).$$

Then, by semigroup property, we have:

$$\begin{aligned}
\Psi_\varepsilon(t,x) &= \int_{\mathbb{R}^{nd}} dy \int_t^T ds \int_{\mathbb{R}^{nd}} dz \tilde{p}_\alpha^{s+\varepsilon,y}(t,s,x,z) \tilde{p}_\alpha^{s+\varepsilon,y}(s,s+\varepsilon,z,y) h(s,y) \\
&= \int_{\mathbb{R}^{nd}} dy \int_t^T ds \tilde{p}_\alpha^{s+\varepsilon,y}(t,s+\varepsilon,x,y) h(s,y).
\end{aligned}$$

Hence,

$$\begin{aligned}
(\partial_t + L_t)\Psi_\varepsilon(t,x) &= \int_{\mathbb{R}^{nd}} dy (\partial_t + L_t)(G^{\varepsilon,y}\phi^{\varepsilon,y})(t,x) \\
&= \int_{\mathbb{R}^{nd}} dy \{\partial_t G^{\varepsilon,y}\phi^{\varepsilon,y}(t,x) + M_{t,x}^{\varepsilon,y}\phi^{\varepsilon,y}(t,x)\} \\
&\quad + \int_{\mathbb{R}^{nd}} dy \{L_t G^{\varepsilon,y}\phi^{\varepsilon,y}(t,x) - M_{t,x}^{\varepsilon,y}\phi^{\varepsilon,y}(t,x)\} \\
&\overset{(4.4)}{=} -\int_{\mathbb{R}^{nd}} dy \phi^{\varepsilon,y}(t,x) + \int_{\mathbb{R}^{nd}} dy \{L_t G^{\varepsilon,y}\phi^{\varepsilon,y}(t,x) - M_{t,x}^{\varepsilon,y}\phi^{\varepsilon,y}(t,x)\} \\
&= I_1^\varepsilon + I_2^\varepsilon.
\end{aligned}$$

We now need the following lemma whose proof is postponed to the end of Section 5.2.

Lemma 4.1. *For all bounded continuous function $f : \mathbb{R}^{nd} \to \mathbb{R}, x \in \mathbb{R}^{nd}$:*

$$\left| \int_{\mathbb{R}^{nd}} f(y) \tilde{p}_\alpha^{T,y}(t,T,x,y)dy - f(x) \right| \underset{T \downarrow t}{\longrightarrow} 0. \tag{4.5}$$

We emphasize that the above lemma is not a direct consequence of the convergence of the law of the frozen process towards the Dirac mass when $T \downarrow t$. Indeed, the integration parameter is also the freezing parameter which makes things more subtle. Lemma 4.1 yields $I_1^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} -h(t,x)$. On the other hand, we have the following identity:

$$\begin{aligned} I_2^\varepsilon &= \int_t^T ds \int_{\mathbb{R}^{nd}} dy (L_t - \tilde{L}_t^{s+\varepsilon}) \tilde{p}_\alpha^{s+\varepsilon,y}(t, s+\varepsilon, x, y) h(s,y) \\ &= \int_t^T ds \int_{\mathbb{R}^{nd}} dy H(t, s+\varepsilon, x, y) h(s,y). \end{aligned}$$

The bound of Lemma 3.7 now yields:

$$\begin{aligned} |I_2^\varepsilon| &\leq C \int_t^T ds \int_{\mathbb{R}^{nd}} dy \frac{\delta \wedge |x - R_{t,s+\varepsilon}(y)|^{\eta(\alpha \wedge 1)}}{s+\varepsilon-t} \tilde{p}_\alpha^{s+\varepsilon,y}(t, s+\varepsilon, x, y)|h(s,y)| \\ &\leq C|h|_\infty \int_t^T (s+\varepsilon-t)^{\eta(\frac{1}{\alpha} \wedge 1)-1} ds \leq C|h|_\infty [(T-t) \vee \varepsilon]^{\eta(\frac{1}{\alpha} \wedge 1)}. \end{aligned}$$

Hence, we may choose T and ε small enough to obtain

$$|I_2^\varepsilon| \leq 1/2 |h|_\infty. \tag{4.6}$$

Observe now that (4.1) gives $S^\Delta\Big((\partial_\cdot + L_\cdot)\Psi_\varepsilon\Big) = 0$ so that $|S^\Delta(I_1^\varepsilon)| = |S^\Delta(I_2^\varepsilon)|$. From Lemma 4.1 and (4.6) we derive:

$$|S^\Delta h| = \lim_{\varepsilon \to 0} |S^\Delta I_1^\varepsilon| = \lim_{\varepsilon \to 0} |S^\Delta I_2^\varepsilon| \leq \|S^\Delta\| \limsup_{\varepsilon \to 0} |I_2^\varepsilon| \leq 1/2 \|S^\Delta\| |h|_\infty, \ \|S^\Delta\| := \sup_{|f|_\infty \leq 1} |S^\Delta f|.$$

By a monotone class argument, the previous inequality still holds for bounded Borel functions h compactly supported in $[0,T) \times \mathbb{R}^{nd}$. Taking the supremum over $|h|_\infty \leq 1$ leads to $\|S^\Delta\| \leq 1/2 \|S^\Delta\|$. Since $\|S^\Delta\| \leq T-t$, we deduce that $\|S^\Delta\| = 0$ which proves the result on $[0,T]$. Regular conditional probabilities allow to extend the result on $\mathbb{R}^+$, see e.g. Theorem 4, Chapter II, §7, in [29], see also Chapter 6 in [31] and [30].

□

5 Proof of the results involving the Frozen process.

Introduce for a given $t > 0$ and all $s \geq t$ the process:

$$\Lambda_s := \int_t^s R_{s,u} B \sigma_u dZ_u, \tag{5.1}$$

solving $d\Lambda_s = A_s \Lambda_s ds + B\sigma_s dZ_s$, $Z_t = 0$, i.e. Λ_s can be viewed as the process of the iterated integrals of Z weighted by the entries of the resolvent. In (5.1), $(\sigma_u)_{u \geq t}$ is a deterministic $\mathbb{R}^d \otimes \mathbb{R}^d$-valued function s.t. $(\sigma_u \sigma_u^*)_{u \geq t}$ satisfies **[H-3]** (uniform ellipticity). It can be seen from Proposition 3.1 that the frozen process will have a density if and only if Λ does for $s > t$. This is what we establish through Fourier inversion. The structure of the resolvent is crucial: it gives the multi-scale behaviour of the frozen process and allows to prove in Proposition 5.3 that the Fourier transform is integrable. Recalling as well that B stands for the embedding matrix from $\mathbb{R}^d$ into $\mathbb{R}^{nd}$, we observe that only the first d columns of the resolvent are taken into account in (5.1). Reasoning by blocks we rewrite: $R_{s,t} = \begin{pmatrix} R_{s,t}^{1,1} & \cdots & R_{s,t}^{1,n} \\ \vdots & \ddots & \vdots \\ R_{s,t}^{n,1} & \cdots & R_{s,t}^{n,n} \end{pmatrix}$, where the entries $(R_{s,t}^{i,j})_{(i,j) \in [\![1,n]\!]^2}$ belong to $\mathbb{R}^d \otimes \mathbb{R}^d$.

5.1 Analysis of the Resolvent.

Lemma 5.1 (**Form of the Resolvent**). *Let $0 \leq t \leq s \leq T \leq T_0 := T_0(\textbf{\textit{[H]}}) \leq 1$. We can write the first column of the resolvent in the following way:*

$$R^{\cdot,1}_{s,t} = \begin{pmatrix} \bar{R}^1_{s,t} \\ (s-t)\bar{R}^2_{s,t} \\ \vdots \\ \frac{(s-t)^{n-1}}{(n-1)!}\bar{R}^n_{s,t} \end{pmatrix}, \tag{5.2}$$

where the $(\bar{R}^i_{s,t})_{i\in[\![1,n]\!]}$ are non-degenerate and bounded matrices in $\mathbb{R}^d \otimes \mathbb{R}^d$, i.e. $\exists C := C(\textbf{\textit{[H]}}, T_0)$ s.t. for all $\xi \in S^{d-1}$, $C^{-1} \leq |\bar{R}^i_{s,t}\xi| \leq C$.

Proof. We are going to prove the result by induction. Let us first consider the case $n = 2$. We have, for $i \in \{1, 2\}$:

$$\frac{d}{ds}R^{1,1}_{s,t} = a^{1,1}_s R^{1,1}_{s,t} + a^{1,2}_s R^{2,1}_{s,t},\ \frac{d}{ds}R^{2,1}_{s,t} = a^{2,1}_s R^{1,1}_{s,t} + a^{2,2}_s R^{2,1}_{s,t}.$$

In order to obtain, for $i \in \{1, 2\}$, a semi-integrated representation of the entry $R^{i,1}_{s,t}$, we use the resolvent $\Gamma^i_{u,v}$ satisfying $\frac{d}{du}\Gamma^i_{u,v} = a^{i,i}_u \Gamma^i_{u,v},\ \Gamma^i_{v,v} = I_d$. This yields:

$$R^{1,1}_{s,t} = \Gamma^1_{s,t} + \int_t^s \Gamma^1_{s,u} a^{1,2}_u R^{2,1}_{u,t} du,\ R^{2,1}_{s,t} = \int_t^s \Gamma^2_{s,u}\left\{a^{2,1}_u R^{1,1}_{u,t}\right\} du.$$

Hence for all $0 \leq t \leq s \leq T$:

$$\begin{aligned} R^{1,1}_{s,t} &= \Gamma^1_{s,t} + \int_t^s \Gamma^1_{s,u} a^{1,2}_u \left\{\int_t^u \Gamma^2_{u,v}\left\{a^{2,1}_v R^{1,1}_{v,t}\right\} dv\right\} du, \\ |R^{1,1}_{s,t}| &\leq C_T(1 + \int_t^s |R^{1,1}_{v,t}|(s-t)dv) \leq C_T,\ |R^{2,1}_{s,t}| \leq C_T(s-t), \end{aligned}$$

using Gronwall's lemma for the last but one inequality. This in particular yields

$$R^{2,1}_{s,t} = \int_t^s \Gamma^2_{s,u} a^{2,1}_u (\Gamma^1_{u,t} + O((u-t)^2)) du.$$

From the non-degeneracy of $a^{2,1}$ (Hörmander like assumption **[H-4]**) and the resolvents on a compact set we derive that for T small enough $R^{2,1}_{s,t} = (t-s)\bar{R}^2_{s,t}$ where $\bar{R}^2_{s,t}$ is non-degenerate and bounded. Rewriting $R^{1,1}_{s,t} = \Gamma^1_{s,t} + O((s-t)^2)$ we derive similarly that $R^{1,1}_{s,t} = \bar{R}^1_{s,t}$, $\bar{R}^1_{s,t}$ being non-degenerate and bounded. This proves (5.2) for $n = 2$. Let us now assume that (5.2) holds for a given $n \geq 2$ and let us prove it for $n + 1$.

We first need to introduce some notations to keep track of the induction hypothesis. To this end, we denote by $A^{n+1}_t := A_t$ and $R^{n+1}_{s,t} := R_{s,t}$ the matrices in $\mathbb{R}^{(n+1)d} \otimes \mathbb{R}^{(n+1)d}$ associated with the linear system $\frac{d}{ds}R_{s,t} = A_t R_{s,t},\ R_{t,t} = I_{(n+1)d}$. Observe now that:

$$A^{n+1}_t = \left(\begin{array}{c|ccc} a^{1,1}_t & \cdots & \cdots & a^{1,n+1}_t \\ \hline a^{2,1}_t & & & \\ 0 & & & \\ \vdots & & A^n_t & \\ 0 & & & \end{array}\right),$$

where A^n_t is an $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ matrix satisfying **[H-4]**. Hence, denoting by $R^n_{s,t}$ the associated resolvent, i.e. $\frac{d}{ds}R^n_{s,t} = A^n_s R^n_{s,t},\ R^n_{t,t} = I_{nd}$, $R^n_{s,t}$ satisfies (5.2) from the induction hypothesis, so that

$$\forall i \in [\![1, n]\!],\ \forall 0 \leq t \leq s \leq T,\ (R^n_{s,t})^{i,1} = \frac{(s-t)^{i-1}}{(i-1)!}\bar{R}^{i,n}_{s,t},$$

where the $(\bar{R}^{i,n}_{s,t})_{i\in[\![1,n]\!]}$ are non-degenerate and bounded. Let us now observe that the differential dynamics of $(R^{n+1}_{s,t})^{2:n+1,1} := \left((R^{n+1}_{s,t})^{2,1}, \cdots, (R^{n+1}_{s,t})^{n+1,1}\right)^*$ writes:

$$\frac{d}{ds}(R^{n+1}_{s,t})^{2:n+1,1} = A^n_s (R^{n+1}_{s,t})^{2:n+1,1} + G^{n+1}_{s,t},\ G^{n+1}_{s,t} := \left(a^{2,1}_s (R^{n+1}_{s,t})^{1,1}\ 0_{n\times n}\ \cdots\ 0_{n\times n}\right)^*,$$

where

$$(R^{n+1}_{s,t})^{1,1} = \Gamma^{n+1,1}_{s,t} + \int_t^s \Gamma^{n+1,1}_{s,u} \left\{ \sum_{j=2}^{n+1} a^{1,j}_u (R^{n+1}_{u,t})^{j,1} \right\} du, \tag{5.3}$$

$\Gamma^{n+1,1}$ standing for the resolvent associated with a^{11}. Using now the resolvent $R^n_{s,t}$, the above equation can be integrated. We get:

$$\left(R^{n+1}_{s,t}\right)^{2:n+1,1} = \int_t^s R^n_{s,u} G^{n+1}_{u,t} du. \tag{5.4}$$

From the above representation, using the induction assumption, (5.3) and Gronwall's lemma we derive:

$$|(R^{n+1}_{s,t})^{n+1,1}| \leq C_T \int_t^s \frac{(s-u)^{n-1}}{(n-1)!} \left\{ 1 + \int_t^u \sum_{j=2}^{n} |(R^{n+1}_{v,t})^{j,1}| dv \right\} du.$$

By induction one also derives for all $i \in [\![2, n+1]\!]$:

$$|(R^{n+1}_{s,t})^{i,1}| \leq C_T \int_t^s \frac{(s-u)^{i-2}}{(i-2)!} \left\{ 1 + \int_t^u \sum_{j=2}^{i-1} |(R^{n+1}_{v,t})^{j,1}| dv \right\} du,$$

up to modifications of C_T at each step. These controls yield that for all $i \in [\![2, n]\!]$, $0 \leq t \leq s \leq T$:

$$|(R^{n+1}_{s,t})^{i,1}| = O((s-t)^{i-1}). \tag{5.5}$$

Now from (5.4), (5.3) and the induction assumption, we obtain, for all $i \in [\![2, n]\!]$, $0 \leq t \leq s \leq T$:

$$(R^{n+1}_{s,t})^{i,1} = \int_t^s \frac{(s-u)^{i-2}}{(i-2)!} \bar{R}^{i-1,n}_{s,u} a^{2,1}_u \{ \Gamma^{n+1,1}_{u,t} + \int_t^u \Gamma^{n+1,1}_{u,v} \{ \sum_{j=2}^{n+1} a^{1,j}_v (R^{n+1}_{v,t})^{j,1} dv \} \} du.$$

From the non degeneracy of $\bar{R}^{i-1,n}, a^{2,1}, \Gamma^{n+1,1}$ and (5.5), we can conclude as for the case $n = 2$. □

To conclude our analysis of the resolvent $R_{s,t}$, we give here a technical lemma that will be useful for the controls of Section 6.

Lemma 5.2 (**Scaling Lemma**). *Under **[H-4]**, the resolvent $(R_{s,T})_{s\in[t,T]}$, for $0 \leq t < T$ associated with the linear system $\frac{d}{ds} R_{s,T} = A_s R_{s,T}$, $R_{T,T} = I$ can be written as*

$$R_{s,T} = \mathbb{T}^\alpha_{T-t} \hat{R}^{t,T}_{\frac{s-t}{T-t}} (\mathbb{T}^\alpha_{T-t})^{-1},$$

where $\hat{R}^{t,T}_{\frac{s-t}{T-t}}$ is non-degenerate and bounded uniformly on $s \in [t, T]$ with constants depending on T.

Proof. The proof of the above statement follows from the structure of the matrix A_t (Assumption **H-4**), setting for all $u \in [0, 1]$, $\hat{R}^{t,T}_u := (\mathbb{T}^\alpha_{T-t})^{-1} R_{t+u(T-t),T} \mathbb{T}^\alpha_{T-t}$ and differentiating:

$$\begin{aligned} \partial_u \hat{R}^{t,T}_u &= (T-t)(\mathbb{T}^\alpha_{T-t})^{-1} A_{t+u(T-t)} R_{t+u(T-t),T} \mathbb{T}^\alpha_{T-t} \\ &= \left((T-t)(\mathbb{T}^\alpha_{T-t})^{-1} A_{t+u(T-t)} \mathbb{T}^\alpha_{T-t} \right) \hat{R}^{t,T}_u := A^{t,T}_u \hat{R}^{t,T}_u. \end{aligned}$$

□

Remark 5.1. Let us observe that the scaling Lemma already gives the right orders for the entries $(R^{i,1}_{t,s})_{i\in[\![1,n]\!]}$ of the resolvent. However for the analysis of the Fourier transform of Λ, we explicitly need that those entries write in the form of equation (5.2).

5.2 Estimates on the frozen density

5.2.1 Existence and first estimates

The main result of this section is the following.

Proposition 5.3 (**Existence of the density**). *Let $T_0 := T_0(\textbf{\textit{[H]}})$ be as in Lemma 5.1. The process $(\Lambda_s)_{s\in[t,t+T_0]}$, $t \geq 0$, defined in (5.1) has for all $s \in (t, t+T_0]$ a density p_{Λ_s} given for all $z \in \mathbb{R}^{nd}$ by:*

$$p_{\Lambda_s}(z) \quad = \quad \frac{\det(\mathbb{T}^{\alpha}_{s-t})^{-1}}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} e^{-i\langle q,(\mathbb{T}^{\alpha}_{s-t})^{-1}z\rangle} \exp\left(-\int_{S^{nd-1}} |\langle q,\eta\rangle|^{\alpha}\mu^*_S(d\eta)\right) dq,$$

*where $\mu^*_S := \mu^*_S(t,T,s,\sigma)$ is a symmetric measure on S^{nd-1} satisfying **[H-2]** uniformly in $s \in (t, t+T_0]$. As a consequence of this representation, we get the following global (diagonal) estimate:*

$$\exists C := C(\textbf{\textit{[H]}}, T_0),\ \forall s \in (t, t+T_0],\ \forall z \in \mathbb{R}^{nd},\ p_{\Lambda_s}(z) \leq C \det(\mathbb{T}^{\alpha}_{s-t})^{-1}. \tag{5.6}$$

Remark 5.2. The previous result emphasizes that the process $(\Lambda_s)_{s\in[t,t+T_0]}$ can actually be seen as an α-stable symmetric process in dimension nd, with non-degenerate spectral measure, (left) multiplied by the intrinsic scale factor $(\mathbb{T}^{\alpha}_{s-t})_{s\in[t,t+T_0]}$.

Proof. The proof is divided into two steps:

- The first step is to compute the Fourier transform.

Starting from the representation (5.1), we write the integral as a limit of its increments. Let $\tau_n := \{(t_i)_{i\in[\![0,n]\!]}; t = t_0 < t_1 < \cdots < t_n = s\}$ be a subdivision of $[t,s]$, whose mesh $|\tau_n| := \max_{i\in[\![0,n-1]\!]} |t_{i+1} - t_i|$ tends to zero when $n \to \infty$. Write now for all $p \in \mathbb{R}^{nd}$:

$$\langle p, \Lambda_s\rangle \quad = \quad \lim_{|\tau_n|\to 0} \sum_{i=0}^{n-1} \langle p, R_{s,t_i} B\sigma_{t_i}(Z_{t_{i+1}} - Z_{t_i})\rangle = \lim_{|\tau_n|\to 0} \sum_{i=0}^{n-1} \langle \sigma^*_{t_i} B^* R^*_{s,t_i} p, (Z_{t_{i+1}} - Z_{t_i})\rangle.$$

Recalling that μ is the spectral measure of Z which has independent increments, and using the bounded convergence theorem, we get that:

$$\forall p \in \mathbb{R}^{nd},\ \varphi_{\Lambda_s}(p) := \mathbb{E}(e^{i\langle p,\Lambda_s\rangle}) = \exp\left(-\int_t^s \int_{S^{d-1}} |\langle p, R_{s,u} B\sigma_u\varsigma\rangle|^{\alpha}\mu(d\varsigma)du\right). \tag{5.7}$$

- The second one is to prove its integrability.

Setting $v = (s-u)/(s-t)$ and denoting $u(v) := s - v(s-t)$, the exponent in (5.7) writes:

$$\int_t^s \int_{S^{d-1}} |\langle p, R_{s,u} B\sigma_u\varsigma\rangle|^{\alpha}\mu(d\varsigma)du = \int_0^1 \int_{S^{d-1}} |\langle p, (s-t)^{\frac{1}{\alpha}} R^{\cdot,1}_{s,u(v)}\sigma_{u(v)}\varsigma\rangle|^{\alpha}\mu(d\varsigma)dv.$$

Now, from Lemma 5.1, we have the identity

$$(s-t)^{\frac{1}{\alpha}} R^{\cdot,1}_{s,u(v)} = \mathbb{T}^{\alpha}_{s-t}\bar{R}_v,$$

setting with a slight abuse of notation $\bar{R}_v = \begin{pmatrix} \bar{R}^1_v \\ v\bar{R}^2_v \\ \vdots \\ \frac{v^{n-1}}{(n-1)!}\bar{R}^n_v \end{pmatrix}$, where the $(\bar{R}^k_v)_{k\in[\![1,n]\!]} \in \mathbb{R}^d \otimes \mathbb{R}^d$ are non-degenerate and bounded. The exponent in (5.7) thus rewrites:

$$\int_t^s \int_{S^{d-1}} |\langle p, R_{s,u} B\sigma_u\varsigma\rangle|^{\alpha}\mu(d\varsigma)du = \int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^{\alpha}_{s-t}p, \bar{R}_v\sigma_{u(v)}\varsigma\rangle|^{\alpha}\mu(d\varsigma)dv.$$

From the non degeneracy of μ and the uniform ellipticity of σ in assumptions **[H]**, we can conclude that:

$$\int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^{\alpha}_{s-t}p, \bar{R}_v\sigma_{u(v)}\varsigma\rangle|^{\alpha}\mu(d\varsigma)dv \geq C\int_0^1 |\sigma^*_{u(v)}\bar{R}^*_v\mathbb{T}^{\alpha}_{s-t}p|^{\alpha}dv \geq C\int_0^1 |\bar{R}^*_v\mathbb{T}^{\alpha}_{s-t}p|^{\alpha}dv.$$

The lower bound of the following lemma, whose proof is postponed to Subsection 5.2.3, gives that $\varphi_{\Lambda_s} \in L^1(\mathbb{R}^{nd})$ and therefore yields the existence of the density.

Lemma 5.4. *There exists a constant $C_{5.4} := C_{5.4}(\textbf{[H]}, T_0) > 0$, such that for all $s \in [t, t+T_0]$:*

$$\int_0^1 |\bar{R}_v^* \mathbb{T}^\alpha_{s-t} p|^\alpha dv \geq C_{5.4} |\mathbb{T}^\alpha_{s-t} p|^\alpha. \tag{5.8}$$

Since φ_{Λ_s} is integrable, we can write by Fourier inversion that for all $z \in \mathbb{R}^{nd}$:

$$p_{\Lambda_s}(z) = \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp e^{-i\langle p,z\rangle} \exp\left(- \int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^\alpha_{s-t} p, \bar{R}_v \sigma_{u(v)} \varsigma \rangle|^\alpha \mu(d\varsigma) dv \right). \tag{5.9}$$

We now define the function

$$\begin{array}{rccc} f: & [0,1] \times S^{d-1} & \longrightarrow & S^{nd-1} \\ & (v, \varsigma) & \longmapsto & \frac{\bar{R}_v \sigma_{u(v)} \varsigma}{|\bar{R}_v \sigma_{u(v)} \varsigma|}, \end{array}$$

and on $[0,1] \times S^{d-1}$ the measure:

$$m_\alpha(dv, d\varsigma) = |\bar{R}_v \sigma_{u(v)} \varsigma|^\alpha \mu(d\varsigma) dv.$$

The exponent in (5.9) thus rewrites:

$$\int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^\alpha_{s-t} p, \bar{R}_v \sigma_{u(v)} \varsigma \rangle|^\alpha \mu(d\varsigma) dv = \int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^\alpha_{s-t} p, f(v,\varsigma) \rangle|^\alpha m_\alpha(dv, d\varsigma).$$

Denoting by μ^* the image measure of m_α by f (which is a measure on S^{nd-1}), we have:

$$\int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^\alpha_{s-t} p, \bar{R}_v \sigma_{u(v)} \varsigma \rangle|^\alpha \mu(d\varsigma) dv = \int_{S^{nd-1}} |\langle \mathbb{T}^\alpha_{s-t} p, \eta \rangle|^\alpha \mu^*(d\eta).$$

Symmetrizing μ^* as follows: $\mu^*_S(A) = \frac{\mu^*(A) + \mu^*(-A)}{2}$, and using the fact that $\eta \mapsto |\langle \mathbb{T}^\alpha_{s-t} p, \eta \rangle|^\alpha$ is symmetric as well, we can write the exponent as:

$$\int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}^\alpha_{s-t} p, \bar{R}_v \sigma_{u(v)} \varsigma \rangle|^\alpha \mu(d\varsigma) dv = \int_{S^{nd-1}} |\langle \mathbb{T}^\alpha_{t-s} p, \eta \rangle|^\alpha \mu^*_S(d\eta).$$

Lemma 5.4 directly yields that μ^*_S satisfies **[H-2]**. Plugging this equality into (5.9) and setting $q = \mathbb{T}^\alpha_{t-s} p$ leads to the announced expression for the density. The global upper bound then readily follows.

□

5.2.2 Final derivation of the density bounds

Let us fix a threshold $K > 0$. Starting from the representation formula in Proposition (5.3), we can derive following the computations of [14] that for $(\Lambda_s)_{s \in [t, t+T_0]}$, $t \geq 0$ defined in (5.1), there exists $C := C(\textbf{[H]}, T_0, K) \geq 1$, s.t. for all $z \in \mathbb{R}^{nd}$:

$$\frac{C^{-1} \det(\mathbb{T}^\alpha_{s-t})^{-1}}{\{K \vee |\mathbb{T}^\alpha_{s-t} z|\}^{nd+\alpha}} \leq p_{\Lambda_s}(z) \leq \frac{C \det(\mathbb{T}^\alpha_{s-t})^{-1}}{\{K \vee |\mathbb{T}^\alpha_{s-t} z|\}^{nd+\alpha}}.$$

For the sake of completeness, we sketch the proof in Appendix A. The result follows directly from the asymptotic representation of Propositions A.1 and A.2 that respectively give the *diagonal* and *off-diagonal* expansions of the density.

From those expansions, considering for given $T \in (0, T_0], y \in \mathbb{R}^{nd}$, $\sigma_u := \sigma(u, R_{u,T}(y)), u \in [t, T]$ in the definition of $(\Lambda_s)_{s \in [t,T]}$ in (5.1), we derive from (3.2) that $X_T^{t,x,T,y} := R_{T,t} x + \Lambda_T$ so that $\tilde{p}_\alpha^{T,y}(t, T, x, y) = p_{\Lambda_T}(y - R_{T,t}(x))$. The above control then gives the important bounds of Proposition 3.3 in *small time*. The results can then be generalized for an arbitrary fixed $T > 0$ by convolution arguments. For $T = t + 2T_0$ we have:

$$\begin{aligned}
\tilde{p}_\alpha^{T,y}(t, t+2T_0, x, y) &= \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha^{T,y}(t, t+T_0, x, z) \tilde{p}_\alpha^{T,y}(t+T_0, t+2T_0, z, y) dz \\
&\overset{Prop.\ 3.3}{\asymp} \int_{\mathbb{R}^{nd}} \frac{\det(\mathbb{T}^\alpha_{T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{T_0})^{-1}(R_{t+T_0,t}(x) - z)|^{nd+\alpha}} \frac{\det(\mathbb{T}^\alpha_{T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{T_0})^{-1}(R_{t+2T_0,t+T_0}(z) - y)|^{nd+\alpha}} dz \\
&\overset{Lemma\ 5.2}{\asymp} \int_{\mathbb{R}^{nd}} \frac{\det(\mathbb{T}^\alpha_{T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{T_0})^{-1}(R_{t+T_0,t}(x) - z)|^{nd+\alpha}} \frac{\det(\mathbb{T}^\alpha_{T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{T_0})^{-1}(z - R_{t+T_0,t+2T_0}(y))|^{nd+\alpha}} dz \\
&\asymp \frac{\det(\mathbb{T}^\alpha_{2T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{2T_0})^{-1}(R_{t+T_0,t}(x) - R_{t+T_0,t+2T_0}(y))|^{nd+\alpha}} \\
&\overset{Lemma\ 5.2}{\asymp} \frac{\det(\mathbb{T}^\alpha_{2T_0})^{-1}}{K \vee |(\mathbb{T}^\alpha_{2T_0})^{-1}(R_{t+2T_0,t}(x) - y)|^{nd+\alpha}},
\end{aligned}$$

using stable convolution arguments for the last but one equivalence. The result follows by induction.

5.2.3 Proof of Lemma 5.4.

It is enough to show that there exists $C_{5.4} := C_{5.4}(\mathbf{[H]}, T_0)$, s.t. for any $\theta \in S^{nd-1}$, $\int_0^1 |\bar{R}_v^* \theta|^\alpha dv \geq C_{5.4}$. We define

$$\bar{C} := \inf_{\theta \in S^{nd-1}} \int_0^1 |\bar{R}_v^* \theta|^\alpha dv.$$

By continuity of the involved functions and compactness of S^{nd-1}, the infimum is actually a minimum. We need to show that this quantity is not zero. We proceed by contradiction. Assume that $\bar{C} = 0$. Then, there exists $\theta_0 \in S^{nd-1}$ such that for almost all $v \in [0,1]$, $|\bar{R}_v^* \theta_0| = 0$. But since $\bar{R}_v^*$ is a continuous function in v, the previous statement holds for all $v \in [0,1]$, i.e. $\exists \theta_0 \in S^{nd-1}, \forall v \in [0,1], |\bar{R}_v^* \theta_0| = 0$, or equivalently, that $\exists \theta_0 \in S^{nd-1}, \forall v \in [0,1], \theta_0 \in Ker(\bar{R}_v^*)$. Take now arbitrary $(v_i)_{i \in [\![1,n]\!]}$ in $[0,1]$. We have for each $i \in [\![1,n]\!]$:

$$\begin{pmatrix} (\bar{R}^1_{v_i})^* & v_i(\bar{R}^2_{v_i})^* & \cdots & \frac{v_i^n}{(n-1)!}(\bar{R}^n_{v_i})^* \end{pmatrix} \begin{pmatrix} \theta_0^1 \\ \vdots \\ \theta_0^n \end{pmatrix} = 0_{\mathbb{R}^d}.$$

This equivalently writes in matrix form:

$$\begin{pmatrix} (\bar{R}^1_{v_1})^* & v_1(\bar{R}^2_{v_1})^* & \cdots & \frac{v_1^n}{(n-1)!}(\bar{R}^n_{v_1})^* \\ \vdots & \vdots & & \vdots \\ (\bar{R}^1_{v_n})^* & v_n(\bar{R}^2_{v_n})^* & \cdots & \frac{v_n^n}{(n-1)!}(\bar{R}^n_{v_n})^* \end{pmatrix} \begin{pmatrix} \theta_0^1 \\ \vdots \\ \theta_0^n \end{pmatrix} = 0_{\mathbb{R}^{nd}}.$$

Now, taking $v_1 \to 0$ in the first line yields $(\bar{R}^1_{v_1})^* \theta_0^1 = 0_{\mathbb{R}^d}$. Since the $(\bar{R}^i_v)_{i \in [\![1,n]\!]}$ are from Lemma 5.1 non degenerate, we have that $\theta_0^1 = 0_{\mathbb{R}^d}$. Hence, the second line becomes:

$$v_2(\bar{R}^2_{v_2})^* \theta_0^2 + \cdots + \frac{v_2^n}{(n-1)!}(\bar{R}^n_{v_2})^* \theta_0^n = 0_{\mathbb{R}^d}.$$

Dividing by v_2, and taking $v_2 \to 0$, we get $(\bar{R}^2_{v_2})^* \theta_0^2 = 0_{\mathbb{R}^d}$. Hence, $\theta_0^2 = 0_{\mathbb{R}^d}$. By induction, we have that all components $\theta_0^i = 0_{\mathbb{R}^d}$, but this contradicts $\theta_0 \in S^{nd-1}$. This yields $\bar{C} := C_{5.4} > 0$, which concludes the proof. □

Remark 5.3. In the previous argument, the fact that the powers are increasing plays a key-role. Indeed, we rely on the multi-scale property reflected by the scale matrix $\mathbb{T}^\alpha$.

5.2.4 Proof of Lemma 4.1.

Let us write:

$$\begin{aligned} \int_{\mathbb{R}^{nd}} f(y) \tilde{p}_\alpha^{T,y}(t,T,x,y) dy - f(x) &= \int_{\mathbb{R}^{nd}} f(y) \Big(\tilde{p}_\alpha^{T,y}(t,T,x,y) - \tilde{p}_\alpha^{T,R_{T,t}(x)}(t,T,x,y) \Big) dy \\ &\quad + \int_{\mathbb{R}^{nd}} f(y) \Big(\tilde{p}_\alpha^{T,R_{T,t}(x)}(t,T,x,y) \Big) dy - f(x). \end{aligned}$$

From Proposition 3.2, the second term tends to zero as T tends to t. Let us discuss the first term. Define:

$$I = \int_{\mathbb{R}^{nd}} f(y) \Big(\tilde{p}_\alpha^{T,y}(t,T,x,y) - \tilde{p}_\alpha^{T,R_{T,t}(x)}(t,T,x,y) \Big) dy. \tag{5.10}$$

For a given threshold $K > 0$ and a certain $\beta > 0$ to be specified, we split $\mathbb{R}^{nd}$ into $D_1 \cup D_2$ where:

$$D_1 = \{ y \in \mathbb{R}^{nd}; |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \leq K(T-t)^{-\beta} \},$$

$$D_2 = \{ y \in \mathbb{R}^{nd}; |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| > K(T-t)^{-\beta} \}.$$

From Proposition 3.3, the two densities in (5.10) are equivalent to $\dfrac{\det(\mathbb{T}^\alpha_{T-t})^{-1}}{K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|^{nd+\alpha}}$. The idea is that on D_2 they are both in the *off-diagonal* regime so that tail estimates can be used. On the other hand,

we will explicitly exploit the compatibility between the spectral measures and the Fourier transform on D_1. Set for $i \in \{1,2\}$, $I_{D_i} := \int_{D_i} f(y)\Big(\tilde{p}_\alpha^{T,y}(t,T,x,y) - \tilde{p}_\alpha^{T,R_{T,t}(x)}(t,T,x,y)\Big)dy$. We derive:

$$|I_{D_2}| \le C|f|_\infty \int_{D_2} \frac{\det(\mathbb{T}_{T-t}^\alpha)^{-1}}{K \vee |(\mathbb{T}_{T-t}^\alpha)^{-1}(y - R_{T,t}(x))|^{nd+\alpha}} dy = C|f|_\infty \int_{K(T-t)^{-\beta}}^{+\infty} \frac{r^{nd-1}}{K \vee r^{nd+\alpha}} \le C(T-t)^{\beta\alpha}.$$

Thus, for $\beta > 0$, $I_{D_2} \underset{T\downarrow t}{\longrightarrow} 0$. On D_1, we will start from the inverse Fourier representation of $\tilde{p}_\alpha^{T,z}$ deriving from (5.9), for $z = y$ or $R_{T,t}(x)$. The Fourier exponent writes:

$$\forall (p,z) \in (\mathbb{R}^{nd})^2,\ F(p,z) = -\int_0^1 \int_{S^{d-1}} |\langle \mathbb{T}_{T-t}^\alpha p, \bar{R}_v \sigma(u(v), R_{u(v),T}(z))\varsigma\rangle|^\alpha \mu(d\varsigma)dv.$$

We thus rewrite:

$$\Big(\tilde{p}_\alpha^{T,y} - \tilde{p}_\alpha^{T,R_{T,t}(x)}\Big)(t,T,x,y) = \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y - R_{T,t}(x)\rangle} \Big((e^{F(p,y)} - e^{F(p,R_{T,t}(x))}\Big)$$

$$= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y - R_{T,t}(x)\rangle} \int_0^1 d\lambda \Big(F(p,y) - F(p,R_{T,t}(x))\Big) e^{(\lambda F(p,y) + (1-\lambda)F(p,R_{T,t}(x)))}.$$

The key point is now to observe that from **[H-3]** and the bound of Lemma 5.4, we have:

$$\forall (p,z) \in (\mathbb{R}^{nd})^2,\ F(p,z) \le -C_{5.4}|\mathbb{T}_{T-t}^\alpha p|^\alpha.$$

Hence, $\exp(\lambda F(p,y) + (1-\lambda)F(p,R_{T,t}(x))) \le \exp(-C_{5.4}|\mathbb{T}_{T-t}^\alpha p|^\alpha)$, independently on $\lambda \in [0,1]$. On the other hand, since σ is η-Hölder continuous in its second variable (see **[H-2]**), we have:

$$|F(p,y) - F(p,R_{T,t}(x))|$$

$$\le \int_0^1 \int_{S^{d-1}} \Big| |\langle \mathbb{T}_{T-t}^\alpha p, \bar{R}_v \sigma(u(v), R_{u(v),T}(y))\varsigma\rangle|^\alpha - |\langle \mathbb{T}_{T-t}^\alpha p, \bar{R}_v \sigma(u(v), R_{u(v),t}(x))\varsigma\rangle|^\alpha \Big| \mu(d\varsigma)dv$$

$$\le C|\mathbb{T}_{T-t}^\alpha p|^\alpha |y - R_{T,t}(x)|^{\eta(\alpha\wedge 1)},$$

using the Lipschitz property of the flow for the last inequality. To summarize, we get:

$$\begin{aligned} |I_{D_1}| &\le |f|_\infty \int_{D_1} dy \,|\tilde{p}_\alpha^{T,y}(t,T,x,y) - \tilde{p}_\alpha^{T,x}(t,T,x,y)| \\ &\le C \int_{D_1} dy \int_{\mathbb{R}^{nd}} dp |\mathbb{T}_{T-t}^\alpha p|^\alpha |y - R_{T,t}(x)|^{\eta(\alpha\wedge 1)} e^{-C5.4^{|\mathbb{T}_{T-t}^\alpha p|^\alpha}}. \end{aligned}$$

Changing variables, and integrating over p yields

$$\begin{aligned} |I_{D_1}| &\le C\det(\mathbb{T}_{T-t}^\alpha)^{-1} \int_{\{|(\mathbb{T}_{T-t}^\alpha)^{-1}(y - R_{T,t}(x))| \le K(T-t)^{-\beta}\}} dy |y - R_{T,t}(x)|^{\eta(\alpha\wedge 1)} \\ &\le C \int_{\{|Y| \le K(T-t)^{-\beta}\}} dY |\mathbb{T}_{T-t}^\alpha Y|^{\eta(\alpha\wedge 1)} \le C(T-t)^{\eta(1/\alpha\wedge 1) - \beta(nd + \eta(\alpha\wedge 1))}. \end{aligned}$$

Choosing now $\frac{\eta(1/\alpha\wedge 1)}{nd+\eta(\alpha\wedge 1)} > \beta > 0$ gives that $|I_{D_1}| \underset{T\downarrow t}{\longrightarrow} 0$, which concludes the proof. □

5.3 Estimates on the convolution kernel H.

In order to derive pointwise bounds on the kernel $H(t,T,x,y) := (L_t - \tilde{L}_t^{T,y})\tilde{p}_\alpha^{T,y}(t,T,x,y)$, it is convenient, since $\tilde{p}_\alpha^{T,y}$ is given in terms of Fourier inversion, to compute the symbols of the operators $L_t, \tilde{L}_t^{T,y}$. Precisely, we denote by $l_t(p,x)$ (resp. $\tilde{l}_t^{T,y}(p,x)$) the functions of $(p,x) \in (\mathbb{R}^{nd})^2$ s.t.

$$\begin{aligned} \forall \phi \in C_0^1(\mathbb{R}^{nd}),\ \forall x \in \mathbb{R}^{nd},\ L_t\phi(x) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp \exp(-i\langle p,x\rangle) l_t(p,x)\hat{\phi}(p), \\ \tilde{L}_t^{T,y}\phi(x) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp \exp(-i\langle p,x\rangle) \tilde{l}_t^{T,y}(p,x)\hat{\phi}(p). \end{aligned}$$

We refer to Jacob [12] for further properties of the symbols associated to an integro-differential operator. From usual properties of the (inverse) Fourier transform and the polar decomposition of the Lévy measure ν (see Section 2), we derive the following expressions.

Lemma 5.5. *Let $(p,x)\in(\mathbb{R}^{nd})^2$ be given. Recalling that B stands for the injection matrix of $\mathbb{R}^d$ into $\mathbb{R}^{nd}$, we have:*

$$\begin{aligned} l_t(p,x) &= \langle p, A_t x\rangle + \int_{\mathbb{R}^d} e^{-i\langle p, B\sigma(t,x)z\rangle} - 1 - i\frac{\langle p, B\sigma(t,x)z\rangle}{1+|z|^2}\nu(dz) \\ &= \langle p, A_t x\rangle - \int_{S^{d-1}} |\langle p^1, \sigma(t,x)\varsigma\rangle|^\alpha \mu(d\varsigma), \\ \tilde{l}_t^{T,y}(p,x) &= \langle p, A_t x\rangle + \int_{\mathbb{R}^d} e^{-i\langle p, B\sigma(t,R_{t,T}(y))z\rangle} - 1 - i\frac{\langle p, B\sigma(t,R_{t,T}(y))z\rangle}{1+|z|^2}\nu(dz) \\ &= \langle p, A_t x\rangle - \int_{S^{d-1}} |\langle p^1, \sigma(t,R_{t,T}(y))\varsigma\rangle|^\alpha \mu(d\varsigma). \end{aligned}$$

From Lemma 5.5 we rewrite:

$$\begin{aligned} H(t,T,x,y) &= \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y - R_{T,t}(x)\rangle}\left(\int_{S^{d-1}}\Big(|\langle p, B\sigma(t,R_{t,T}(y))\varsigma\rangle|^\alpha - |\langle p, B\sigma(t,x)\varsigma\rangle|^\alpha\Big)\mu(d\varsigma)\right) \\ &\quad\times\exp\left(-\int_t^T\int_{S^{d-1}}|\langle p, R^{1,\cdot}_{T,u}\sigma(u,R_{u,T}(y))\varsigma\rangle|^\alpha\mu(d\varsigma)\right) \\ &= \frac{\delta\wedge|R_{t,T}(y)-x|^{\eta(\alpha\wedge 1)}}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y - R_{T,t}(x)\rangle}\left(\int_{S^{d-1}}\frac{|\langle p, B\sigma(t,R_{t,T}(y))\varsigma\rangle|^\alpha - |\langle p, B\sigma(t,x)\varsigma\rangle|^\alpha}{\delta\wedge|R_{t,T}(y)-x|^{\eta(\alpha\wedge1)}}\mu(d\varsigma)\right) \\ &\quad\times\exp\left(-\int_t^T\int_{S^{d-1}}|\langle p, R^{1,\cdot}_{T,u}\sigma(u,R_{u,T}(y))\varsigma\rangle|^\alpha\mu(d\varsigma)\right), \end{aligned}$$

if $R_{t,T}(y)\neq x$, and where we can chose $\delta := H\wedge 2|\sigma|_\infty$ (H being the Hölder modulus of σ, see **[H-1]**).

Remark 5.4. Observe the interesting fact that since the drift is linear, it disappears in the difference of the generators.

Recalling that σ is a bounded, Hölder continuous function of its second variable (**[H-1]**, **[H-3]**), we also get:

$$\left|\int_{S^{d-1}}\frac{|\langle p, B\sigma(t,R_{t,T}(y))\varsigma\rangle|^\alpha - |\langle p, B\sigma(t,x)\varsigma\rangle|^\alpha}{\delta\wedge|R_{t,T}(y)-x|^{\eta(\alpha\wedge1)}}\mu(d\varsigma)\right| \leq \frac{C}{T-t}\{(T-t)|p^1|^\alpha\},$$

where p^1 stands for the d first entries of $p=(p^1,\cdots,p^n)\in\mathbb{R}^{nd}$. Formally, the contribution $(T-t)|p^1|^\alpha$ is homogeneous to the contributions associated with p^1 in the exponential (see **[H-3]** and Lemmas 5.1, 5.4). This explains heuristically why we obtain the control:

$$|H(t,T,x,y)|\leq C\frac{\delta\wedge|x-R_{t,T}(y)|^{\eta(\alpha\wedge1)}}{T-t}\tilde{p}_\alpha^{T,y}(t,T,x,y).$$

A precise proof is given in Appendix B. The technique is quite similar to the one giving the density bounds on the frozen density.

Remark 5.5. We emphasize here that we could also consider an additional bounded drift term in the first d components when $\alpha>1$. Denoting this term by $b:\mathbb{R}^+\times\mathbb{R}^{nd}\to\mathbb{R}^d$, we could still use the previous frozen process as proxy. Exploiting the above symbol representation, the additional term coming from the difference of the generators would write

$$\begin{aligned} \langle b(t,x), \nabla_{x_1}\tilde{p}_\alpha(t,T,x,y)\rangle &= \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y-R_{T,t}(x)\rangle}\langle b(t,x), p^1\rangle \\ &\quad\times\exp\left(-\int_t^T\int_{S^{d-1}}|\langle p, R^{1,\cdot}_{T,u}\sigma(u,R_{u,T}(y))\varsigma\rangle|^\alpha\mu(d\varsigma)\right), \end{aligned}$$

where ∇_{x_1} stands for the derivative w.r.t. to the first d components. Following the previous heuristics it can be observed that $|p_1|(T-t)^{1/\alpha}$ is homogeneous to the the contributions associated with p^1 in the exponential. This actually yields:

$$|\langle b(t,x), \nabla_{x_1}\tilde{p}_\alpha(t,T,x,y)\rangle|\leq\frac{|b|_\infty}{(T-t)^{1/\alpha}}\tilde{p}_\alpha(t,T,x,y),$$

which for $\alpha>1$ gives an integrable singularity in time.

6 Controls of the convolutions.

In this section we assume w.l.o.g. that $T \leq T_0 = T_0(\mathbf{[H]}) \leq 1$, as in Lemma 5.1. We first prove Lemma 3.7 that emphasizes how the spatial contribution in the r.h.s. of (3.17) yields, once integrated, a regularizing effect in time.

6.1 Proof of Lemma 3.7.

We prove the first estimate only, the other is obtained similarly. Let us naturally split the space according to the regimes of $\tilde{p}_\alpha^{T,y}$. With the notations of Proposition 3.3 we introduce the partition:

$$D_1 = \{z \in \mathbb{R}^{nd}; |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))| \leq K\},\ D_2 = \{z \in \mathbb{R}^{nd}; |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))| > K\}.$$

On D_1, the diagonal expansion holds for $\tilde{p}_\alpha^{T,y}$, that is, for $z \in D_1$ and recalling the definition of $\mathbb{T}^\alpha_{T-\tau}$ in Theorem 2.2:

$$\tilde{p}_\alpha^{T,y}(\tau, T, z, y) \leq C_{3.3} \det(\mathbb{T}^\alpha_{T-\tau})^{-1} = C_{3.3}(T-\tau)^{-d(n/\alpha + n(n-1)/2)}.$$

On the other hand, denoting by $\|\cdot\|$ the matricial norm, we have:

$$|z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \quad \leq \quad \|R_{\tau,T}\|^{\eta(\alpha\wedge 1)} \|\mathbb{T}^\alpha_{T-\tau}\|^{\eta(\alpha\wedge 1)} |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|^{\eta(\alpha\wedge 1)} \leq C(T-\tau)^{\eta(1/\alpha\wedge 1)},$$

where the last inequality follows from the boundedness of the resolvent on compact sets and the definition of $\mathbb{T}^\alpha_{T-\tau}$.

Besides, the Lebesgue measure of the set D_1 is bounded by $C \det(\mathbb{T}^\alpha_{T-\tau})$, compensating exactly the time singularity appearing in $\tilde{p}_\alpha^{T,y}$. In conclusion, we obtained on D_1:

$$\int_{D_1} \delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha^{T,y}(\tau, T, z, y) dz \leq (T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)}.$$

Similarly, for $z \in D_2$, the off-diagonal bound holds for $\tilde{p}_\alpha^{T,y}$, i.e.:

$$\tilde{p}_\alpha^{T,y}(\tau, T, z, y) \leq C \frac{\det(\mathbb{T}^\alpha_{T-\tau})^{-1}}{|(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|^{nd+\alpha}}.$$

From the Lipschitz property of the flow we derive $|z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \leq C|y - R_{T,\tau}(z)|^{\eta(\alpha\wedge 1)} \leq C(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|^{\eta(\alpha\wedge 1)}$. Hence setting $\xi := |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|$ we derive

$$\int_{D_2} \delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha^{T,y}(\tau, T, z, y) dz \leq C \int_{\xi > K} \big(\delta \wedge [(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \xi^{\eta(\alpha\wedge 1)}]\big) \xi^{nd-1} \frac{d\xi}{\xi^{nd+\alpha}}. \tag{6.1}$$

Observe now that for $\eta \in (0,1),\ \alpha > \eta(\alpha \wedge 1)$. Hence, we directly get $\int_{\xi>K} \big(\delta \wedge [(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \xi^{\eta(\alpha\wedge 1)}]\big) \frac{d\xi}{\xi^{1+\alpha}} \leq (T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \int_{\xi>K} \frac{d\xi}{\xi^{1+\alpha-\eta(\alpha\wedge 1)}} := C_2(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)}$. When $\alpha \leq \eta(\alpha \wedge 1)$, which happens for $\eta = 1,\ \alpha \leq 1$, we have to be more subtle. We refine the partition introducing:

$$D_{2,1} = \{\xi \in \mathbb{R}^{nd}; K \leq \xi \leq K(T-\tau)^{-1/\alpha}\},\ D_{2,2} = \{\xi \in \mathbb{R}^{nd}; \xi > K(T-\tau)^{-1/\alpha}\}.$$

On $D_{2,1}$, writing $\delta \wedge [(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \xi^{\eta(\alpha\wedge 1)}] \leq [(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \xi^{\eta(\alpha\wedge 1)}]$ we get: $(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \int_{\xi\in D_{2,1}} d\xi \xi^{\eta(\alpha\wedge 1)-\alpha-1} \leq C(T-\tau)$. On $D_{2,2}$, using $\delta \wedge [(T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)} \xi^{\eta(\alpha\wedge 1)}] \leq \delta$ we derive $\int_{\xi\in D_{2,2}} \frac{d\xi}{\xi^{1+\alpha}} \leq C_\delta(T-\tau)$. Plugging the above controls in (6.1) yields the result. □

A useful extension of the previous result is the following lemma involving an additional logarithmic contribution which is *explosive* in the off-diagonal regime. This anyhow does not affect *much* the smoothing effect.

Lemma 6.1. *For a given $\varepsilon \in (0, \alpha)$, there exists $C_{6.1} := C_{6.1}(\mathbf{[H]}, T_0, \varepsilon) > 0$ s.t. for all $T \in (0, T_0], (x,y) \in (\mathbb{R}^{nd})^2$, $\tau \in (t, T)$:*

$$\int_{\mathbb{R}^{nd}} \log(K \vee |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|) \big\{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}\big\} \tilde{p}_\alpha^{T,y}(\tau, T, z, y) dz \quad \leq \quad C_{6.1}(T-\tau)^{\eta(\frac{1}{\alpha-\varepsilon}\wedge 1)},$$

$$\int_{\mathbb{R}^{nd}} \log(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|) \big\{\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)}\big\} \tilde{p}_\alpha^{T,y}(t, \tau, x, z) dz \quad \leq \quad C_{6.1}(\tau-t)^{\eta(\frac{1}{\alpha-\varepsilon}\wedge 1)}.$$

Proof. The proof does not change much from the previous one. Observe also that, from the supremum in the logarithm, the only difference arises for *off-diagonal* regimes, that is, for $z \in D_2$ referring to the partition in the previous proof. The argument in the logarithm is however the same as the denominator of the *off-diagonal estimate*. Now, for any $\varepsilon \in (0,\alpha)$, there exists $C_\varepsilon > 0$ s.t. for all $\tau \in (t,T]$, $(x,z) \in (\mathbb{R}^{nd})^2$: $\log(K \vee |(\mathbb{T}_{\tau-t})^{-1}(z - R_{\tau,t}(x))|) \le C_\varepsilon \Big(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|\Big)^\varepsilon$. For $z \in D_2$, we get the following bound:

$$\log(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|)\tilde{p}^{T,y}_\alpha(t,\tau,x,z) \le \frac{\det(\mathbb{T}^\alpha_{\tau-t})^{-1}}{|(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|^{nd+\alpha-\varepsilon}}.$$

This bound allows to proceed as in the proof of Lemma 3.7. □

We now state a key lemma for our analysis. It gives a control for the first convolution between the frozen density $\tilde{p}_\alpha$ and the parametrix kernel H. The result differs here from the expected one: we get an additional logarithmic factor, w.r.t. the bounds established for this quantity in [8] for the Gaussian degenerate case, or [14] for the stable non-degenerate case.

Lemma 6.2 (First Step Convolution.). *There exist $C_{6.2} := C_{6.2}(\mathbf{[H]}) > 0$, $\omega := \omega(\mathbf{[H]}) \in (0,1]$ s.t. for all $T \in (0,T_0]$, $T_0 := T_0(\mathbf{[H]}) \le 1, (x,y) \in (\mathbb{R}^{nd})^2$, $t \in [0,T)$,*

$$|\tilde{p}_\alpha \otimes H|(t,T,x,y) \le C_{6.2}\tilde{p}_\alpha(t,T,x,y)\Big((T-t)^\omega + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} + \log(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|)\Big).$$

Suppose now that $n = 2$. We can then improve the previous bound and derive:

$$|\tilde{p}_\alpha \otimes H|(t,T,x,y) \quad \le \quad C_{6.2}\Big((T-t)^\omega \tilde{p}_\alpha(t,T,x,y) + \bar{q}_\alpha(t,T,x,y)\Big), \tag{6.2}$$

where we denote:

$$\bar{q}_\alpha(t,T,x,y) = \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}\tilde{p}_\alpha(t,T,x,y)\left(1 + \log\left[K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\right]\right).$$

Remark 6.1. The first part of the Lemma gives the bound of Lemma 3.8. Let us emphasize, as we have already mentioned in Remark 3.4, that this bound is not sufficient to derive the convergence of the parametrix series (3.11). The second control of the lemma might be seen as a *slight* improvement but is actually sufficient to imply the convergence when $d = 1, n = 2$. It gives the first statement in Lemma 3.9.

Proof. To perform the analysis, we first bound H using (3.17). We thus obtain:

$$|\tilde{p}_\alpha \otimes H|(t,T,x,y) \le C\int_t^T d\tau \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha(t,\tau,x,z)\frac{\delta \wedge |z - R_{\tau,T}(y)|^\eta}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y)dz. \tag{6.3}$$

For the proof it will be convenient to split the time interval $[t,T]$ into two subintervals $I_1 := [t, \frac{t+T}{2}], I_2 := [\frac{t+T}{2}, T]$. We observe that for $\tau \in I_1$, $T - \tau \asymp T - t$ whereas for $\tau \in I_2$, $\tau - t \asymp T - t$.

The leading idea for the proof is to partition the space in order to say that one of the two densities involved in (6.3) is homogeneous to the global one $\tilde{p}_\alpha(t,T,x,y)$, and to get some *regularization* from the other contribution, using thoroughly Lemma 3.7.

Diagonal Estimates. When the global diagonal regime holds, i.e. $|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x) - y)| \le K$, we will prove the following global diagonal estimate:

$$|\tilde{p}_\alpha \otimes H|(t,T,x,y) \le C\Big((T-t)^\omega + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}\Big)\tilde{p}_\alpha(t,T,x,y). \tag{6.4}$$

Indeed, on I_1, if $|(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))| \le K$, from Proposition 3.3 the diagonal estimate holds for $\tilde{p}_\alpha(\tau,T,z,y)$. Since $T - \tau \asymp T - t$, we have:

$$\tilde{p}_\alpha(\tau,T,z,y) \le C\det(\mathbb{T}^\alpha_{T-\tau})^{-1} \le C\det(\mathbb{T}^\alpha_{T-t})^{-1} \le C\tilde{p}_\alpha(t,T,x,y).$$

On the other hand, if $|(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))| > K$, the off-diagonal expansion holds for $\tilde{p}_\alpha(\tau,T,z,y)$ and from Proposition 3.3:

$$\tilde{p}_\alpha(\tau,T,z,y) \le C\frac{\det(\mathbb{T}^\alpha_{T-\tau})^{-1}}{|(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))|^{nd+\alpha}} \le C\det(\mathbb{T}^\alpha_{T-\tau})^{-1} \le C\det(\mathbb{T}^\alpha_{T-t})^{-1} \le C\tilde{p}_\alpha(t,T,x,y)^1.$$

[1] Observe that we could have used here that the diagonal control is a global bound. We introduced the dichotomy on the regime to emphasize that it is a crucial argument in this section.

Additionally, the boundedness of the resolvent yields:

$$|z - R_{\tau,T}(y)| \leq |z - R_{\tau,t}(x)| + |R_{\tau,t}(x) - R_{\tau,T}(y)| \leq C\Big(|z - R_{\tau,t}(x)| + |x - R_{t,T}(y)|\Big). \tag{6.5}$$

Denoting by $\otimes_{|I_1}$ the time-space convolution, where the time parameter is restricted to the interval I_1, we have from (6.3), (6.5) and Lemma 3.7:

$$\begin{aligned}
|\tilde{p}_\alpha \otimes_{|I_1} H|(t,T,x,y) &\leq C\tilde{p}_\alpha(t,T,x,y)\int_{I_1} d\tau \int_{\mathbb{R}^{nd}} \tilde{p}_\alpha(t,\tau,x,z)\left(\frac{\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)}}{\tau - t}\right.\\
&\quad \left. + \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}}{T-t}\right) dz\\
&\leq C\tilde{p}_\alpha(t,T,x,y)\int_{I_1} d\tau \left((\tau - t)^{\eta(\frac{1}{\alpha}\wedge 1)-1} + \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}}{T-t}\right)\\
&\leq C\tilde{p}_\alpha(t,T,x,y)((T-t)^\omega + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}).
\end{aligned} \tag{6.6}$$

Now, when $\tau \in I_2$, we have $\tilde{p}_\alpha(t,\tau,x,z) \leq \tilde{p}_\alpha(t,T,x,y)$, so that from Lemma 3.7:

$$\begin{aligned}
|\tilde{p}_\alpha \otimes_{|I_2} H|(t,T,x,y) &\leq C\tilde{p}_\alpha(t,T,x,y)\int_{I_2} d\tau \int_{\mathbb{R}^{nd}} \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y)dz\\
&\leq C\tilde{p}_\alpha(t,T,x,y)\int_{I_2} d\tau (T-\tau)^{\eta(\frac{1}{\alpha}\wedge 1)-1} \leq C(T-t)^\omega \tilde{p}_\alpha(t,T,x,y).
\end{aligned}$$

Thus, when the global diagonal estimate holds, the bound is true with $\omega = \eta(\frac{1}{\alpha} \wedge 1)$.

Off-Diagonal Estimates. We consider here the case $|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \geq K$, i.e. the *off-diagonal* estimate holds for $\tilde{p}_\alpha(t,T,x,y)$. In our current degenerate setting, several scales are involved in the term $|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|$. The *slow* time scales, associated to the first $\lfloor \frac{\alpha + nd(n+1)/2}{nd+\alpha} \rfloor$ components of the process, induce in the off-diagonal regime additional time singularities in the density w.r.t. to the non-degenerate case. We thus need to be very careful when comparing the two densities appearing in the convolution $\tilde{p}_\alpha \otimes H$. Observe anyhow from the scaling Lemma 5.2 that:

$$\begin{aligned}
|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| &\leq |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,\tau}(z))| + |(\mathbb{T}^\alpha_{T-t})^{-1}(\mathbb{T}^\alpha_{T-t}\hat{R}^{t,T}_{\frac{\tau-t}{T-t}}(\mathbb{T}^\alpha_{T-t})^{-1}\{z - R_{\tau,t}(x)\})|\\
&\leq |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,\tau}(z))| + C|(\mathbb{T}^\alpha_{T-t})^{-1}(z - R_{\tau,t}(x))|\\
&\leq |(\mathbb{T}^\alpha_{T-\tau})^{-1}(y - R_{T,\tau}(z))| + C|(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|,\ C := C(\mathbf{[H]}, T_0).
\end{aligned} \tag{6.7}$$

Hence, at least one of the two densities involved in the convolution is off-diagonal. As emphasized below, the main difficulty w.r.t. the non degenerate case consists in suitably controlling the multi-scale effects that prevent from handling directly the time singularity of H in the convolution $\tilde{p}_\alpha \otimes H$, see e.g. Proposition 3.2 in Kolokoltsov [14]. Assume now that the component number k dominates in the global density $\tilde{p}_\alpha(t,T,x,y)$ when considering the flow at the current time τ of the convolution, the off-diagonal estimate becomes:

$$\tilde{p}^{T,y}_\alpha(t,T,x,y) \overset{Lemma\ 5.2}{\asymp} \frac{(\det(\mathbb{T}^\alpha_{T-t}))^{-1}}{|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{\tau,t}x - R_{\tau,T}y)|^{nd+\alpha}} \asymp \frac{(T-t)^{nd(k-(n+1)/2)+\alpha(k-1)+1}}{|R^k_{\tau,t}(x) - R^k_{\tau,T}(y)|^{nd+\alpha}}.$$

According to the sign of the power to $T-t$, two cases arise. We call *fast* components those for which the exponent $nd(k-(n+1)/2)+\alpha(k-1)$ is non negative. The *slow* ones are those for which the exponent is negative. This is the aforementioned *slow/fast* dichotomy.

- When a fast component dominates, as the off-diagonal estimates are not singular in time anymore, no major problem arises. We refine (6.7) in the following sense:

$$K(T-t)^{(k-1)+\frac{1}{\alpha}} \leq |R^k_{\tau,T}(y) - R^k_{\tau,t}(x)| \leq |R^k_{\tau,T}(y) - z^k| + |z^k - R^k_{\tau,t}(x)|.$$

Thus, at least one of the two densities in (6.3) is off-diagonal through a fast component. On the one hand, if $1/2|R^k_{\tau,T}(y) - R^k_{\tau,t}(x)| \leq |z^k - R^k_{\tau,t}(x)|$,

$$\begin{aligned}
\tilde{p}_\alpha(t,\tau,x,z) &\leq C\frac{\det(\mathbb{T}^\alpha_{\tau-t})^{-1}}{|(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|^{nd+\alpha}} \leq C\frac{(\tau-t)^{nd(k-(n+1)/2)+\alpha(k-1)+1}}{|z^k - R^k_{\tau,t}(x)|^{nd+\alpha}}\\
&\leq C\frac{(T-t)^{nd(k-(n+1)/2)+\alpha(k-1)+1}}{|R^k_{\tau,t}(x) - R^k_{\tau,T}(y)|^{nd+\alpha}}.
\end{aligned}$$

On the other hand, if $1/2|R^k_{\tau,T}(y) - R^k_{\tau,t}(x)| \le |R^k_{\tau,T}(y) - z^k|$,

$$\frac{1}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y) \le C\frac{(T-\tau)^{nd(k-(n+1)/2)+\alpha(k-1)}}{|z^k - R^k_{\tau,T}(y)|^{nd+\alpha}} \le \frac{C}{T-t}\frac{(T-t)^{nd(k-(n+1)/2)+\alpha(k-1)+1}}{|R^k_{\tau,T}(y) - R^k_{\tau,t}(x)|^{nd+\alpha}}.$$

In both cases, we are in position to apply Lemma 3.7, directly in the first case, similarly to (6.6) in the second one. The proof is then the same as in Kolokoltsov [14]. Observe that in the second case, we have compensated the singularity induced by the kernel H, independently of the position of the time parameter τ.

- We now focus on the second case, that is when the dominating component is such that $nd(k-(n+1)/2)+\alpha(k-1)$ is negative. We consider the partition $[t,T] = I_1 \cup I_2$ and start with $\tau \in I_2$. In this case, we have $T-t \asymp \tau - t$. In other words, this is the case where the singularity induced by the kernel H is the worst.

We split $\mathbb{R}^{nd}$ into

$$\begin{aligned} D_1 &:= \{z \in \mathbb{R}^{nd}; (T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \le |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|\}, \\ D_2 &:= \{z \in \mathbb{R}^{nd}; (T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| > |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|\}, \end{aligned} \tag{6.8}$$

for a parameter $\beta > 0$ to be specified later on. We define accordingly, for $i \in \{1,2\}$:

$$\tilde{p}_\alpha \otimes_{|I_2,D_i} |H|(t,T,x,y) := \int_{I_2} d\tau \int_{D_i} \tilde{p}_\alpha(t,\tau,x,z)\frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y)dz. \tag{6.9}$$

Let us first deal with $z \in D_1$. Since $\tau \in I_2$, we have:

$$\tilde{p}_\alpha(t,\tau,x,z) \le C\frac{\det(\mathbb{T}^\alpha_{\tau-t})^{-1}}{|(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|^{nd+\alpha}} \le C\frac{\det(\mathbb{T}^\alpha_{T-t})^{-1}}{(T-\tau)^{\beta(nd+\alpha)}|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|^{nd+\alpha}}.$$

Hence, as we did in the first part of the proof, we take out $\tilde{p}_\alpha(t,\tau,x,z)$ off the integral (6.9). This is done here up to the additional singular coefficient $(T-\tau)^{-\beta(nd+\alpha)}$. Still from Lemma 3.7, we get:

$$\tilde{p}_\alpha \otimes_{|I_2,D_1} |H|(t,T,x,y) \le C\tilde{p}_\alpha(t,T,x,y)\int_{I_2} d\tau (T-\tau)^{\eta(1/\alpha\wedge 1)-\beta(nd+\alpha)-1}.$$

Then, in order to get an integrable bound, we must take:

$$0 < \beta < \frac{\eta(1/\alpha \wedge 1)}{nd+\alpha}. \tag{6.10}$$

On D_2, we have to be more subtle. From the previous partition, the idea is to say that if $\tau \in [\tau_0, T]$ for τ_0 *close enough* to T, then the diagonal bound holds for the first density on D_2. In such cases we manage to get the global expected bound in the convolution. However, the previous τ_0 will highly depend on the global off-diagonal estimate $|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x) - y)|$, and for $\tau \in I_2$, $\tau \le \tau_0$, we did not succeed to do better than integrating the singularity in $(T-\tau)^{-1}$ yielding the logarithmic contribution.

- Let us fix $\delta_0 \in (0,K)$. Observe that for fixed (t,T,x,y), if $\tau \ge \tau_0 := T - \left(\frac{\delta_0}{|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|}\right)^{\frac{1}{\beta}}$ then $\delta_0 \ge (T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|$. Then, since $z \in D_2$, we have $\delta_0 \ge |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|$, and the diagonal estimate holds for $\tilde{p}_\alpha(t,\tau,x,z)$. We write:

$$\begin{aligned} \tilde{p}_\alpha \otimes_{|I_2\cap\{\tau\ge\tau_0\},D_2} |H|(t,T,x,y) &\le C\int_{I_2\cap\{\tau\ge\tau_0\}} d\tau \det(\mathbb{T}^\alpha_{\tau-t})^{-1}\int_{D_2}\frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y)dz \\ &\overset{Lemma\ 3.7}{\le} C\int_{I_2\cap\{\tau\ge\tau_0\}} d\tau \det(\mathbb{T}^\alpha_{\tau-t})^{-1}(T-\tau)^{\eta(1/\alpha\wedge 1)-1}. \end{aligned}$$

Now $\delta_0^{nd+\alpha} \ge (T-\tau)^{\beta(nd+\alpha)}|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|^{nd+\alpha}$, so that:

$$\tilde{p}_\alpha \otimes_{|I_2\cap\{\tau\ge\tau_0\},D_2} |H|(t,T,x,y) \le \int_{I_2} d\tau \det(\mathbb{T}^\alpha_{T-t})^{-1}(T-\tau)^{\eta(1/\alpha\wedge 1)-\beta(nd+\alpha)-1}\frac{\delta_0^{nd+\alpha}}{|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|^{nd+\alpha}}.$$

Thus, as long as β satisfies (6.10), $\tilde{p}_\alpha \otimes_{|I_2\cap\{\tau\ge\tau_0\},D_2} |H|(t,T,x,y) \le (T-t)^\omega \tilde{p}_\alpha(t,T,x,y)$, $\omega := \eta(1/\alpha\wedge 1) - \beta(nd+\alpha)$.

• Assume now that $\tau < \tau_0 = T - \left(\frac{\delta_0}{|(\mathbb{T}^\alpha_{T-t})^{-1}(y-R_{T,t}(x))|}\right)^{\frac{1}{\beta}}$. The singularity induced by H is then integrable, and yields the logarithmic contribution. Specifically:

$$\begin{aligned}\tilde{p}_\alpha \otimes_{|I_2\cap\{\tau<\tau_0\},D_2} |H|(t,T,x,y) &\leq C\int_{I_2} d\tau \mathbf{1}_{\tau\leq\tau_0}\int_{D_2}\tilde{p}_\alpha(t,\tau,x,z)\frac{\delta\wedge|z-R_{\tau,T}(y)|^\eta}{T-\tau}\tilde{p}_\alpha(\tau,T,z,y)dz,\\ &\leq C\int_{I_2} d\tau\frac{1}{T-\tau}\mathbf{1}_{\tau\leq\tau_0}\int_{D_2}\tilde{p}_\alpha(t,\tau,x,z)\tilde{p}_\alpha(\tau,T,z,y)dz\\ &\leq C\log(|(\mathbb{T}^\alpha_{T-t})^{-1}(y-R_{T,t}(x))|)\tilde{p}_\alpha(t,T,x,y).\end{aligned}$$

To get to the last equation, we used the semigroup property (3.7) in Remark 3.2, and the expression of τ_0. Observe also that since this term only appears when $|(\mathbb{T}^\alpha_{T-t})^{-1}(y-R_{T,t}(x))|\geq K$, one can add w.l.o.g. a $K\vee\cdot$ inside the logarithm. This exactly gives the first control of the current lemma and completes the proof of Lemma 3.8.

Let us now focus on the proof of (6.2), when $n=2$. The key-point to get a smoothing effect is to keep the $\delta\wedge|x-R_{t,T}(y)|^{\eta(\alpha\wedge1)}$ part in the control of the convolution. In order to keep track of this term, we need to determine which component dominates in $|x-R_{t,T}(y)|$. This can be rather intricate in the multi-scale setting. In the case $n=2$, the only *slow* component is the first one. Saying that it dominates at a given integration time τ is asking:

$$|R^2_{\tau,T}(y)-R^2_{\tau,t}(x)|\leq(T-t)|R^1_{\tau,T}(y)-R^1_{\tau,t}(x)|. \tag{6.11}$$

Furthermore, we can write:

$$|R^1_{T,t}(x)-y^1| \geq |R^1_{\tau,t}(x)-R^1_{\tau,T}(y)|-\|R_{T,\tau}-I\||R_{\tau,t}(x)-R_{\tau,T}(y)|.$$

From Lemma 5.1, and observing from its proof that we could also establish that $\sum_{j=1}^2\|(R_{T,\tau}-I)^{j,2}\|+\|(R_{T,\tau}-I)^{1,1}\|\leq C(T-\tau)$, $C:=C(\mathbf{[H]},T_0)$, $T_0\leq1$ we get using (6.11):

$$|R^1_{T,t}(x)-y^1| \geq |R^1_{\tau,t}(x)-R^1_{\tau,T}(y)|(1-C(T-\tau)).$$

Thus, for T small enough we get: $(T-t)|R^1_{T,t}(x)-y^1|\geq\frac{T-t}{2}|R^1_{\tau,t}(x)-R^1_{\tau,T}(y)|\overset{(6.11)}{\geq}\frac{1}{2}|R^2_{\tau,t}(x)-R^2_{\tau,T}(y)|$. We then derive similarly that:

$$\begin{aligned}|R^2_{\tau,t}(x)-R^2_{\tau,T}(y)| &\geq |R^2_{T,t}(x)-y^2|-\|R_{\tau,T}-I\||R_{T,t}(x)-y|\\ &\geq \frac{|R^2_{T,t}(x)-y^2|}{2}-C(T-\tau)|R^1_{T,t}(x)-y^1|.\end{aligned}$$

This finally yields that

$$(T-t)|R^1_{T,t}(x)-y^1|\geq\frac{|R^2_{T,t}(x)-y^2|}{4(1+C)}, \tag{6.12}$$

that is, the first component dominates in the contribution $|(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x)-y)|$ appearing in D_2. Write now:

$$|z-R_{\tau,T}(y)|\leq|z^1-R^1_{\tau,t}(x)|+|z^2-R^2_{\tau,t}(x)|+|R_{\tau,t}(x)-R_{\tau,T}(y)|. \tag{6.13}$$

◇ Suppose first that $(\tau-t)|z^1-R^1_{\tau,t}(x)|\leq|z^2-R^2_{\tau,t}(x)|$. Since $z\in D_2$, we have from (6.12):

$$|z^2-R^2_{\tau,t}(x)|\leq C(\tau-t)(T-\tau)^\beta|R^1_{T,t}(x)-y^1|.$$

Consequently, plugging the last two inequalities into (6.13), we get:

$$\begin{aligned}|z-R_{\tau,T}(y)| &\leq \left(\frac{1}{\tau-t}+1\right)|z^2-R^2_{\tau,t}(x)|+|R_{\tau,t}(x)-R_{\tau,T}(y)|\\ &\leq \Big(1+(\tau-t)\Big)(T-\tau)^\beta|R^1_{T,t}(x)-y^1|+|R_{\tau,t}(x)-R_{\tau,T}(y)|\\ &\leq C|x-R_{t,T}(y)|,\end{aligned}$$

using the Lipschitz property of the flow for the last inequality.

$\diamond$ Assume now that $|z^2 - R^2_{\tau,t}(x)| \le (\tau - t)|z^1 - R^1_{\tau,t}(x)| \le |z^1 - R^1_{\tau,t}(x)|$. We exploit that $z \in D_2$ and (6.12) to write:

$$|z^1 - R^1_{\tau,t}(x)| \le C(T-\tau)^\beta |R^1_{T,t}(x) - y^1|.$$

Plugging the last two inequalities into (6.13) yields:

$$\begin{aligned} |z - R_{\tau,T}(y)| &\le 2|z^1 - R^1_{\tau,t}(x)| + |R_{\tau,t}(x) - R_{\tau,T}(y)| \\ &\le 2C(T-\tau)^\beta |R^1_{T,t}(x) - y^1| + |R_{\tau,t}(x) - R_{\tau,T}(y)| \le C|x - R_{t,T}(y)|, \end{aligned}$$

using again the Lipschitz property of the flow for the last inequality.

Thus, in both cases,

$$|z - R_{\tau,T}(y)| \le C|x - R_{t,T}(y)| \Rightarrow \delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \le C\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}. \tag{6.14}$$

Taking out this contribution from the spatial integral we get:

$$\begin{aligned} \tilde{p}_\alpha \otimes_{|I_2 \cap \{\tau \le \tau_0\}, D_2} |H|(t,T,x,y) &\le C \int_{I_2} d\tau \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \mathbf{1}_{\tau \le \tau_0} \int \tilde{p}_\alpha(t,\tau,x,z)\tilde{p}_\alpha(\tau,T,z,y)dz \\ &\le C\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \log\Big(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\Big)\tilde{p}_\alpha(t,T,x,y), \end{aligned}$$

using the semigroup property (3.7) for the last inequality.

To complete the proof, it remains to consider the case $\tau \in I_1$. In this case, $T - t \asymp T - \tau$, and we have by triangle inequality:

$$\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \le C\left(\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)} + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}\right).$$

Recalling that $T - \tau$ is not singular and splitting $\tilde{p}_\alpha \otimes_{|I_1} |H|(t,T,x,y)$ accordingly yields:

$$\begin{aligned} \tilde{p}_\alpha \otimes_{|I_1} |H|(t,T,x,y) \le C \int_{I_1} d\tau \int_{\mathbb{R}^{nd}} dz \tilde{p}_\alpha(t,\tau,x,z) \frac{\delta \wedge |x - R_{t,\tau}(z)|^{\eta(\alpha\wedge 1)}}{\tau - t} \tilde{p}_\alpha(\tau,T,z,y) \\ + C\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha(t,T,x,y), \end{aligned}$$

where we used the semigroup property (3.7) for the last term in the r.h.s. Now, for the first term in the above r.h.s., the previous arguments apply. Similarly to (6.7) one of the two terms $|(\mathbb{T}^\alpha_{\tau-t})^{-1}(R_{\tau,t}(x) - z)|$, $|(\mathbb{T}^\alpha_{T-\tau})^{-1}(R_{T,\tau}(z) - y)|$ is in the *off-diagonal* regime. If it is the second one, then $\tilde{p}_\alpha(\tau,T,z,y) \le C\tilde{p}_\alpha(t,T,x,y)$ and we conclude using Lemma 3.7. If it is the first term, then we can still perform the previous dichotomy along the dominating component in $|(\mathbb{T}^\alpha_{\tau-t})^{-1}(R_{\tau,t}(x) - z)|$. If the fast component dominates, the density is not singular. When the first component dominates, we modify the previous partition $(D_i)_{i\in\{1,2\}}$, considering:

$$D_1 = \{z \in \mathbb{R}^{nd}; (\tau - t)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \le |(\mathbb{T}^\alpha_{T-\tau})^{-1}(z - R_{\tau,T}(y))|\},$$

$$D_2 = \{z \in \mathbb{R}^{nd}; (\tau - t)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| > |(\mathbb{T}^\alpha_{T-\tau})^{-1}(z - R_{\tau,T}(y))|\}.$$

From this point on, the proof is similar: on D_1, we compensate the singularity, as long as β is like in (6.10). When $z \in D_2$, we subdivide along $\delta_0 \le$ or $> (\tau - t)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|$. The first case is dealt as above. In the second case, we can integrate the time singularity.

□

Remark 6.2. The first term in the r.h.s. of (6.2) yields a *smoothing* effect in time. The second is similar to the control obtained in [14]. However, the very last part is unexpected. The logarithmic contribution is specific to the degenerate framework. The multi-scale stable process lacks integrability to compensate entirely the singularities induced by the kernel H.

Remark 6.3. Let us mention that the previous proof still holds for $n = 3$ if $d = 1$ for all $\alpha \in (0,2)$.

The convergence of the parametrix series (3.11) will now follow from controls involving the convolutions of H with the last term $\bar{q}_\alpha(t,T,x,y)$. The following lemma completes the proof of Lemma 3.9.

Lemma 6.3. *There exist* $C_{6.3} := C_{6.3}(\textbf{\textit{[H]}}) > 0$, $\omega := \omega(\textbf{\textit{[H]}}) \in (0,1]$ *s.t. for all* $T \in (0, T_0]$, $T_0 := T_0(\textbf{\textit{[H]}}) \leq 1$, $(x,y) \in (\mathbb{R}^{nd})^2$, $t \in [0,T)$,

$$\begin{aligned}|\bar{q}_\alpha \otimes H|(t,T,x,y) \quad &\leq \quad C(T-t)^\omega \Big(\tilde{p}_\alpha(t,T,x,y) \\ &\quad + \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \log\Big(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\Big) \tilde{p}_\alpha(t,T,x,y)\Big).\end{aligned}$$

Proof. Recall that $\bar{q}_\alpha(t,T,x,y)$ writes as the sum of

$$q_\alpha(t,T,x,y) := \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha(t,T,x,y)$$

and

$$\rho_\alpha(t,T,x,y) := \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \log\Big(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\Big) \tilde{p}_\alpha(t,T,x,y).$$

Though the lines of the proof are similar to those of Lemma 6.2, we treat the two convolutions separately, to emphasize the difficulties induced by the logarithmic factor. First, for $|q_\alpha \otimes H|(t,T,x,y)$, we bound $|H|$ using Lemma 3.6, to get:

$$|q_\alpha \otimes H|(t,T,x,y) \leq C \int_t^T d\tau \int_{\mathbb{R}^{nd}} \delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha(t,\tau,x,z) \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \tilde{p}_\alpha(\tau,T,z,y).$$

The above contribution can be handled as in Lemma 6.2, in the *diagonal* case $|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \leq K$, or in the *off-diagonal* case $|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| > K$ when for a given integration time $\tau \in [t,T]$ the *fast component* dominates, i.e. $|R^2_{\tau,T}(y) - R^2_{\tau,t}(x)| \geq (T-t)|R^1_{\tau,T}(y) - R^1_{\tau,t}(x)|$. The only difference is that we do not need to use the triangle inequality in order to apply Lemma 3.7. Indeed, the regularizing terms $\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}$, $\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)}$ already appear for both densities.

When $|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| > K$ and $|R^2_{\tau,T}(y) - R^2_{\tau,t}(x)| \leq (T-t)|R^1_{\tau,T}(y) - R^1_{\tau,t}(x)|$, we split as in the previous proof the time interval into $I_1 \cup I_2 := [t, \frac{T+t}{2}] \cup [\frac{T+t}{2}, T]$. Suppose $\tau \in I_2$, we consider the spatial partition introduced in (6.8).

For $z \in D_1$, we have $\tilde{p}_\alpha(t,\tau,x,z) \leq (T-\tau)^{-\beta(nd+\alpha)} \tilde{p}_\alpha(t,T,x,y)$. This yields a regularization property from Lemma 3.7 when β satisfies (6.10). For $z \in D_2$ and a given $\delta_0 > 0$, we use again the partition $(T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \geq$ or $< \delta_0$. The case $(T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \leq \delta_0$ yields a regularization in time similarly to the previous proof.

In order for $(T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|$ to exceed δ_0, we see that τ must be lower than $\tau_0 := T - \Big(\frac{\delta_0}{|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|}\Big)^{\frac{1}{\beta}}$. In that case, the time singularity is still logarithmically explosive but integrable. We are led to consider:

$$\int_{I_2} d\tau \frac{1}{T-\tau} \mathbf{1}_{\tau \leq \tau_0} \int \delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)} \tilde{p}(t,\tau,x,z) \delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}(\tau,T,z,y) dz. \tag{6.15}$$

Using iteratively the scaling Lemma 5.2 we derive:

$$\begin{aligned}\frac{|y^1 - R^1_{T,\tau}(z)|}{(T-t)^{\frac{1}{\alpha}}} + \frac{|y^2 - R^2_{T,\tau}(z)|}{(T-t)^{1+\frac{1}{\alpha}}} &\geq c_2 |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,\tau}(z))| \\ &\geq c_2 C^{-1} \left\{ |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{\tau,t}(x) - R_{\tau,T}(y))| - |(\mathbb{T}^\alpha_{T-t})^{-1}(z - R_{\tau,t}(x))| \right\} \\ &\geq c_2 \{ C^{-1} |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{\tau,t}(x) - R_{\tau,T}(y))| - C^{-1}(T-\tau)^\beta |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{T,t}(x) - y)| \} \\ &\geq c_2 \{ C^{-1} - (T-\tau)^\beta \} |(\mathbb{T}^\alpha_{T-t})^{-1}(R_{\tau,t}(x) - R_{\tau,T}(y))|, \; c_2 > 0, C := C(T) \geq 1,\end{aligned}$$

recalling that $z \in D_2$ for the last but one inequality. Thus, for T small enough and up to a modification of C, we have either $|y^1 - R^1_{T,\tau}(z)| \geq C|R^1_{\tau,t}(x) - R^1_{\tau,T}(y)|$, or $|y^2 - R^2_{T,\tau}(z)| \geq C(T-t)|R^1_{\tau,t}(x) - R^1_{\tau,T}(y)|$. In both cases, $\tilde{p}_\alpha(\tau,T,z,y) \leq \frac{1}{|R^1_{\tau,t}(x) - R^1_{\tau,T}(y)|^{2+\alpha}}$, which is **in dimension one** the off-diagonal two-sided estimate for $\tilde{p}_\alpha(t,T,x,y)$. We emphasize that the technical restriction leading to consider the scalar case appears exactly here. Indeed, for $n = 2$ and an arbitrary $d \geq 1$, when the first component dominates, we have from Proposition 3.3 that the two-sided off-diagonal estimate for $\tilde{p}_\alpha(\tau,T,z,y)$ reads:

$$(T-\tau)^{-(d-1)} |R^1_{T,\tau}(z) - y^1|^{-(2d+\alpha)}.$$

Hence, we can get rid off the remaining diagonal singularity only if $d = 1$. This yields $\tilde{p}_\alpha(\tau,T,z,y) \leq \tilde{p}_\alpha(t,T,x,y)$. In our current case, we then derive from (6.14) that:

$$\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha(\tau,T,z,y) \leq \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \tilde{p}_\alpha(t,T,x,y).$$

Consequently, we can bound (6.15) by:

$$\int_{I_2} d\tau (\tau - t)^{\omega} \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha \wedge 1)}}{T - \tau} \mathbf{1}_{\tau \leq \tau_0} \tilde{p}_{\alpha}(t, T, x, y).$$

Note that the case $\tau \in I_1$ could be handled similarly, see Lemma 6.2. Once integrated in time, the controls become :

$$q_{\alpha} \otimes |H|(t, T, x, y) \leq C(T - t)^{\omega} \left(\tilde{p}_{\alpha}(t, T, x, y) + \log \left(K \vee |(\mathbb{T}^{\alpha}_{T-t})^{-1}(y - R_{T,t}(x))| \right) q_{\alpha}(t, T, x, y) \right). \tag{6.16}$$

We point out that the important contribution in the above equation is the factor $(T - t)^{\omega}$, whose power will grow at each iteration. This key feature gives the convergence of the series (3.11).

Now, for $\rho_{\alpha} \otimes |H|(t, T, x, y)$, we still bound $|H|$ using Lemma 3.6:

$$\begin{aligned} \rho_{\alpha} \otimes |H|(t, T, x, y) \leq C \int_t^T \int_{\mathbb{R}^{nd}} \log \left(K \vee |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \right) \delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha \wedge 1)} \tilde{p}_{\alpha}(t, \tau, x, z) \\ \times \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha \wedge 1)}}{T - \tau} \tilde{p}_{\alpha}(\tau, T, z, y). \end{aligned} \tag{6.17}$$

W.r.t. the previous contribution, the main difference comes from the logarithm. However, the lines of the proof remain the same. Suppose first that $|(\mathbb{T}^{\alpha}_{T-t})^{-1}(y - R_{T,t}(x))| \leq K$. Depending on the time parameter τ, we can show that we always have either $\tilde{p}_{\alpha}(t, \tau, x, z) \leq C\tilde{p}_{\alpha}(t, T, x, y)$ or $\tilde{p}_{\alpha}(\tau, T, z, y) \leq C\tilde{p}_{\alpha}(t, T, x, y)$. The second case occurs when $\tau \in I_1$. Using the notations of the previous proof, this yields:

$$\begin{aligned} \rho_{\alpha} \otimes_{|I_1} |H|(t, T, x, y) \quad \leq \quad & C\tilde{p}_{\alpha}(t, T, x, y) \int_{I_1} d\tau \int_{\mathbb{R}^{nd}} \log \left(K \vee |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \right) \\ & \times \frac{\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha \wedge 1)}}{\tau - t} \tilde{p}_{\alpha}(t, \tau, x, z) dz, \end{aligned}$$

and we conclude by Lemma 6.1. In the case when $\tilde{p}_{\alpha}(t, \tau, x, z) \leq C\tilde{p}_{\alpha}(t, T, x, y)$, which happens for $\tau \in I_2$, we have:

$$|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C(|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,t}(x))| + |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,\tau}(z))|) \leq C(K + |(\mathbb{T}^{\alpha}_{T-\tau})^{-1}(y - R_{T,\tau}(z))|).$$

Plugging this inequality into the logarithm and taking out the first density, we can bound:

$$\begin{aligned} \rho_{\alpha} \otimes_{|I_2} |H|(t, T, x, y) \quad \leq \quad & C\tilde{p}_{\alpha}(t, T, x, y) \int_{I_2} d\tau \int_{\mathbb{R}^{nd}} \log \left(K \vee |(\mathbb{T}^{\alpha}_{T-\tau})^{-1}(y - R_{T,\tau}(z))| \right) \\ & \times \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha \wedge 1)}}{T - \tau} \tilde{p}_{\alpha}(\tau, T, z, y) dz, \end{aligned}$$

and once again, we conclude by Lemma 6.1. Thus, we have so far managed to show that in the global diagonal regime, $\rho_{\alpha} \otimes |H|(t, T, x, y) \leq C(T - t)^{\omega} \tilde{p}_{\alpha}(t, T, x, y)$.

It remains to deal with the case when $|(\mathbb{T}^{\alpha}_{T-t})^{-1}(y - R_{T,t}(x))| \geq K$. Suppose first that $\tau \in I_2$, and that the first component dominates in the global action $|(\mathbb{T}^{\alpha}_{T-t})^{-1}(y - R_{T,t}(x))|$, i.e. $|(\mathbb{T}^{\alpha}_{T-t})^{-1}(y - R_{T,t}(x))| \asymp \frac{|y^1 - R^1_{T,t}(x)|}{(T-t)^{1/\alpha}}$. We still consider the partition in (6.8).

When $z \in D_1$, we can bound

$$\tilde{p}_{\alpha}(t, \tau, x, z) \leq C(T - \tau)^{-\beta(2+\alpha)} \tilde{p}(t, T, x, y). \tag{6.18}$$

On the other hand, the triangle inequality and the scaling Lemma 5.2 yield:

$$|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C\Big(|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,\tau}(z))| + |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,t}(x))| \Big).$$

Consequently, up to a modification of C, we have either:

$$|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,t}(x))| \text{ or } |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,\tau}(z))|.$$

Define accordingly,

$$D_{1,1} = \{z \in D_1; |(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C|(\mathbb{T}^{\alpha}_{\tau-t})^{-1}(y - R_{T,t}(x))|\},$$

$$D_{1,2} = \{z \in D_1; |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))| \leq C|(\mathbb{T}^\alpha_{\tau-t})^{-1}(y - R_{T,\tau}(z))|\}.$$

Observe that with this definition, $D_{1,1}$ and $D_{1,2}$ is not a partition of D_1. However, $D_1 \subset D_{1,1} \cup D_{1,2}$.

When $z \in D_{1,1}$, we can bound

$$\log\left(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|\right) \leq \log\left(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\right) + C.$$

On the other hand, for $\tau \in I_2$, we get from the definition of $D_{1,1}$:

$$\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)} \leq C(\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}).$$

From (6.18), we thus have:

$$\begin{aligned}
\rho_\alpha \otimes_{|I_2,D_{1,1}} |H|(t,T,x,y) &\leq C\left(\log\left(K \vee |(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|\right) + 1\right) \delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)} \\
&\quad \times \int_{I_2} d\tau \int_{D_{1,1}} \tilde{p}_\alpha(t,\tau,x,z) \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \tilde{p}_\alpha(\tau,T,z,y)dz \\
&\leq (T-t)^\omega(\rho_\alpha + q_\alpha)(t,T,x,y),
\end{aligned}$$

choosing β satisfying (6.10).

When $z \in D_{1,2}$, we can bound:

$$\log\left(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(z - R_{\tau,t}(x))|\right) \leq \log\left(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(y - R_{T,\tau}(z))|\right) + C.$$

Bounding also roughly $\delta \wedge |z - R_{\tau,t}(x)|^{\eta(\alpha\wedge 1)} \leq \delta$, and using the bound (6.18), we can write:

$$\begin{aligned}
\rho_\alpha \otimes_{|I_2,D_{1,2}} |H|(t,T,x,y) &\leq C \int_{I_2} d\tau \int_{D_{1,2}} \tilde{p}_\alpha(t,\tau,x,z) \left(\log\left(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(y - R_{T,\tau}(z))|\right) + 1\right) \\
&\quad \times \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \tilde{p}_\alpha(\tau,T,z,y)dz \\
&\leq C\tilde{p}_\alpha(t,T,x,y) \int_{I_2} d\tau (T-\tau)^{-\beta(2+\alpha)} \\
&\quad \times \int_{D_{1,2}} \left(\log\left(K \vee |(\mathbb{T}^\alpha_{\tau-t})^{-1}(y - R_{T,\tau}(z))|\right) + 1\right) \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \tilde{p}_\alpha(\tau,T,z,y).
\end{aligned}$$

Thus, using Lemma 6.1, we have $\rho_\alpha \otimes_{|I_2,D_{1,2}} |H|(t,T,x,y) \leq (T-t)^\omega \tilde{p}_\alpha(t,T,x,y)$.

We have to deal with $z \in D_2$. In this case, **and because** $d = 1$, $\tilde{p}_\alpha(\tau,T,z,y) \leq C\tilde{p}_\alpha(t,T,x,y)$. As above, we split for a given $\delta_0 > 0$, the time interval I_2 in $(T-\tau)^\beta|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \geq \delta_0$ and $(T-\tau)^\beta|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| < \delta_0$.

Assume first that $(T-\tau)^\beta|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \leq \delta_0$. Then, taking $\delta_0 \leq K$ gives that the first density is diagonal. Hence, the logarithm part disappears, and we have to deal with:

$$\begin{aligned}
\rho_\alpha \otimes_{I_2,D_2} |H|(t,T,x,y) &\leq C \int_{\tau_0}^T \int_{D_2} \frac{1}{(T-t)^{2/\alpha+1}} \frac{\delta \wedge |z - R_{\tau,T}(y)|^{\eta(\alpha\wedge 1)}}{T-\tau} \tilde{p}_\alpha(\tau,T,z,y)dz \\
&\overset{Lemma\ 3.7}{\leq} \frac{\delta_0^{2+\alpha}}{(T-t)^{2/\alpha+1}|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))|^{2+\alpha}} \int_{\tau_0}^T d\tau (T-\tau)^{\eta(1/\alpha\wedge 1)-1-\beta(2+\alpha)} \\
&\leq (T-t)^\omega \tilde{p}_\alpha(t,T,x,y).
\end{aligned}$$

Finally, we have to deal with the case $(T-\tau)^\beta|(\mathbb{T}^\alpha_{T-t})^{-1}(y - R_{T,t}(x))| \geq \delta_0$. Observe that, on I_2, this imposes that $\tau \in [\frac{T+t}{2}, \tau_0]$, with τ_0 defined above. In the considered set, we have from (6.14):

$$|z - R_{\tau,T}(y)| \leq |z - R_{\tau,t}(x)| + C|x - R_{t,T}(y)| \leq C(1 + (T-\tau)^\beta)|x - R_{t,T}(y)|.$$

Plugging this estimate into the convolution and recalling for $z \in D_2$, $\tilde{p}_\alpha(\tau,T,z,y) \leq C\tilde{p}_\alpha(t,T,x,y)$, we obtain from Lemma 6.1:

$$\rho_\alpha \otimes_{I_2,D_2} |H|(t,T,x,y) \leq C(\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)})\tilde{p}_\alpha(t,T,x,y) \int_{\frac{T+t}{2}}^{\tau_0} d\tau \frac{1}{T-\tau}(\tau - t)^\omega.$$

Hence, integrating over τ yields the logarithmic contribution:

$$\rho_\alpha \otimes_{I_2,D_2} |H|(t,T,x,y) \leq C(T-t)^\omega \rho_\alpha(t,T,x,y).$$

In order to complete the proof, we have to specify how to proceed in the remaining cases, that is when $\tau \in I_2$ and the second component dominates or when $\tau \in I_1$. When a fast component dominates, as we have seen in the previous proof, we can compensate the singularities brought by the kernel H, and conclude directly with Lemmas 3.7 and 6.1. When $\tau \in I_1$, we can adapt the previous strategy following the procedure described in Lemma 6.2. □

Using the previous lemmas, we get the following result.

Corollary 6.4. *Let $T_0 := T_0(\boldsymbol{[H]}) \leq 1$ be as in Lemma 6.2. There exists $C_{6.4} := 4C_{3.9} > 0$, s.t. for all $T \in (0,T_0]$, $(x,y) \in (\mathbb{R}^{nd})^2$, $t \in [0,T)$, $\forall k \in \mathbb{N}$:*

$$|\tilde{p}_\alpha \otimes H^{(2k)}(t,T,x,y)| \leq C_{6.4}^{2k}(T-t)^{k\omega}\Big((T-t)^{k\omega}\tilde{p}_\alpha(t,T,x,y) + (\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\Big)$$
$$|\tilde{p}_\alpha \otimes H^{(2k+1)}(t,T,x,y)| \leq C_{6.4}^{2k+1}(T-t)^{k\omega}\Big((T-t)^{(k+1)\omega}\tilde{p}_\alpha + (T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha) + \bar{q}_\alpha\Big)(t,T,x,y).$$

Proof. We prove the estimate by induction. The idea is to use the controls of Lemmas 6.2 and 6.3 gathered in Lemma 3.9 to get from an estimate to the following one. The bounds may not be very precise, as we will sometimes bound $(T-t)^{k\omega} \leq 1$, but they are sufficient to prove the convergence of the Parametrix series (3.11).

Initialization:
Since $(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha) \geq 0$, we clearly have:

$$|\tilde{p}_\alpha \otimes H(t,T,x,y)| \leq C_{3.9}\Big((T-t)^\omega\tilde{p}_\alpha + \bar{q}_\alpha + (T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha)\Big)(t,T,x,y).$$

Now, using Lemmas 6.2 and 6.3, we have:

$$\begin{aligned}
|\tilde{p}_\alpha \otimes H^{(2)}(t,T,x,y)| &\leq C_{3.9}\Big((T-t)^\omega|\tilde{p}_\alpha \otimes H| + |\bar{q}_\alpha \otimes H|\Big)(t,T,x,y)\\
&\leq C_{3.9}\Big(C_{3.9}(T-t)^{2\omega}\tilde{p}_\alpha + C_{3.9}(T-t)^\omega\bar{q}_\alpha + C_{3.9}(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha)\Big)(t,T,x,y)\\
&\leq (2C_{3.9})^2(T-t)^\omega\Big((T-t)^\omega\tilde{p}_\alpha + (\tilde{p}_\alpha + \bar{q}_\alpha)\Big)(t,T,x,y).
\end{aligned}$$

Induction:
Suppose that the estimate for $2k$ holds. Let us prove the estimate for $2k+1$.

$$\begin{aligned}
|\tilde{p}_\alpha \otimes H^{(2k+1)}|(t,T,x,y) &\leq (4C_{3.9})^{2k}(T-t)^{k\omega}\Big((T-t)^{k\omega}|\tilde{p}_\alpha \otimes H|(t,T,x,y) + |(\tilde{p}_\alpha + \bar{q}_\alpha) \otimes H|(t,T,x,y)\Big)\\
&\leq (4C_{3.9})^{2k}(T-t)^{k\omega}\Big(C_{3.9}(T-t)^{k\omega}((T-t)^\omega\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\\
&\quad + C_{3.9}((T-t)^\omega\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y) + C_{3.9}(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha)(t,T,x,y)\Big).
\end{aligned}$$

Recalling that $T-t \leq 1$, we have $(T-t)^{k\omega}\bar{q}_\alpha \leq (T-t)^\omega\bar{q}_\alpha$. Thus:

$$\begin{aligned}
&|\tilde{p}_\alpha \otimes H^{(2k+1)}|(t,T,x,y)\\
&\leq (4C_{3.9})^{2k}(T-t)^{k\omega}\Big(C_{3.9}(T-t)^{(k+1)\omega}\tilde{p}_\alpha + 2C_{3.9}(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha) + C_{3.9}\bar{q}_\alpha\Big)(t,T,x,y)\\
&\leq (4C_{3.9})^{2k}(2C_{3.9})(T-t)^{k\omega}\Big((T-t)^{(k+1)\omega}\tilde{p}_\alpha + (T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha) + \bar{q}_\alpha)\Big)(t,T,x,y),
\end{aligned}$$

which gives the announced estimate.
Suppose now that the estimate for $2k+1$ holds. Let us prove the estimate for $2k+2$.

$$\begin{aligned}
&|\tilde{p}_\alpha \otimes H^{(2k+2)}(t,T,x,y)|\\
&\leq (4C_{3.9})^{2k+1}(T-t)^{k\omega}\Big((T-t)^{(k+1)\omega}|\tilde{p}_\alpha \otimes H| + (T-t)^\omega|(\tilde{p}_\alpha + \bar{q}_\alpha) \otimes H| + |\bar{q}_\alpha \otimes H|\Big)(t,T,x,y)\\
&\leq (4C_{3.9})^{2k+1}(T-t)^{k\omega}\Big(C_{3.9}(T-t)^{(k+1)\omega}[(T-t)^\omega\tilde{p}_\alpha + \bar{q}_\alpha]\\
&\quad + C_{3.9}(T-t)^\omega[\{(T-t)^\omega\tilde{p}_\alpha + \bar{q}\} + C_{3.9}(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha)] + C_{3.9}(T-t)^\omega(\tilde{p}_\alpha + \bar{q}_\alpha)\Big)(t,T,x,y)\\
&\leq (4C_{3.9})^{2k+2}(T-t)^{(k+1)\omega}\Big((T-t)^{(k+1)\omega}\tilde{p}_\alpha + (\tilde{p}_\alpha + \bar{q}_\alpha)\Big)(t,T,x,y),
\end{aligned}$$

where to get to the last equation, we used the fact that $(T-t)^\omega\tilde{p}_\alpha \leq \tilde{p}_\alpha$, and $(T-t)^{k\omega}\bar{q}_\alpha \leq \bar{q}_\alpha$.

□

The following part is rather technical and given for the sake of completeness. The proofs below do not differ much from those developed in [14] for the non degenerate case. We refer to that work for further details.

A Proof of the estimates on the frozen density.

In this section we derive the so-called *diagonal* and *off-diagonal* expansions for the frozen density. Recall from Proposition 5.3, that the frozen density p_{Λ_s} is given for all $z \in \mathbb{R}^{nd}$ by:

$$p_{\Lambda_s}(z) = \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} e^{-i\langle q,(\mathbb{T}^\alpha_{s-t})^{-1}z\rangle} \exp\left(-\int_{S^{nd-1}} |\langle q,\eta\rangle|^\alpha \mu_S^*(d\eta)\right) dq.$$

The complex exponential can be written as a cosine. Denoting $\bar{x}$ the projection of $x \in \mathbb{R}^{nd}$ on the sphere, we change variable to the polar coordinates by setting $q = |q|\bar{q}$, where $(|q|,\bar{q}) \in \mathbb{R}_+ \times S^{nd-1}$. Also, we take a parametrization of the sphere by setting $\bar{q} = (\theta,\phi) \in [0,\pi] \times S^{nd-2}$, along the axis directed by $(\mathbb{T}^\alpha_{s-t})^{-1}z$. Finally, let $\tau = \cos(\theta)$, the density writes:

$$\begin{aligned} p_{\Lambda_s}(z) &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}} \int_0^{+\infty} d|q||q|^{nd-1} \int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}} \int_{S^{nd-2}} d\phi \\ &\quad \cos(|q||(\mathbb{T}^\alpha_{s-t})^{-1}z|\tau) \exp\left(-|q|^\alpha \int_{S^{nd-1}} |\langle \bar{q},\eta\rangle|^\alpha \mu_S^*(d\eta)\right). \end{aligned} \tag{A.1}$$

The idea is as follows: if $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ is small, we can expand the cosine and show that the first term is positive, giving the two-sided diagonal estimate. If on the contrary, $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ is large, we set $y = |q||(\mathbb{T}^\alpha_{s-t})^{-1}z|$, which yields:

$$\begin{aligned} p_{\Lambda_s}(z) &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}|(\mathbb{T}^\alpha_{s-t})^{-1}z|} \int_0^{+\infty} dy y^{nd-1} \int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}} \int_{S^{nd-2}} d\phi \\ &\quad \cos(y\tau) \exp\left(-\frac{y^\alpha}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|^\alpha} \int_{S^{nd-1}} |\langle \bar{q},\eta\rangle|^\alpha \mu_S^*(d\eta)\right), \end{aligned} \tag{A.2}$$

and we can expand the exponential, giving the off-diagonal estimate.

Proposition A.1 (**Diagonal expansion**). *For small $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, the following expansion holds:*

$$p_{\Lambda_s}(z) = \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}} \sum_{k=0}^N a_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) \frac{(-1)^k}{(2k)!} |(\mathbb{T}^\alpha_{s-t})^{-1}z|^{2k} + R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|),$$

where $|(\mathbb{T}^\alpha_{s-t})^{-1}z|^N R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)$ tends to zero as $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ tends to zero, and $a_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z})$ is defined as:

$$a_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) = \int_0^{+\infty} d|q||q|^{2k+nd-1} \int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}} \tau^{2k} \int_{S^{nd-2}} d\phi \exp\left(-|q|^\alpha \int_{S^{nd-1}} |\langle \bar{q},\varsigma\rangle|^\alpha \mu_S^*(d\varsigma)\right).$$

Also, using Lemma 5.4 to bound the spectral measure μ_S^*, the following estimate holds:

$$\alpha^{-1}A_{nd-2}B\Big(d-\frac{1}{2},k+\frac{1}{2}\Big)\Gamma\Big(\frac{nd+2k}{\alpha}\Big)C_{5.4}^{\frac{2k+nd}{\alpha}} \le a_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) \le \alpha^{-1}A_{nd-2}B\Big(d-\frac{1}{2},k+\frac{1}{2}\Big)\Gamma\Big(\frac{nd+2k}{\alpha}\Big)C_{5.4}^{-\frac{2k+nd}{\alpha}},$$

where B stands for the β function.

Proof. There is no difference with the non degenerate case for the diagonal expansion. For small $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, we use Taylor's formula to expand $\cos(|q||(\mathbb{T}^\alpha_{s-t})^{-1}z|\tau)$ in equation (A.1):

$$\begin{aligned} p_{\Lambda_s}(z) &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}} \int_0^{+\infty} d|q||q|^{nd-1} \int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}} \left(\sum_{k=0}^N \frac{(-1)^k}{(2k)!}|q|^{2k}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{2k}\tau^{2k} + \tilde{R}_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)\right) \\ &\quad \times \int_{S^{nd-2}} d\phi \exp\left(-|q|^\alpha \int_{S^{nd-1}} |\langle \bar{q},\varsigma\rangle|^\alpha \mu_S^*(d\varsigma)\right) \\ &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}} \sum_{k=0}^N \frac{(-1)^k}{(2k)!}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{2k} \int_0^{+\infty} d|q||q|^{2k+nd-1} \int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}}\tau^{2k} \\ &\quad \times \int_{S^{nd-2}} d\phi \exp\left(-|q|^\alpha \int_{S^{nd-1}} |\langle \bar{q},\varsigma\rangle|^\alpha \mu_S^*(d\varsigma)\right) + R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|). \end{aligned}$$

The estimate on the coefficient also serves to estimate the remainder $R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)$. To bound the coefficient, we use the bound of Lemma 5.4 to get rid-off the integral on S^{nd-1} in the exponent. The estimate readily follows form the definition of Euler's Beta and Gamma function. Note that the coefficient $a_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z})$ depends on $\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}$ because of the choice of the parametrization of the sphere S^{nd-2}.

□

We now turn to large distance estimates.

Proposition A.2 (**Off-diagonal expansion**). *For large $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, the following expansion holds:*

$$p_{\Lambda_s}(z) = \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{nd}} \sum_{k=1}^{N} b_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) \frac{(-1)^{k+1}}{k!} \left(\frac{1}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|}\right)^{\alpha k} + R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|),$$

where $|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{-N}R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)$ tends to zero as $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ tends to infinity.

Also, the coefficients b_k is the sum of $b^1_k + b^2_k$, with :

$$\begin{aligned}
b^1_k(\overline{(\mathbb{T}^\alpha_{s-t})}^{-1}z) &= \Re\Big[\int_0^{+\infty} y^{\alpha k+nd-1}\int_{-\infty}^{+\infty} e^{-i\tau y}(1-\tau^2)^{\frac{nd-3}{2}}\chi(\tau)\int_{S^{nd-2}}\Big(\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\Big)^k dy\Big],\\
b^2_k(\overline{(\mathbb{T}^\alpha_{s-t})}^{-1}z) &= \Re\Big[e^{-i\frac{\pi}{2}(\alpha k+nd)}\int_0^{+\infty} d\xi\xi^{\alpha k+nd-1}\int_{1-2\varepsilon}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}}(1-\chi(\tau))e^{-\tau\xi}\\
&\qquad\times\int_{S^{nd-2}}d\phi\Big(\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\Big)^k\Big],
\end{aligned}$$

where χ is a regular version of the indicator of $[-1,1]$. Eventually, $b_1(\overline{(\mathbb{T}^\alpha_{s-t})}^{-1}z)$ is positive.

Proof. We start with the expression (A.2) for the density. As we mentioned above, the idea is to expand the exponential to derive the off-diagonal expansion. To do so, define $\chi : \mathbb{R} \to [0,1]$, a regular version of the indicator of $[-1,1]$, such that $\chi(\tau) = 1$ when $|\tau| \le 1-2\varepsilon$, and $\chi(\tau) = 0$ when $|\tau| \ge 1-\varepsilon$. We split $p_{\Lambda_s} = p_1(z) + p_2(z)$, with, for $k = 1, 2$:

$$\begin{aligned}
p_k(z) &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{nd}}\int_0^{+\infty} dy y^{nd-1}\int_{-1}^1 d\tau f_k(\tau)\cos(\tau y)\\
&\qquad\int_{S^{nd-2}}d\phi\exp\left(-\frac{y^\alpha}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|^\alpha}\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\right),
\end{aligned}$$

where $f_1(\tau) = (1-\tau^2)^{\frac{nd-3}{2}}\chi(\tau)$ and $f_2(\tau) = (1-\tau^2)^{\frac{nd-3}{2}}(1-\chi(\tau))$.

For $p_1(z)$, we can expand the exponential directly. Since χ is of compact support, we can write the integral in τ as an integral over the whole line $\mathbb{R}$.

$$\begin{aligned}
p_1(z) &= \frac{\det(\mathbb{T}^\alpha_{s-t})^{-1}}{(2\pi)^{nd}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{nd}}\Re\Big(\int_0^{+\infty} dy y^{nd-1}\int_{-\infty}^{+\infty} d\tau f_1(\tau)e^{i\tau y}\\
&\qquad\int_{S^{nd-2}}d\phi\exp\left(-\frac{y^\alpha}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|^\alpha}\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\right)\Big)\\
&= \frac{\det\mathbb{T}}{(2\pi)^{nd}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{nd}}\sum_{k=0}^{+\infty} b^1_k(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z})\frac{(-1)^{k+1}}{k!}|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{-\alpha k},
\end{aligned}$$

where

$$b^1_k(\overline{(\mathbb{T}^\alpha_{s-t})}^{-1}z) = \Re\left[\int_0^{+\infty} y^{\alpha k+nd-1}\int_{-\infty}^{+\infty} e^{-i\tau y}f_1(\tau)\int_{S^{nd-2}}\Big(\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\Big)^k dy\right].$$

Since f_1 is of compact support and as a function of τ the integrand is regular, the function of y defined by

$$\int_{-\infty}^{+\infty} e^{i\tau y}f_1(\tau)\int_{S^{nd-2}}\Big(\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu^*_S(d\varsigma)\Big)^k dy$$

is in the Schwartz space, as the Fourier transform of a function $\mathcal{C}^\infty$ with compact support. Consequently, the coefficient $b_k^1(\overline{(\mathbb{T}_{s-t}^\alpha)}^{-1}z)$ is well defined.

For $p_2(z)$, we have to be more subtle, as the previous expansion fails in this case. The function of y defined by:

$$y^{nd-1}\int_{-1}^1 d\tau f_2(\tau)\cos(\tau y)\int_{S^{nd-2}} d\phi \exp\left(-\frac{y^\alpha}{|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^\alpha}\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right),$$

is holomorphic, and we can show that its integral on the arc $\gamma_R(t)=Re^{-it}, t\in[0,\frac{\pi}{2}]$ if $\alpha\in(0,1]$, and $t\in[0,\frac{\pi}{2\alpha}]$ if $\alpha\in(1,2)$, tends to zero as R tends to infinity. Hence, from Cauchy's theorem, we can rotate the positive half-line accordingly and expand the exponential (roughly speaking, we change variables to $y=-i\xi$ if $\alpha\in(0,1]$ and $y=\xi\exp(-i\frac{\pi}{2\alpha})$ if $\alpha\in(1,2)$):

$$\begin{aligned}
p_2(z) &= \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{(2\pi)^{nd}|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^{nd}}\Re\Big(\int_0^{+\infty}d\xi(-i)^{nd}\xi^{nd-1}\int_{1-2\varepsilon}^1 d\tau f_2(\tau)e^{-\tau\xi}\\
&\quad\int_{S^{nd-2}}d\phi\exp\left(-\frac{(-i\xi)^\alpha}{|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^\alpha}\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right)\Big)\\
&= \frac{\det\mathbb{T}_{s-t}^{\alpha\,-1}}{(2\pi)^{nd}|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^{nd}}\sum_{k=0}^{+\infty}b_k^2(\overline{(\mathbb{T}_{s-t}^\alpha)}^{-1}z)\frac{(-1)^k}{k!}|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^{-\alpha k},
\end{aligned}$$

with coefficients defined as:

$$b_k^2(\overline{(\mathbb{T}_{s-t}^\alpha)}^{-1}z)=\Re\left[e^{-i\frac{\pi}{2}(\alpha k+nd)}\int_0^{+\infty}d\xi\xi^{\alpha k+nd-1}\int_{1-2\varepsilon}^1 d\tau f_2(\tau)e^{-\tau\xi}\int_{S^{nd-2}}d\phi\left(\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right)^k\right].$$

To give an estimate on this coefficient, since the variable τ is not zero, we can change variables to $\zeta=\tau\xi$ and bound $\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)$ by some constant, so that after integration,

$$|b_k^2(\overline{(\mathbb{T}_{s-t}^\alpha)}^{-1}z)|\le C_2 2\varepsilon\Gamma(\alpha k+nd)A_{nd-1}.$$

As per usual, the estimate on the coefficient serves as an estimate for the remainder. To get the announced expansion, we define $b_k(\overline{(\mathbb{T}_{s-t}^\alpha)^{-1}z})=b_k^1(\overline{(\mathbb{T}_{s-t}^\alpha)^{-1}z})+b_k^2(\overline{(\mathbb{T}_{s-t}^\alpha)^{-1}z})$.

It remains to show that the b_0 vanishes. Indeed, to show that $b_0^1=0$, we use symmetry arguments. Also $b_0^2=0$, as it is the real part of a purely imaginary number.

□

B Proof of the Estimates on the Kernel H

In this section, we establish the bound (3.17) for the convolution kernel H. To this end, we define for a measure ν on S^{d-1}, not necessarily positive, but such that $|\nu|$ is finite, the quantity:

$$\phi_{\Lambda_s}^\nu(z) := \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}dp e^{-i\langle p,z\rangle}\int_{S^{d-1}}|\langle p^1,\varsigma\rangle|^\alpha\nu(d\varsigma)\exp\left(-\int_{S^{nd-1}}|\langle\mathbb{T}_{s-t}^\alpha p,\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right).$$

The calculations we did on p_{Λ_s} can be carried out for $\phi_{\Lambda_s}^\nu$. In particular, we can change variables in the exponent and define $p=(\mathbb{T}_{s-t}^\alpha)^{-1}q$ to get

$$\phi_{\Lambda_s}^\nu(z) = \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{(2\pi)^{nd}(s-t)}\int_{\mathbb{R}^{nd}}dq e^{-i\langle q,(\mathbb{T}_{s-t}^\alpha)^{-1}z\rangle}|q|^\alpha\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\tilde{\nu}(d\varsigma)\exp\left(-|q|^\alpha\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right),$$

with $\int_{S^{nd-1}}|\langle q,\varsigma\rangle|^\alpha\tilde{\nu}(d\varsigma)=\int_{S^{d-1}}|\langle q_1,\varsigma\rangle|^\alpha\nu(d\varsigma)$. We use the same change of variable as for p_{Λ_s} to get:

$$\begin{aligned}
\phi_{\Lambda_s}^\nu(z) &= \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{(2\pi)^{nd}(s-t)}\int_0^{+\infty}d|q||q|^{nd+\alpha-1}\int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}}\cos(|(\mathbb{T}_{s-t}^\alpha)^{-1}z||q|\tau)\\
&\quad\times\int_{S^{nd-2}}d\phi\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\tilde{\nu}(d\varsigma)\exp\left(-|q|^\alpha\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right)\\
&= \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{(2\pi)^{nd}(s-t)|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^{nd}}\int_0^{+\infty}dy y^{nd+\alpha-1}\int_{-1}^1 d\tau(1-\tau^2)^{\frac{nd-3}{2}}\cos(\tau y)\\
&\quad\times\int_{S^{nd-2}}d\phi\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\tilde{\nu}(d\varsigma)\exp\left(-\frac{y^\alpha}{|(\mathbb{T}_{s-t}^\alpha)^{-1}z|^\alpha}\int_{S^{nd-1}}|\langle\bar{q},\varsigma\rangle|^\alpha\mu_S^*(d\varsigma)\right).
\end{aligned}$$

As we can see, the only difference between $\phi^\nu_{\Lambda_s}$ and p_{Λ_s} comes from the integral $\int_{S^{nd-1}} |\langle \bar{q}, \varsigma\rangle|^\alpha \tilde{\nu}(d\varsigma)$ and the additional multiplier $|q|^\alpha$. Those additional terms will not change much the expansions.

Proposition B.1. *For small $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, the following expansion holds:*

$$\phi^\nu_{\Lambda_s}(z) = \frac{\det \mathbb{T}}{(2\pi)^{nd}(s-t)} \sum_{k=0}^{N} a_k^\phi(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) \frac{(-1)^k}{(2k)!} |(\mathbb{T}^\alpha_{s-t})^{-1}z|^{2k} + R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|),$$

where $|(\mathbb{T}^\alpha_{s-t})^{-1}z|^N R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)$ tends to zero as $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ tends to zero, and $a_k^\phi(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z})$ is defined as:

$$\begin{aligned} a_k^\phi(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z}) &= \int_0^{+\infty} d|q||q|^{2k+\alpha+nd-1} \int_{-1}^{1} d\tau (1-\tau^2)^{\frac{nd-3}{2}} \tau^{2k} \\ &\quad \int_{S^{nd-2}} d\phi \int_{S^{nd-1}} |\langle \bar{q}, \varsigma\rangle|^\alpha \tilde{\nu}(d\varsigma) \exp\left(-|q|^\alpha \int_{S^{nd-1}} |\langle \bar{q}, \varsigma\rangle|^\alpha \mu^*_S(d\varsigma) \right). \end{aligned}$$

Also, the following estimate holds:

$$|a_k^\phi(\overline{(\mathbb{T}^\alpha_{s-t})^{-1}z})| \leq \alpha^{-1} A_{nd-2} B\Big(d-\frac{1}{2}, k+\frac{1}{2}\Big) \Gamma\Big(\frac{nd+2k}{\alpha}+1\Big) C^{-\frac{2k+nd}{\alpha}-1}.$$

For large $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, the following expansion holds:

$$\phi^\nu_{\Lambda_s}(z) = \frac{\det \mathbb{T}^\alpha_{s-t}{}^{-1}}{(2\pi)^{nd}(s-t)|(\mathbb{T}^\alpha_{s-t})^{-1}z|^{nd}} \sum_{k=0}^{N} b_k^\phi(\overline{(\mathbb{T}^\alpha_{s-t})}^{-1}z) \frac{(-1)^k}{k!} \left(\frac{1}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|} \right)^{\alpha k} + R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|),$$

where $\frac{R_N(|(\mathbb{T}^\alpha_{s-t})^{-1}z|)}{|(\mathbb{T}^\alpha_{s-t})^{-1}z|^N}$ tends to zero as $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$ tends to infinity.

Proof. The proof is fairly the same, up to the additional multipliers. For small $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, the expansion is straight forward. To get the estimate, on the coefficient, we observe that $\int_{S^{nd-1}} |\langle \bar{q}, \varsigma\rangle|^\alpha \tilde{\nu}(d\varsigma)$ is bounded by some constant, and does not alter whatsoever the boundedness of the integrals. For large $|(\mathbb{T}^\alpha_{s-t})^{-1}z|$, we split as in the proof of Proposition A.2, the additional terms do not change the definition of the coefficient. However, the first term is no more zero, due to the presence of $\int_{S^{nd-1}} |\langle \bar{q}, \varsigma\rangle|^\alpha \tilde{\nu}(d\varsigma)$ under the integral, even for $k = 0$. □

Comparing the major term in each expansion on $\phi^\nu_{\Lambda_s}$ to the corresponding term in p_{Λ_s}, we get the following corollary:

Corollary B.2. *There is a positive constant C such that, for all $s \in [t,T]$, for all $z \in \mathbb{R}^{nd}$, the following bound holds:*

$$|\phi^\nu_{\Lambda_s}(z)| \leq \frac{C}{s-t} p_{\Lambda_s}(z).$$

We now turn to the control of the convolution kernel H. Recall from Section 5.3, the expression of H:

$$\begin{aligned} H(t,T,x,y) &= \frac{\delta \wedge |R_{t,T}(y)-x|^{\eta(\alpha\wedge 1)}}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} dp e^{-i\langle p, y - R_{T,t}(x)\rangle} \\ &\quad \times \left(\int_{S^{d-1}} \frac{|\langle p, B\sigma(t, R_{t,T}(y))\varsigma\rangle|^\alpha - |\langle p, B\sigma(t,x)\varsigma\rangle|^\alpha}{\delta \wedge |R_{t,T}(y)-x|^{\eta(\alpha\wedge 1)}} \mu(d\varsigma) \right) \\ &\quad \times \exp\left(-\int_t^T \int_{S^{d-1}} |\langle p, R^{1,\cdot}_{T,u} \sigma(u, R_{u,T}(y))\varsigma\rangle|^\alpha \mu(d\varsigma) \right). \end{aligned}$$

Observe that after a change of variables, we can write

$$\int_{S^{d-1}} \frac{|\langle p, B\sigma(t, R_{t,T}(y))\varsigma\rangle|^\alpha - |\langle p, B\sigma(t,x)\varsigma\rangle|^\alpha}{\delta \wedge |R_{t,T}(y)-x|^{\eta(\alpha\wedge 1)}} \mu(d\varsigma) = \int_{S^{d-1}} |\langle p^1, \varsigma\rangle|^\alpha m(d\varsigma),$$

where m is a signed measure such that $|m|$ is finite. Thus, from the above definition of $\phi^m_{\Lambda_s}(z)$, we have:

$$H(t,T,x,y) = \delta \wedge |R_{t,T}(y)-x|^{\eta(\alpha\wedge 1)} \phi^m_{\Lambda_s}(y - R_{T,t}(x)).$$

Applying the controls we obtained for $\phi^m_{\Lambda_s}$, we get the upper bound:

$$|H(t,T,x,y)| \leq C \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha\wedge 1)}}{T-t} \tilde{p}^{T,y}_\alpha(t,T,x,y). \tag{B.1}$$

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