A Notion of Compliance Robustness in Topology Optimization

Samuel Amstutz, Marc Ciligot-Travain

A NOTION OF COMPLIANCE ROBUSTNESS IN TOPOLOGY OPTIMIZATION

SAMUEL AMSTUTZ AND MARC CILIGOT-TRAVAIN

ABSTRACT. The goal of this paper is twofold. On one hand, our work revisits the minimization of the robust compliance in shape optimization, with a more natural and more general approach than what has been done before. On the other hand, following a more recent viewpoint on robust optimization, we study the maximization of the so-called stability radius for a fixed maximal compliance. We provide theorical as well as numerical results.

INTRODUCTION

The compliance $\mathcal{C}(\Omega, f_\Omega)$ of a linear elastic structure occupying a domain Ω and submitted to a load f_Ω is defined as the work done by the load, or equivalently as the stored elastic energy. Minimizing the compliance for a fixed load is a very standard shape optimization problem, for which a wide range of methods have been developed, see e.g. [1, 7] and the references therein. However, it often occurs that the load is not known exactly. In this work we suppose that it takes the form $\bar{f}_\Omega + B_\Omega \xi$, $\xi \in r\mathbb{B}$, with $r > 0$ and $\mathbb{B}$ the closed unit ball of a Hibert space. The robust compliance (also called principal compliance) is then defined by

$$J_{WC}(\Omega) = \sup_{\xi \in r\mathbb{B}} \mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega \xi).$$

The robust compliance may replace the compliance when the load is uncertain, so that minimizing the robust compliance is just minimizing the compliance 'in the worst case'. The way from compliance to robust compliance is just an illustration of the transition from optimization to robust optimization. The robust compliance has been first studied in [9], see also [10].

The goal of this paper is twofold. On one hand, our work revisits the paper by De Gournay et al [11] about the minimization of the robust compliance, with a more natural (and more general) approach. In [11], the authors clearly announced that they renounced to follow this way because they did not see how to proceed, but we show in the present article how to overcome these difficulties.

On the other hand, following another and more recent viewpoint on robust optimization, we study the maximization of the so-called stability radius for a fixed maximal compliance. This is actually the main purpose of this paper, and we shall enter a little more into details. Let $\bar{f}_\Omega$ be the nominal load, i.e., the load which is expected, and suppose that it is desired that the compliance do not exceed α. Then

---

Date: 13/02/2014.

2000 *Mathematics Subject Classification.* 49Q10, 49M29, 74P05, 74P10, 74P15, 90C20.

Key words and phrases. Robustness, Stability radius, Compliance, Topological derivative, Topology Optimization.

the stability radius relatively to the level-set $[\mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega .) \leq \alpha]$ is defined as

$$\sqrt{2J_{SR}(\Omega)} = \sup\left\{r > 0 \,\middle|\, r\mathbb{B} \subset [\mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega .) \leq \alpha]\right\} = \operatorname{dist}(0, [\mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega .) > \alpha]).$$

Then it is natural to look for the domain Ω which maximizes the stability radius $\sqrt{2J_{SR}(\Omega)}$. In other words, we seek the shape which tolerates the greatest deviation from the nominal load, in the sense that the compliance remains below α. This is a kind of robustness optimization. Actually, we will not necessarily find globally optimal shapes. It is more exact to say that we show how to improve the stability radius of a given shape.

The notion of stability radius appeared in robust control (see, e.g., [17, 18]), and has been developed in its full generality, but not from a very mathematical point of view, at the end of the 90's by Ben–Haim (see [6] and the references therein). Compared to the worst case approach, this one avoids fixing r a priori, which is not necessarily easy and natural in some circumstances. We think that in many situations, it is more natural to fix an upper bound α for the objective function, here the compliance. This amounts somehow to fixing some specifications.

The paper is organized as follows. In section 1, we describe the general mathematical setting of our problems. In section 2, we first show that the two robust criteria, namely $J_{WC}(\Omega)$ and $J_{SR}(\Omega)$, are the value functions of some quadratic programs with equality constraints. More precisely, the objective function of each subproblem is a quadratic functional and the equality constraint is associated with another quadratic functional. Such problems are known in the literature as trust–region subproblems, and have been extensively studied in the finite dimensional setting, see e.g. [12, 14, 16, 21, 13, 23, 26, 27, 28]. Then, in arbitrary dimension and for both problems, we prove the existence of critical loads (i.e. solutions of the subproblems). Finally, using a strong duality argument for Lagrangian duality extending known results in finite dimension, we show the existence of a unique solution of the dual problem and give a complete description of the critical loads based on this solution. In section 3, for both problems again, we give an expression of the Hadamard semiderivative of the two criteria relatively to the quadratic functionals depending on Ω, based on the Lagrangian and the solution of the dual problem. Sections 4 through 6 specifically deal with the optimization problem with respect the shape Ω. To keep concise and avoid repetitions, we concentrate on the maximization of the stability radius. Our procedure is based on the concept of topological derivative [15, 22, 25], which evaluates the variation of the objective functional with respect to small topological perturbations. In section 4, we deduce from section 3 the expression of the topological derivative of the stability radius. The optimization algorithm is described in section 5, while section 6 reports on some numerical computations.

1. General setting

We denote by Ω the domain to be optimized, and by $\mathscr{E}$ the set of admissible domains. For each $\Omega \in \mathscr{E}$ we are given a reflexive Banach space $\mathcal{V}_\Omega$. We denote by $\|.\|$ the norm on $\mathcal{V}_\Omega$ and by $\langle .,. \rangle$ the duality pairing between $\mathcal{V}'_\Omega$ and $\mathcal{V}_\Omega$, where $\mathcal{V}'_\Omega$ stands for the continuous dual space of $\mathcal{V}_\Omega$. We also consider a continuous and self-adjoint positive definite isomorphism A_Ω from $\mathcal{V}_\Omega$ into $\mathcal{V}'_\Omega$. In particular, there exist two constants $\gamma_\Omega^+, \gamma_\Omega^- > 0$ such that

$$\gamma_\Omega^- \|u\|^2 \leq \langle A_\Omega u, u\rangle \leq \gamma_\Omega^+ \|u\|^2 \qquad \forall u \in \mathcal{V}_\Omega.$$

We associate to each $f \in \mathcal{V}'_\Omega$ the vector $u_{\Omega,f} = A_\Omega^{-1} f \in \mathcal{V}_\Omega$ and the scalar

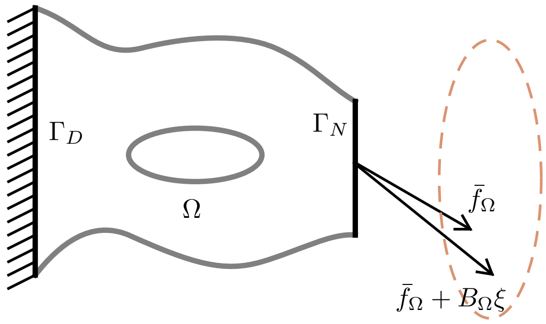
$$\mathcal{C}(\Omega, f_\Omega) = \frac{1}{2}\langle f_\Omega, u_{\Omega,f_\Omega}\rangle = \frac{1}{2}\langle f_\Omega, A_\Omega^{-1} f_\Omega\rangle.$$

Refering to the context of structural mechanics, we will subsequently call f_Ω the load, u_{Ω,f_Ω} the displacement field, and $\mathcal{C}(\Omega, f_\Omega)$ the compliance (actually the half compliance). We will consider a parameterized family of loads of the form

$$f_\Omega = \bar{f}_\Omega + B_\Omega \xi, \qquad \xi \in \mathcal{H},$$

where $\bar{f}_\Omega \in \mathcal{V}'_\Omega$ is a nominal load, $\mathcal{H}$ is a fixed (i.e., independent of Ω) separable Hilbert space and $B_\Omega : \mathcal{H} \to \mathcal{V}'_\Omega$ is a linear, compact and injective operator. We set

$$C(\Omega, \xi) = \mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega \xi).$$



Before continuing with the abtract framework, let us give a typical concrete example. We consider the problem of compliance minimization for a structure submitted to an uncertain load. The structure is represented by a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$), whose boundary is split into three disjoint subsets Γ, Γ_D and Γ_N with $meas(\Gamma_D) > 0$. Homogeneous Dirichlet boudary conditions are prescribed on Γ_D, and Neumann boundary conditions are prescribed on $\Gamma \cup \Gamma_N$, with zero force on Γ. We denote by $H^1_D(\Omega)^d$ the space of vector fields belonging to $H^1(\Omega)^d$ with vanishing trace on Γ_D. For a given load $f_\Omega \in H^1_D(\Omega)^d$ and a given displacement $u \in H^1_D(\Omega)^d$, the elastic energy is the negative quadratic functional in u defined by

$$E_\Omega(u, f_\Omega) = -\frac{1}{2}\int_\Omega He(u) : e(u) + \langle f_\Omega, u\rangle \tag{1.1}$$

where $e(u) = (\nabla u + \nabla u^T)/2$ is the strain tensor and H is the fourth–order elasticity tensor (Hooke's tensor) such that $He(u)$ is the stress tensor. The compliance is defined as

$$\mathcal{C}(\Omega, f_\Omega) = \max_{u \in H^1_D(\Omega)^d} E(u, f_\Omega). \tag{1.2}$$

Therefore, setting $\mathcal{V}_\Omega = H^1_D(\Omega)^d$, the operator $A_\Omega : \mathcal{V}_\Omega \to \mathcal{V}'_\Omega$ is defined by

$$\langle A_\Omega u, v\rangle = \int_\Omega He(u) : e(v) \tag{1.3}$$

and the compliance admits the expression

$$\mathcal{C}(\Omega, f_\Omega) = \frac{1}{2}\langle f_\Omega, A_\Omega^{-1} f_\Omega\rangle. \tag{1.4}$$

We choose $\mathcal{H}$ as a closed subspace of $L^2(\Gamma_N)^d$, and we define $B_\Omega : \mathcal{H} \to \mathcal{V}'_\Omega$ by

$$B_\Omega\xi : u \in \mathcal{V}_\Omega \mapsto \int_{\Gamma_N} \xi.u|_{\Gamma_N}. \tag{1.5}$$

The compactness of B_Ω is due to the compactness of the Sobolev embedding $H^{1/2}(\Gamma_N) \to L^2(\Gamma_N)$. A very standard problem in optimal design consists in minimizing the compliance with a fixed load $\bar{f}_\Omega$, i.e.,

$$\text{Minimize} \quad C(\Omega, 0) = \mathcal{C}(\Omega, \bar{f}_\Omega), \quad \Omega \in \mathscr{E},$$

where the set $\mathscr{E}$ can include constraints. One speaks of robust compliance minimization when, at the same time, perturbations of the load are considered.

Let us now come back to the general case. In this paper we will investigate two notions of robustness and the associated optimization problems.

(1) **Stability radius as compliance robustness.** Given a threshold $\alpha > C_\Omega(0) = \mathcal{C}_\Omega(\bar{f}_\Omega)$, the stability radius is defined by

$$J_{SR}(\Omega) = \frac{1}{2}\operatorname{dist}(0, [C(\Omega, .) \geq \alpha])^2 = \inf_{C(\Omega,\xi)\geq\alpha} \frac{1}{2}\|\xi\|^2 = \inf_{\mathcal{C}(\Omega,\bar{f}_\Omega+B_\Omega\xi)\geq\alpha} \frac{1}{2}\|\xi\|^2. \tag{1.6}$$

Increasing this value amounts to increasing the distance to unfeasability, where unfeasability means that the compliance greater than α. This leads to considering the optimization problem:

$$\text{Maximize } J_{SR}(\Omega), \quad \Omega \in \mathscr{E}, \tag{1.7}$$

hence one speaks of robustness maximization.

(2) **Robust compliance in the worst case sense.** Given a radius $r > 0$, the worst case compliance is defined by

$$J_{WC}(\Omega) = \sup_{\|\xi\|\leq r} C(\Omega, \xi) = \sup_{\|\xi\|\leq r} \mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega\xi). \tag{1.8}$$

This is the maximal compliance obtained for a given family of loads. One naturally wants to minimize this quantity, leading to the so-called worst case compliance minimization problem:

$$\text{Minimize } J_{WC}(\Omega), \quad \Omega \in \mathscr{E}. \tag{1.9}$$

In fact it is easily checked that the inequalities in (1.6) and (1.8) can be replaced by equalities, i.e. we have

$$J_{SR}(\Omega) = \frac{1}{2}\operatorname{dist}(0, [C(\Omega, .) = \alpha])^2 = \inf_{C(\Omega,\xi)=\alpha} \frac{1}{2}\|\xi\|^2 = \inf_{\mathcal{C}(\Omega,\bar{f}_\Omega+B_\Omega\xi)=\alpha} \frac{1}{2}\|\xi\|^2. \tag{1.10}$$

$$J_{WC}(\Omega) = \sup_{\|\xi\|=r} C(\Omega, \xi) = \sup_{\|\xi\|=r} \mathcal{C}(\Omega, \bar{f}_\Omega + B_\Omega\xi). \tag{1.11}$$

2. Expression of the robust criteria

In this section the domain Ω is fixed. We shall give practical procedures to compute the values of $J_{SR}(\Omega)$ and $J_{WC}(\Omega)$.

2.1. **Notation.** The two optimization problems appearing in (1.10) and (1.11) can be formulated in the form

$$(\mathscr{P}) \qquad \text{Minimize}\, q_1(\xi), \qquad \xi \in \mathcal{H} \text{ subject to } q_2(\xi) = 0,$$

with two quadratic functionals q_1 and q_2 written as

$$q_1(\xi) = \frac{1}{2}\langle Q_1\xi, \xi\rangle + \langle b_1, \xi\rangle + c_1, \quad q_2(\xi) = \frac{1}{2}\langle Q_2\xi, \xi\rangle + \langle b_2, \xi\rangle + c_2.$$

Above, for each $i = 1, 2$, $Q_i : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear continuous operator, $b_i \in \mathcal{H}$, $c_i \in \mathbb{R}$. Specifically, these quantities are defined as follows.

(1) Compliance robustness :

$$q_1(\xi) = \frac{1}{2}\|\xi\|^2, \quad q_2(\xi) = C(\Omega, \xi) - \alpha, \tag{2.1}$$

i.e.,

$$Q_1 = I, \qquad b_1 = 0, \qquad c_1 = 0,$$

$$Q_2 = B_\Omega^* A_\Omega^{-1} B_\Omega, \qquad b_2 = B_\Omega^* A_\Omega^{-1} \bar{f}_\Omega, \qquad c_2 = \frac{1}{2}\langle \bar{f}_\Omega, A_\Omega^{-1}\bar{f}_\Omega\rangle - \alpha.$$

(2) Worst case robust compliance :

$$q_1(\xi) = -C(\Omega, \xi), \quad q_2(\xi) = \frac{1}{2}\|\xi\|^2 - \frac{1}{2}r^2, \tag{2.2}$$

i.e.,

$$Q_1 = -B_\Omega^* A_\Omega^{-1} B_\Omega, \qquad b_1 = -B_\Omega^* A_\Omega^{-1} \bar{f}_\Omega, \qquad c_1 = -\frac{1}{2}\langle \bar{f}_\Omega, A_\Omega^{-1}\bar{f}_\Omega\rangle,$$

$$Q_2 = I, \qquad b_2 = 0, \qquad c_2 = -r^2/2.$$

Problems of form $(\mathscr{P})$ are known in the literature as trust region problems (or subproblems). They have been extensively studied, but almost always in the case of a finite dimensional space $\mathcal{H}$, see e.g. [12, 14, 16, 21, 13, 23, 26, 27, 28].

2.2. **Existence of critical loads.**

Theorem 2.1. *Assume one of the following hypotheses hold :*

(1) *either Q_1 is compact, negative semi-definite and Q_2 is positive definite,*
(2) *or Q_1 is positive definite and Q_2 is compact.*

Then $(\mathscr{P})$ admits at least a solution.
In particular, the optimal values in (1.10) *and* (1.11) *are attained.*

Proof. Consider the first case. Let (ξ_n) be a minimizing sequence of the problem $\inf_{[q_2\leq 0]} q_1$. Since Q_1 is compact, we have $q_1(\xi) = \lim q_1(\xi_n) = \inf_{[q_2\leq 0]} q_1$. As q_2 is convex and continuous, it is weakly lower-semicontinuous, thus $q_2(\xi) \leq \liminf q_2(\xi_n) \leq 0$. Therefore, the infimum $\inf_{[q_2\leq 0]} q_1$ is attained at some points of $[q_2 \leq 0]$. Finally, as q_1 is concave, at least one of these points can be found in $[q_2 = 0]$.

Let us turn to the second case. Let again (ξ_n) be a minimizing sequence. For any $\beta \in \mathbb{R}$, the level set $[q_1 \leq \beta]$ is bounded. Thus, up to a subsequence, there exists $\xi \in \mathcal{H}$ such that $\xi_n \rightharpoonup \xi$. By compactness of Q_2, we have $q_2(\xi) = \lim q_2(\xi_n) = 0$. Since q_1 is convex and continuous, it is weakly lower-semicontinuous, thus $q_1(\xi) \leq \liminf q_1(\xi_n) = \inf_{[q_2=0]} q_1$.

Obviously, for the functionals (2.1) and (2.2), items (1) and (2) are fulfilled, respectively. $\square$

2.3. **Dual formulation: general framework.**

2.3.1. *Dual problem.* For all $(\xi,\mu)\in\mathcal{H}\times\mathbb{R}$ we define the Lagrangian

$$L(\xi,\mu)=q_1(\xi)+\mu q_2(\xi)=\frac{1}{2}\langle(Q_1+\mu Q_2)\xi,\xi\rangle+\langle b_1+\mu b_2,\xi\rangle+c_1+\mu c_2.$$

The dual criterion is

$$\hat{\psi}(\mu)=\inf_{\xi\in\mathcal{H}}L(\xi,\mu).$$

For any self-adjoint linear continuous operator $T:\mathcal{H}\to\mathcal{H}$, we have

$$\mathcal{H}=\ker T\overset{\perp}{\oplus}\overline{\operatorname{im}T},$$

hence the restriction $T_{|\overline{\operatorname{im}(T)}}:\overline{\operatorname{im}(T)}\to\operatorname{im}T$ is a bijection. We denote by $T^\dagger:=(T_{|\overline{\operatorname{im}(T)}})^{-1}:\operatorname{im}T\to\overline{\operatorname{im}(T)}$ the inverse operator, which, by virtue of the open mapping theorem, is continuous as soon as $\operatorname{im}T$ is closed. For all $\mu\in\mathbb{R}$ we set

$$\psi(\mu)=-\frac{1}{2}\langle(Q_1+\mu Q_2)^\dagger(b_1+\mu b_2),b_1+\mu b_2\rangle+c_1+\mu c_2.$$

Throughout all this section 2.3, we assume that

$$\forall\mu\in\mathbb{R},\qquad Q_1+\mu Q_2\geq 0\Longrightarrow\operatorname{im}(Q_1+\mu Q_2)\text{ is closed.}\tag{2.3}$$

Note that this assumption is fulfilled in the two cases under study. Indeed, in the first case, we have $Q_1+\mu Q_2=I+2\mu B_\Omega^*A_\Omega^{-1}B_\Omega$, whose image is always closed since $B_\Omega^*A_\Omega^{-1}B_\Omega$ is compact. In the second case we have $Q_1+\mu Q_2=-2B_\Omega^*A_\Omega^{-1}B_\Omega+\mu I$, and $Q_1+\mu Q_2\geq 0$ implies $\mu>0$, whereby we conclude as before.

Lemma 2.2. *Under Assumption* (2.3)*, the dual criterion is expressed by*

$$\hat{\psi}(\mu)=\begin{cases}-\infty & \text{if } Q_1+\mu Q_2\not\geq 0,\\ -\infty & \text{if } Q_1+\mu Q_2\geq 0 \text{ and } b_1+\mu b_2\notin\operatorname{im}(Q_1+\mu Q_2),\\ \psi(\mu) & \text{if } Q_1+\mu Q_2\geq 0 \text{ and } b_1+\mu b_2\in\operatorname{im}(Q_1+\mu Q_2).\end{cases}$$

Proof. For simplicity, we set $Q_\mu=Q_1+\mu Q_2$, $b_\mu=b_1+\mu b_2$, $c_\mu=c_1+\mu c_2$, so that

$$\hat{\psi}(\mu)=\inf_{\xi\in\mathcal{H}}q_\mu(\xi):=\frac{1}{2}\langle Q_\mu\xi,\xi\rangle+\langle b_\mu,\xi\rangle+c_\mu.$$

If $Q_\mu\not\geq 0$, it is clear that $\hat{\psi}(\mu)=-\infty$. Therefore we assume now that $Q_\mu\geq 0$. By Assumption (2.3), $\operatorname{im}Q_\mu$ is closed, thus

$$\mathcal{H}=\ker Q_\mu\overset{\perp}{\oplus}\operatorname{im}Q_\mu.$$

For all $\xi\in\mathcal{H}$, we make the decomposition $\xi=\xi_1+\xi_2$, with $\xi_1\in\ker Q_\mu$ and $\xi_2\in\operatorname{im}Q_\mu$. We get

$$q_\mu(\xi)=q_\mu(\xi_2)+\langle b_\mu,\xi_1\rangle.$$

Two cases can arise.

(1) If $b_\mu\notin\operatorname{im}Q_\mu$, choosing $\xi=t\tilde{b}_\mu$ with $t\in\mathbb{R}$ and $\tilde{b}_\mu$ the orthogonal projection of b_μ onto $\ker Q_\mu$, we obtain

$$q_\mu(\xi)=t\|b_\mu\|^2.$$

Letting t go to $-\infty$ yields $\hat{\psi}(\mu)=-\infty$.

(2) If $b_\mu \in \operatorname{im} Q_\mu$, we have $q_\mu(\xi) = q_\mu(\xi_2)$ for all $\xi \in \mathcal{H}$, and

$$\hat{\psi}(\mu) = \inf_{\xi_2 \in \operatorname{im} Q_\mu} q_\mu(\xi_2) := \frac{1}{2}\langle Q_\mu \xi_2, \xi_2\rangle + \langle b_\mu, \xi_2\rangle + c_\mu.$$

The unique minimizer of this quadratic problem is $\xi_2 = -Q_\mu^\dagger b_\mu$, and the value of the minimum is $-\frac{1}{2}\langle Q_\mu^\dagger b_\mu, b_\mu\rangle + c_\mu$, i.e., $\psi(\mu)$.

□

The dual problem is then

$$(\mathscr{D}) \qquad \operatorname*{Maximize}_{\mu \in \mathbb{R}} \psi(\mu) \qquad \text{subject to } \begin{cases} Q_1 + \mu Q_2 \geq 0, \\ b_1 + \mu b_2 \in \operatorname{im}(Q_1 + \mu Q_2). \end{cases}$$

2.3.2. *Optimality conditions.* The following result is an adaptation of Theorem 2.1 of [28]. Due to its importance in the sequel, we nevertheless give a proof.

Theorem 2.3. *Let $\bar{\xi} \in \mathcal{H}$ be such that*

$$\nabla q_2(\bar{\xi}) = Q_2 \bar{\xi} + b_2 \neq 0. \tag{2.4}$$

The following statements are equivalent:

(1) *$\bar{\xi}$ is a (global) minimizer of ($\mathscr{P}$);*

(2) *$q_2(\bar{\xi}) = 0$ and there exists $\bar{\mu} \in \mathbb{R}$ such that*

$$\partial_\xi L(\bar{\xi}, \bar{\mu}) = (Q_1 + \bar{\mu} Q_2)\bar{\xi} + (b_1 + \bar{\mu} b_2) = 0, \tag{2.5}$$

$$\partial^2_{\xi\xi} L(\bar{\xi}, \bar{\mu}) = Q_1 + \bar{\mu} Q_2 \geq 0; \tag{2.6}$$

(3) *there exists $\bar{\mu} \in \mathbb{R}$ such that*

$$\bar{\xi} \in \operatorname{argmin} L(., \bar{\mu}) \cap [q_2 = 0]. \tag{2.7}$$

Proof. We shall prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1).
First step. Let us assume that $\bar{\xi}$ is a minimizer of ($\mathscr{P}$). By the first order necessary optimality condition, there exists $\bar{\mu} \in \mathbb{R}$ such that

$$\partial_\xi L(\bar{\xi}, \bar{\mu}) = 0.$$

The second order necessary optimality condition reads

$$\langle \partial^2_{\xi\xi} L(\bar{\xi}, \bar{\mu})\zeta, \zeta\rangle \geq 0 \qquad \forall \zeta \in T_{\bar{\xi}},$$
$$T_{\bar{\xi}} = \{\zeta \in \mathcal{H}, \langle Q_2\bar{\xi} + b_2, \zeta\rangle = 0\}.$$

Now, suppose that $\zeta \notin T_{\bar{\xi}}$. We assume first that $Q_2\zeta \neq 0$. Since $Q_2 \geq 0$, this entails $\langle Q_2\zeta, \zeta\rangle > 0$. Set

$$\bar{t} = -2\frac{\langle Q_2\bar{\xi} + b_2, \zeta\rangle}{\langle Q_2\zeta, \zeta\rangle} \neq 0, \qquad \xi = \bar{\xi} + \bar{t}\zeta.$$

After calculation we find that $q_2(\xi) = q_2(\bar{\xi}) = 0$. This implies

$$L(\xi, \bar{\mu}) - L(\bar{\xi}, \bar{\mu}) = q_1(\xi) - q_1(\bar{\xi}) \geq 0. \tag{2.8}$$

Yet we have

$$L(\xi, \bar{\mu}) - L(\bar{\xi}, \bar{\mu}) = \bar{t}\langle (Q_1 + \bar{\mu}Q_2)\bar{\xi} + b_1 + \bar{\mu} b_2, \zeta\rangle + \frac{\bar{t}^2}{2}\langle (Q_1 + \bar{\mu}Q_2)\zeta, \zeta\rangle. \tag{2.9}$$

Combining (2.5), (2.8), (2.9) and $\bar{t} \neq 0$, we derive that $\langle (Q_1 + \bar{\mu}Q_2)\zeta, \zeta\rangle \geq 0$. Assume now that $Q_2\zeta = 0$. As $Q_2 \neq 0$, we choose some $\xi_0 \in \mathcal{H}$ such that $\xi_0 \neq 0$. Let (t_n) be an arbitrary sequence of positive real numbers such that $t_n \to 0$ as

$n \to \infty$. We set $\zeta_n = \zeta + t_n \xi_0$. For all n we have $Q_2 \zeta_n = t_n Q_2 \zeta_0 \neq 0$, hence $\langle (Q_1 + \bar{\mu} Q_2)\zeta_n, \zeta_n \rangle \geq 0$. Passing to the limit yields $\langle (Q_1 + \bar{\mu} Q_2)\zeta, \zeta \rangle \geq 0$. We have proved that $Q_1 + \bar{\mu} Q_2 \geq 0$.

Second step. Since the function $\xi \mapsto L(\xi, \bar{\mu})$ is quadratic, a Taylor expansion immediately shows that the conditions (2.5) and (2.6) imply $\bar{\xi} \in \operatorname{argmin} L(., \bar{\mu})$. As $q_2(\bar{\xi}) = 0$, we have $L(\bar{\xi}, \mu) = q_1(\bar{\xi})$ for all $\mu \in \mathbb{R}$.

Third step. We remark that, for all $\mu \in \mathbb{R}$, it holds

$$[L(., \mu) = q_1 + \delta_{[q_2=0]}] = [q_2 = 0], \tag{2.10}$$

with δ_C the indicator function of the set C. By (A.2) in the appendix, we have $\bar{\xi} \in \operatorname{argmin} L(., \bar{\mu}) \cap [q_2 = 0] \subset \operatorname{argmin} q_1 + \delta_{[q_2=0]}$. □

2.3.3. *Strong duality.* The following Theorem is a consequence of Theorem 2.3 together with general results on duality (cf. appendix A).

Theorem 2.4. *Suppose that* ($\mathscr{P}$) *admits a solution* $\bar{\xi}$ *which satisfies the constraint qualification* (2.4). *Then the primal problem* ($\mathscr{P}$) *and the dual problem* ($\mathscr{D}$) *have the same optimal values. In addition,* ($\mathscr{D}$) *admits solutions, and, if* $\bar{\mu}$ *is one of these solutions, we have*

$$\operatorname*{argmin}_{q_2(\xi)=0} q_1(\xi) = \operatorname*{argmin}_{\xi \in \mathcal{H}} L(\xi, \bar{\mu}) \cap [q_2 = 0].$$

Proof. In view of (2.7) and (2.10), one uses successively (A.5), (A.3) and (A.6). □

2.4. **Expression of the critical loads for the compliance robustness.** In this case, the primal problem ($\mathscr{P}$) and the dual problem ($\mathscr{D}$) read, respectively,

$$\operatorname*{Minimize}_{\xi \in \mathcal{H}} q(\xi) = \frac{1}{2}\|\xi\|^2 \qquad \text{subject to } \frac{1}{2}\langle Q\xi, \xi \rangle + \langle b, \xi \rangle + c - \alpha = 0, \tag{2.11}$$

(2.12)

$$\operatorname*{Maximize}_{\mu \in \mathbb{R}} \psi(\mu) = -\frac{\mu^2}{2}\langle (I + \mu Q)^{\dagger} b, b \rangle + \mu(c - \alpha) \, \text{subject to } \begin{cases} I + \mu Q \geq 0, \\ \mu b \in \operatorname{im}(I + \mu Q), \end{cases}$$

with

$$Q = B_\Omega^* A_\Omega^{-1} B_\Omega, \qquad b = B_\Omega^* A_\Omega^{-1} \bar{f}_\Omega, \qquad c = \frac{1}{2}\langle \bar{f}_\Omega, A_\Omega^{-1} \bar{f}_\Omega \rangle. \tag{2.13}$$

The operator $Q : \mathcal{H} \to \mathcal{H}$ is self-adjoint positive semi-definite and compact. We denote by λ_{max} the largest eigenvalue of Q.

The largest eigenvalue of Q is denoted by λ_{max}. The next theorem refines Theorem 2.4 and uses the same terminology as [28].

Theorem 2.5. *The primal problem* (2.11) *and the dual problem* (2.12) *have the same optimal values. The dual problem* (2.12) *admits a unique solution* $\bar{\mu} \in \mathbb{R}$, *which can be computed in the following way.*

- *Easy case:* $b \notin \ker(Q - \lambda_{max} I)^{\perp}$. *Then* $\bar{\mu}$ *is the unique solution in* $]-1/\lambda_{max}, 0[$ *of the equation*

$$\frac{1}{2}\langle (I + \mu Q)^{-2} Q^{-1} b, b \rangle - \frac{1}{2}\langle Q^{-1} b, b \rangle + c - \alpha = 0. \tag{2.14}$$

- *Hard case I:* $b \in \ker(Q - \lambda_{max} I)^{\perp}$ *and*

$$\Delta := \frac{1}{2}\langle \lambda_{max}^2 Q^{-1} (Q - \lambda_{max} I)^{\dagger\dagger} b, b \rangle - \frac{1}{2}\langle Q^{-1} b, b \rangle + c - \alpha > 0. \tag{2.15}$$

  Then $\bar{\mu}$ *is also the unique solution in* $]-1/\lambda_{max}, 0[$ *of* (2.14).

- *Hard case II:* $b \in \ker(Q-\lambda_{max}I)^\perp$ *and* $\Delta \leq 0$. *Then* $\bar\mu = -1/\lambda_{max}$.

The set Ξ *of solutions of the primal problem* (2.27) *is given by the following expressions.*

- *Easy case and Hard case I. There is a unique critical load given by*
$$\Xi = \{-\bar\mu(I+\bar\mu Q)^{-1}b\}.$$
- *Hard case II. The set of critical loads is*
$$\Xi = \Big[\{-(Q-\lambda_{max}I)^\dagger b\} + \ker(Q-\lambda_{max}I)\Big] \cap S_r,$$
with $S_r = \{\xi \in \mathcal{H}, q(\xi)=0\}$.

Proof. By Theorem 2.1, there exists at least a solution $\bar\xi \in \mathcal{H}$ to (2.11). The constraint qualification (2.4) is $Q\bar\xi + b \neq 0$, which is fulfilled since otherwise the constraint in (2.11) would yield $\langle Q\bar\xi,\bar\xi\rangle = 2(c-\alpha) < 0$. Hence Theorem 2.4 applies. Let us reformulate the constraints of the dual problem (2.12). The first one is equivalent to $\mu \geq -1/\lambda_{max}$ and we have $\mathrm{im}(I+\mu Q) = \ker(I+\mu Q)^\perp = \mathcal{H}$ as soon as $\mu > -1/\lambda_{max}$. Therefore we have
$$\left\{\begin{array}{l} I+\mu Q \geq 0 \\ \mu b \in \mathrm{im}(I+\mu Q)\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \mu > -1/\lambda_{max} \text{ if } b \notin \ker(Q-\lambda_{max}I)^\perp \\ \mu \geq -1/\lambda_{max} \text{ if } b \in \ker(Q-\lambda_{max}I)^\perp \end{array}\right\}.$$
If $-1/\lambda_{max} < \mu \leq 0$ we have
$$\psi(\mu) = -\frac{\mu^2}{2}\langle (I+\mu Q)^{-1}b, b\rangle + \mu(c-\alpha).$$
On writing $\mu^2\langle (I+\mu Q)^{-1}b,b\rangle = \langle (I+\mu Q)^{-1}Q^{-2}b, \mu^2Q^2b\rangle = \langle (I+\mu Q)^{-1}Q^{-2}b,$ $I+(\mu Q-I)(\mu Q+I)b\rangle$ we arrive at the more convenient expression
$$\psi(\mu) = -\frac{1}{2}\langle (I+\mu Q)^{-1}Q^{-2}b,b\rangle - \frac{\mu}{2}\langle Q^{-1}b,b\rangle + \frac{1}{2}\langle Q^{-2}b,b\rangle + \mu(c-\alpha). \tag{2.16}$$
Differentiating entails
$$\psi'(\mu) = \frac{1}{2}\langle (I+\mu Q)^{-2}Q^{-1}b,b\rangle - \frac{1}{2}\langle Q^{-1}b,b\rangle + (c-\alpha), \tag{2.17}$$
$$\psi''(\mu) = -\langle (I+\mu Q)^{-3}b,b\rangle < 0. \tag{2.18}$$
We infer that ψ is strictly concave on $]-1/\lambda_{max},0]$ and $\psi'(0) = c-\alpha < 0$. Again we decompose b as $b = \sum_i b_i$ with $b_i \in \ker(Q-\lambda_iI)$ and λ_i the distinct eigenvalues of Q in decreasing order such that $\lambda_0 = \lambda_{max}$. We have for all $\mu \in]-1/\lambda_{max},0]$
$$\psi'(\mu) = \sum_i \frac{1}{\lambda_i(1+\mu\lambda_i)^2}\|b_i\|^2 - \frac{1}{2}\langle Q^{-1}b,b\rangle + c - \alpha.$$

- *Easy case:* $b \notin \ker(Q-\lambda_{max}I)^\perp$. Then $b_0 \neq 0$ and $\lim_{\mu\to(-1/\lambda_{max})^+}\psi'(\mu) = +\infty$. It follows that ψ admits a unique maximizer $\bar\mu \in]-1/\lambda_{max},0[$ characterized by $\psi'(\bar\mu)=0$.
- *Hard case:* $b \in \ker(Q-\lambda_{max}I)^\perp$. Then the expressions (2.16)-(2.18) remain true for $\mu = -1/\lambda_{max}$, provided that the restriction of $I+\mu Q$ to $\ker(Q-\lambda_{max}I)^\perp$ is considered. In particular, $\psi'(-1/\lambda_{max})$ is finite.

  If $\psi'(-1/\lambda_{max}) > 0$ (hard case I), then ψ admits a unique maximizer $\bar\mu$ in $]-1/\lambda_{max},0[$, characterized by $\psi'(\bar\mu)=0$. If $\psi'(-1/\lambda_{max}) \leq 0$ (hard case II), then $-1/\lambda_{max}$ is the unique maximizer of ψ in $[-1/\lambda_{max},0[$.

By Theorem 2.4, the set of solutions of the primal problem is given by

$$\Xi = \operatorname*{argmin}_{\xi\in\mathcal{H}} L(\xi,\bar\mu) \cap S_r.$$

Here the Lagrangian is

$$L(\xi,\bar\mu) = \frac{1}{2}\langle (I+\bar\mu Q)\xi,\xi\rangle + \bar\mu\langle b,\xi\rangle + \bar\mu(c-\alpha).$$

In the easy case and the hard case I, $L(.,\bar\mu)$ is strictly convex since $\bar\mu > -1/\lambda_{max}$. It admits as unique minimizer

$$\bar\xi = -\bar\mu(I+\bar\mu Q)^{-1}b.$$

From (2.17) and $\psi'(\bar\mu) = 0$ we derive after some algebra that $q(\bar\xi) = \frac{1}{2}\langle Q\bar\xi,\bar\xi\rangle + \langle b,\bar\xi\rangle + c - \alpha = 0$.
In the hard case II, we have $\bar\mu = -1/\lambda_{max}$ and

$$\operatorname*{argmin}_{\xi\in\mathcal{H}} L(\xi,\bar\mu) = \{\xi\in\mathcal{H}, (Q-\lambda_{max}I)\xi = -b\}.$$

Using that $b\in\ker(Q-\lambda_{max}I)^\perp = \operatorname{im}(Q-\lambda_{max}I)$ we obtain

$$\operatorname*{argmin}_{\xi\in\mathcal{H}} L(\xi,\bar\mu) = \{-(Q-\lambda_{max}I)^\dagger b\} + \ker(Q-\lambda_{max}I).$$

□

Remark 2.6. If $\bar f\in\operatorname{im}(B_\Omega)$, then the expressions (2.14) and (2.15) can be simplified thanks to the equality:

$$-\frac{1}{2}\langle Q^{-1}b,b\rangle + c = 0.$$

Indeed, writing $\bar f = B_\Omega\bar\xi$ yields $b = Q\bar\xi$ as well as $c = \frac{1}{2}\langle Q\bar\xi,\bar\xi\rangle$.

To conclude this section, we will present an interesting reformulation of the dual problem as a semidefinite programming problem. Again, this kind of approach is standard in finite dimension. Let us go back to the primal problem (2.11):

$$\text{(2.19)}\qquad \operatorname*{Minimize}_{\xi\in\mathcal{H}} q(\xi) = \frac{1}{2}\|\xi\|^2 \qquad \text{subject to } g(\xi) := \frac{1}{2}\langle Q\xi,\xi\rangle + \langle b,\xi\rangle + c = \alpha.$$

If $\bar\xi$ is a solution of this problem then $\|\bar\xi\| = \operatorname{dist}(0,[g=\alpha]) = \operatorname{dist}(0,[g\ge\alpha])$. It follows that $\|\bar\xi\|$ is a solution of

$$\text{(2.20)}\qquad \text{Maximize } \rho,$$
$$\text{(2.21)}\qquad \text{s.t. } \rho\mathbb{B}\subset[g\le\alpha],$$
$$\text{(2.22)}\qquad \rho\in\mathbb{R}_+^\star.$$

Yet we have for $\rho > 0$

$$\begin{aligned}\rho\mathbb{B}\subset[g\le\alpha] &\iff \forall\xi\in\mathcal{H},\ \|\xi\|\le\rho \Rightarrow g(\xi)\le\alpha,\\ &\iff \forall\xi\in\mathcal{H},\ \|\xi\|\le 1 \Rightarrow g(\rho\xi)\le\alpha,\\ &\iff \forall\xi\in\mathcal{H},\ 1-\|\xi\|^2\ge 0 \Rightarrow \alpha - g(\rho\xi)\ge 0.\end{aligned}$$

As $1-\|\xi\|^2 = 1 > 0$ for $\xi = 0$ and $\alpha - g(\rho\xi) = \alpha - c > 0$ for $\xi = 0$, using the S–lemma, one obtains

$$\rho\mathbb{B}\subset[g\le\alpha] \iff \exists\lambda > 0, \forall\xi\in\mathcal{H},\ \alpha - g(\rho\xi) - \lambda[1-\|\xi\|^2]\ge 0.$$

But $g(\xi)$ can be expressed as

$$g(\xi) = \max_{v\in\mathcal{V}_\Omega} -\frac{1}{2}\langle A_\Omega v, v\rangle + \langle \bar{f}_\Omega + B_\Omega\xi, v\rangle,$$

whereby

$$\begin{aligned}&\alpha - g(\rho\xi) - \lambda[1-\|\xi\|^2] \geq 0 \iff \\ &\qquad \forall v \in \mathcal{V}_\Omega,\ \frac{1}{2}\langle A_\Omega v, v\rangle + \lambda\|\xi\|^2 - \rho\langle B_\Omega\xi, v\rangle - s\langle \bar{f}_\Omega, v) + (\alpha-\lambda) \geq 0\end{aligned}$$

Changing (ξ, v) into $(s^{-1}\xi, s^{-1}v)$ for any $s \neq 0$, one also has

$$\begin{aligned}&\forall\xi\in\mathcal{H},\ \alpha - g(\rho\xi) - \lambda[1-\|\xi\|^2] \geq 0\\ &\iff \forall(v,\xi,s)\in\mathcal{V}_\Omega\times\mathcal{H}\times\mathbb{R},\\ &\tfrac{1}{2}\langle A_\Omega v, v\rangle + \lambda\|\xi\|^2 - \rho\langle B_\Omega\xi, v\rangle - s\langle\bar{f}_\Omega, v) + (\alpha-\lambda)s^2 \geq 0\\ &\iff \begin{bmatrix} A_\Omega & \rho B_\Omega & \bar{f}^* \\ \rho B_\Omega^* & \lambda I & 0 \\ \bar{f} & 0 & 2\alpha-\lambda \end{bmatrix} \geq 0.\end{aligned}$$

Therefore, if $\bar{\xi}$ is a solution of the primal problem, then $\|\bar{\xi}\|$ is a solution of

$$\text{Maximize } \rho, \tag{2.23}$$

$$\text{s.t.} \begin{bmatrix} A_\Omega & \rho B_\Omega & \bar{f}^* \\ \rho B_\Omega^* & \lambda I & 0 \\ \bar{f} & 0 & 2\alpha-\lambda \end{bmatrix} \geq 0, \tag{2.24}$$

$$\bar{f}\rho, \lambda \in \mathbb{R}_+^\star. \tag{2.25}$$

Note that the above matrix is affine in (ρ,λ).

Proposition 2.7. *Let $(\bar{\rho},\bar{\lambda})$ be a solution of (2.23)-(2.25), then $-\bar{\rho}^2/2\bar{\lambda}$ is the solution of the dual problem ($\mathscr{D}$).*

Proof. We have

$$\forall\xi\in\mathcal{H}, \quad \alpha - g(\bar{\rho}\xi) - \bar{\lambda}[1-\|\xi\|^2] \geq 0. \tag{2.26}$$

Let $\bar{\xi}$ be a solution of the primal problem. Substituting ξ for $\bar{\rho}\xi$ in (2.26) and multiplying by $\bar{\rho}^2/2\bar{\lambda}$ we arrive at

$$\forall\xi\in\mathcal{H}, \quad \frac{1}{2}\|\xi\|^2 - \frac{\bar{\rho}^2}{2\bar{\lambda}}[g(\xi)-\alpha] \geq \frac{1}{2}\bar{\rho}^2 = \frac{1}{2}\|\bar{\xi}\|^2.$$

As $g(\bar{\xi}) = \alpha$ we can write

$$\forall\xi\in\mathcal{H}, \quad \frac{1}{2}\|\xi\|^2 - \frac{\bar{\rho}^2}{2\bar{\lambda}}[g(\xi)-\alpha] \geq \frac{1}{2}\|\bar{\xi}^2\| - \frac{\bar{\rho}^2}{2\bar{\lambda}}[g(\bar{\xi})-\alpha].$$

It follows that $\bar{\xi} \in \operatorname{argmin} L(., -\bar{\rho}^2/2\bar{\lambda}) \cap [L(., -\bar{\rho}^2/2\bar{\lambda}) = q] \neq \emptyset$ thus, by virtue of (A.7), $-\bar{\rho}^2/2\bar{\lambda}$ is the solution of the dual problem ($\mathscr{D}$). ☐

2.5. **Expression of the critical loads for the robust compliance in the worst case sense.** Here, using again the notation (2.13), the primal problem ($\mathscr{P}$) and the dual problem ($\mathscr{D}$) read, respectively:

$$\operatorname*{Minimize}_{\xi\in\mathcal{H}} q(\xi) = -\frac{1}{2}\langle Q\xi,\xi\rangle - \langle b,\xi\rangle - c \qquad \text{subject to } \|\xi\| = r, \tag{2.27}$$

(2.28)
$$\operatorname*{Maximize}_{\mu\in\mathbb{R}} \psi(\mu) = \frac{1}{2}\langle (Q-\mu I)^{\dagger} b, b\rangle - c - \mu\frac{r^2}{2} \qquad \text{subject to } \left\{ \begin{array}{l} -Q+\mu I \geq 0, \\ b \in \operatorname{im}(Q-\mu I). \end{array} \right.$$

The largest eigenvalue of Q is still denoted by λ_{max}. Applying Theorem 2.4 similarly to Theorem 2.5 provides the following result.

Theorem 2.8. *The primal problem* (2.27) *and the dual problem* (2.28) *have the same optimal values. The dual problem* (2.28) *admits a unique solution* $\bar{\mu} \in \mathbb{R}$, *which can be computed in the following way.*

- *Easy case:* $b \notin \ker(Q-\lambda_{max} I)^{\perp}$. *Then* $\bar{\mu}$ *is the unique solution in* $]\lambda_{max}, +\infty[$ *of the equation*

$$\langle (Q-\mu I)^{-2} b, b\rangle = r^2. \tag{2.29}$$

- *Hard case I:* $b \in \ker(Q-\lambda_{max} I)^{\perp}$ *and* $\langle (Q-\lambda_{max} I)^{\dagger\dagger} b, b\rangle > r^2$. *Then* $\bar{\mu}$ *is also the unique solution in* $]\lambda_{max}, +\infty[$ *of* (2.29).
- *Hard case II:* $b \in \ker(Q-\lambda_{max} I)^{\perp}$ *and* $\langle (Q-\lambda_{max} I)^{\dagger\dagger} b, b\rangle \leq r^2$. *Then* $\bar{\mu} = \lambda_{max}$.

The set Ξ *of solutions of the primal problem* (2.27) *is given by the following expressions.*

- *Easy case and Hard case I. There is a unique critical load given by*
$$\Xi = \{(-Q+\bar{\mu} I)^{-1} b\}.$$
- *Hard case II. The set of critical loads is*
$$\Xi = \Big[\{(Q-\lambda_{max} I)^{\dagger} b\} + \ker(Q-\lambda_{max} I) \Big] \cap S_r,$$
where $S_r = \{\xi \in \mathcal{H}, \|\xi\| = r\}$.

3. Hadamard semiderivative of the robust criteria

3.1. **Notation.** Let X be a Banach space and consider a fonction $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. If $z, d \in X$ and $f(z) \in \mathbb{R}$ the upper and lower Hadamard semiderivatives of f at point z in direction d are defined, respectively, as
$$f'_{+}(z,d) = \limsup_{\substack{t\downarrow 0 \\ v\to d}} \frac{f(z+tv)-f(z)}{t}, \qquad f'_{-}(z,d) = \liminf_{\substack{t\downarrow 0 \\ v\to d}} \frac{f(z+tv)-f(z)}{t}.$$
If $f'_{+}(z,d) = f'_{-}(z,d)$, their common value is called the Hadamard semiderivative of f at point z in direction d, denoted by $f'(z,d)$.

This section aims at calculating the Hadamard semiderivative of the robust compliance, if it exists, with respect to the parameter
$$w = (Q,b,c) \in W := \mathcal{K}_s(\mathcal{H}) \times \mathcal{H} \times \mathbb{R}$$
defined by (2.13). Here, $\mathcal{K}_s(\mathcal{H})$ stands for the set of self-adjoint compact linear operators from $\mathcal{H}$ into itself. The sensitivity of w with respect to Ω, specifically its topological derivative, will be studied in Section 4.

Our approach specializes and complements, by checking some technical assumptions, the proof of Theorem 4.24 in [8]. For the reader's convenience, we give the full proof in details.

3.2. **Hadamard semiderivative of the compliance robustness.**

3.2.1. *Notations and preliminaries.* The objective function is the stability radius

$$\varphi(w) = \inf_{q(w,\xi)=0} \frac{1}{2}\|\xi\|^2, \tag{3.1}$$

with

$$q(w,\xi) = \frac{1}{2}\langle Q\xi, \xi\rangle + \langle b, \xi\rangle + c - \alpha. \tag{3.2}$$

We denote by $\Xi(w)$ the minimizing set of (3.1), which is nonempty and whose composition has been obtained in Theorem 2.5. Recall that the Lagrangian of this problem is

$$L(w,\xi,\mu) = \frac{1}{2}\|\xi\|^2 + \mu q(w,\xi), \tag{3.3}$$

and that the dual criterion is

$$\hat{\psi}(w,\mu) = \inf_{\xi\in\mathcal{H}} L(w,\xi,\mu).$$

We call $\Lambda(w)$ the solution set of the dual problem, i.e.,

$$\Lambda(w) = \operatorname*{argmax}_{\mu\in\mathbb{R}} \hat{\psi}(w,\mu),$$

which has been shown to be a singleton.

Lemma 3.1. *Let $(\bar{w},\bar{\xi}) \in W\times\mathcal{H}$ be such that $q(\bar{w},\bar{\xi}) = 0$. There exists a neighborhood $\mathcal{W}$ of $\bar{w}$ and a function $s : \mathcal{W} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ such that $s(\bar{w}) = 1$ and*

$$q(w, s(w)\bar{\xi}) = 0 \qquad \forall w \in \mathcal{W}.$$

Proof. Consider the function $F : (w,s) \in W\times\mathbb{R} \mapsto q(w, s\bar{\xi})$, which is clearly of class $\mathcal{C}^\infty$. We have

$$\partial_s F(\bar{w},1) = \langle Q\bar{\xi},\bar{\xi}\rangle + \langle b,\bar{\xi}\rangle = \frac{1}{2}\langle Q\bar{\xi},\bar{\xi}\rangle - (c-\alpha),$$

due to $q(\bar{w},\bar{\xi}) = 0$. Using that $Q \geq 0$ and $c < \alpha$ we infer $\partial_s F(\bar{w},1) > 0$. The implicit function theorem leads to the result. $\square$

In order to emphasize the generality of the following derivations, we collect below the different properties we will need to obtain the Hadamard semiderivative of the function φ. Property (1) is a straightforward consequence of Lemma 3.1, with $S_{\bar{w},\bar{\xi}}(w) = s(w)\bar{\xi}$. Properties (2), (3) and (4) have been proved in Section 2, see in particular Theorems 2.1, 2.4 and 2.5.

Assumption 3.2.

(1) If $(\bar{w},\bar{\xi}) \in W\times\mathcal{H}$ satisfy $q(\bar{w},\bar{\xi}) = 0$, then there exists a neighborhood $\mathcal{W}$ of $\bar{w}$ and a function $S_{\bar{w},\bar{\xi}} : \mathcal{W} \to \mathcal{H}$ of class $\mathcal{C}^\infty$ such that $S_{\bar{w},\bar{\xi}}(\bar{w}) = \bar{\xi}$ and

$$q(w, S_{\bar{w},\bar{\xi}}(w)) = 0 \qquad \forall w \in \mathcal{W}.$$

(2) For all $\bar{w} \in W$ the sets $\Xi(\bar{w})$ and $\Lambda(\bar{w})$ are nonempty.
(3) For all $\bar{w} \in W$ and $\bar{\xi} \in \Xi(\bar{w})$ there exists $\bar{\mu} \in \Lambda(\bar{w})$ such that $\bar{\xi} \in \operatorname{argmin}_{\xi\in\mathcal{H}} L(\bar{w},\xi,\bar{\mu})$.
(4) There is no duality gap, i.e., it holds $\varphi(\bar{w}) = \sup_{\mu\in\mathbb{R}} \hat{\psi}(\bar{w},\mu)$ for all $\bar{w} \in W$.

3.2.2. *Upper bound for the upper Hadamard semiderivative.*

Lemma 3.3. *Suppose that items (1), (2) and (3) of Assumption 3.2 hold. For any $\bar{w}, \bar{h} \in W$ we have*

$$\varphi'_+(\bar{w}, \bar{h}) \leq \inf_{\bar{\xi} \in \Xi(\bar{w})} \sup_{\bar{\mu} \in \Lambda(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

Proof. Choose an arbitrary $\bar{\xi} \in \Xi(\bar{w})$. Let $(t_n, h_n) \in \mathbb{R}^*_+ \times W$ be such that $t_n \to 0$, $h_n \to \bar{h}$, and

$$\varphi'_+(\bar{w}, \bar{h}) = \lim_{n \to +\infty} \frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n}.$$

We assume that n is large enough so that $\bar{w} + t_n h_n \in \mathcal{W}$. Denoting $\xi_n := S_{\bar{w},\bar{\xi}}(\bar{w} + t_n h_n)$, we have $q(\bar{w} + t_n h_n, \xi_n) = 0$, hence (3.1) entails

$$\varphi(\bar{w} + t_n h_n) \leq \frac{1}{2}\|\xi_n\|^2 = L(\bar{w} + t_n h_n, \xi_n, \mu),$$

for any $\mu \in \mathbb{R}$. As $\bar{\xi} \in \Xi(\bar{w})$, (3.1) also yields

$$\varphi(\bar{w}) = \frac{1}{2}\|\bar{\xi}\|^2 = L(\bar{w}, \bar{\xi}, \mu).$$

Therefore we have

$$\frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n} \leq \frac{L(\bar{w} + t_n h_n, \xi_n, \mu) - L(\bar{w}, \bar{\xi}, \mu)}{t_n}.$$

For all $\tilde{w}$ in a neighborhood of 0 we set

$$\Phi(\tilde{w}) = L(\bar{w} + \tilde{w}, S_{\bar{w},\bar{\xi}}(\bar{w} + \tilde{w}), \mu).$$

We have

$$L(\bar{w} + t_n h_n, \xi_n, \mu) - L(\bar{w}, \bar{\xi}, \mu) = \Phi(t_n h_n) - \Phi(0) = d\Phi(0)(t_n h_n) + o_{n \to +\infty}(t_n h_n),$$

since Φ is Fréchet differentiable by composition. The chain rule gives

$$d\Phi(0)(\tilde{w}) = \partial_w L(\bar{w}, \bar{\xi}, \mu)\tilde{w} + \partial_\xi L(\bar{w}, \bar{\xi}, \mu)(dS_{\bar{w},\bar{\xi}}(\bar{w})\tilde{w}).$$

By item (3) of Assumption 3.2, we can choose $\mu = \bar{\mu} \in \Lambda(\bar{w})$ such that $\bar{\xi} \in \operatorname{argmin}_{\xi \in \mathcal{H}} L(\bar{w}, \xi, \bar{\mu})$, which in turn implies $\partial_\xi L(\bar{w}, \bar{\xi}, \bar{\mu}) = 0$. We arrive at $d\Phi(0)(\tilde{w}) = \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\tilde{w}$, and

$$\frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n} \leq \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})h_n + o(1).$$

Passing to the limit yields

$$\varphi'_+(\bar{w}, \bar{h}) \leq \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h} \leq \sup_{\mu \in \Lambda(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \mu)\bar{h}.$$

This being true for any $\bar{\xi} \in \Xi(\bar{w})$, we arrive at the desired result. $\square$

3.2.3. *Lower bound for the lower Hadamard semiderivative.*

Lemma 3.4. *Suppose that items (1), (2) and (4) of Assumption 3.2 hold. For any $\bar{w}, \bar{h} \in W$ we have*

$$\varphi'_-(\bar{w}, \bar{h}) \geq \inf_{\bar{\xi} \in \Xi(\bar{w})} \sup_{\bar{\mu} \in \Lambda(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

Proof. Let $(t_n, h_n) \in \mathbb{R}_+^* \times W$ be such that $t_n \to 0$, $h_n \to \bar{h}$, and

$$\varphi'_-(\bar{w}, \bar{h}) = \lim_{n \to +\infty} \frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n}.$$

For all n we set $w_n = \bar{w} + t_n h_n$ and choose some $\xi_n \in \Xi(w_n)$.
Step 1. Let $\hat{\xi} \in \Xi(\bar{w})$ be arbitrary. By item (1) of Assumption 3.2 there exists a neighborhood $\mathcal{W}$ of $\bar{w}$ and a function $S_{\bar{w},\hat{\xi}} : \mathcal{W} \to \mathcal{H}$ of class $\mathcal{C}^\infty$ such that $S_{\bar{w},\hat{\xi}}(\bar{w}) = \hat{\xi}$ and

$$q(w, S_{\bar{w},\hat{\xi}}(w)) = 0 \qquad \forall w \in \mathcal{W}.$$

Since $w_n \to \bar{w}$, it holds for n large enough $q(w_n, S_{\bar{w},\hat{\xi}}(w_n)) = 0$, hence

$$\frac{1}{2}\|\xi_n\|^2 \leq \frac{1}{2}\|S_{\bar{w},\hat{\xi}}(w_n)\|^2.$$

This shows that the sequence (ξ_n) is bounded. Therefore there exists $\bar{\xi} \in \mathcal{H}$ such that $\xi_n \rightharpoonup \bar{\xi}$ weakly for some non-relabeled subsequence.
Step 2. We shall show that $\bar{\xi} \in \Xi(\bar{w})$. >From $w_n = \bar{w} + t_n h_n$ and $q(w_n, \xi_n) = 0$, denoting $\bar{w} = (\bar{Q}, \bar{b}, \bar{c})$ and $h_n = (Q_n, b_n, c_n)$, we obtain

$$\frac{1}{2}\langle(\bar{Q} + t_n Q_n)\xi_n, \xi_n\rangle + \langle\bar{b} + t_n b_n, \xi_n\rangle + \bar{c} + t_n c_n - \alpha = 0.$$

A rearrangement yields

$$\frac{1}{2}\langle\bar{Q}\xi_n, \xi_n\rangle + \langle\bar{b}, \xi_n\rangle + \bar{c} - \alpha = -t_n \left(\frac{1}{2}\langle Q_n \xi_n, \xi_n\rangle + \langle b_n, \xi_n\rangle + c_n\right). \tag{3.4}$$

By boundedness of ξ_n and h_n, the right hand side of (3.4) tends to 0. By compactness of $\bar{Q}$, we have $\bar{Q}\xi_n \to \bar{Q}\bar{\xi}$ strongly, thus $\frac{1}{2}\langle\bar{Q}\xi_n, \xi_n\rangle \to \frac{1}{2}\langle\bar{Q}\bar{\xi}, \bar{\xi}\rangle$. We infer that the left hand side of (3.4) tends to

$$\frac{1}{2}\langle\bar{Q}\bar{\xi}, \bar{\xi}\rangle + \langle\bar{b}, \bar{\xi}\rangle + \bar{c} - \alpha = q(\bar{w}, \bar{\xi}),$$

and subsequently that $q(\bar{w}, \bar{\xi}) = 0$. Consider now an arbitrary $\xi \in \mathcal{H}$ such that $q(\bar{w}, \xi) = 0$. Using again item (1) of Assumption 3.2 there exists a neighborhood of $\bar{w}$, which we can assume equal to $\mathcal{W}$, and a function $S_{\bar{w},\xi} : \mathcal{W} \to \mathcal{H}$ of class $\mathcal{C}^\infty$ such that $S_{\bar{w},\xi}(\bar{w}) = \xi$ and

$$q(w, S_{\bar{w},\xi}(w)) = 0 \qquad \forall w \in \mathcal{W}.$$

As previously this implies for n large enough $q(w_n, S_{\bar{w},\xi}(w_n)) = 0$, and thus

$$\frac{1}{2}\|\xi_n\|^2 \leq \frac{1}{2}\|S_{\bar{w},\xi}(w_n)\|^2.$$

Since $\xi_n \rightharpoonup \bar{\xi}$, this entails

$$\frac{1}{2}\|\bar{\xi}\|^2 \leq \frac{1}{2}\liminf_{n \to +\infty} \|\xi_n\|^2 \leq \frac{1}{2}\liminf_{n \to +\infty} \|S_{\bar{w},\xi}(w_n)\|^2,$$

and, using that $S_{\bar{w},\xi}(w_n) \to S_{\bar{w},\xi}(\bar{w}) = \xi$,

$$\frac{1}{2}\|\bar{\xi}\|^2 \leq \frac{1}{2}\|\xi\|^2.$$

This concludes the proof of the fact that $\bar{\xi} \in \Xi(\bar{w})$.
Step 3. Let $\bar{\mu} \in \Lambda(\bar{w})$. >From item (4) of Assumption 3.2, we have $\varphi(\bar{w}) = \hat{\psi}(\bar{w}, \bar{\mu}) = \inf_{\xi \in \mathcal{H}} L(\bar{w}, \xi, \bar{\mu})$. It follows that $\varphi(\bar{w}) \leq L(\bar{w}, \xi_n, \bar{\mu})$. As $\xi_n \in \Xi(\bar{w} + t_n h_n)$, we have $\varphi(\bar{w} + t_n h_n) = \frac{1}{2}\|\xi_n\|^2 = L(w_n + t_n h_n, \xi_n, \bar{\mu})$. We arrive at

$$\frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n} \geq \frac{L(\bar{w} + t_n h_n, \xi_n, \bar{\mu}) - L(\bar{w}, \xi_n, \bar{\mu})}{t_n}.$$

As L is twice differentiable, the Taylor-Lagrange inequality yields for n large enough

$$|L(\bar{w} + t_n h_n, \xi_n, \bar{\mu}) - L(\bar{w}, \xi_n, \bar{\mu}) - \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu}) t_n h_n| \leq c\|t_n h_n\|^2,$$

with c a positive constant independent of n. It follows that

$$\varphi'_-(\bar{w}, \bar{h}) = \lim_{n \to +\infty} \frac{\varphi(\bar{w} + t_n h_n) - \varphi(\bar{w})}{t_n} \geq \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

This being true for any $\bar{\mu} \in \Lambda(\bar{w})$ and some $\bar{\xi} \in \Xi(\bar{w})$, we obtain the desired result. ☐

3.2.4. *Hadamard semiderivative of the robust criterion in the abstract setting.* Combining Lemmas 3.3 and 3.4 leads to the following result.

Proposition 3.5. *Suppose that items (1), (2), (3) and (4) of Assumption 3.2 hold. For any $\bar{w}, \bar{h} \in W$ the function φ admits a Hadamard semiderivative at point $\bar{w}$ in the direction $\bar{h}$ given by*

$$\varphi'(\bar{w}, \bar{h}) = \inf_{\bar{\xi} \in \Xi(\bar{w})} \sup_{\bar{\mu} \in \Lambda(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

3.2.5. *Hadamard semiderivative of the compliance robustness.* We now come back to the specific case of the stability radius. As said before, all items of Assumption 3.2 are satisfied. In addition, we have seen that for any $\bar{w} \in W$ the set of Lagrange multipliers $\Lambda(\bar{w})$ is a singleton. We arrive at the following theorem.

Theorem 3.6. *For any $\bar{w}, \bar{h} \in W$ the stability radius φ admits a Hadamard semiderivative at point $\bar{w}$ in the direction $\bar{h}$ given by*

$$\varphi'(\bar{w}, \bar{h}) = \inf_{\bar{\xi} \in \Xi(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h},$$

where $\{\bar{\mu}\} = \Lambda(\bar{w})$ is the solution set of the dual poblem (2.12).

3.3. **Hadamard semiderivative of the robust compliance in the worst case sense.** The objective function is now

$$\varphi(w) = \inf_{\|\xi\| = r} q(w, \xi), \tag{3.5}$$

with

$$q(w, \xi) = -\frac{1}{2}\langle Q\xi, \xi\rangle - \langle b, \xi\rangle - c. \tag{3.6}$$

We denote by $\Xi(w)$ the minimizing set of (3.5), which has been obtained in Theorem 2.8.

Slightly adapting the proof of Theorem 4.13 in [8], one obtains the following result.

Theorem 3.7. *For any $\bar{w}, \bar{h} \in W$ with $\bar{w} = (\bar{Q}, \bar{b}, \bar{c})$ and $\bar{Q}$ definite negative the worst case functional φ admits a Hadamard semiderivative at point $\bar{w}$ in the direction $\bar{h}$ given by*

$$\varphi'(\bar{w}, \bar{h}) = \inf_{\bar{\xi} \in \Xi(\bar{w})} \partial_w q(\bar{w}, \bar{\xi})\bar{h}.$$

4. Topological derivative of the robust criteria

Consider a reference domain $\Omega = \Omega_0 \in \mathcal{E}$ and a family of perturbed domains $(\Omega_t)_{t>0}$ such that, for all t small enough, $\Omega_t \in \mathcal{E}$. We choose a nominal load of the form $\bar{f} = B_\Omega \bar{\xi} \in \mathcal{V}'_\Omega$ with $\bar{\xi} \in \mathcal{H}$. We set $w_\Omega = (Q_\Omega, b_\Omega, c_\Omega)$ with

$$Q_\Omega = 2B^*_\Omega A^{-1}_\Omega B_\Omega, \quad b_\Omega = 2B^*_\Omega A^{-1}_\Omega B_\Omega \bar{\xi} = Q_\Omega \bar{\xi}, c_\Omega = \langle B^*_\Omega A^{-1}_\Omega B_\Omega \bar{\xi}, \bar{\xi} \rangle = \langle Q_\Omega \bar{\xi}, \bar{\xi} \rangle.$$

We make the following assumption, which will be verified for specific problems in Section 5.

Assumption 4.1. There exists $\delta > 0$ and a self-adjoint linear operator $G_\Omega : \mathcal{H} \to \mathcal{H}$ such that

$$\langle f, Q_{\Omega_t} f \rangle - \langle f, Q_\Omega f \rangle = t \langle f, G_\Omega f \rangle + O(t^{1+\delta}) \qquad \forall f \in \mathcal{H}.$$

Lemma 4.2. *The function $t \in \mathbb{R}_+ \mapsto Q_{\Omega_t} \in \mathcal{L}(\mathcal{H})$ admits a right derivative at 0 given by*

$$\frac{d}{dt}[Q_{\Omega_t}]_{|t=0} = G_\Omega.$$

Proof. Assumption 4.1 and the polarization identity entail

$$\langle f, Q_{\Omega_t} g \rangle - \langle f, Q_\Omega g \rangle = t \langle f, G_\Omega g \rangle + O(t^{1+\delta}) \qquad \forall f, g \in \mathcal{H},$$

that is,

$$\forall f, g \in \mathcal{V}', \qquad \sup_{t>0} t^{-\delta} \left| \left\langle f, \left(\frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right) g \right\rangle \right| < +\infty.$$

By the Banach-Steinhaus theorem we obtain

$$\forall g \in \mathcal{V}', \qquad \sup_{t>0} t^{-\delta} \left\| \left(\frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right) g \right\|_{\mathcal{H}} < +\infty.$$

Another application of the Banach-Steinhaus theorem yields

$$\sup_{t>0} t^{-\delta} \left\| \frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

In particular we have

$$\lim_{t \downarrow 0} \left\| \frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right\|_{\mathcal{L}(\mathcal{H})} = 0,$$

and the proof is achieved. □

The following theorem states the right derivative of the stability radius. A similar result holds for the worst case compliance, which is left to the reader.

Theorem 4.3. *The function $t \mapsto J_{SR}(\Omega_t)$ admits a right derivative at 0 given by*

$$\frac{d}{dt}[J_{SR}(\Omega_t)]_{|t=0} = \inf_{\xi \in \Xi} \bar{\mu} \langle \bar{\xi} + \xi, G_\Omega(\bar{\xi} + \xi) \rangle, \tag{4.1}$$

where Ξ is the set of solutions of the primal problem (2.11) and $\bar{\mu}$ is the solution of the dual poblem (2.12).

Proof. By Lemma 4.2, we get that the map $t \mapsto w_{\Omega_t}$ admits a right derivative at 0 given by

$$\frac{d}{dt}[Q_{\Omega_t}]_{|t=0} = 2G_\Omega, \qquad \frac{d}{dt}[b_{\Omega_t}]_{|t=0} = 2G_\Omega\bar{\xi}, \qquad \frac{d}{dt}[c_{\Omega_t}]_{|t=0} = \langle \bar{\xi}, G_\Omega\bar{\xi}\rangle. \tag{4.2}$$

Next, with the notation of Section 3, we have

$$J_{SR}(\Omega_t) = \varphi(w_{\Omega_t}).$$

By composition (see, e.g., [8] Proposition 2.47), the function $t \mapsto J_{SR}(\Omega_t)$ admits a right derivative at 0 given by

$$\frac{d}{dt}\left[J_{SR}(\Omega_t)\right]_{|t=0} = \varphi'\left(w_\Omega, \frac{d}{dt}[w_{\Omega_t}]_{|t=0}\right).$$

Theorem 3.6 yields

$$\frac{d}{dt}\left[J_{SR}(\Omega_t)\right]_{|t=0} = \inf_{\bar{\xi}\in\Xi} \frac{\partial L}{\partial Q}(w_\Omega, \bar{\xi}, \bar{\mu})\frac{d}{dt}[Q_{\Omega_t}]_{|t=0} + \frac{\partial L}{\partial b}(w_\Omega, \bar{\xi}, \bar{\mu})\frac{d}{dt}[b_{\Omega_t}]_{|t=0} + \frac{\partial L}{\partial c}(w_\Omega, \bar{\xi}, \bar{\mu})\frac{d}{dt}[c_{\Omega_t}]_{|t=0}.$$

Using the expressions (3.3), (3.2) we obtain

$$\frac{\partial L}{\partial Q}(w, \xi, \mu)\tilde{Q} = \mu\frac{1}{2}\langle \tilde{Q}\xi, \xi\rangle, \quad \frac{\partial L}{\partial b}(w, \xi, \mu)\tilde{b} = \mu\langle \tilde{b}, \xi\rangle, \quad \frac{\partial L}{\partial c}(w, \xi, \mu)\tilde{c} = \mu\tilde{c}.$$

Using (4.2) we arrive at

$$\frac{d}{dt}\left[J_{SR}(\Omega_t)\right]_{|t=0} = \inf_{\xi\in\Xi} \bar{\mu}\left[\langle G_\Omega\xi, \xi\rangle + \langle 2G_\Omega\bar{\xi}, \xi\rangle + \langle \bar{\xi}, G_\Omega\bar{\xi}\rangle\right].$$

A rearrangement completes the proof. □

5. Algorithm

5.1. **Problem setting.** In the examples we will present we want to minimize

$$\mathcal{J}(\Omega) := \Phi(J_{SR}(\Omega)) + \ell|\Omega|,$$

where $\Phi : \mathbb{R}_+^* \to \mathbb{R}$ is a smooth and decreasing function, ℓ is a user-given Lagrange multiplier, and $|\Omega|$ is the Lebesgue measure of Ω. In our computations we have used $\Phi(t) = -\log t$.

We choose $\mathcal{E}$ as the set of all subdomains of a fixed "hold-all" domain $D \subset \mathbb{R}^N$. Our model problem is that of linear elasticity, with the following standard framework. The domain Ω is occupied by an elastic material of unitary Young modulus, and its complement $D \setminus \Omega$ is filled with a weak phase, i.e., a fictitious material with small Young modulus ε. This permits to formulate the equilibrium equations, represented by the operator A_Ω, in the fixed domain D. Therefore the function space $\mathcal{V}_\Omega$ is the subspace $H^1_D(D)^N$ including the Dirichlet boundary condition on the appropriate part Γ_D of $\partial\Omega$.

For some $\hat{x} \in D \setminus \partial\Omega$, we consider the *topological* perturbation

$$\Omega_t = \begin{cases} \Omega \setminus \overline{\mathbb{B}(\hat{x}, \rho(t))} & \text{if } \hat{x} \in \Omega, \\ (\Omega \cup \mathbb{B}(\hat{x}, \rho(t))) \cap D & \text{if } \hat{x} \in D \setminus \overline{\Omega}, \end{cases}$$

with $\rho(t) = t^{1/N}$.

5.2. **Optimality condition.** The derivative of $\mathcal{J}(\Omega_t)$ with respect to t is called topological derivative. It is given by the chain rule:

$$g_\Omega(\hat{x}) := \frac{d}{dt}[\mathcal{J}(\Omega_t)]_{|t=0} = \Phi'(J_{SR}(\Omega))\frac{d}{dt}[J_{SR}(\Omega_t) + \ell|\Omega_t|]_{|t=0}.$$

Of course, this is only valid if the topological derivatives $\frac{d}{dt}[J_{SR}(\Omega_t)]_{|t=0}$ and $\frac{d}{dt}[|\Omega_t|]_{|t=0}$ exist. For this later one this is obviously true, as one has

$$\frac{d}{dt}[|\Omega_t|]_{|t=0} = \begin{cases} -\pi & \text{if} \quad \hat{x} \in \Omega, \\ \pi & \text{if} \quad \hat{x} \in D \setminus \overline{\Omega}. \end{cases}$$

The expression of $\frac{d}{dt}[J_{SR}(\Omega_t)]_{|t=0}$ has been obtained in Theorem 4.3 upon Assumption 4.1. The operator G_Ω that satisfies Assumption 4.1 is associated to the topological derivative of the classical compliance. Its expression is known as (see [2, 4]):

$$\langle f, G_\Omega f\rangle = -\pi\frac{r-1}{\kappa r+1}\frac{\kappa+1}{2}\left[2\sigma : e + \frac{(\varepsilon-1)(\kappa-2)}{\kappa+2\varepsilon-1}\operatorname{tr}\sigma\operatorname{tr}e\right],$$

with

$$r = \begin{cases} \varepsilon & \text{if} \quad \hat{x} \in \Omega, \\ \varepsilon^{-1} & \text{if} \quad \hat{x} \in D \setminus \overline{\Omega}, \end{cases}$$

$\kappa = (\lambda_L + 3\mu_L)/(\lambda_L + \mu_L)$, λ_L, μ_L the Lamé coefficients of the material, and (σ, e) the stress and strain tensors at point $\hat{x}$, respectively.

In order to solve the minimization problem in (4.1), we make the simplifying assumption that the largest eigenvalue λ_{max} of Q is simple, which we have always encountered in our test cases. Therefore, Ξ is either a singleton (Easy case and Hard case I) or a pair (Hard case II), hence the minimization is trivial.

A necessary optimality condition for this class of perturbations is clearly

$$g_\Omega(\hat{x}) \geq 0 \qquad \forall \hat{x} \in D, \tag{5.1}$$

which is the starting point of our algorithm.

5.3. **Description of the algorithm.** In order to solve (5.1) we use the algorithm introduced in [4] and further analyzed in [3]. We recall its main features. Each domain Ω is represented by a smooth function $\psi_\Omega : D \to \mathbb{R}$ such that

$$\Omega = \{x \in D, \psi_\Omega(x) < 0\}.$$

We define the signed topological derivative as

$$\tilde{g}_\Omega(\hat{x}) = \begin{cases} -g_\Omega(\hat{x}) & \text{if} \quad \hat{x} \in \Omega, \\ g_\Omega(\hat{x}) & \text{if} \quad \hat{x} \notin \Omega. \end{cases}$$

Therefore (5.1) will be solved as soon as

$$\tilde{g}_\Omega \sim \psi_\Omega, \tag{5.2}$$

with the equivalence relation $\sim$ defined by

$$\psi_1 \sim \psi_2 \iff \exists \alpha > 0 | \psi_1 \sim \alpha\psi_2.$$

We apply to (5.2) the fixed point iteration with relaxation, i.e., the update of the function ψ_Ω at iteration k is

$$\psi_{\Omega_{k+1}} \sim (1-\omega_k)\psi_{\Omega_k} + \omega_k\tilde{g}_{\Omega_k}.$$

The parameter $\omega_k \in (0, 1]$ acts as step size and is fixed at every iteration by a line search.

Remark 5.1. Consider a combination of disjoint topological perturbations, such as for instance

$$\Omega_t = \Omega \setminus \bigcup_{i=1}^{n} \overline{B}(\hat{x}_i, \rho(t)).$$

The topological derivative of the compliance is additive with respect to the perturbation (see [5]), i.e.,

$$G_\Omega = \sum_{i=1}^{n} G_{\Omega,i},$$

where $G_{\Omega,i}$ is the topological derivative for a single perturbation. The topological derivative of the stability radius for the combination of perturbations is then

$$\frac{d}{dt}[J_{SR}(\Omega_t)]_{|t=0} = \inf_{\bar\xi\in\Xi} \bar\mu \sum_{i=1}^{n} \langle \bar f + B\bar\xi, G_{\Omega,i}(\bar f + B\bar\xi)\rangle \geq \sum_{i=1}^{n} \inf_{\bar\xi\in\Xi} \bar\mu \langle \bar f + B\bar\xi, G_{\Omega,i}(\bar f + B\bar\xi)\rangle.$$

This means that the stability radius is superadditive with respect to the perturbation, hence a combination of descent directions for the functional $\mathcal{J}$ still provides a descent direction (recall that Φ is decreasing).

6. Numerical examples

6.1. **Beam.** The hold all domain D is the unit square $(0,1)\times(0,1)$, with a Dirichlet boundary condition on the left side. We denote by p the middle of the right side and by ϕ_1 the unit horizontal force applied at p. The nominal load $\bar f_\Omega$ corresponds to ϕ_1.

At first we consider a space of perturbations of dimension 1, for which the value $\xi = 1$ of the parameter corresponds to the force ϕ_1. The threshold α is chosen as 10 times the compliance of the initial domain, which is the band $(0,1) \times (0.4, 0.6)$, under the nominal load. The Lagrange multiplier for the area is fixed to $\ell = 10$. The optimized domain is represented in Figure 1, left.

Next we add a unit vertical force ϕ_2, still applied at point p, and represented by the parameter $\xi = (0,1)$. The force ϕ_1, represented by $\xi = (1,0)$, remains the nominal load. The other data are unchanged. The optimized domain is given in Figure 1, right. For comparison, note that its area (0.168) is close to the area obtained in the previous case (0.160).

6.2. **Mast.** For this problem the hold-all domain D is the union of the rectangles $(-1,1) \times (0,4)$ and $(-2,2) \times (4,6)$. A Dirichlet boundary condition is applied at the bottom side. We consider 4 forces:

- ϕ_1 is a unit vertical force applied at point $p_1 = (-1,4)$,
- ϕ_2 is a unit vertical force applied at point $p_1 = (1,4)$,
- ϕ_3 is a unit horizontal force applied at point p_1,
- ϕ_4 is a unit horizontal force applied at point p_2.

The nominal load corresponds to the forces ϕ_1 and ϕ_2 applied simultaneously.

As first case we again consider a one-dimensional space of perturbations, spanned by the nominal load. The forces ϕ_3 and ϕ_4 are not taken into account. The initialization is the full domain D, and α is chosen as 10 times the compliance of this domain under the nominal load. The Lagrange multiplier ℓ is fixed to 0.8. The optimized domain is represented in Figure 2, left.

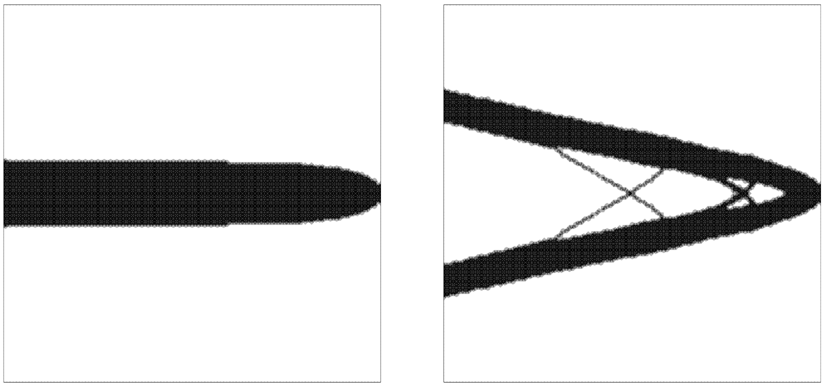


Figure 1. Beam: obtained domains for one load (left) and two loads (right)

Then we consider a two-dimensional space of perturbations, spanned by the forces ϕ_1 and ϕ_2 applied independently. All the other data are unchanged. The optimized domain is represented in Figure 2, middle.

Finally we consider four independent perturbations, given by the forces ϕ_1, ϕ_2, ϕ_3 and ϕ_4. We obtain the domain represented in Figure 2, right.

From the first case to the last one, we clearly observe, first, a stiffening under non-symmetric vertical load, then, a stiffening under horizontal load.

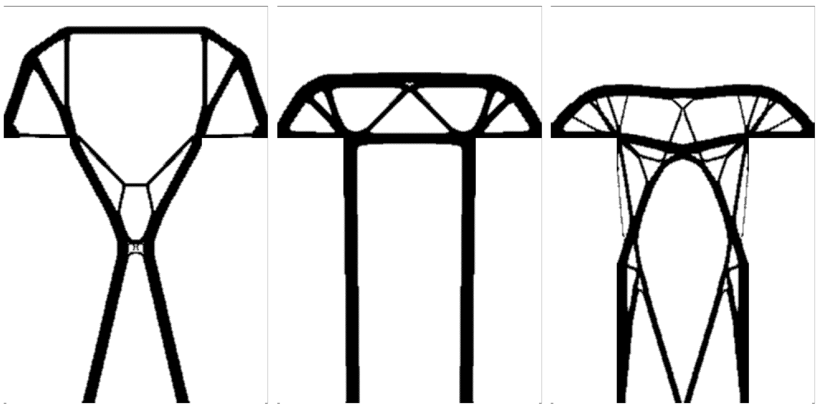


Figure 2. Mast: obtained domains for one load (left), two loads (middle) and four loads (right)

Appendix A. Lagrangian duality

Here we gather useful results on general Lagrangian duality theory, which are essentially reformulations of classical results found in [24, 19, 20]. We nevertheless provide concise proofs for completeness.

Let X, Y be two sets and $L : X \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be an application, called the *Lagrangian*. We define

$$f(x) = \sup_{y \in Y} L(x, y), \qquad x \in X,$$

$$g(y) = \inf_{x \in X} L(x, y), \qquad y \in Y.$$

The duality theory aims at finding relations between the *primal problem*

$$(\mathscr{P}) \qquad \text{Minimize } f(x), \quad x \in X.$$

and the so-called *dual problem*

$$(\mathscr{D}) \qquad \text{Maximize } g(y), \quad y \in Y.$$

We denote by $v(\mathscr{P}) = \inf_{x \in X} f(x)$ and $v(\mathscr{D}) = \sup_{y \in Y} g(y)$ the values of the primal and the dual problems, respectively.

Moreover, for any $y \in Y$, we consider the problem

$$(\mathscr{L}_y) \qquad \text{Minimize } L(x, y), \quad x \in X,$$

with value $v(\mathscr{L}_y) = \inf_{x \in X} L(x, y) = g(y)$. We always have (weak duality):

$$\forall y \in Y, \qquad v(\mathscr{L}_y) \le v(\mathscr{D}) \le v(\mathscr{P}). \tag{A.1}$$

Theorem A.1. (1) *For all $y \in Y$ it holds*

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \subset \operatorname{argmin}(\mathscr{P}) \tag{A.2}$$

(2) *For all $y \in Y$ it holds*

$$v(\mathscr{P}) = v(\mathscr{L}_y) \Longleftrightarrow \begin{cases} y \in \operatorname{argmax}(\mathscr{D}), \\ v(\mathscr{D}) = v(\mathscr{P}). \end{cases} \tag{A.3}$$

(3) *We have the relations:*

(A.4)

$$\left.\begin{matrix} y \in \operatorname{argmax}(\mathscr{D}) \\ v(\mathscr{D}) = v(\mathscr{P}) \end{matrix}\right\} \Longleftrightarrow v(\mathscr{P}) = v(\mathscr{L}_y) \Longrightarrow \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] = \operatorname{argmin}(\mathscr{P}).$$

(4) *For all $y \in Y$ it holds*

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Longrightarrow v(\mathscr{L}_y) = v(\mathscr{P}), \tag{A.5}$$

hence

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Longrightarrow \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] = \operatorname{argmin}(\mathscr{P}). \tag{A.6}$$

(5) *We always have*

$$\Big\{ y \in Y \,\Big|\, \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Big\} \subset \operatorname{argmax}(\mathscr{D}). \tag{A.7}$$

(6) *We have*

$$\Big\{ y \in Y \,\Big|\, \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Big\} \neq \emptyset \Longrightarrow v(\mathscr{D}) = v(\mathscr{P}). \tag{A.8}$$

(7) *We have*

$$\begin{aligned} &\Big\{ y \in Y \,\Big|\, \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Big\} \neq \emptyset \Longrightarrow \\ &\qquad \Big\{ y \in Y \,\Big|\, \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Big\} = \operatorname{argmax}(\mathscr{D}). \end{aligned} \tag{A.9}$$

Proof. (1) If $\bar{x} \in \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f]$ then, for all $x \in X$, $f(\bar{x}) = L(\bar{x}, y) \le L(x, y) \le f(x)$.

(2) If $v(\mathscr{L}_y) = v(\mathscr{P})$, using (A.1), one obtains $v(\mathscr{D}) = v(\mathscr{P}) = v(\mathscr{L}_y) = g(y)$ and $y \in \operatorname{argmax}(\mathscr{D})$. If $y \in \operatorname{argmax}(\mathscr{D})$ and $v(\mathscr{D}) = v(\mathscr{P})$ then $v(\mathscr{L}_y) = g(y) = v(\mathscr{D}) = v(\mathscr{P})$.

(3) If $\bar{x} \in \mathrm{argmin}(\mathscr{P})$ then $L(\bar{x}, y) \le f(\bar{x}) = v(\mathscr{P}) = v(\mathscr{L}_y) \le L(x, y)$ for all $x \in X$. In particular $L(\bar{x}, y) = f(\bar{x})$ and $\bar{x} \in \mathrm{argmin}(\mathscr{L}_y)$.

(4) If $\bar{x} \in \mathrm{argmin}(\mathscr{L}_y) \cap [L(., y) = f]$ then

$$v(\mathscr{P}) = \inf_{x \in X} f(x) \le f(\bar{x}) = L(\bar{x}, y) = v(\mathscr{L}_y) = \inf_{x \in X} L(x, y) \le \inf_{x \in X} f(x) = v(\mathscr{P}).$$

(5) If $\mathrm{argmin}(\mathscr{L}_y) \cap [L(., y) = f] \neq \emptyset$, then using (A.5), one infers $v(\mathscr{L}_y) = v(\mathscr{P})$. We conclude using (A.3).

(6) This stems from the second assertion of (A.3).

(7) Let $y \in \mathrm{argmax}(\mathscr{D})$. Using (A.8) we obtain $v(\mathscr{D}) = v(\mathscr{P})$, and from (A.4), we arrive at

$$\mathrm{argmin}(\mathscr{L}_y) \cap [L(., y) = f] = \mathrm{argmin}(\mathscr{P}).$$

Now there exists $y_0 \in Y$ such that $\mathrm{argmin}(\mathscr{L}_{y_0}) \cap [L(., y_0) = f] \neq \emptyset$, and (A.2) implies

$$\mathrm{argmin}(\mathscr{L}_{y_0}) \cap [L(., y_0) = f] \subset \mathrm{argmin}(\mathscr{P}).$$

Therefore $\mathrm{argmin}(\mathscr{L}_y) \cap [L(., y) = f] \neq \emptyset$.

□

References

[1] G. Allaire. *Conception optimale de structures*, volume 58 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2007. With the collaboration of Marc Schoenauer (INRIA) in the writing of Chapter 8.

[2] S. Amstutz. Sensitivity analysis with respect to a local perturbation of the material property. *Asymptot. Anal.*, 49(1-2):87–108, 2006.

[3] S. Amstutz. Analysis of a level set method for topology optimization. *Optim. Methods Softw.*, 26(4-5):555–573, 2011.

[4] S. Amstutz and H. Andrä. A new algorithm for topology optimization using a level-set method. *J. Comput. Phys.*, 216(2):573–588, 2006.

[5] S. Amstutz and M. Ciligot-Travain. Optimality conditions for shape and topology optimization subject to a cone constraint. *SIAM J. Control Optim.*, 48(6):4056–4077, 2010.

[6] Y. Ben-Haim. *Information-gap decision theory*. Series on Decision and Risk. Academic Press Inc., San Diego, CA, 2001. Decisions under severe uncertainty.

[7] M. P. Bendsøe and O. Sigmund. *Topology optimization*. Springer-Verlag, Berlin, 2003. Theory, methods and applications.

[8] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, New York, 2000.

[9] A. Cherkaev and E. Cherkaeva. Optimal design for uncertain loading condition. In *Homogenization*, volume 50 of *Ser. Adv. Math. Appl. Sci.*, pages 193–213. World Sci. Publ., River Edge, NJ, 1999.

[10] E. Cherkaev and A. Cherkaev. Principal compliance and robust optimal design. *J. Elasticity*, 72(1-3):71–98, 2003. Essays and papers dedicated to the memory of Clifford Ambrose Truesdell III. Vol. III.

[11] F. de Gournay, G. Allaire, and F. Jouve. Shape and topology optimization of the robust compliance via the level set method. *ESAIM Control Optim. Calc. Var.*, 14(1):43–70, 2008.

[12] J.-M. Feng, G.-X. Lin, R.-L. Sheu, and Y. Xia. Duality and solutions for quadratic programming over single non-homogeneous quadratic constraint. *J. Global Optim.*, 54(2):275–293, 2012.

[13] O. Flippo and B. Jansen. Duality and sensitivity in nonconvex quadratic optimization over an ellipsoid. *European Journal of Operational Research*, 94(1):167–178, 1996.

[14] C. Fortin and H. Wolkowicz. The trust region subproblem and semidefinite programming. *Optim. Methods Softw.*, 19(1):41–67, 2004.

[15] S. Garreau, P. Guillaume, and M. Masmoudi. The topological asymptotic for PDE systems: the elasticity case. *SIAM J. Control Optim.*, 39(6):1756–1778 (electronic), 2001.

[16] D. M. Gay. Computing optimal locally constrained steps. *SIAM J. Sci. Statist. Comput.*, 2(2):186–197, 1981.

[17] D. Hinrichsen and A. J. Pritchard. Real and complex stability radii: a survey. In *Control of uncertain systems (Bremen, 1989)*, volume 6 of *Progr. Systems Control Theory*, pages 119–162. Birkhäuser Boston, Boston, MA, 1990.

[18] D. Hinrichsen, A. J. Pritchard, and S. B. Townley. Riccati equation approach to maximizing the complex stability radius by state feedback. *Internat. J. Control*, 52(4):769–794, 1990.

[19] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms. II*, volume 306 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993. Advanced theory and bundle methods.

[20] C. Lemaréchal. The omnipresence of Lagrange. *Ann. Oper. Res.*, 153:9–27, 2007.

[21] J. J. More. Generalizations of the trust region problem. *Optimization Methods and Software*, 2(3-4):189–209, 1993.

[22] A. A. Novotny and J. Sokołowski. *Topological derivatives in shape optimization*. Interaction of Mechanics and Mathematics. Springer, Heidelberg, 2013.

[23] F. Rendl and H. Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Math. Programming*, 77(2, Ser. B):273–299, 1997. Semidefinite programming.

[24] R. T. Rockafellar. *Conjugate duality and optimization*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974. Lectures given at the Johns Hopkins University, Baltimore, Md., June, 1973, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 16.

[25] J. Sokołowski and A. Żochowski. On the topological derivative in shape optimization. *SIAM J. Control Optim.*, 37(4):1251–1272 (electronic), 1999.

[26] D. C. Sorensen. Newton's method with a model trust region modification. *SIAM J. Numer. Anal.*, 19(2):409–426, 1982.

[27] R. J. Stern and H. Wolkowicz. Trust region problems and nonsymmetric eigenvalue perturbations. *SIAM J. Matrix Anal. Appl.*, 15(3):755–778, 1994.

[28] R. J. Stern and H. Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM J. Optim.*, 5(2):286–313, 1995.

Laboratoire de Mathématiques d'Avignon, Faculté des Sciences, 33 rue Pasteur, 84000 Avignon, France.

Laboratoire de Mathématiques d'Avignon, Faculté des Sciences, 33 rue Pasteur, 84000 Avignon, France.