Energetic Order for Optimization on Hierarchies of Partitions: Continuous hierarchy and Lagrange optimization

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Continuous hierarchy and Lagrange optimization *

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Abstract

In the current technical note we provide a topological generalization of hierarchy of partitions(HOP) structure, and the implications over the axioms of h-increasingness and scale increasingness [13]. Further in this study we will explicit the Lagrange optimization in the optimal cuts problem and the conditions necessary on the energy to obtain a global optimum using the a dynamic program. Further a general multi-constraint optimization problem is considered with multiple Lagrangian multipliers, leading to a general version of scale increasingness that orders cuts, by ordered tuples of multipliers. The report also differentiates Inf-Modularity and Submodularity and their space of application. The final demonstration on *wavering hierarchies* show how one can relax conditions on the hierarchical structure.

1 Introduction

What follows aims to build up a basic theory for optimizing the cuts in a hierarchy of partitions of the space, and study in particular the Lagrange type methods. The need for such hierarchies appears frequently in image processing, where segmentation (i.e. partitioning the space of definition of a function) is a major goal. Image compression by wavelets also belongs to the same sort of questions. In all these issues, the level of details to keep vary from place to place. In the photograph of a face, the features of the eyes carry more information than the wallpaper of the background. It can be convenient to keep all levels of details by generating a sequence of segmentations from fine to coarse. In a second step, one then chooses and picks up the good details at the good place. It is this second step that we would like to formalize here.

Most of the methods proposed in literature in image processing refer to the famous Lagrange approach and to the Lagrange-Euler Equation. Its links optimal segmentation of an image, considered as a numerical function in $\mathbb{R}^2$, have been studied for the first time by Mumford and Shah in their classical paper [17], followed by an abundant literature

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on the theme, e.g. the theoretical basis proposed in the monograph [16] by Morel and Solimini.

The idea of using a hierarchy of partitions for solving the Lagrange problem is developed [20], where Salembier and Garrido find the optimal cut for a given λ, but not the whole hierarchy of potential optimal cuts when the cost varies. In contrast, Guigues et Al. establish at once the series of optimal cuts, which permit them to solve the Lagrange problem [2] [11]. These authors focus on linear energies, which is a particular case, and introduce some singularity without telling what is really necessary with it.

Further Mumford-Shah functional minimization has been studied in the hierarchical space of component trees by [10],[5] and over a more general space of, hierarchies of partitions in [9] by Caselles et Al and has been further extended for level line selection in [24]. Lagrange optimization has also been extended to sets in [19], [12], [3], but, as far as we know, neither to partitions nor to hierarchies of partitions, except for a linear version cited above.

Two troubles If we want to build up a theory for Lagrange minimizations on partitions, two difficulties arise. Firstly, the number of ways to partition a set increases exponentially with its number of elements, and is approximately multiplied by ten for each new element added. According to a classical formula due to E.T. Bell [6], a small square image of 5 by 5 pixels can partitioned in half a billion of billion of different manners [1]! Even if we make the range of energies extend from one to one million, which is probably too much, even then, each energy corresponds on average to 500 billions of different segmentations of the small square. We are very far away from the classical Lagrange situation of minimizing a function $f(x, y)$ represented by a derivable surface in $\mathbb{R}^3$, and we can wonder which surreptitious assumption underlies the approaches which work (Salembier, Guigues, etc.).

The second trouble is the concern of convexity. A numerical function f defined on a vector space V is convex when, in the product space $V \times \mathbb{R}$, the set of points above the graph $\{x, f(x)\}$ is convex. Such a function admits a unique minimum, hence its interest in optimization problems. But there is no such thing as a vector space for partitions. What could be the weighted average of two partitions? This useful tool for grasping minima in optimizations vanishes here... We must build up new concepts, adapted to partitions.

Lagrange's optimization for point functions In the classical point-wise Lagrange optimization, the purpose is to find the points $\mathbf{x} \in \mathbb{R}^n$ of coordinates $\mathbf{x} = \{x_1..., x_n\}$ where the numerical function $\omega_\varphi(\mathbf{x})$ is minimal, when $\mathbf{x}$ are constrained by the relations

$$\begin{aligned} & \underset{x}{\text{minimize}} && \omega_\varphi(\mathbf{x}) \\ & \text{subject to} && \omega_\partial(\mathbf{x}), \omega_\epsilon(\mathbf{x}) \leq 0, \theta(\mathbf{x}) = 0. \end{aligned} \tag{1}$$

ω_φ is called the objective function, ω_∂, are inequality constraints, and θ an equality constraint. One can prove [8] that the $\mathbf{x}$ with Lagrange multipliers λ, ν, and ρ are then solutions of the system composed by the equations (1) plus the n derivatives

$$\nabla\omega_\varphi(\mathbf{x}) = \nabla\omega_\partial(\mathbf{x}) + \lambda\nabla\theta(\mathbf{x}) + \nu\nabla\omega_\epsilon(\mathbf{x}) \tag{2}$$

[1] One often estimates at 10^{80} the number of particles of the universe. Thus the number of ways to partition a 250×250 square, which is higher than 10^{200} equals the particles of billions of billions of universes.

The expression $\omega_\varphi + \lambda\omega_\partial + \nu\omega_\epsilon + \rho\theta$ is called Lagrangian, and the numbers λ, ν, and ρ are the Karush Kuhn Tucker KKT(and Lagrangian for equality constraint) multipliers. There is expansive work on the conditions on functions $\omega_\partial(\mathbf{x}), \omega_\epsilon(\mathbf{x}), \theta(\mathbf{x})$ to ensure convexity [8].

Lagrange type approach for partitions Given an energy ω_φ, we also would like to find a cut which minimizes ω_φ on $\Pi_H(E)$ under some energy constraint ω_∂. In contrast with the classical Lagrange formalism, there are no longer points here (they are replaced by the leaves), and the space E, does not always lend itself to derivatives. Moreover, as we saw, the idea of convexity which underlies most of the methods in optimization [8] does not apply here, unfortunately.

Since Lagrange's starting points seems unrealistic here, why not to start from his arrival point, i.e. from Lagrangians of the type (2)? For the convenience, we begin with one constraint only, and introduce the energy ω_λ

$$\omega_\lambda(\pi) = \omega_\varphi(\pi) + \lambda\omega_\partial(\pi) \qquad \pi \in \Pi_H(E). \tag{3}$$

Plan of the paper From this starting point a twofold approach can be sketched. Independently of the Lagrange model (3), which is rather particular, we must firstly

1. give to "minimal cut" a sense which ensures that such a cut exists and is unique;
2. characterize the families $\{\omega_\lambda\}$ convenient for the optimization purposes;
3. obtain explicitly the minimal cuts π_λ^*.

In a second part, we shall come back to the Lagrange model, and specify the conditions under which a (non constrained) minimal cut π_λ^* relative to energy ω_λ is also the cut which minimizes ω_φ under the constraint ω_∂. Before entering the issue of constrained Lagrange energy, we thus present the three major notions of a singular, of a scale increasing energy, and of a h-increasing. Already partly introduced in [13] in a finite framework, they are re-formulated below in a broader context, and with additional properties.

2 Hierarchies of partitions

This section extends to the continuous case the basic notions on hierarchies of partitions of [13]. We keep the same notation and vocabulary (cut, class, etc.).Consider an arbitrary space E, which may be discrete or not, finite or not, topological or not. $\mathcal{P}(E)$ stands for its power set. A partition of E into classes S is an extensive mapping $S : E \to \mathcal{P}(E)$ such that

$$x, y \in E \quad \Rightarrow \quad S(x) = S(y) \quad \text{or} \quad S(x) \cap S(y) = \varnothing \tag{4}$$

Below, the partitions of E are given the generic symbol π. The set of all partitions of E forms a complete lattice for the partial ordering of the refinement, where $\pi_i \leq \pi_j$ when the class $S_i(x)$ of π_i at any point $x \in E$ is included in the class $S_j(x)$ of π_j at the same point:

$$\pi_i \leq \pi_j \quad \Leftrightarrow \quad S_i(x) \subseteq S_j(x). \tag{5}$$

The refinement infimum of a family $\{\pi_i, i \in I \subseteq \mathbb{R}\}$ in $\Pi(E)$ is the partition π whose class at point x is $\cap S_i(x)$, and the refinement supremum is the finest partition π' such that $S_i(x) \subseteq S'(x)$ for all $i \in I$ and $x \in E$.

Hierarchy We will define a hierarchy as a family of partitions characterized by three axioms:

Definition 1 *A family $\{\pi_i, i \in I \subseteq \mathbb{R}\}$ of partitions of E defines a hierarchy when*

(i) the partitions π_i form a chain for the refinement ordering, i.e.

$$H = \{\pi_i, i \in I\} \quad \textit{with} \quad i \leq k \;\Rightarrow\; \pi_i \leq \pi_k, \qquad I \subseteq \mathbb{R}, \tag{6}$$

(ii) the chain is closed. Its finest partition π_0 is called the leaves and its coarsest one $\{E\}$ is called the root,

$$\pi_0 \leq \bigwedge\{\pi_i, i \in J\} \textit{ with } \pi_0 \in H \tag{7}$$

(iii) the map $i \to \pi_i$ is $\uparrow$ continuous, i.e. for any sub family $J \subseteq I$

$$i_0 = \bigvee\{i \in J\} \quad \Rightarrow \quad \pi_{i_0} = \bigvee\{\pi_i, i \in J\}. \tag{8}$$

One writes $i \uparrow i_0 \;\Rightarrow\; \pi_i \uparrow \pi_{i_0}$. The notation $\{E\}$ stands for the partition of E in a single class. Relation (8) makes more precise the merging process in continuous cases. Consider, for example, a hierarchy H of two partitions and a range of scales $I = [0, 1]$. For $i < 0.5$ one has the finer partition π_1, and for $i > 0.5$ the larger one π_2. The axiom (8) states that one has π_1 for $i = 0.5$. In particular, the axiom (8) is automatically satisfied when the set I is finite. In this case, the first axiom suffices for characterizing hierarchies [7], [4]. But set I may be infinite, even in discrete backgrounds. Then the axiom (7) cannot be dropped. For example, the nested partitions of $\overline{\mathbb{Z}}$:

$$\begin{array}{ll}
i = 0 & E = [-\infty, +\infty] \\
i = 1 & \pi_1 = [-\infty, -1]; \{0\}; [1, +\infty] \\
i = 2 & \pi_2 = [-\infty, -2]; \{-1\}; \{0\}; \{+1\}; [2, +\infty] \\
i = 3 & \pi_3 = [-\infty, -3]; \{-2\}\{-1\}; \{0\}; \{+1\}; \{+2\}; \{+3\}; [3, +\infty] \\
\ldots\ldots\ldots &
\end{array}$$

for all integers $i \geq 0$, does not suffice for generating a hierarchy, though it satisfies Rel.(6) up to a change of sign, and needs to be completed by the leaves level $\wedge \pi_i$.

Classes A hierarchy can be described from its classes, or nodes. The axiom (6) means that at each point $x \in E$ the family of all classes $S_i(x)$ containing x forms a closed chain of nested elements in $\mathcal{P}(E)$, from $S_0(x)$ to E. Let $\mathcal{S}$ =$\{S_i(x), x \in E, i \in I\}$ be the family of all classes of H. One directly extends to $\mathcal{S}$ the characterization (4) of a partition by its classes by putting

$$i \leq j \text{ and } x, y \in E \quad \Rightarrow \quad S_i(x) \subseteq S_j(y), \text{ or } S_i(x) \supseteq S_j(y), \text{ or } S_i(x) \cap S_j(y) = \varnothing. \tag{9}$$

One can also transpose the axiom (7) in terms of classes:

$$x \in E \text{ and } S_i(x) \in \mathcal{S} \Rightarrow \quad S = \cap\{S_i(x), i \in I\} \in \mathcal{S}, \tag{10}$$

and as well Axiom (8), which becomes

$$i \uparrow i_0 \text{ , } x \in E, \text{ and } S_i(x) \in \mathcal{S} \quad \Rightarrow S_i(x) \uparrow S_{i_0}(x)), \text{ with } S_{i_0}(x) \in \mathcal{S}. \tag{11}$$

Below, the symbol $\sqcup$ is used for expressing that groups of classes are concatenated, i.e.

$$S = S_1 \sqcup S_2 \Leftrightarrow S = S_1 \cup S_2 \textit{ and } S_1 \cap S_2 = \varnothing.$$

Cuts and energy A *cut* of H is a partition of the space E into classes taken in $\mathcal{S}$. The symbol $\Pi_H(E)$ stands for the set of all cuts of H. The notion of a partition applies also to any set $A \in \mathcal{P}(E)$. One then speaks of partial partition (in short p.p.) of support A [18]. If $A \in \mathcal{S}(H)$, then $\Pi_H(A)$ denotes the family of all partial partitions of A whose classes are in $\mathcal{S}(H)$. In particular, $\Pi_H(A)$ contains the one class p.p. denoted by $\{A\}$. An *energy* ω is a non negative function over the set $\mathcal{D}(E)$ of the partial partitions of E.

3 Singular energies

First of all, what does "minimal cut" mean? A cut of minimal energy? If so, how to go from energies to cuts? Remember the instructive result given introduction (a small set of only 25 leaves generates 0.5×10^{18} different partitions). We propose to drop the lattice of the positive numbers, and to act directly on some lattice of *cuts*, which should involve the energy ω by some modalities. Then the existence and unicity of minimal cuts will be ensured by this lattice structure. In fact, we do not need more than a semi-lattice, i.e. an ordered set where every family of elements admits a unique infimum (maximizations and always possible by changing ω into $K - \omega$, but a semi-lattice does not permit to work on suprema and infima at the same time).

The key notion required here is that of *singularity.*

Definition 2 *Let $\mathcal{D}(E)$ be the set of the partial partitions of E. A energy ω on $\mathcal{D}(E)$ is singular when*

i) the energy $\omega(\{S\})$ of every one class partition $\{S\}$ differs from the energy of any p.p. $\pi(S)$ of S, i.e.

$$\pi(S) \in \Pi_H(S) \;\Rightarrow\; \omega(\{S\}) \neq \omega(\pi(S)), \tag{12}$$

ii) the inequality of energies passes at the limit:

$$S_i \uparrow S_{i_0} \quad \textit{and} \quad \omega(\{S_i\}) < \omega(\pi(S_i)) \Rightarrow \omega(\{S_{i_0}\}) < \omega(\pi(S_{i_0})). \tag{13}$$

The second axiom, always satisfied in finite cases, has to be demanded otherwise. Since we are dealing with hierarchies, the monotone convergence (which does not require a topology) is sufficient. As a counter example, take for E the segment $[0,1]$, and for stack of classes S_i at point x the sequence $]2^{-1},1[,]2^{-2},1[, ..]2^{-i},1[...]0,1[$. For an energy ω which equals zero when the length ℓ of $S_i < 1$ and 1 when not, Axiom (13) is not satisfied, though the energy $\omega(S_i) = \ell(S_i)$ is acceptable.

3.1 Energetic ordering and lattices

Consider now two cuts $\pi, \pi' \in \Pi_H(E)$ of a hierarchy H, and the two classes $S \in \pi$ and $S' \in \pi'$ which contain the point $x \in E$. We have either $S \subseteq S'$, or $S' \subseteq S$. In the first case, S' is the support of a certain p.p. of π, a say, and in the second case, S is the support of a p.p. of π', a' say (see Figure 1). Intuitively, one may assess that π is less energetic than π' in $S \cup S'$ when the binary relation defined by

$$\textit{i)} \text{ either } S \subset S' \text{ and then } \omega(a) < \omega(\{S'\}), \tag{14}$$

$$\textit{ii)} \quad \text{or } S' \subset S \text{ and then } \omega(\{S\}) < \omega(a'), \tag{15}$$

$$\textit{iii)} \quad \text{or } S' = S \text{ and then } \omega(\{S\}) = \omega(\{S'\}) \tag{16}$$

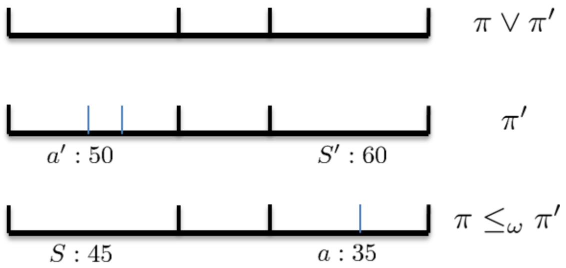

Figure 1: An example of energetic ordering: $\pi \preceq_\omega \pi'$ because in each class of $\pi \vee \pi'$, the ω energy is smaller or equal.

is true. Let us denote it by $S \preceq_\omega S'$. Which condition on the energy ω makes $\preceq_\omega$ an ordering relation on cuts? The answer is given by the following result (proof given in [13] and reproduce in annex):

Proposition 3 *Let H be a hierarchy of partitions of E, and π, π' be two cuts of H. The implication*

$$S \in \pi \text{ and } S' \in \pi' \ \Rightarrow S \preceq_\omega S', \tag{17}$$

for all pairs of classes of S and S', induces an ordering on the set $\Pi_H(E)$ of all cuts of H if and only if ω is singular. One calls it energetic ordering w.r.t. ω, and one writes $\pi \preceq_\omega \pi'$, $\pi, \pi' \in \Pi_H(E)$.

The energetic ordering induces an inf semi-lattice on the set $\Pi_H(E)$ of all cuts of H :

Theorem 4 *The set $\Pi_H(E)$ of all cuts of $H(E)$ forms an inf semi-lattice for the energetic ordering $\preceq_\omega$. Given a family $\{\pi_j, j \in J\}$ of cuts in $\Pi_H(E)$, the class $T(x)$ at point $x \in E$ of the infimum $\pi = \wedge_\omega \pi_j$ is the largest class of the π_j to be less energetic than the partial partitions of the π_j whose $T(x)$ is the support. .*

Proof. For the sake of clarity, we firstly give the proof for finite families, and next we extend it to the infinite case. Let $\mathcal{S}(x) = \{S_j(x), j \in J\}$ be set of all classes of the finite family $\{\pi_j, j \in J\}$ at point x. Consider the class $S_j(x)$. Some partitions π_k of the family may cut up, in $S_j(x)$, a p.p. $a_{k,j}(x) = \pi_k \sqcap \{S_j(x)\}$. If $\{S_j(x)\}$ is less energetic than $a_{k,j}(x)$, then we pick it out as a candidate, and we take the largest candidate $T(x)$, as j spans J, to be the class at x of the cut $\pi = \sqcup\{T(x), x \in E\}$. If, whatever $j \in J$ the class $S_j(x)$ is never cut up by an internal p.p. $\pi_k \sqcap \{S_j(x)\}$ more energetic than $S_j(x)$, then we take the smallest element $\cap\{S_j(x), j \in J\}$ as class $T(x)$ at point x in the cut π. The partition π is $\preceq_\omega$than all π_j since, on the one hand, each class $T(x)$ is $\preceq_\omega$ than the p.p. induced on it by a π_j, and, on the other hand, each class $S_j(x)$ of a π_j that contains $T(x)$ is more energetic than the p.p. $\pi_j \sqcap \{T(x)\}$.

Moreover, cut π is the largest lower bound of the family $\{\pi_j, j \in J\}$. If another lower bound $\widetilde{\pi}$ is not $\preceq_\omega \pi$, two possibilities arise. Either, for some x, the class $T(x)$ is the support of a p.p. of $\widetilde{\pi}$ both more energetic than $T(x)$ and less than the $\pi_j \sqcap T(x)$, which is impossible by definition of $T(x)$, or for some $y \in E$, a class $\widetilde{S(y)}$ of $\widetilde{\pi}$ covers $T(x)$ and is more energetic than the restriction of π to $\widetilde{S(y)}$ but less than the $\pi_j \sqcap \widetilde{S(y)}$. Again this is impossible, because the π_j classes $\supseteq T(y)$ are more energetic than $T(y)$. Therefore the finite family $\{\pi_j, j \in J\}$ admits π as a unique largest lower bound, i.e. $\pi = \wedge_\omega \pi_j$.

Suppose now J infinite. The class $T(x)$ of the previous proof was obtained by a finite union of classes of H which now may become infinite. But Rel. (8) ensures us that $T(x)$ is still a class of H (possibly not in the family $\{\pi_j, j \in J\}$). In addition, axiom (13) of the singular energies ensures us that the energy of $T(x)$ is minimal like it was in the finite case, which achieves the proof. ■

In particular the universal infimum of the inf semi-lattice $\Pi_H(E)$ is denoted by $\pi^* = \wedge_\omega\{\pi, \pi \in \Pi_H(E)\}$. It is the unique cut of H smaller than all the other cuts of $\Pi_H(E)$ for the ordering $\preceq_\omega$.

For illustrating the theorem, let us give the two examples of the finite case and of that of open classes.

Finite case In the applications, the number of leaves is finite, thus also is the number of possible cuts. In this finite case, the axiom (13), which intervenes in the infinite case only, has no longer reason for being. Clearly, the rest of the proof of the theorem may be presented, just as well, in terms of suprema, which leads to [13]:

Corollary 5 *When the number of leaves of H is finite, then the energetic ordering $\preceq_\omega$ induces a lattice on the cuts $\Pi_H(E)$.*

We can also remark that, in the finite case, the proof of the theorem provides a climbing procedure for finding π^* which is passably combinatorial. But the h-increasingness of section 5 reduces it to a greedy algorithm.

Figure 2 depicts a toy example of the energetic infimum $\wedge_\omega$. At point x the larger class to be less energetic than its internal p.p. is that of π_2 and at point y it is the class of π_1. The energetic infimum $\wedge_\omega$ is the partition drawn at the bottom. If we replace 14 by 11 in the energy of π_3, the $\wedge_\omega$infimum becomes the one class partition π_3.

Topological case Though the above algebraic framework suffices for the main theorem 4, it tolerates many "pathological" partitions. But we can try and restrict it to a topological structure more adapted to model the segmentations met in practice. The space E under study is now topological, and $\mathcal{G}(E)$ stands for the set of all open sets of E plus all points of E. We consider the set $\Pi(\mathcal{G})$ of all partitions of E into classes of $\mathcal{G}(E)$. Such partitions describe for example the geometry of the Voronoi polygons in $\mathbb{R}^2$, or the skeletons by zone of influence [21], where a locally finite number of open classes are separated by simple arcs. The set $\mathcal{G}(E)$ forms a complete lattice for the partial ordering of the refinement, where $\pi_1 \leq \pi_1$ when the open classes of π_1 are contained in those of π_2. The infimum becomes now the interior of $\cap S_i(x)$, and the supremum the interior of the smaller upper bound. The definition of an energy now includes that the points classes of $\mathcal{G}(E)$ have always a zero weight. The definition of a singular energy is the same as previously.

If we focus on a hierarchy H and a singular energy ω, the proposition 3 on the energetic order and the theorem 4 on the energetic lattices remain valid. In what follows, all notions (scale increasingness, h-increasingness, inf-modularity, Lagrange families) and theoretical results apply for both families $\Pi(E)$ and $\mathcal{G}(E)$. For the presentation, we keep the algebraic framework of $\Pi(E)$.

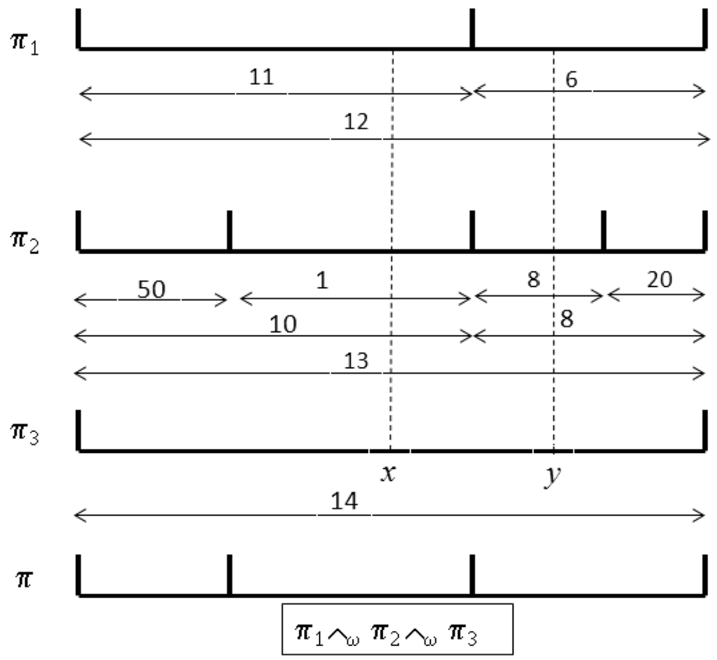

Figure 2: Energetic infimum of three partitions

3.2 The three lattices

We see, finally, that a hierarchy H admits a unique minimal cut for the energetic ordering $\preceq_\omega$ if and only if ω is singular. As we purpose to compare minimal cuts depending on a scale parameter λ, we must be sure that each of them is unique, and thus suppose that the energy $\omega(\lambda)$ is singular. If we drop this assumption, the hierarchy of minimal cuts of Theorem 6 no longer exist, and we also loose the theorem 16, hence the solution to the Lagrange minimization we contemplate. Fortunately, the singularity hypothesis is not very restricting in practice, where most of the energies admit a singular version, up to minor changes.

In the notation, one distinguishes the refinement lattice from the ω-semi-lattice by using for the former the three symbols $\leq, \vee$, and $\wedge$, without ω subscript. Below, the expression "minimal cuts" always refers to energy infima $\wedge_\omega$, the only ones for which the expression makes sense. This meaning is indeed twofold: it associates an energetic minimum with *each class* of $\pi \vee \pi'$, and also *globally*, to the whole cuts π.

Three lattice interact in the present study. There is the numerical one ($\leq, \vee$, and $\wedge$) for energies, that of the refinement ($\leq, \vee$, and $\wedge$) for partitions, and the energetic $\wedge_\omega$ semi-lattice, again for partitions. The relations between the last two form the matter of the next section, and the relations between the first and the third will be investigated afterwards.

4 Scale increasing families of energies

We thus begin by comparing the energetic ordering $\preceq_\omega$ with that $\leq$ of the refinement, when both apply on the partitions $\Pi_H(E)$. At a first glance, no relation between them sticks out a mile. They do not hold on the same features of the partitions. But we can enlarge the scope and consider a family $\{\omega_\lambda, \lambda \geq 0\}$ of singular energies which act on the same partitions $\Pi_H(E)$. Each energy ω_λ induces a minimal cut π^*_λ. Would it be possible to order them for the refinement, so that the relation (19) be true?

By setting the problem this way, we automatically reduce the set of p.p. under study. We only need to deal with the p.p. that generate the family $\{\pi^*_\lambda, \lambda \geq 0\}$ of the minimal cuts, a collection incomparably smaller than all the p.p. of H, as involved in Theorem 4. We will denote that family by $\mathcal{D}^*$

$$\mathcal{D}^* = \{\pi, \pi \in \mathcal{D}(E), \pi \sqsubseteq \pi^*_\lambda, \lambda > 0\}$$

and denote by

$$\mathcal{S}^* = \{S, \{S\} \sqsubseteq \pi^*_\lambda, \lambda > 0\}$$

the family of all classes of the minimal cuts. A convenient starting point is provided by the notion of *scale increasingness* of the family $\{\omega_\lambda\}$[13], i.e. by the axiom (18):

$$\lambda \leq \mu \text{ and } \omega_\lambda(\{S\}) \leq \omega_\lambda(\pi) \Rightarrow \omega_\mu(\{S\}) \leq \omega_\mu(\pi), \qquad S \in \mathcal{S}^*,\ \pi \in \mathcal{D}^*,\ \pi \sqsubseteq \{S\} \quad (18)$$

These inequalities become strict when the scale increasing ω_λ are singular energies. Then the minimal cuts form a hierarchy:

Theorem 6 *Let $\{\omega_\lambda, \lambda \geq 0\}$ be a family of singular energies acting on a hierarchy H. Their minimal cuts $\{\pi^*_\lambda, \lambda \geq 0\}$ for the ω_λ-lattices are ordered by refinement, i.e.:*

$$\lambda \leq \mu \quad \Rightarrow \quad \pi^*_\lambda \leq \pi^*_\mu \qquad \lambda, \mu \geq 0. \quad (19)$$

if and only if the family $\{\omega_\lambda, \lambda \geq 0\}$ is scale increasing.

Proof. Suppose that the family is scale increasing, and denote by S_λ (resp. S_μ) the class of π^*_λ (resp. π^*_μ) that contains the point $x \in E$. As H is a hierarchy, have either $S_\lambda \subseteq S_\mu$, or $S_\mu \subseteq S_\lambda$. If $S_\mu \subset S_\lambda$ there exists a partial partition a_μ of π^*_μ of support S_λ, whose ω-energy is $> \omega_\lambda(\{S_\lambda\})$, because π^*_λ is the $\wedge_{\omega(\lambda)}$infimum of the cuts of H. Then by scale increasingness $\omega_\mu(\{S_\lambda\}) \leq \omega_\mu(a_\mu)$. In Besides, as π^*_μ is the minimal cut for $\wedge_{\omega(\mu)}$, the p.p. a_μ is strictly less ω_μ-energetic than its support $\{S_\lambda\}$, i.e. $\omega_\mu(a_\mu) < \omega_\mu(\{S_\lambda\})$, which contradicts the previous inequality. Thus $S_\lambda \subseteq S_\mu$, and as this inclusion is valid for all $x \in E$, we finally obtain $\pi^*_\lambda \leq \pi^*_\mu$.

Conversely, suppose that there exists a set $S \subseteq E$ for which $\omega_\lambda(\{S\}) \leq \omega_\lambda(\pi), \pi \in \Pi_H(S)$ does not imply $\omega_\mu(\{S\}) \leq \omega_\mu(\pi)$, hence implies $\omega_\mu(\{S\}) > \omega_\mu(\pi)$. It means that $\{S\}$, which is a class of π^*_λ, is replaced by π in π^*_μ, so that $\pi^*_\lambda \nleq \pi^*_\mu$, which achieves the proof. ■

In addition, we also have $\pi^*_\lambda \preceq_{\omega_\lambda} \pi^*_\mu$ by scale increasingness (18). In the continuous cases, an additional condition of monotony can make the theorem more precise. Indeed, it would be nice to know whether $\pi^*_\mu \downarrow \pi^*_\lambda$ as $\mu \downarrow \lambda$. Let $S_\mu = S_\mu(x)$ the class of π^*_μ at point x, and $S'_\lambda = \cap\{S_\mu, \mu \geq \lambda\}$. We add to the axiom (18) of scale increasingness the following condition of monotony

$$\omega_\mu(\{S_\mu\}) < \omega_\mu(\{S_\mu\} \sqcap \pi) \text{ for all } \mu > \lambda \ \Rightarrow \ \omega_\lambda(\{S'_\lambda\}) < \ \omega_\lambda(\{S'_\lambda\} \sqcap \pi) \quad (20)$$

Suppose that $\pi^*_\mu \not\downarrow \pi^*_\lambda$. Then there exists a class S_λ of π^*_λ such that $S_\lambda \subset S'_\lambda$, which implies $\omega_\lambda(\{S'_\lambda\} \sqcap \pi^*_\lambda) \leq \omega_\lambda(S'_\lambda)$ since $\{S'_\lambda\} \sqcap \pi^*_\lambda$ is a p.p. of the minimal cut for the energy ω_λ. Besides, we draw from (20) that $\omega_\lambda(S'_\lambda) < \omega_\lambda(\{S'_\lambda\} \sqcap \pi)$ a contradiction which means that $\pi^*_\mu \downarrow \pi^*_\lambda$, so that we can state

Corollary 7 *When the family $\{\omega_\lambda, \lambda \geq 0\}$ is monotone in the sense of the condition (20), then the increasing mapping $\lambda \to \pi^*_\lambda$ is right continuous, i.e.*

$$\mu \downarrow \lambda \quad \Rightarrow \quad \pi^*_\mu \downarrow \pi^*_\lambda \tag{21}$$

In finite model, the conditions (20) and (21) are always satified, since λ is the smallest of the μ. In the topological model, one must take the interior of the set $S'_\lambda = \cap\{S_\mu, \mu \geq \lambda\}$ to express the monotone convergence $\pi^*_\mu \downarrow \pi^*_\lambda$.

The following corollary shows how to easily construct scale increasing families:

Corollary 8 *When the map $\lambda \to \omega_\lambda$ is increasing, then the family $\{\omega_\lambda\}$ is scale increasing.*

Proof. For $\lambda \leq \mu$ and $\pi \in \Pi_H(S)$, we have $\omega_\lambda(S) \leq \omega_\mu(S)$ and $\omega_\lambda(\pi) \leq \omega_\mu(\pi)$. By difference, it comes $\omega_\lambda(\pi) - \omega_\lambda(S) \leq \omega_\mu(\pi) - \omega_\mu(S)$. Hence, when $\omega_\lambda(\pi) - \omega_\lambda(S) \geq 0$, then $\omega_\mu(\pi) - \omega_\mu(S) \geq 0$, i.e. the axiom (18). $\blacksquare$

Usual energies, like $\omega_\lambda = \omega_\varphi + \lambda\omega_\partial$, $\omega_\lambda = \omega_\varphi \vee \lambda\omega_\partial$, or $\omega_\lambda = \omega_\varphi \wedge \lambda\omega_\partial$ lead thus to hierarchies of minimal cuts. This nice property can be used for compressing a hierarchy by reducing the number levels in a significant manner.

5 h-increasing energies

This section is devoted to the links between an energetic ordering $\preceq_\omega$ on the cuts $\Pi_H(E)$ and the numerical ordering of the energies of these cuts. The question is set at the level of one energy ω, and no longer for a family $\{\omega_\lambda\}$ as previously, and scale increasingness is not assumed anymore in this section.

The theorem 4 says nothing about the energy of a minimal cut, and does not tell whether the energetic ordering $\pi \preceq_\omega \pi'$ between two cuts implies the same sense of variation for the energies themselves, i.e. $\omega(\pi) \leq \omega(\pi')$. Indeed, one easily sees that it is not always the case. For example, take for singular energy $\omega(\pi) = 0$ (resp. 1) when the number of classes of the p.p. π is odd (resp. even). Then, in Figure 1, $\pi \preceq_\omega \pi'$ whereas $\omega(\pi) = 1$ and $\omega(\pi') = 0$. A new axiom is needed, namely the *h-increasingness:*

Definition 9 *Let (a_i, a'_i) be two different p.p. of the same support S_i, and $\{S_i(x), x \in E, i \in I\}$ a family of disjoint supports. A singular energy ω on the partial partitions $\mathcal{D}(E)$ is h-increasing when for every triplet $\{a_i, a'_i,\ S_i(x), x \in E, i \in I\}$ one has*

$$\omega(a_i) \leq \omega(a'_i) \text{ for all } i \in I \quad \Rightarrow \quad \omega(\sqcup a_i) \leq \omega(\sqcup a'_i). \tag{22}$$

For example, a linear energy, i.e. an energy where $\omega(\sqcup a_i)$ is the sum of the $\omega(a_i)$ is h-increasing, an even *strictly* h-increasing since

$$\omega(a_i) < \omega(a'_i) \text{ for all } i \in I \quad \Rightarrow \quad \omega(\sqcup a_i) < \omega(\sqcup a'_i). \tag{23}$$

Unlike, the h-increasing energy $\omega(\sqcup a_i) = \sum \omega(a_i)$ when $\sum \omega(a_i) < K$ and $= K$ when not, is not strictly h-increasing.

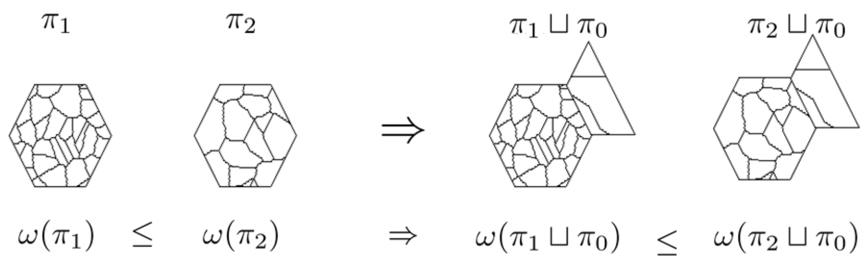

Figure 3: h-increasingness for the finite case Eqn(23).

The two orderings $\preceq$ and $\leq$ *h-increasingness* bridges the gap between the two orderings $\preceq$ and $\leq$ for partitions. Consider two cuts π and π' of a hierarchy H, and denote by $\{S_i, i \in I\}$ the set of all classes of $\pi \vee \pi'$. If a_i and a'_i stand for the p.p. of support S_i of π and π' respectively, and ω for a h-increasing energy, then the left member of (22) implies $\omega(\pi) \leq \omega(\pi')$, hence:

$$\pi \preceq_\omega \pi' \quad \Rightarrow \quad \omega(\pi) \leq \omega(\pi'). \tag{24}$$

with in particular

$$\pi^* \leq \wedge_\omega \{\pi \in \Pi_H(E)\} \quad \Rightarrow \quad \omega(\pi^*) \leq \omega(\pi) \qquad \forall \pi \in \Pi(H)$$

Additive energies, where $\omega(\sqcup_i a_i) = \sum \omega(a_i)$, and energies composed by supremum, where $\omega(\sqcup_i a_i) = \bigvee \omega(a_i)$, which are the two most popular ones, satisfy both the Rel.(22).

The axiom of *h-increasingness* has already been introduced in [13] for the case of a finite number of classes by the Rel.(25) below. The above definition 9 generalizes it to infinite situations:

Proposition 10 *When the family $\{a_i, a'_i \in \mathcal{D}(E), i \in I\}$ of Definition 9 is finite, then the h-increasingness is equivalent to:*

$$\omega(a) \leq \omega(a') \quad \Rightarrow \quad \omega(a \sqcup a_0) \leq \omega(a' \sqcup a_0), \quad a, a' \in \Pi_H(S) \tag{25}$$

where a and a' are two p.p. of same support S, and where a_0 is a p.p. of support S_0 disjoint of S.

Proof. The implication (22) $\Rightarrow$ (25) is obvious. For the reverse sense, consider the two pairs (a_1, a'_1) and (a_2, a'_2). The relation (25) allows us to write

$$\begin{aligned} \omega(a_1) \leq \omega(a'_1) &\quad \Rightarrow \quad \omega(a_1 \sqcup a_2) \leq \omega(a'_1 \sqcup a_2) \\ \omega(a_2) \leq \omega(a'_2) &\quad \Rightarrow \quad \omega(a'_1 \sqcup a_2) \leq \omega(a'_1 \sqcup a'_2) \end{aligned}$$

hence $\omega(a_1 \sqcup a_2) < \omega(a'_1 \sqcup a'_2)$. Under iteration, this inequality extends to any finite family $\{\omega(a_i), \omega(a'_i), i \in I\}$, i.e. to Relation (22). ■

Minimal cut and h-increasingness The finite definition (25) is the one introduced in [13].where one proves that it yields a greedy algorithm for scanning the classes of H only once ([2] and Proposition 4.3. in [13])

Proposition 11 *When ω is h-increasing and singular, then, in each sub hierarchy of H of root S, the unique cut of minimal energy is either $\{S\}$, or concatenation $\pi_1^* \sqcup \pi_2^* ... \sqcup \pi_p^*$ of the minimal cuts of the sons $T_1, T_2, ...T_p$ of S.*

For comparing the energy of the one class partition $\{S\}$ to the energies of all its descendants, it suffices to compare S to its sons. It results indeed in a simplified version of the algorithm given in the proof of Theorem 4 (which is established without assuming h-increasingness). The further descendants do not intervene. Moreover, if ω is not singular, one can always decide to choose $\omega(\{S\})$ when $\omega(\{S\}) = \omega(\pi), \pi \in \Pi_H(S)$. This choice makes ω singular and preserves its h-increasingness (Proposition 4.4 of [13]).

Discussion In fact, the energy ω is never minimized. It only intervenes to define energetic lattice $\wedge_\omega$ associated with each hierarchy, and the minimizations hold on the cuts of this lattice $\wedge_\omega$. The same occurs for the sequences $\{\omega_\lambda, \lambda \geq 0\}$ of scale increasing energies: the minimal cuts π_λ^* do not increase with the energy ω_λ, but with the *parameter* λ of this energy. Similarly, the h-increasingness gives a way from the partitions to their energies in Rel.(24), but not the way back, because this should validate a bijection between orderings of cuts and of energies. Now we saw that the number of partitions is incomparably higher than the possible energies. Everything has been settled indeed to get round this lack of bijection.

Roughly, the scale increasingness plays the role of a derivation, and the singularity plus h-increasingness compensate the lack of convexity. And at this theoretical level, no use is made of notions such as connectivity, saliency, or ultra-metrics.

6 Energies $\omega_\lambda = \omega_\varphi + \lambda\omega_\partial$

The above notions were the concern of any type of energy. We now concentrate on the energies of the type $\omega_\lambda = \omega_\varphi + \lambda\omega_\partial$. The intuition which drives us in the problem of minimizing ω_φ under the constraint ω_∂ is the following. As ω_λ is singular, it characterizes a unique minimal cut π_λ^* of hierarchy H, thus a specific pair $\omega_\varphi(\pi_\lambda^*)$, $\omega_\partial(\pi_\lambda^*)$ of energies. If the first one increases with λ and the second decreases, then the Lagrange problem will be solved. For a given cost C there will exist a smallest λ, λ_0 say, such that $\omega_\partial(\pi_{\lambda_0}^*) \leq C$ and the associated objective energy $\omega_\varphi(\pi_{\lambda_0}^*)$ will be the minimal energy which induces a cut $\pi_{\lambda_0}^*$ satisfying the cost constraint $\omega_\partial(\pi_{\lambda_0}^*) \leq C$.

The studies [20], [11] show the soundness of this approach when the energies ω_φ and ω_∂ are linear, i.e. are additive functions of the classes of the partial partitions. But their results rest on the linearity assumption, which is in fact a particular case: one finds in the literature energies which involve other operations, like suprema or infima [22], [1]. We must try and find a more comprehensive approach

6.1 Inf-modularity

We saw that when the family $\{\omega_\lambda\}$ is scale increasing, then the optimal cuts for $\wedge_{\omega(\lambda)}$ form a hierarchy (Rel.(19)). In the present case, the structure of ω_λ being more precisely defined by (3), can we hope stronger properties?

Definition 12 *An energy $\omega_\partial : \mathcal{D}(E) \to \mathbb{R}^+$ is said inf-modular when for each p.p. π of support $S \in \mathcal{P}(E)$ we have*

$$\omega_\partial(\{S\}) \leq \omega_\partial(\pi) \qquad \pi \in \Pi_H(S), S \in \mathcal{P}(E). \tag{26}$$

Notice that $\omega_\partial(\{S\}) \leq \bigwedge\{\omega_\partial(\pi), \pi \in \Pi_H(S)\backslash\{S\}\}$, which explains the terminology of inf-modularity.

inf-modularity and scale increasingness For the Lagrange type energies given by equation (3), the two notions of scale increasingness and of inf-modularity coincide, but the latter applies to ω_∂ only. More precisely, we can state:

Proposition 13 *The family $\{\omega_\lambda = \omega_\varphi + \lambda\omega_\partial, \lambda \geq 0\}$ is scale increasing if and only if ω_∂ is inf-modular.*

Proof. If ω_∂ is inf-modular, and $\mu > \lambda$, we have $(\mu - \lambda)\omega_\partial(\{S\}) \leq (\mu - \lambda)\omega_\partial(\pi)$, If in addition $\omega_\lambda(\{S\}) \leq \omega_\lambda(\pi)$, then by summing the two inequalities, we obtain $\omega_\mu(\{S\}) \leq \omega_\mu(\pi)$, and the scale increasingness is satisfied. Conversely, if the implication (18) holds, then by taking the difference between $\omega_\varphi(\{S\}) + \mu\omega_\partial(\{S\}) < \omega_\varphi(\pi) + \mu\omega_\partial(\pi)$ and $\omega_\varphi(\{S\}) + \lambda\omega_\partial(\{S\}) < \omega_\varphi(\pi) + \lambda\omega_\partial(\pi)$, we find $(\mu - \lambda)\omega_\partial(\{S\}) \leq (\mu - \lambda)\omega_\partial(\pi)$, i.e. Rel.(26), which achieves the proof. ■

The "only if" part of Proposition 13 is specific of the $\omega_\varphi + \lambda\omega_\partial$ type energies. For a family such as $\{\omega_\lambda = \omega_\varphi \vee \lambda\omega_\partial\}$ for example, the inf-modularity of ω_∂ implies the scale increasingness of the $\{\omega_\lambda\}$, but the converse is false.

inf-modularity and sub-modularity The concept of inf-modularity we just introduced is to be compared with the more classical notions of sub-modularity and sub-additivity, which appear in discrete optimizations, in graph cuts[15]. As these notions hold on sets of E, whereas the above inf-modularity is relative to the partial partitions of E, we firstly need to introduce some energy ω'_∂ on sets that corresponds to ω_∂, by putting

$$\omega'_\partial(S) = \omega_\partial(\{S\}), \qquad A \in \mathcal{P}(E), \quad \{A\} \in \mathcal{D}(E)$$

with $\omega'_\partial(\varnothing) = 0$. Then we must match sets and partial partitions in some sense, and the first rule which comes to the mind is the comparison of a p.p. π with its classes $T_j, 1 \leq j \leq p$. If we take

$$\omega_\partial(\pi) \leq \sum_{j=1}^{j=p} \omega_\partial(\{T_j\}), \tag{27}$$

then the inf-modularity of ω_∂ yields inequality

$$\omega'_\partial(S) = \omega_\partial(\{S\}) \leq \omega_\partial(\pi) \leq \sum_{j=1}^{j=p} \omega_\partial(\{T_j\}) = \sum_{j=1}^{j=p} \omega'_\partial(T_j),$$

with $\pi \in \Pi_H(S)$, which is nothing but the sub-modularity of ω'_∂ (i.e. the relation $\omega'_\partial(A \cup B)+ \omega'_\partial(A \cap B) \leq \omega'_\partial(A)+ \omega'_\partial(B)$ with here $A \cap B = \varnothing$). It is as well the sub-additivity condition. As one can observe, we passed from partial partitions to sets by the relation (27), which restricts the approach by a sub-linear condition. The way followed here, based on partial partitions (and no longer on sets) and on the inf-modularity, frees ourselves from this limitation. Moreover, unlike graph cuts, the method proposed here is not discrete, and its implementation not combinatorial (in case of h-increasingness, at least).

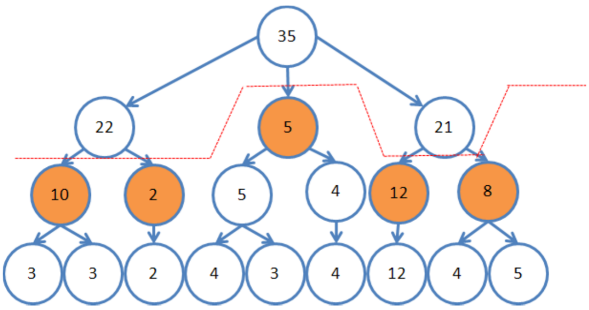

Figure 4: An example of minimization by inf-modularity (P. Soille).

An example of inf-modularity The example which follows was proposed by P. Soille and J. Grazzini for segmenting air-born and satellite images [22], [23]. They have indicated several variants, which all rest on a same idea. A family of previous segmentations of a $2-D$ function f led to hierarchy H. One wants to take the largest classes where function f is constant enough. Each node S is given the energy $\omega(S) = \max\{f(S)\} - \min\{f(S)\}$. The values of $f(S)$ obviously increase as going up in the hierarchy. A node S is kept when $\omega(S) \leq 20$). The minimal cut is then the union of the largest remaining nodes.

By so doing, we implicitly weight each partial partition by the supremum of the energies of its classes. Scan top-down the hierarchy. If at class S the energy is $\omega(S) > 20$, then one goes down to the sons T_k of S and look at $\sup\{\omega(T_k)\}$. If this supremum is ≤ 20, one stops, and all T_k belong to the minimal cut, if not one continues to go downwards.

Such a use of the suprema is relatively frequent. One may find another example in [1], and other ones based on combinations by infima in [14].

7 Lagrange optimization on hierarchies of partitions

We now analyze the properties of the minimal cuts for $\omega_\lambda = \omega_\varphi + \lambda\omega_\partial$ type energies.

Definition 14 *One calls* one parameter Lagrange family *any family* $\{\omega_\lambda = \omega_\varphi + \lambda\omega_\partial, \lambda \geq 0\}$ *where* ω_λ *is singular,* ω_φ *is h-increasing, and* ω_∂ *is inf-modular and h-increasing. Moreover, we suppose that the mappings* ω_φ *and* ω_∂*, as functions of* λ*, are right continuous.*

Given a hierarchy H, a Lagrange family provides a unique minimal cut π^*_λ of H with each λ, since ω_λ is singular. Moreover the inf-modularity of ω_∂ shows that these minimal cuts π^*_λ enlarge as λ increases (Proposition 13 and Theorem 6).

7.1 Minimal cuts

The h-increasingness of ω_φ and ω_∂ allow us to improve these first results.

Proposition 15 *Let* $\{\omega_\lambda = \omega_\varphi + \lambda\omega_\partial\}$ *be a one parameter Lagrange family, and let* H *be a hierarchy. The sequence of energies* $\{\omega_\partial(\pi^*_\lambda)\}$ *decreases with* λ*, and the sequence*

$\{\omega_\varphi(\pi^*_\lambda)\}$ *increases with* λ, *with the monotone convergence*

$$\pi^*_\mu \downarrow \pi^*_\lambda \quad \Rightarrow \quad \omega_\varphi(\pi^*_\lambda) \uparrow \omega_\varphi(\pi^*_\mu) \ \ and \ \ \omega_\partial(\pi^*_\lambda) \downarrow \omega_\partial(\pi^*_\mu) \tag{28}$$

Proof. We have to prove that $\omega_\partial(\pi^*_\lambda)$ is an decreasing function of λ, and that $\omega_\varphi(\pi^*_\lambda)$ is an increasing one. Then the right contin uity of these two energies will lead to the implication (28). Suppose $\lambda < \mu$ and let S_μ be a class of π^*_μ. As ω_∂ is inf-modular, the energy ω_λ is scale increasing. Theorem 6 applies, and $\pi^*_\lambda \leq \pi^*_\mu$. The class S_μ is thus the support of a p.p. a_λ of π^*_λ, and we can write

$$\lambda < \mu \quad \Rightarrow \ [\text{either } a_\lambda = \{S_\mu\} \ \text{ or } \ \omega_\partial(a_\lambda) > \omega_\partial(\{S_\mu\})] \quad \Rightarrow \ \pi^*_\mu \preceq_{\omega_\partial} \pi^*_\lambda,$$

which leads to $\omega_\partial(\pi^*_\lambda) \geq \omega_\partial(\pi^*_\mu)$ by h-increasingness of ω_∂. On the other hand, as π^*_λ is minimal cut in the energetic lattice $\wedge_{\omega_\lambda}$, we have either $a_\lambda = \{S_\mu\}$, or

$$\omega_\varphi(a_\lambda) + \lambda\omega_\partial(a_\lambda) < \omega_\varphi(\{S_\mu\}) + \lambda\omega_\partial(\{S_\mu\}),$$

i.e.

$$\omega_\varphi(\{S_\mu\}) - \omega_\varphi(a_\lambda) > \lambda[\omega_\partial(a_\lambda) - \omega_\partial(\{S_\mu\})] > 0,$$

hence

$$\lambda < \mu \quad \Rightarrow \ [\text{either } a_\lambda = \{S_\mu\} \ \text{ or } \ \omega_\varphi(a_\lambda) < \omega_\varphi(\{S_\mu\})] \quad \Rightarrow \ \pi^*_\lambda \preceq_{\omega_\varphi} \pi^*_\mu \ . \tag{29}$$

which leads to $\omega_\varphi(\pi^*_\lambda) \leq \omega_\varphi(\pi^*_\mu)$ by h-increasingness of ω_φ, and achieves the proof. ■

Hence, the two energies ω_∂ and ω_φ vary in opposite sense on the minimal cuts.

7.2 Constrained optimization on Lagrange families

The energies of the minimal cuts in a Lagrange family allow us to solve the problem of minimizing ω_φ under a cost constraint holding on ω_∂:

Theorem 16 *Let* $\{\omega_\lambda = \omega_\varphi + \lambda\omega_\partial, \lambda \geq 0\}$ *be a one parameter Lagrange family acting on a hierarchy* H *of partitions of set* E. *As* λ *varies, let* $\{\pi^*_\lambda\}$ *be the minimal cuts of* H *w.r.t. the* ω_λ. *For a given cost* C, *when there exists no* λ *such that* $\omega_\partial(\pi^*_\lambda) \leq C$, *the constrained minimization is impossible. When not, then there exists a cut* $\pi^*_{\lambda_0}$ *of minimal energy* $\omega_\varphi(\pi^*_{\lambda_0})$ *under the constraint* $\omega_\partial(\pi^*_\lambda) \leq C$, *with*

$$\lambda_0 = \inf\{\lambda \mid \omega_\partial(\pi^*_\lambda) \leq C\}. \tag{30}$$

If $\omega_\varphi(\pi^*_\lambda) > \omega_\varphi(\pi^*_{\lambda_0})$ *for* $\lambda > \lambda_0$, *this cut is unique . If not, all* π^*_λ, $\lambda_0 \leq \lambda < \lambda_1$, *with*

$$\lambda_1 = \sup\{\lambda \mid \omega_\varphi(\pi^*_\lambda) = \omega_\varphi(\pi^*_{\lambda_0})\}, \tag{31}$$

are minimal constrained cuts.

Proof. Suppose that there exists a λ such that $\omega_\partial(\pi^*_\lambda) \leq C$. When $\lambda \downarrow \lambda_0$ the monotone continuity $\pi^*_\lambda \downarrow \pi^*_{\lambda_0}$ (Relation 21) and the monotone decreasingness of $\omega_\partial(\pi^*_\lambda)$ (Relation 28) show that $\omega_\partial(\pi^*_{\lambda_0}) \leq C$, and that for $\lambda < \lambda_0$ the constraint $\omega_\partial(\pi^*_\lambda) \leq C$ is not satisfied. Moreover for $\lambda > \lambda_0$ we have $\omega_\varphi(\pi^*_\lambda) \geq \omega_\varphi(\pi^*_{\lambda_0})$. If $\omega_\varphi(\pi^*_\lambda) = \omega_\varphi(\pi^*_{\lambda_0})$ with $\pi^*_\lambda \neq \pi^*_{\lambda_0}$, then by scale increasingness we have $\pi^*_\lambda > \pi^*_{\lambda_0}$. This determines the upper bound λ_1 and achieves the proof. ■

Interestingly, the only property demanded for the energy ω_φ is h-increasingness, already often required in practice for computational reasons. Note also that the two energies ω_φ and ω_∂ of the Lagrange family may vary in the same sense. The proposition 14 uniquely holds on the senses of variation of ω_φ and ω_∂ w.r.t. λ for the optimal cuts.

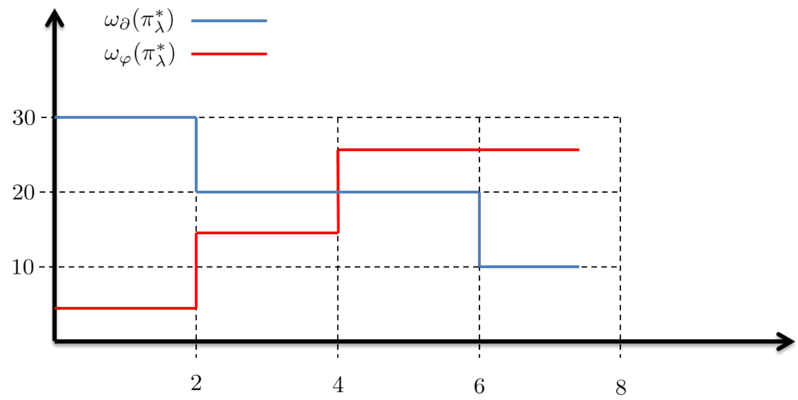

Figure 5: Plots of the objective function $\omega_\varphi(\pi^*_\lambda)$ and of constraint $\omega_\partial(\pi^*_\lambda)$ in function of λ.

7.3 Discussion

If C stands for a cost, the possible constancy of a sequence of cuts risks to give no solution to the equation $\omega_\partial(\pi^*_\lambda) = C$, which imitates the constraint k of the above Relation (1).

Here is a toy example of this phenomenon. Consider a hierarchy which provides three minimal cuts only. Take the following values for their energies :

	$\omega_\varphi(\pi^*_\lambda)$	$\omega_\partial(\pi^*_\lambda)$	$\omega_\lambda(\pi^*_\lambda)$
$0 \leq \lambda < 2$	5	30	$5 + 30\lambda$
$2 \leq \lambda < 4$	15	20	$15 + 20\lambda$
$4 \leq \lambda < 6$	25	20	$25 + 20\lambda$
$6 \leq \lambda$	25	10	$25 + 10\lambda$

This table is plotted in Figure5, both $\omega_\partial(\pi^*_\lambda)$ and $\omega_\varphi(\pi^*_\lambda)$ are piecewise constant functions of λ. If the cost $C = 24$ for example, the λ_0 of equation (30) equals 2, and the λ domain of the minimal π^*_λ cut is $[\lambda_0, \lambda_1[= [2, 4[$. If now $C = 5$ the minimization has no solution.

This type of plot is general. The pairs "father/son" in all partial partitions of a hierarchy H are similar to the two partitions of Figure 5. By h-increasingness, the function $\lambda \to \omega_\partial(\pi^*_\lambda)$ is also piecewise constant, with more discontinuities than in case of Figure 5, so that an impossible cost equation may again occur. This explains why we must slightly relax the cost condition in Theorem 16 by demanding only constraints like that of ω_φ or ω_∂, but not θ, in Equations (1). This relaxed constraint is classically called KKT [8].

8 Multi-constrained Lagrange optimization

The above minimization extends to situations when several constraints interact. For avoiding heavy notation, we restrict the number of parameters to two, and consider the families $\{\omega_{\lambda,\nu}, \lambda \geq 0, \mu \geq 0\}$ such that

$$\lambda \leq \mu,\ \nu \leq \rho, \text{ and } \omega_{\lambda,\nu}(\{S\}) \leq \omega_{\lambda,\nu}(\pi) \ \Rightarrow\ \omega_{\mu,\rho}(\{S\}) \leq \omega_{\mu,\rho}(\pi), \qquad \pi \in \Pi_H(S), \quad (32)$$

a relation which generalizes the scale increasingness. For such multi scale families the theorem 6 becomes:

Theorem 17 *When a family $\{\omega_{\lambda,\nu}, \lambda \geq 0, \mu \geq 0\}$ of singular energies satisfies the axiom (32) of scale increasingness, then the minimal cuts of any hierarchy H enlarge as λ and μ increase:*

$$\lambda \leq \mu \ , \ \nu \leq \rho \quad \Rightarrow \quad \pi^*_{\lambda,\nu} \leq \pi^*_{\mu,\rho} \qquad \lambda, \mu \geq 0. \tag{33}$$

Proof. Fix the parameter ν. By applying Theorem 6 to the variation in λ, we obtain $\lambda \leq \mu \ \Rightarrow \pi^*_{\lambda,\nu} \leq \pi^*_{\mu,\nu}$. Fix now the first parameter at the value μ and apply again Theorem 6. We finally get $\nu \leq \rho \Rightarrow \ \pi^*_{\mu,\nu} \leq \pi^*_{\mu,\rho}$, and thus Rel. (33) by composing the two inequalities. ■

Similarly, the corollary 8 is re-written "all families $\{\omega_{\lambda,\nu}\}$ which increase with (λ, ν) are scale increasing". It is in particular the case for the families $\{\omega_{\lambda,\nu} = \omega_\varphi + \lambda\omega_\partial + \nu\omega_\kappa\}$, $\{\omega_{\lambda,\nu} = \omega_\varphi + (\lambda\omega_\partial \vee \nu\omega_\kappa)\}$, $\{\omega_{\lambda,\nu} = \omega_\varphi + (\lambda\omega_\partial \wedge \nu\omega_\kappa)\}$, $\{\omega_{\lambda,\nu} = \omega_\varphi \vee (\lambda\omega_\partial + \nu\omega_\kappa)\}$, $\{\omega_{\lambda,\nu} = \omega_\varphi \wedge (\lambda\omega_\partial + \nu\omega_\kappa)\}$, etc..

We now focus the two-parameters families of the type

$$\{\omega_{\lambda,\nu} = \omega_\varphi + \lambda\omega_\partial + \nu\omega_\kappa, \ \lambda, \nu \geq 0\}.$$

One easily verify that the axiom (32) of twofold scale increasingness is equivalent to the inf-modularities of both ω_∂ and ω_κ. A two-parameters family is said to be Lagrange when $\omega_{\lambda,\nu}$ is singular, ω_φ, ω_∂ and ω_κ are h-increasing, and when the axiom (32) is satisfied. In this case the proposition 15 becomes:

Proposition 18 *Let $\{\omega_{\lambda,\nu}\}$ be a two-parameters Lagrange family of energies acting on a hierarchy H. The energy and $\omega_\varphi(\pi^*_{\lambda,\nu})$ increase with (λ, ν) whereas the energies $\omega_\partial(\pi^*_{\lambda,\nu})$ and $\omega_\kappa(\pi^*_{\lambda,\nu})$ decrease with (λ, ν).*

Proof. The two energies $\omega_\partial(\pi^*_{\lambda,\nu})$ and $\omega_\kappa(\pi^*_{\lambda,\nu})$ decrease with (λ, ν) because they are inf-modular. Il follows that

$$\lambda \leq \mu \text{ and } \nu \leq \rho \quad \Rightarrow \ \lambda[\omega_\partial(a_{\lambda,\nu}) - \omega_\partial(\{S\}_{\mu,\rho})] + \nu[\omega_\kappa(a_{\lambda,\mu}) - \omega_\kappa(\{S\}_{\mu,\rho})] > 0 \tag{34}$$

We can also write by scale increasingness of $\omega_{\lambda,\nu}$

$$\omega_\varphi(a_{\lambda,\nu}) + \lambda\omega_\partial(a_{\lambda,\nu}) + \nu\omega(a_{\lambda,\nu}) \leq \omega_\varphi(\{S\}_{\mu,\rho}) + \lambda\omega_\partial(\{S\}_{\mu,\rho}) + \nu\omega_\kappa(\{S\}_{\mu,\rho}).$$

By taking Rel. (34) into account, we finally obtain:

$$\omega_\varphi(\{S\}_{\mu,\rho}) - \omega_\varphi(a_{\lambda,\nu}) > 0.$$

The proof is achieved as for Proposition 15. ■

Introduce now two costs C and D, for the two conditions $\omega_\partial(\pi^*_{\lambda,\nu}) \leq C$ and $\omega_\kappa(\pi^*_{\lambda,\nu}) \leq D$, and the two infima

$$\lambda_0 = \inf\{\lambda \mid \omega_\partial(\pi^*_{\lambda,\nu}) \leq C\} \text{ and } \nu_0 = \inf\{\nu \mid \omega_\kappa(\pi^*_{\lambda,\nu}) \leq D\}. \tag{35}$$

As the number of cuts of H is finite, so is the number of labels (λ, ν) of minimal cuts. There exists thus a minimal cut $\pi^*_{\lambda_0,\nu_0}$. It is unique because the $\omega_{\lambda,\nu}$ are singular, and this gives the solution of the multi constrained minimization:

Theorem 19 *Let $\omega_{\lambda,\nu} = \omega_\varphi + \lambda\omega_\partial + \nu\omega_\kappa, \lambda, \nu \geq 0$ be a two parameters Lagrange family, let C and D be two costs, and H a finite hierarchy of partitions of set E. As λ varies, let $\{\pi_\lambda^*\}$ be the minimal cuts of H w.r.t. the ω_λ. Then there exist two doublets (λ_0, ν_0) and (λ_1, ν_1) which generate respectively the smallest and the largest cuts of minimal energy $\omega_\varphi(\pi_\lambda^*)$ under the two constraints $\omega_\partial(\pi_{\lambda,\nu}^*) \leq C$ and $\omega_{\partial k}(\pi_{\lambda,\nu}^*) \leq D$.*

9 The model $\omega_\lambda = \omega_\varphi \vee \lambda\omega_\partial$

We started from energies of the type of Equation 3 because it is the relation between the objective function ω_φ and its constraint ω_∂ which appears in the classical Lagrange formalism. And hopefully this model also worked for minimizing partitions. Are there others such nice starting points ?

Consider the family

$$\{\omega_\lambda = \omega_\varphi \vee \lambda\omega_\partial, \lambda > 0\} \tag{36}$$

and suppose that ω_λ and ω_∂ fulfill the same conditions as in a Lagrange family (Definition 14).

Proposition 7, about the hierarchy of minimal cuts, is now one way ($\wedge$-modularity => scale increasingness). For showing that, it suffices to use

$$\omega_\lambda(\{S\}) = \omega_\varphi(\{S\}) \vee \lambda\omega_\partial(\{S\}) \leq \omega_\varphi(\pi) \vee \lambda\omega_\partial(\pi) \qquad \pi \in \Pi_H(S)$$

and

$$\mu\omega_\partial(\{S\}) \leq \mu\omega_\partial(\pi)$$

and to take the supremum of these new inequalities.

10 From hierarchies to lattices of partitions

One easily imagine, between the lattice $\mathcal{L}$ of all partitions of a finite set E of leaves, and a chain, or hierarchy H in $\mathcal{L}$, many other intermediary families in $\mathcal{L}$ which borrow their features to both structures. They all range from the leaves π_0 to the root E, but when one labels their elements by real numbers $i \in I$, the inequality $i \leq j$ does not imply $\pi_i \leq \pi_j$, as in hierarchies. Mathematically speaking, they constitute the set $\mathcal{L}'$ of the sub lattices of $\mathcal{L}$, since they share with $\mathcal{L}$ the same $\leq$, $\vee$, and $\wedge$ of the refinement, and the same extremal elements. Among others, they include the hierarchies of leaves π_0, and one can wonder if the previous analysis, for hierarchies, can extend to $\mathcal{L}'$ and how.

We will adopt the finite model here, though it is probably not necessary, because the question involves interweaving sequences of labels, which can become inextricable when their number is infinite. For the same sake of simplicity, we suppose the energies ω singular, scale increasing and h-increasing.

Consider a sub-lattice $K \in \mathcal{L}'$, such as Figure 6. It deviates from a hierarchy on account of a leakage effect. The two braces that contain point x designate two p.p. whose $\sqcup$ is the class S_1. The same occurs with the two classes of π_2 and π_3 at point y whose $\sqcup$ gives the class S_1'. But S_1 and S_1', which intersect, cannot be taken as classes of some underlying hierarchy, and we must go up by one more level, etc..This leakage weakens the hierarchical properties of K, and thus the relevance of the above method for describing it. But this also suggests a typology of the sub-lattices $K \in \mathcal{L}'$. Start

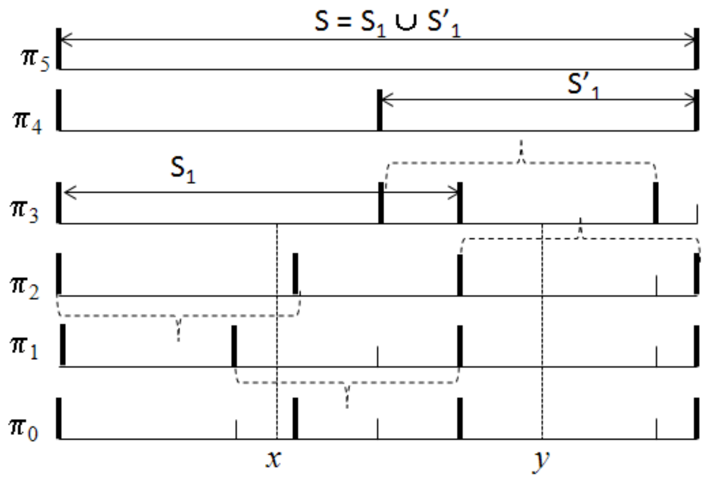

Figure 6: An example of leakage effect.

from a sub-lattices K whose the even levels form a hierarchy. Several p.p. compose the intermediary odd levels, but if the class S at level $i+2$ is the support of the p.p. $\pi(S)$ at level i, then $\{S\}$ is the supremum of all the p.p. candidates for occupying the level $i+1$ below S:

$$\pi(S) \leq \{\pi_j...\pi_r\} \leq \vee\{\pi_j...\pi_r\} = S$$

Figure 7 depicts the situation at two the places of S and S'. We call *wavering hierarchy of order one* such a sub-lattices K.

The order two is depicted in Figure 6. In this case, the underlying hierarchy corresponds to the levels which are a multiple of three. If i is the level of π_0 and $i+3$ that of π_5, then S_1 in π_3 and S_1' in π_4 are intermediary suprema of p.p.. But they overlap, which requires to go up once more for getting the convenient class $S = S_1 \cup S_1'$.

Let us find the minimal cut of a wavering hierarchy of order one w.r.t. a singular energy ω. An additional assumption is required here, for choosing among the intermediary $\{\pi_j...\pi_r\}$. Their energies must be different form $\omega(\{S\})$, which is the singularity, but we assume also that, if some are smaller than $\omega(\{S\})$ then they cannot be equal. With this condition, which generalizes the singularity, the cuts of K become a partial ordered set for the ω-energetic ordering of Proposition 3, and an ω-energetic lattice in the sense of Theorem 4. As the ω-energetic ordering is a piecewise relation, each class of level $i+2$ can be studied individually. One take for p.p. at level $i+1$ below S the less energetic of the $\{\pi_j...\pi_r\}$ when $\wedge\{\omega(\pi_j)...\omega(\pi_r)\} < \omega(S)$, and $\{S\}$ when not. An example is given in Figure 8.

By applying the same local minimization to all father classes, we obtain a hierarchy $H(K)$ with odd and even levels. The minimal cuts of K and $H(K)$ are the same, and therefore if $\{\omega_\lambda\}$ is a scale increasing family, the minimal cuts of K increase with λ. Moreover, if the family $\{\omega_\lambda = \omega_\varphi + \lambda\omega_\partial\}$ of energies is Lagrange, then the constrained optimization of Theorem 19 still applies.

The approach and these results extend to the wavering hierarchy of order two or more,

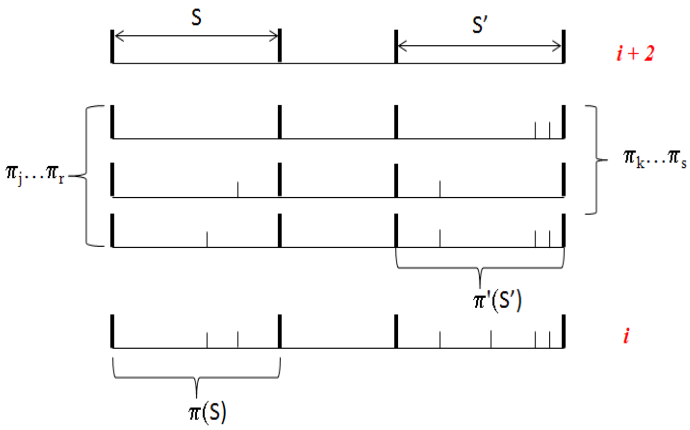

Figure 7: An example of a wavering hierarchy.

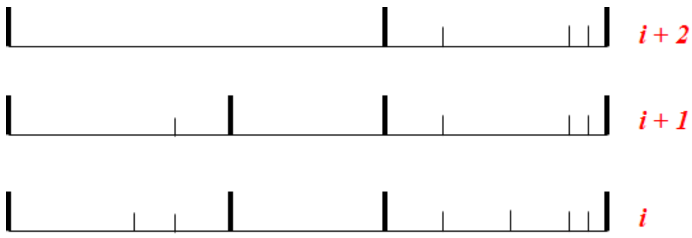

Figure 8: Hierarchy derived from that of Figure 7.

but they become more and more combinatorial.

11 Conclusion

This report presented the topologically continuous hierarchy of partitions, which does not assume any finite leaves any more [13]. It further generalized the concepts of scale-increasingness to define Inf-Modularity, which provides the axiomatic on the constraint in Lagrangian optimization problems, so that one can obtain a unique minimum in the energetic lattice. The scale-increasingness was further generalized for multi-constraint Lagrangian optimization, by introducing an order on the Lagrangian multipliers.

References

[1] Huseyin Gokhan Akcay and Selim Aksoy. Automatic detection of geospatial objects using multiple hierarchical segmentations. *IEEE T. Geoscience and Remote Sensing*, 46(7):2097–2111, 2008.

[2] Laurent Guigues and. Modéles multi-échelles pour la segmentation d'images. *Thése doctorale Université de Cergy-Pontoise*, December 2003.

[3] Andreas Lohne Andreas H. Hamel. Lagrange duality in set optimization. *Journal of Optimization Theory and Applications*, pages 1–30, 2013.

[4] Pablo Arbelaez, Michael Maire, Charless Fowlkes, and Jitendra Malik. Contour detection and hierarchical image segmentation. *IEEE Trans. Pattern Anal. Mach. Intell.*, 33(5):898–916, May 2011.

[5] Coloma Ballester, Vicent Caselles, Laura Igual, and Luis Garrido. Level lines selection with variational models for segmentation and encoding. *J. Math. Imaging Vis.*, 27(1):5–27, January 2007.

[6] E T Bell. Exponential numbers. In *The American Mathematical Monthly*, volume 41, pages 411–419, 1934.

[7] J. P. Benzécri. *L'Analyse des Données. Tome I : La Taxinomie.* 4eme edition, 1984.

[8] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization.* Cambridge University Press, New York, NY, USA, 2004.

[9] Juan Cardelino, Vicent Caselles, Marcelo BertalmÃo, and Gregory Randall. A contrario selection of optimal partitions for image segmentation. *SIAM J. Imaging Sciences*, 6(3):1274–1317, 2013.

[10] Vicent Caselles, Bartomeu Coll, and Jean-Michel Morel. Topographic maps and local contrast changes in natural images. *International Journal of Computer Vision*, 33(1):5–27, 1999.

[11] Laurent Guigues, Jean Pierre Cocquerez, and Hervé Men. Scale-sets image analysis. *Int. J. Comput. Vision*, 68(3):289–317, July 2006.

[12] Rodriguez-Marin L. Hernández E. Lagrangian duality in set-valued optimization. *Journal of Optimization Theory and Applications*, 134(1):119–134, 2007.

[13] B. Ravi Kiran and Jean Serra. Global-local optimizations by hierarchical cuts and climbing energies. *Pattern Recognition*, 47(1):12 – 24, 2014.

[14] Bangalore Ravi Kiran and Jean Serra. Ground truth energies for hierarchies of segmentations. In Cris Luengo Hendriks, Gunilla Borgefors, and Robin Strand, editors, *Mathematical Morphology and Its Applications to Signal and Image Processing*, volume 7883 of *Lecture Notes in Computer Science*, pages 123–134. Springer Berlin Heidelberg, 2013.

[15] Vladimir Kolmogorov and Ramin Zabih. What energy functions can be minimized via graph cuts? *IEEE TPAMI*, 26(2):147–59, 2004.

[16] Jean Michel Morel and Sergio Solimini. *Variational methods in image segmentation.* Birkhauser Boston Inc., Cambridge, MA, USA, 1995.

[17] D. Mumford and J. Shah. Boundary detection by minimizing functionals. In *IEEE Conference on Computer Vision and Pattern Recognition*, 1985.

[18] Christian Ronse. Partial partitions, partial connections and connective segmentation. *J. Math. Imaging Vis.*, 32(2):97–125, October 2008.

[19] Fujishige S. *Submodular Functions and Optimization, Annals of Discrete Mathematics*, volume 58. Elsevier Science, 2005.

[20] Philippe Salembier and Luis Garrido. Binary partition tree as an efficient representation for image processing, segmentation, and information retrieval. *IEEE Transactions on Image Processing*, 9(4):561–576, 2000.

[21] Jean Serra. *Image Analysis and Mathematical Morphology.* Academic Press, Inc., Orlando, FL, USA, 1983.

[22] Pierre Soille. Constrained connectivity for hierarchical image partitioning and simplification. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 30(7):1132–1145, 2008.

[23] Pierre Soille and Jacopo Grazzini. Constrained connectivity and transition regions. In MichaelH.F. Wilkinson and Jos B.T.M. Roerdink, editors, *Mathematical Morphology and Its Application to Signal and Image Processing*, volume 5720 of *Lecture Notes in Computer Science*, pages 59–69. Springer Berlin Heidelberg, 2009.

[24] Yongchao Xu, Thierry Géraud, and Laurent Najman. Salient level lines selection using the mumford-shah functional. In *Proceedings of the 20th International Conference on Image Processing (ICIP)*, Melbourne, Australia, September 2013. IEEE, IEEE.

ANNEX

(Extract from [13])

Theorem 20 *Given a hierarchy H, an energy ω induces an energetic ordering on the set $\Pi(E)$ of all cuts of H, if and only if ω is singular. In this ordering, cut $\pi \in \Pi(E)$ is less energetic than cut $\pi' \in \Pi(E)$ w.r.t. ω, and one writes $\pi \leq_\omega \pi'$, when in each class S of the supremum by refinement $\pi \vee \pi'$ the p.p. of π inside S has an energy smaller or equal to that of π inside S. Equivalently, for each leaf $x \in E$*

a) either the class $S(x)$ of π is the support of a p.p. χ' of π' and $\omega(\{S\} \leq \omega(\chi')$,

b) or the class $S'(x)$ of π' is the support of a p.p. χ of π and $\omega(\chi) \leq \omega(\{S'\})$.

Proof. The equivalence of the two formulations is a consequence of Rel. 9, which shows that each class of $\pi \vee \pi'$ is either a class of π or of π'. The reflexivity, in statements *a)* and *b)* is obvious. For the transitivity, consider $\pi_1, \pi_2, \pi_3 \in \Pi$, with $\pi_1 \leq_\omega \pi_2$ and $\pi_2 \leq_\omega \pi_3$. At leaf x, their three classes are S_1, S_2, and S_3 respectively. If $S_1 = S_2$ or $S_2 = S_3$, the theorem is locally satisfied. If not, one cannot have $S_1 \cup S_3 \subseteq S_2$. Indeed, if $S_3 \subseteq S_2$, there exists a p.p. χ with $\{S_3\} \sqcup \chi = \{S_2\}$, and the assumption $\pi_2 \leq_\omega \pi_3$ implies, by *a)*, that $\omega(\{S_2\} \leq \omega(\{S_3\} \sqcup \chi)$. If in addition $S_1 \subseteq S_2$, i.e. $\{S_1\} \sqcup \chi' = \{S_2\}$, we see similarly that $\omega(\{S_1\} \sqcup \chi') \leq \omega(\{S_2\})$, which contradicts the singularity. Therefore, the three classes S_1, S_2, and S_3 can be ordered in two ways only, namely

$$\begin{aligned} &i) \quad S_1 \subseteq S_3 \quad \text{and} \quad S_2 \subseteq S_3, \\ &ii) \quad S_3 \subseteq S_1 \quad \text{and} \quad S_2 \subseteq S_1. \end{aligned}$$

In case i), there exist two p.p. ζ and ζ' with $\{S_1\} \sqcup \zeta\} = \{S_3\}$ and $\{S_2\} \sqcup \zeta'\} = \{S_3\}$. As $\pi_2 \leq_\omega \pi_3$, we have, by *a)*, $\omega(\{S_2\} \sqcup \zeta') \leq \omega(\{S_3\})$. Therefore, by singularity, all p.p. of $\{S_3\}$ have energies $\leq \omega(\{S_3\})$. In particular $\omega(\{S_1\} \sqcup \zeta) \leq \omega(\{S_3\})$, which shows that transitivity is fulfilled at leave x. In case ii), a similar proof yields the same conclusion, so that finally $\pi_1 \leq_\omega \pi_3$.

For the anti-symmetry, we must prove that $\pi \leq_\omega \pi'$ and $\pi' \leq_\omega \pi$ imply that $\pi = \pi'$. Suppose that the class $S'(x)$ of π' is the support of a p.p. χ made of more than one class of π. By applying the case *b)* of the theorem to the inequality $\pi \leq_\omega \pi'$, we have $\omega(\chi) \leq \omega(\{S'\})$. But we are also in case *a)* for $\pi' \leq_\omega \pi$, hence $\omega(\chi) \geq \omega(\{S'\})$, which implies the equality of the two members. But this contradicts the singularity of ω, so that S' is partitioned into a unique class of π, namely S. If we reverse the roles of π and π', we obtain the same result, which is also independent of the choice of the leave x in E. This achieves the proof of anti-symmetry.

Conversely, consider an ordering $\leq_\omega$whose energy would be non singular, and two cuts π and π' identical everywhere except in the class $S'(x)$ of π, where π is locally the p.p. χ. Supposed that $\omega(\chi) = \omega(S'(x))$. This implies $\pi \leq_\omega \pi'$ and also $\pi' \leq_\omega \pi$. However we do not have $\pi' = \pi$ since $\chi \neq S'(x)$. Thus singularity is needed, which achieves the proof. ∎