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INTEGRAL REPRESENTATIONS OF THE DERIVATIVES IN $\mathcal{H}(b)$ SPACES

EMMANUEL FRICAIN, JAVAD MASHREGHI

ABSTRACT. In this survey, we provide integral representations for the boundary values of derivatives of functions in the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is in the unit ball of H^∞ . To achieve this goal, we need to study several properties of Blaschke products which are interesting in their own right.

1. INTRODUCTION

Let $H^p(\mathbb{D})$, $0 < p \leq \infty$, denote the classical Hardy space of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. As usual, we also treat $H^p(\mathbb{D})$ as a closed subspace of $L^p(\mathbb{T}, m)$, where $\mathbb{T} = \partial\mathbb{D}$ and m is the normalized arc length measure on \mathbb{T} . Let b be in the unit ball of $H^\infty(\mathbb{D})$. Then the canonical factorization of b is $b = BF$, where

$$B(z) = \gamma \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}, \quad (z \in \mathbb{D}),$$

is the Blaschke product with zeros $a_n \in \mathbb{D}$ satisfying the Blaschke condition

$$\sum_n (1 - |a_n|) < \infty,$$

γ is a constant of modulus one, and F is of the form

$$F(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\sigma(\zeta)\right), \quad (z \in \mathbb{D}),$$

where $d\sigma = -\log|b| dm + d\mu$ and $d\mu$ is a positive singular measure on \mathbb{T} . In the definition of B , we assume that $|a_n|/a_n = 1$ whenever $a_n = 0$.

In this paper, we study some aspects of the de Branges–Rovnyak spaces

$$\mathcal{H}(b) = (Id - T_b T_{\overline{b}})^{1/2} H^2(\mathbb{D}).$$

Here T_φ denotes the Toeplitz operator defined on $H^2(\mathbb{D})$ by $T_\varphi(f) = P_+(\varphi f)$, where P_+ is the (Riesz) orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. In general, $\mathcal{H}(b)$ is not closed with respect to the norm of $H^2(\mathbb{D})$. However, it is a Hilbert space when equipped with the inner product

$$\langle (Id - T_b T_{\overline{b}})^{1/2} f, (Id - T_b T_{\overline{b}})^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

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where f and g are chosen so that

$$f, g \perp \ker (Id - T_b T_{\bar{b}})^{1/2}.$$

As a very special case, if $|b| = 1$ a.e. on \mathbb{T} , or equivalently when b is an inner function for the unit disc, then $Id - T_b T_{\bar{b}}$ is an orthogonal projection and the $\mathcal{H}(b)$ norm coincides with the $H^2(\mathbb{D})$ norm. In this case, $\mathcal{H}(b)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{D})$, which coincides with the shift-coinvariant subspace $K_b := H^2(\mathbb{D}) \ominus bH^2(\mathbb{D})$.

These spaces (and more precisely their general vector-valued version) appeared first in L. de Branges and J. Rovnyak [10, 11] as universal model spaces for Hilbert space contractions. In particular, this model says that if T is a contraction on a Hilbert space \mathcal{H} such that $I - TT^*$ and $I - T^*T$ are of rank one and that

$$\|T^n f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}, \quad (n \geq 1) \implies f = 0,$$

then there exists a function b which is an extreme point of the closed unit ball of H^∞ such that T is unitarily equivalent to the restriction of the backward shift operator on $\mathcal{H}(b)$. Despite the operator model theory for Hilbert space contractions and in particular the invariant subspace problem, the motivation of de Branges–Rovnyak for the study of these spaces seems to be the quantum scattering theory. The connection with this topic had to do with using the machinery of Hilbert spaces of analytic functions to set up a formalism for the study of the perturbation theory of self-adjoint operators, an important subject in wave-operator approach to scattering theory. Since the work of de Branges–Rovnyak, one discovered that $\mathcal{H}(b)$ spaces have an important role to play in numerous questions of complex analysis and function theory. In particular, the notion of complementary space of a Hilbert space contractively contained into another, which is hidden in the definition of $\mathcal{H}(b)$ space we give here, is one of the key notion involved in the solution of the Bieberbach conjecture given by de Branges [9]. Sarason also pointed out a connection with the problem of rigid functions or equivalently exposed points of the unit ball of $H^1(\mathbb{D})$. See [27, 28]. They also play an important role in operator theory. Sarason [25, 29] used $\mathcal{H}(b)$ spaces in his study of the kernel of Toeplitz operators. Jury in [19] found a nice proof of the classical result which says that any composition operator is bounded on H^2 and his proof is only based on the reproducing kernel theory and particularly those of $\mathcal{H}(b)$ spaces. Finally, let us mention the theory of linear systems for which $\mathcal{H}(b)$ spaces represent a useful tool and a natural framework. See for instance the survey of D. Alpay [3] and the recent work of Ball–Bolotnikov [4] in this direction. For further information on the internal and still mysterious structure of these spaces, see [16, 26]. Let us mention for instance the natural question of multipliers of $\mathcal{H}(b)$ spaces which is far from being understood despite numerous work from Davis, Lotto, McCarthy, Sarason, Suarez [12, 20, 21, 22, 23, 32].

In the case where b is an inner function, H. Helson [18] studied the problem of analytic continuation across the boundary for functions in K_b . Then, still when b is an inner function, P. Ahern and D. Clark [1, 2] characterized those points of the boundary at which every function $f \in K_b$ and all its derivatives up to order n have a radial limit. Recently, we gave an extension of the preceding results of Helson and of Ahern–Clark to $\mathcal{H}(b)$ spaces [6, 14, 15].

Deriving the representation formulas for higher derivatives requires a lot of background on function theory. As a matter of fact, sometimes even tracing the right result to apply is not that easy. Moreover, in certain cases, one needs to combine two results in different papers to get the required conclusion. These facts made us to write this survey. On one hand, the results about boundary behaviour of Blaschke product are interesting in their own right. We gave them with concrete proofs. On the other hand, we use them to obtain our representation formulas for the derivatives of functions in $\mathcal{H}(b)$ spaces.

2. PRELIMINARIES

We first recall some basic well-known facts concerning reproducing kernels in $\mathcal{H}(b)$. For any $w \in \mathbb{D}$, the linear functional $f \mapsto f(w)$ is bounded on $H^2(\mathbb{D})$ and thus, by Riesz' theorem, it is induced by a unique element k_w of $H^2(\mathbb{D})$. On the other hand, by Cauchy's formula, we have

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\vartheta})}{1 - we^{-i\vartheta}} d\vartheta, \quad (f \in H^2(\mathbb{D}), w \in \mathbb{D}),$$

and thus k_w is the so called cauchy kernel

$$k_w(z) = \frac{1}{1 - \bar{w}z}, \quad (z \in \mathbb{D}).$$

Now, since $\mathcal{H}(b)$ is contained contractively in $H^2(\mathbb{D})$, the restriction to $\mathcal{H}(\mathbb{D})$ of the evaluation functional at $w \in \mathbb{D}$ is a bounded linear functional on $\mathcal{H}(\mathbb{D})$. Hence, relative to the inner product in $\mathcal{H}(b)$, it is induced by a vector k_w^b in $\mathcal{H}(b)$. In other words, for all $f \in \mathcal{H}(b)$, we have

$$f(w) = \langle f, k_w^b \rangle_b.$$

But if $f = (Id - T_b T_{\bar{b}})^{1/2} f_1 \in \mathcal{H}(b)$, we have

$$\langle f, (Id - T_b T_{\bar{b}}) k_w \rangle_b = \langle f_1, (Id - T_b T_{\bar{b}})^{1/2} k_w \rangle_2 = \langle f, k_w \rangle_2 = f(w),$$

which implies that

$$k_w^b = (Id - T_b T_{\bar{b}}) k_w.$$

Finally, using the well known result $T_{\bar{b}} k_w = \overline{b(w)} k_w$, we obtain

$$k_w^b(z) = \frac{1 - \overline{b(w)} b(z)}{1 - \bar{w}z}, \quad (z \in \mathbb{D}).$$

We know that $\mathcal{H}(b)$ is invariant under the backward shift operator S^* and, in the following, we use extensively the contraction $X_b = X = S^*|_{\mathcal{H}(b)}$. Its adjoint satisfies the important formula

$$(2.1) \quad X^* h = S h - \langle h, S^* b \rangle_b b,$$

for all $h \in \mathcal{H}(b)$. See [11, Theorem 13] for the original proof and [26, pages 11-12] for another proof.

A point $w \in \overline{\mathbb{D}}$ is said to be regular (for b) if either $w \in \mathbb{D}$ and $b(w) \neq 0$, or $w \in \mathbb{T}$ and b admits an analytic continuation across a neighbourhood $V_w = \{z : |z - w| < \varepsilon\}$ of w with

$|b| = 1$ on $V_w \cap \mathbb{T}$. The spectrum of b , denoted by $\sigma(b)$, is then defined as the complement in $\overline{\mathbb{D}}$ of all regular points of b .

For $f \in H^2(\mathbb{D})$, we have

$$f \in \mathcal{H}(b) \iff T_{\bar{b}}f \in \mathcal{H}(\bar{b}).$$

Moreover, if $f_1, f_2 \in \mathcal{H}(b)$, then

$$(2.2) \quad \langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle T_{\bar{b}}f_1, T_{\bar{b}}f_2 \rangle_{\bar{b}}.$$

See Lotto–Sarason [22, Lemma 2.2].

We also mention an integral representation for functions in $\mathcal{H}(\bar{b})$ [26, page 16]. Let $\rho(\zeta) = 1 - |b(\zeta)|^2$ a.e. on \mathbb{T} , and let $L^2(\rho)$ stand for the usual Hilbert space of measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ with $\|f\|_\rho < \infty$, where

$$\|f\|_\rho^2 = \int_{\mathbb{T}} |f|^2 \rho \, dm.$$

For each $w \in \mathbb{D}$, the Cauchy kernel k_w belongs to $L^2(\rho)$. Hence, we define $H^2(\rho)$ to be the span in $L^2(\rho)$ of the functions k_w ($w \in \mathbb{D}$). If q is a function in $L^2(\rho)$, then $q\rho$ is in $L^2(\mathbb{T})$, being the product of $q\rho^{1/2} \in L^2(\mathbb{T})$ and the bounded function $\rho^{1/2}$. Finally, we define the operator $C_\rho : L^2(\rho) \rightarrow H^2(\mathbb{D})$ by

$$C_\rho(q) = P_+(q\rho).$$

Then C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ whose initial space equals to $H^2(\rho)$ and it is an isometry if and only if b is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. Recall that by a well-known result of de Leeuw and W. Rudin, b is an extreme point of the closed unit ball of $H^\infty(\mathbb{D})$ if and only if $\log(1 - |b|) \notin L^1(\mathbb{T})$.

3. DERIVATIVES OF BLASCHKE PRODUCTS

Let $(a_n)_{n \geq 1}$ be a Blaschke sequence in \mathbb{D} , and let B be the corresponding Blaschke product. Fix a point ζ on the boundary \mathbb{T} . If ζ is not an accumulation point of the sequence $(a_n)_{n \geq 1}$, then B is actually analytic at this point and hence, in particular, for any value of $j \geq 0$, both limits

$$\lim_{r \rightarrow 1^-} B^{(j)}(r\zeta) \quad \text{and} \quad \lim_{R \rightarrow 1^+} B^{(j)}(R\zeta)$$

exist and are equal. What is more interesting is that ζ might be an accumulation point of the sequence $(a_n)_{n \geq 1}$ and yet some of the above properties still hold.

The case $N = 0$ and $N = 1$ of the following result is due to Frostman [17]. This is in fact the most crucial case. Frostman result was generalized by Cargo [7]. Then the most general version was obtained by Ahern and Clark [1, 2].

Theorem 3.1 (Frostman–Cargo–Ahern–Clark). *Let $(a_n)_{n \geq 1}$ be a Blaschke sequence in \mathbb{D} , and let B be the corresponding Blaschke product. Assume that for an integer $N \geq 0$ and a point $\zeta \in \mathbb{T}$ we have*

$$(3.1) \quad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|^{N+1}} \leq A.$$

Then the following hold.

(i) For each $0 \leq j \leq N$, both limits

$$B^{(j)}(\zeta) := \lim_{r \rightarrow 1^-} B^{(j)}(r\zeta) \quad \text{and} \quad \lim_{R \rightarrow 1^+} B^{(j)}(R\zeta)$$

exist and are equal.

(ii) There is a constant $C = C(N, A)$ such that the estimation

$$|B^{(j)}(r\zeta)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

Proof. The essential case is $N = 0$. The rest follows by induction.

Case $N = 0$: Our strategy is to show that, under the proposed condition, $|B(r\zeta)|$ and $\arg B(r\zeta)$ have both finite limits as r tends to 1^- . For the simplicity of notations, without loss of generality, assume that $\zeta = 1$.

In the course of proof, we repeatedly use the inequalities

$$|1 - \bar{a}_n r| > 1 - r \quad \text{and} \quad |1 - \bar{a}_n r| > \frac{1}{2}|1 - a_n|,$$

for $r \in (0, 1)$, which are elementary to establish. As the first application, note that

$$\frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} \leq 2 \frac{(1 - r^2)(1 - |a_n|^2)}{(1 - r)|1 - a_n|} \leq 8 \frac{1 - |a_n|}{|1 - a_n|}.$$

Therefore, the Weierstrass M-test shows that the series

$$\sum_{n \geq 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2}$$

converges uniformly in $r \in [0, 1]$, and thus

$$\lim_{r \rightarrow 1^-} \sum_{n \geq 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} = 0.$$

But, we have

$$\begin{aligned} |B(r)|^2 &= \prod_{n \geq 1} \frac{|a_n - r|^2}{|1 - \bar{a}_n r|^2} \\ &= \prod_{n \geq 1} \left(1 - \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} \right) \\ &\geq 1 - \sum_{n \geq 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2}, \end{aligned}$$

and this estimation enables us to deduce

$$\liminf_{r \rightarrow 1^-} |B(r)|^2 \geq 1 - \lim_{r \rightarrow 1^-} \sum_{n \geq 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} = 1.$$

Since $|B(z)| < 1$, we conclude that

$$\lim_{r \rightarrow 1^-} |B(r)| = 1.$$

To deal with the argument, write

$$\frac{\bar{a}_n}{|a_n|} \frac{a_n - r}{1 - \bar{a}_n r} = \frac{1}{|a_n|} \frac{|a_n|^2 - 1 + 1 - r\bar{a}_n}{1 - \bar{a}_n r} = \frac{1}{|a_n|} \left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r} \right).$$

Thus

$$\arg \left(\frac{\bar{a}_n}{|a_n|} \frac{a_n - r}{1 - \bar{a}_n r} \right) = \arg \left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r} \right).$$

and, for large enough n for which the combination $(1 - |a_n|)/(|1 - a_n|)$ is small, we have

$$\left| \arg \left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r} \right) \right| \leq M \frac{1 - |a_n|^2}{|1 - \bar{a}_n r|} \leq 4M \frac{1 - |a_n|}{|1 - a_n|},$$

where M is a positive constant. Thus the series

$$\arg B(r) = \sum_{n \geq 1} \arg \left(1 - \frac{1 - |a_n|^2}{1 - \bar{a}_n r} \right)$$

converges absolutely and uniformly on $[0, 1]$, which proves that $\lim_{r \rightarrow 1^-} \arg B(r)$ exists.

The preceding two discussions together show that $L = \lim_{r \rightarrow 1^-} B(r)$ exists and has modulus one, i.e $|L| = 1$. The estimation in part (ii) trivially holds with $C = 1$. Finally, the Blaschke product satisfies the functional equation

$$B(z) \overline{B(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R \rightarrow 1^+} B(R) = \frac{1}{\lim_{R \rightarrow 1^+} \overline{B(1/R)}} = \frac{1}{\lim_{r \rightarrow 1^-} \overline{B(r)}} = \frac{1}{\overline{L}} = L.$$

This argument also shows that if $\varepsilon > 0$ is such that $[1 - \varepsilon, 1)$ is free from the zeros of B , then B is actually continuous on $[1 - \varepsilon, 1 + \varepsilon]$.

Case $N \geq 1$: Fix $1 \leq j \leq N$, and suppose that the result holds for $0, 1, \dots, j - 1$. Using the formula of B and taking the logarithmic derivative of both sides gives us

$$(3.2) \quad \frac{B'(z)}{B(z)} = \sum_{n \geq 1} \frac{(1 - |a_n|^2)}{(z - a_n)(1 - \bar{a}_n z)}.$$

Thus,

$$(3.3) \quad B'(z) = \sum_{n \geq 1} B_n(z) \frac{(1 - |a_n|^2)}{(1 - \bar{a}_n z)^2},$$

where

$$(3.4) \quad B_n(z) = \frac{B(z)(1 - \bar{a}_n z)}{(z - a_n)}, \quad (n \geq 1),$$

is the subproduct formed with all zeros except a_n . Now, we use the formula for B' and take the derivative of both sides $j - 1$ times. Leibnitz's formula tells us

$$B^{(j)}(z) = \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n \geq 1} B_n^{(j-1-k)}(z) \frac{(k+1)! \bar{a}_n^k (1 - |a_n|^2)}{(1 - \bar{a}_n z)^{k+2}}.$$

Note that on the right side we have $B_n^{(\ell)}$, where ℓ runs between 0 and $j - 1$. Hence, the induction hypothesis applies. To deal with the other term, we consider $r < 1$ and $R > 1$ separately.

If $r < 1$, then

$$\begin{aligned} \left| \frac{(k+1)! \bar{a}_n^k (1 - |a_n|^2)}{(1 - \bar{a}_n r)^{k+2}} \right| &\leq \frac{(k+1)! (1 - |a_n|^2)}{|(1 - a_n)/2|^{k+2}} \\ &\leq \frac{2(k+1)! (1 - |a_n|)}{|(1 - a_n)/2|^{N+1}} \\ &= 2^{N+2} (k+1)! \frac{(1 - |a_n|)}{|1 - a_n|^{N+1}}. \end{aligned}$$

But, for $R > 1$, we have

$$\begin{aligned} \left| \frac{(k+1)! \bar{a}_n^k (1 - |a_n|^2)}{(1 - \bar{a}_n R)^{k+2}} \right| &\leq \frac{(k+1)! (1 - |a_n|^2)}{|R^{-1} - a_n|^{k+2}} \\ &\leq M \frac{(1 - |a_n|)}{|1 - a_n|^{N+1}}, \end{aligned}$$

where M is a constant. This is because the condition (3.1) ensures that any Stolz domain anchored at ζ can only contain a finite number of the zeros a_n . Take any of these domains anchored at $\zeta = 1$, e.g. the one with opening $\pi/2$ or more explicitly the domain $|\Im z| \leq 1 - \Re z$. Then, for a_n 's which are not in this domain but are close to $\zeta = 1$, say at a distance at most 1, we have

$$|R^{-1} - a_n| \leq |1 - a_n|/\sqrt{2}.$$

Thus,

$$\begin{aligned} \frac{(k+1)! (1 - |a_n|^2)}{|R^{-1} - a_n|^{k+2}} &\leq \frac{2^{(k+2)/2} (k+1)! (1 - |a_n|^2)}{|1 - a_n|^{k+2}} \\ &\leq \frac{2^{(N+1)/2} N! (1 - |a_n|^2)}{|1 - a_n|^{N+1}}. \end{aligned}$$

The other points rest at a uniform positive distance from $\zeta = 1$.

Based on the above discussion and the induction hypothesis, if $\delta > 0$ is such that $[1 - \delta, 1)$ is free from the zeros of B , then all the series

$$\sum_{n \geq 1} B_n^{(j-1-k)}(z) \frac{(k+1)! \bar{a}_n^k (1 - |a_n|^2)}{(1 - \bar{a}_n z)^{k+2}}, \quad (0 \leq k \leq j-1),$$

are uniformly and absolutely convergent for $z \in [1 - \delta, 1 + \delta]$. Hence, $B^{(j)}(z)$ is also a continuous function on this interval, which can be equally stated as in the theorem based on the right and left limits at $\zeta = 1$.

Appealing to the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j - 1$. Then the above calculation for $r < 1$ shows that

$$\begin{aligned} |B^{(j)}(r)| &\leq \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n \geq 1} |B_n^{(j-1-k)}(r)| \frac{2^{N+2}(k+1)!(1-|a_n|)}{|1-a_n|^{N+1}} \\ &\leq \left(\sum_{k=0}^{j-1} \binom{j-1}{k} 2^{N+2}(k+1)! \right) CA. \end{aligned}$$

Hence, with a bigger constant the result holds for the derivative of order j . We choose the largest constant corresponding to the derivative of order N as the constant C . \square

The mere usefulness of the estimation in part (ii) Theorem 3.1 is that the constant C does not depend on the distribution of zeros. It just depend on the upper bound A and the integer N . Hence, it is equally valid for all the subproducts of B .

Theorem 3.1 is also valid if $\zeta \in \mathbb{D}$. As a matter of fact, the proof is simpler in this case, since part (i) is trivial. Hence, we can say that if $\zeta \in \overline{\mathbb{D}}$ and

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n\zeta|^{N+1}} \leq A,$$

then there is a constant $C = C(N, A)$ such that the estimation

$$|B^{(j)}(r\zeta)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

Corollary 3.2. *Let $(a_n)_{n \geq 1}$ be a Blaschke sequence in \mathbb{D} , and let B be the corresponding Blaschke product. Let $\zeta \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|^2} < \infty.$$

Then B has derivative in the sense of Carathéodory at ζ and

$$|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\zeta - a_n|^2}.$$

Proof. That B has derivative in the sense of Carathéodory at ζ is a direct consequence of Theorem 3.1. To obtain the formula for $|B'(\zeta)|$, we use (3.3). Note that our condition implies that the subproducts B_n have radial limits at ζ . Hence, we can let $r \rightarrow 1$ in

$$B'(r\zeta) = \sum_{n=1}^{\infty} B_n(r\zeta) \frac{(1-|a_n|^2)}{(1-\bar{a}_nr\zeta)^2}$$

to obtain

$$B'(\zeta) = \sum_{n=1}^{\infty} B_n(\zeta) \frac{(1 - |a_n|^2)}{(1 - \bar{a}_n \zeta)^2}.$$

The upper bound

$$\left| \frac{(1 - |a_n|^2)}{(1 - \bar{a}_n r \zeta)^2} \right| \leq \frac{4(1 - |a_n|)}{|\zeta - a|^2}, \quad (0 < r < 1),$$

allows the passage of limit inside the sum. But, according to (3.4), we have

$$B_n(\zeta) = \frac{B(\zeta)(1 - \bar{a}_n \zeta)}{(\zeta - a_n)}, \quad (n \geq 1).$$

Plugging back this in the formula for $B'(\zeta)$ gives

$$B'(\zeta) = \bar{\zeta} B(\zeta) \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|a_n - \zeta|^2}.$$

By taking the absolute values of both sides the result follows. \square

4. HIGHER DERIVATIVES OF b

Let b be an element in the closed unit ball of H^∞ . According to the canonical factorization theorem, b can be decomposed as

$$(4.1) \quad b(z) = B(z)S(z)O(z), \quad (z \in \mathbb{D}),$$

where

$$B(z) = \gamma \prod_n \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right)$$

and

$$S(z) = \exp \left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\sigma(\zeta) \right)$$

and

$$O(z) = \exp \left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right).$$

We can also extend the function b outside the unit disk by the identity (4.1) and the formulas provided for B , S and O . The extended function is analytic for $|z| > 1$, $z \neq 1/\bar{a}_n$. At $1/\bar{a}_n$ it has a pole of the same order as a_n , as a zero of B . We denote this function also by b and it is easily verified that it satisfies the functional identity

$$(4.2) \quad b(z) \overline{b(1/\bar{z})} = 1.$$

One should be careful in dealing with function b inside and outside the unit disc. For example, if

$$b(z) = \frac{1}{2} z^n, \quad (|z| < 1),$$

it is natural to use the same nice formula for $|z| > 1$. However, the functional equation (4.2) says that

$$b(z) = 2z^n, \quad (|z| > 1).$$

Hence, b and its derivatives up to order n show a different behaviour if we approach a point $\zeta_0 \in \mathbb{T}$ from within \mathbb{D} or from outside. In Theorem 4.1 below, we show that under certain circumstances, this can be avoided.

For our application in this section, we can merge $S(z)$ and $O(z)$ and write

$$(4.3) \quad b(z) = B(z)f(z)$$

where

$$(4.4) \quad f(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\right),$$

and μ is the positive measure $d\mu(\zeta) = -\log|b(\zeta)| dm(\zeta) + d\sigma(\zeta)$. Now, Leibnitz's formula says

$$b^{(j)}(z) = \sum_{k=0}^j B^{(k)}(z)f^{(j-k)}(z).$$

For the derivatives of B on a ray, we already established Theorem 3.1. However, a similar result holds for function f , and thus similar statements actually hold for b , i.e. for any function in the closed unit ball of H^∞ .

A special case of the following result and for $N = 0$ is in [?] without proof. The general version was mentioned in [1, 2], again without proof.

Theorem 4.1. *Let b be a in the closed unit ball of H^∞ with the decomposition (4.1). Assume that, for an integer $N \geq 0$ and a point $\zeta_0 \in \mathbb{T}$, we have*

$$(4.5) \quad \sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta_0-a_n|^{N+1}} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^{N+1}} + \int_0^{2\pi} \frac{|\log|b(\zeta)||}{|\zeta_0-\zeta|^{N+1}} dm(\zeta) \leq A.$$

Then the following hold.

(i) For each $0 \leq j \leq N$, both limits

$$b^{(j)}(\zeta_0) := \lim_{r \rightarrow 1^-} b^{(j)}(r\zeta_0) \quad \text{and} \quad \lim_{R \rightarrow 1^+} b^{(j)}(R\zeta)$$

exist and are equal.

(ii) There is a constant $C = C(N, A)$ such that the estimation

$$|b^{(j)}(r\zeta_0)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

Proof. As discussed before theorem, it is enough to establish the result just for the function $f = SO$ given by (4.4). The proof has the same flavor as the proof of Theorem 3.1. We first do the case is $N = 0$, and then the rest follows by induction.

Case $N = 0$: We show that, under the condition (4.5), which now translates as

$$(4.6) \quad \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0-\zeta|} \leq A,$$

$|f(r\zeta_0)|$ and $\arg f(r\zeta_0)$ have both finite limits as r tends to 1^- . For the simplicity of notations, without loss of generality, assume that $\zeta_0 = 1$.

A simple computation shows that

$$f(r) = \exp\left(-\int_{\mathbb{T}} \frac{1-r^2}{|\zeta-r|^2} d\mu(\zeta)\right) \exp\left(-i \int_{\mathbb{T}} \frac{r\Im(\zeta)}{|\zeta-r|^2} d\mu(\zeta)\right).$$

Therefore, we have explicit formulas for $|f(r)|$ and $\arg f(r)$.

The assumption (4.6) implies that there is no Dirac mass at $\zeta_0 = 1$, i.e. $\mu(\{1\}) = 0$. Therefore,

$$\lim_{r \rightarrow 1^-} \frac{1-r^2}{|\zeta-r|^2} = 0$$

for μ -almost every $\zeta \in \mathbb{T}$. Moreover, we have the upper bound estimation

$$\frac{1-r^2}{|\zeta-r|^2} \leq \frac{2}{|1-\zeta|}, \quad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0, 1)$. The condition (4.6) means that the function on the right-hand side belongs to $L^1(\mu)$. Hence, by the dominated convergence theorem, we get

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \frac{1-r^2}{|\zeta-r|^2} d\mu(\zeta) = 0.$$

In return, this observation implies

$$\lim_{r \rightarrow 1^-} |f(r)| = 1.$$

In a similar manner,

$$\lim_{r \rightarrow 1^-} \frac{r\Im(\zeta)}{|\zeta-r|^2} = \frac{\Im(\zeta)}{|1-\zeta|^2}$$

for μ -almost all $\zeta \in \mathbb{T}$. We also have the upper bound estimation

$$\frac{r|\Im(\zeta)|}{|\zeta-r|^2} \leq \frac{2}{|1-\zeta|}, \quad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0, 1)$. Finally, again by the dominated convergence theorem, we see that the limit

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \frac{r\Im(\zeta)}{|\zeta-r|^2} d\mu(\zeta) = \int_{\mathbb{T}} \frac{\Im(\zeta)}{|1-\zeta|^2} d\mu(\zeta)$$

exists and is a finite real number. In return, this implies

$$\lim_{r \rightarrow 1^-} \arg f(r)$$

also exists and is a finite real number. Therefore, $L := \lim_{r \rightarrow 1^-} f(r)$ exists and, moreover, $|L| = 1$.

Put $L = \lim_{r \rightarrow 1^-} f(r)$. By (4.2), the function f satisfies the functional equation

$$f(z) \overline{f(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R \rightarrow 1^+} f(R) = \frac{1}{\lim_{R \rightarrow 1^+} \overline{f(1/R)}} = \frac{1}{\lim_{r \rightarrow 1^-} f(r)} = \frac{1}{L} = L.$$

This argument also shows that f is actually bounded on $[0, +\infty)$. The estimation in part (ii) trivially holds with $C = 1$.

Case $N \geq 1$: Fix $1 \leq j \leq N$, and suppose that the result holds for $0, 1, \dots, j-1$. The condition (4.5) is rewritten as

$$(4.7) \quad \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1-\zeta|^{N+1}} \leq A.$$

Using the formula of f and taking the derivative of both sides gives us

$$(4.8) \quad f'(z) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta-z)^2} d\mu(\zeta) \right) f(z).$$

Now, take the derivative of both sides $j-1$ times. Leibnitz's formula tells us

$$(4.9) \quad f^{(j)}(z) = \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \frac{-2(k+1)!\zeta}{(\zeta-z)^{k+2}} d\mu(\zeta) \right) f^{(j-1-k)}(z).$$

On the right side we have $f^{(\ell)}$, where ℓ runs between 0 and $j-1$. Hence, the induction hypothesis applies. To deal with the other term, note that for $z = r < 1$ and also $z = R > 1$, we have

$$\frac{1}{|\zeta-z|} \leq \frac{2}{|\zeta-1|}, \quad (\zeta \in \mathbb{T}).$$

Thus, for all $z \in (0, \infty) \setminus \{1\}$ and all k with $0 \leq k \leq j-1 \leq N-1$,

$$(4.10) \quad \begin{aligned} \left| \frac{-2(k+1)!\zeta}{(\zeta-z)^{k+2}} \right| &\leq \frac{2(k+1)!}{|(\zeta-1)/2|^{k+2}} \\ &\leq \frac{2N!}{|(\zeta-1)/2|^{N+1}} \\ &= \frac{2^{N+2}N!}{|\zeta-1|^{N+1}}, \quad (\zeta \in \mathbb{T}). \end{aligned}$$

Therefore, by (4.7), (4.10) and the dominated convergence theorem,

$$\lim_{r \rightarrow 1^\pm} \int_{\mathbb{T}} \frac{-2(k+1)!\zeta}{(\zeta-z)^{k+2}} d\mu(\zeta) = \int_{\mathbb{T}} \frac{-2(k+1)!\zeta}{(\zeta-1)^{k+2}} d\mu(\zeta).$$

Note that we again implicitly used the fact $\mu(\{1\}) = 0$. Thus, by induction hypothesis and (4.9), part (i) follows. Moreover, again by the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j-1$. Then, by (4.9) and (4.10),

$$\begin{aligned} |f^{(j)}(r)| &\leq \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \left| \frac{-2(k+1)!\zeta}{(\zeta-r)^{k+2}} \right| d\mu(\zeta) \right) |f^{(j-1-k)}(r)| \\ &\leq \left(\int_{\mathbb{T}} \frac{2^{N+2}N!}{|\zeta-1|^{N+1}} d\mu(\zeta) \right) \sum_{k=0}^{j-1} |f^{(j-1-k)}(r)| \leq j2^{N+2}N!AC. \end{aligned}$$

Hence, with a bigger constant the result holds for the derivative of order j . We choose the largest constant corresponding to the derivative of order N as the constant C . \square

We highlight one property that explicitly mentioned in the proof of Theorem 3.1 for Blaschke products, but it also holds for an arbitrary b . Under the hypothesis of Theorem 4.1, there is a $\delta > 0$ (which depends on b) such that $b^{(j)}(z)$, for $0 \leq j \leq N$, is a continuous function on the ray $[(1 - \delta)\zeta_0, (1 + \delta)\zeta_0]$.

Corollary 4.2. *Let b be in the closed unit ball of H^∞ with the decomposition (4.1). Let $\zeta_0 \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{|\log |b(\zeta)||}{|\zeta_0 - \zeta|^2} dm(\zeta) < \infty.$$

Then b has derivative in the sense of Carathéodory at ζ_0 and

$$|b'(\zeta_0)| = \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{2d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{2|\log |b(\zeta)||}{|\zeta_0 - \zeta|^2} dm(\zeta).$$

Proof. As we did in (4.3), write $b = Bf$. Corollary 3.2 treats the Blaschke product B and gives a formula (the first term appearing in $|b'(\zeta_0)|$ above). Hence, since on the boundary $|b'| = |B'| + |f'|$, we just need to study f and prove that $|f'(\zeta_0)|$ is precisely the remaining two terms in the formula for $|b'(\zeta_0)|$.

That f has derivative in the sense of Carathéodory at ζ_0 is a direct consequence of Theorem 4.1. To obtain the formula for $|f'(\zeta_0)|$, we use (4.8), i.e.

$$f'(r\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - r\zeta_0)^2} d\mu(\zeta) \right) f(r\zeta_0).$$

Now, $r \rightarrow 1$ to obtain

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - \zeta_0)^2} d\mu(\zeta) \right) f(\zeta_0).$$

The upper bound

$$\frac{1}{|\zeta - r\zeta_0|} \leq \frac{2}{|\zeta - \zeta_0|}, \quad (\zeta \in \mathbb{T}, 0 < r < 1).$$

allows the passage of limit inside the integral. Now, note that

$$\frac{-2\zeta}{(\zeta - \zeta_0)^2} = \frac{2\zeta}{(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)\zeta\bar{\zeta}_0} = \frac{2\bar{\zeta}_0}{|\zeta - \zeta_0|^2}.$$

Hence, we rewrite the formula for $f'(\zeta_0)$ as

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{2}{|\zeta - \zeta_0|^2} d\mu(\zeta) \right) \bar{\zeta}_0 f(\zeta_0).$$

By taking the absolute values of both sides the result follows. \square

5. APPROXIMATION BY BLASCHKE PRODUCTS

According to (4.3), and arbitrary element of the closed unit ball of H^∞ may be decomposed as $b = Bf$, where B is a Blaschke product and f is a nonvanishing function given by (4.4). Generally speaking, since B is given by a product of some simple fraction of the form $(az + b)/(cz + d)$, it is easy to handle and study its properties. That is why in this section we explore the possibility to approximate f by some Blaschke products. This will enable us to establish certain properties for the family of Blaschke products first, and then extend them to the whole closed unit ball of H^∞ .

Given a Blaschke product B with zeros $(a_n)_{n \geq 1}$, we define the measure σ_B on $\overline{\mathbb{D}}$ by

$$\sigma_B = \sum_{n=1}^{\infty} (1 - |a_n|) \delta_{\{a_n\}},$$

where $\delta_{\{z\}}$ is the Dirac measure anchored at the point z . We consider σ_B as an element of $\mathcal{M}(\overline{\mathbb{D}})$, the space of finite complex Borel measures on $\overline{\mathbb{D}}$. This space is the dual of $\mathcal{C}(\overline{\mathbb{D}})$. Hence, we equip it with the weak-star topology. Since $\mathcal{C}(\overline{\mathbb{D}})$ is separable, this topology is first countable on $\mathcal{M}(\overline{\mathbb{D}})$. More specifically, this means that each measure has a countable local basis. Naively speaking, this implies that we just need to consider sequences of measures to study the properties of this topology.

In the following, we assume that the Blaschke products are normalized so that $B(0) > 0$.

Theorem 5.1 (Ahern-Clark [1]). *Let f be given by (4.4), and let $(B_n)_{n \geq 1}$ be a sequence of Blaschke products. Then B_n converges uniformly to f on compact subsets of \mathbb{D} if and only if $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$.*

Proof. Assume that $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$, and denote the zeros of B_n by $(a_{nm})_{m \geq 1}$. Since μ is supported on \mathbb{T} the zeros of B_n must tend to \mathbb{T} . In fact, fix any $r < 1$ and consider a continuous positive function φ which is identically 1 on $|z| \leq r$, and identically 0 on $|z| > (1+r)/2$. In between, it has a continuous transition from 1 to 0. Since $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology, we have

$$\int_{\overline{\mathbb{D}}} \varphi d\sigma_{B_n} \longrightarrow \int_{\overline{\mathbb{D}}} \varphi d\mu = 0.$$

But,

$$\begin{aligned} \int_{\overline{\mathbb{D}}} \varphi d\sigma_{B_n} &\geq \int_{|z| \leq r} \varphi d\sigma_{B_n} \\ &= \sum_{|a_{nm}| \leq r} (1 - |a_{nm}|) \\ &\geq (1 - r) \times \text{Card}\{m : |a_{nm}| \leq r\}. \end{aligned}$$

Therefore, for each $r < 1$, there is an $N = N(r)$ such that

$$(5.1) \quad |a_{nm}| > r, \quad (n \geq N, m \geq 1).$$

We now further explore our assumption to show that

$$(5.2) \quad B_n(0) \longrightarrow f(0).$$

Since $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology, we have

$$\int_{\mathbb{D}} d\sigma_{B_n} \longrightarrow \int_{\mathbb{D}} d\mu.$$

By (4.4), $f(0)$ is a positive real number and

$$\int_{\mathbb{D}} d\mu = -\log f(0).$$

But, the left side is

$$\int_{\mathbb{D}} d\sigma_{B_n} = \sum_{m=1}^{\infty} (1 - |a_{nm}|)$$

which is not precisely $-\log B_n(0)$. The actual formula is

$$-\log B_n(0) = -\sum_{m=1}^{\infty} \log |a_{nm}|.$$

However, thanks to (5.1), this difference can be handled. It is elementary to verify that

$$0 \leq t - 1 - \log t \leq (1 - t)^2, \quad (1/2 \leq t \leq 1).$$

Hence, for $n \geq N(r)$,

$$(1 - |a_{nm}|) \leq -\log |a_{nm}| \leq (1 + r)(1 - |a_{nm}|), \quad (m \geq 1).$$

Summing over m gives

$$(5.3) \quad \int_{\mathbb{D}} d\sigma_{B_n} \leq -\log B_n(0) \leq (1 + r) \int_{\mathbb{D}} d\sigma_{B_n}, \quad (n \geq N(r)).$$

Let $n \rightarrow \infty$ to deduce that

$$-\log f(0) \leq \liminf_{n \rightarrow \infty} -\log B_n(0) \leq \limsup_{n \rightarrow \infty} -\log B_n(0) \leq -(1 + r) \log f(0).$$

Now, let $r \rightarrow 1$ to conclude that

$$\lim_{n \rightarrow \infty} \log B_n(0) = \log f(0).$$

The next step is to show that B_n actually uniformly converges to f on any compact subset of \mathbb{D} . In the language of σ_{B_n} , the formula (3.2) is rewritten as

$$\frac{B'_n(z)}{B_n(z)} = \int_{\mathbb{D}} \frac{(1 + |\zeta|)}{(z - \zeta)(1 - \bar{\zeta}z)} d\sigma_{B_n}(\zeta).$$

In the first glance, it seems that the function

$$\zeta \longmapsto \frac{(1 + |\zeta|)}{(z - \zeta)(1 - \bar{\zeta}z)}$$

is not continuous on $\bar{\mathbb{D}}$ and thus we cannot appeal to the weak-star convergence. However, we fix a compact set $|z| \leq r$ and, as we saw above, after a finite number of indices the support of σ_{B_n} is in $|z| > (1 + r)/2$. Hence, we can multiply the above function by a transient function which is 1 on $|z| \geq (1 + r)/2$ and 0 on $|z| \leq 1$. This operation, on one

hand, will not change the value of integrals and, on the other hand, will create a genuine continuous function on $\overline{\mathbb{D}}$. Therefore, we can surely say

$$\begin{aligned} \int_{\overline{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)} d\sigma_{B_n}(\zeta) &\longrightarrow \int_{\overline{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)} d\mu(\zeta) \\ &= \int_{\mathbb{T}} \frac{-2\zeta}{(\zeta-z)^2} d\mu(\zeta), \end{aligned}$$

which translates as

$$(5.4) \quad \frac{B'_n(z)}{B_n(z)} \longrightarrow \frac{f'(z)}{f(z)}$$

as $n \rightarrow \infty$.

Since $(B_n)_{n \geq 1}$ is uniformly bounded by 1 on \mathbb{D} , it is a normal family. Let g be any pointwise limit of a subsequence of $(B_n)_{n \geq 1}$. Then, by (5.2) and (5.4), we must have

$$g(0) = f(0) \quad \text{and} \quad \frac{f'(z)}{f(z)} = \frac{f'(z)}{f(z)}, \quad (z \in \mathbb{D}).$$

Thus, $g = f$, which means that the whole sequence converges uniformly to f on compact sets.

To prove the other way around, assume that B_n converges uniformly to f on compact sets. Thus, $B_n(0) \rightarrow f(0)$ and since f has no zeros on \mathbb{D} , for each r , (5.1) must hold. Hence, if we let $n \rightarrow \infty$ in (5.3), we obtain

$$\limsup_{n \rightarrow \infty} \int_{\overline{\mathbb{D}}} d\sigma_{B_n} \leq -\log f(0) \leq (1+r) \liminf_{n \rightarrow \infty} \int_{\overline{\mathbb{D}}} d\sigma_{B_n}.$$

Let $r \rightarrow 1$ to deduce that

$$\int_{\overline{\mathbb{D}}} d\sigma_{B_n} \longrightarrow -\log f(0).$$

Hence, $(\sigma_{B_n})_{n \geq 1}$ is a bounded sequence in $\mathcal{M}(\overline{\mathbb{D}})$, and any weak-star limit of this sequence must be a positive measure supported on \mathbb{T} . But, the sequence has just one weak-star limit, i.e. μ . This is because if ν is any weak-star limit of the sequence, the first part of proof shows that a subsequence of B_n converges to f_ν , where f_ν is given by (4.4) (with μ replaced by ν). Therefore, $f_\nu = f$ on \mathbb{D} and, using the uniqueness theorem for Fourier coefficients of measures, we conclude that $\nu = \mu$. \square

To establish our next approximation theorem, we need a result of Frostman which by itself interesting and has numerous other applications. Let Θ be an inner function and, for each $w \in \mathbb{D}$, define

$$\Theta_w(z) = \frac{w - \Theta(z)}{1 - \bar{w}\Theta(z)}.$$

The function Θ_w is called a *Frostman shift* of Θ . It is easy to verify that Θ_w is an inner function for each $w \in \mathbb{D}$. However, a lot more is true. Define the *exceptional set* of Θ to be

$$\mathcal{E}(\Theta) = \{w \in \mathbb{D} : \Theta_w \text{ is not a Blaschke product}\}.$$

Frostman showed that $\mathcal{E}(\Theta)$ is a very small set.

Lemma 5.2 (Frostman [17]). *Let Θ be a non-constant inner function, and let $0 < \rho < 1$. Define*

$$\mathcal{E}_\rho(\Theta) = \{\zeta \in \mathbb{T} : \Theta_{\rho\zeta} \text{ is not a Blaschke product}\}.$$

Then $\mathcal{E}_\rho(\Theta)$ has one dimensional Lebesgue measure zero.

Among numerous applications of Lemma 5.2, we single out the one which states that Blaschke products are uniformly dense in the family of inner functions. A variation of the technic used in the proof of the following result will be exploited in establishing Theorem 5.4.

Corollary 5.3 (Frostman [17]). *Let Θ be an inner function, and let $\varepsilon > 0$. Then there is a Blaschke product B such that*

$$\|\Theta - B\|_\infty < \varepsilon.$$

Proof. We have

$$\Theta(z) + \Theta_w(z) = \Theta(z) + \frac{w - \Theta(z)}{1 - \bar{w}\Theta(z)} = \frac{w - \bar{w}\Theta^2(z)}{1 - \bar{w}\Theta(z)}.$$

Thus,

$$|\Theta(z) + \Theta_w(z)| \leq \frac{2|w|}{1 - |w|}.$$

On one hand, this shows that $-\Theta_w \rightarrow \Theta$ in the H^∞ -norm as $w \rightarrow 0$ and, on the other hand, Lemma 5.2 ensures that there are numerous choices of w for which $-\Theta_w$ is a Blaschke product. \square

In the following result, we again use $\mathcal{M}(\overline{\mathbb{D}})$, equipped with the weak-star topology. We remind that it is first countable, i.e. each point has a countable local basis of open neighborhood.

Theorem 5.4 (Ahern-Clark [1]). *Let $\lambda \in \overline{\mathbb{D}}$, let $N \geq 1$, and let μ be a positive measure on \mathbb{T} such that*

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N} < \infty.$$

Then there is a sequence of Blaschke products $(B_n)_{n \geq 1}$ such that $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$ and, moreover,

$$\sum_{m=1}^{\infty} \frac{1 - |a_{nm}|^2}{|1 - \bar{\lambda}a_{nm}|^N} \rightarrow \int_{\mathbb{T}} \frac{2d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N}$$

as $n \rightarrow \infty$.

Proof. First, note that the growth restriction on μ implies that μ cannot have a Dirac mass at $1/\bar{\lambda}$. Our strategy is to prove the theorem for discrete measures with finitely many Dirac masses and then appeal to a limiting argument to extend it for the general case.

Assume that $\sigma = \alpha\delta_{\{1\}}$, where $\alpha > 0$. Construct f according to the recipe (4.4), i.e.

$$f(z) = \exp\left(-\alpha \frac{1+z}{1-z}\right).$$

By Lemma 5.2, the function

$$B_c(z) = \gamma_c \frac{f(z) - c}{1 - \bar{c}f(z)}$$

is a Blaschke product for values of c through a sequence which tend to zero and avoids the exceptional set of f . The unimodular constant

$$\gamma_c = \frac{f(0) - \bar{c}}{|f(0) - c|} \frac{|1 - \bar{c}f(0)|}{1 - cf(0)}$$

is added to ensure $B_c(0) > 0$. The precise value of γ_c is not used below. We just need to know that $\gamma_c \rightarrow 1$ as $c \rightarrow 0$. The formula for B_c implies

$$|f(z) - B_c(z)| \leq |1 - \gamma_c| + \frac{2|c|}{1 - |c|}, \quad (z \in \mathbb{D}).$$

Thus, B_c converges uniformly to f on \mathbb{D} (even uniform convergence on compact sets is enough for us). Therefore, by Theorem 5.1, σ_{B_c} tends to μ in the weak-star topology. The zeros of B_c are

$$\{z : f(z) = c\} = \left\{ a_{cm} = \frac{\alpha + \log c + i2\pi m\alpha}{-\alpha + \log c + i2\pi m\alpha} : m \in \mathbb{Z} \right\},$$

which clearly cluster at 1 as $c \rightarrow 0$. In this case, $\lambda \neq 1$, and thus the function

$$\zeta \mapsto \frac{1 + |\zeta|}{|1 - \bar{\lambda}\zeta|^N}$$

can be considered as a continuous function on $\overline{\mathbb{D}}$ when we deal with measures μ and σ_{B_c} (at least for small values of c). Hence,

$$\int_{\overline{\mathbb{D}}} \frac{1 + |\zeta|}{|1 - \bar{\lambda}\zeta|^N} d\sigma_{B_c}(\zeta) \longrightarrow \int_{\overline{\mathbb{D}}} \frac{1 + |\zeta|}{|1 - \bar{\lambda}\zeta|^N} d\mu(\zeta).$$

But,

$$\int_{\overline{\mathbb{D}}} \frac{1 + |\zeta|}{|1 - \bar{\lambda}\zeta|^N} d\sigma_{B_c}(\zeta) = \sum_{m=1}^{\infty} \frac{1 - |a_{cm}|^2}{|1 - \bar{\lambda}a_{cm}|^N}$$

and

$$\int_{\overline{\mathbb{D}}} \frac{1 + |\zeta|}{|1 - \bar{\lambda}\zeta|^N} d\mu(\zeta) = \int_{\mathbb{T}} \frac{2}{|1 - \bar{\lambda}\zeta|^N} d\mu(\zeta).$$

Therefore, the result follows.

If μ consists of a finite sum of Dirac masses, the result still holds by induction. Now, we turn to the general situation. Assume that μ is an arbitrary positive Borel measure on \mathbb{T} , fulfilling the above mentioned growth restriction. Put

$$d\tau(z) = \frac{d\mu(z)}{|1 - \bar{\lambda}z|^N}.$$

Again note that μ cannot have a Dirac mass at $1/\bar{\lambda}$, and this property persists for all measures considered below. The family of discrete measures with finite number of Dirac

masses is dense in $\mathcal{M}(\overline{\mathbb{D}})$. Hence, there is a sequence τ_n of such measures such that $\tau_n \rightarrow \tau$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$. Therefore, for each $f \in \mathcal{C}(\overline{\mathbb{D}})$,

$$\int_{\overline{\mathbb{D}}} f(z) |1 - \bar{\lambda}z|^N d\tau_n(z) \longrightarrow \int_{\overline{\mathbb{D}}} f(z) |1 - \bar{\lambda}z|^N d\tau(z) = \int_{\overline{\mathbb{D}}} f(z) d\mu(z).$$

This means that $\sigma_n \rightarrow \mu$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$, where σ_n is the discrete measure

$$d\sigma_n(z) = |1 - \bar{\lambda}z|^N d\tau_n(z), \quad (n \geq 1).$$

We appeal to the first part and find a Blaschke product B_n such that σ_{B_n} is close enough to σ_n in the weak-star topology and also

$$\left| \int_{\overline{\mathbb{D}}} \frac{1 + |z|}{|1 - \bar{\lambda}z|^N} d\sigma_{B_n}(z) - \int_{\mathbb{T}} \frac{2}{|1 - \bar{\lambda}\zeta|^N} d\sigma_n(\zeta) \right| < \frac{1}{n}.$$

The result thus follows. Our choice of B_n also implies that $\sigma_{B_n} \rightarrow \mu$ in the weak-star topology of $\mathcal{M}(\overline{\mathbb{D}})$. \square

6. REPRODUCING KERNELS FOR DERIVATIVES

Let \mathcal{H} be a reproducing kernel of functions which are analytic on the domain Ω . The kernels of evaluation at point $z \in \Omega$ form a two parameter family of functions $k_z^{\mathcal{H}}(w)$, where z and w run through Ω and $k_z(w)^{\mathcal{H}}$ is analytic with respect to w and conjugate analytic with respect to z . The essential property of $k_z^{\mathcal{H}}(z)$ is

$$(6.1) \quad f(z) = \langle f, k_z^{\mathcal{H}} \rangle_{\mathcal{H}}, \quad (f \in \mathcal{H}, z \in \Omega).$$

If we successively take the derivative of f with respect to z , we see that the evaluation functional $f \mapsto f^{(n)}(z)$ is given by

$$(6.2) \quad f^{(n)}(z) = \langle f, \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \rangle_{\mathcal{H}}, \quad (f \in \mathcal{H}).$$

But, we need to show that $\partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \in \mathcal{H}$ and also taking the derivative operator inside the inner product is legitimate. We verify this for $n = 1$. For higher derivative, a similar argument works.

For simplicity, write k_z for $k_z^{\mathcal{H}}$. Put $\delta = (1 - |z|)/2$. Then, for each $f \in \mathcal{H}$ and each Δ with $0 < |\Delta| < \delta$, we have

$$\left| \left\langle f, \frac{k_{z+\Delta} - k_z}{\Delta} \right\rangle_{\mathcal{H}} \right| = \left| \frac{f(z+\Delta) - f(z)}{\Delta} \right| = \left| \frac{1}{\Delta} \int_z^{z+\Delta} f'(\zeta) d\zeta \right| \leq C_f,$$

where C_f is the maximum of f' on the disk with center z and radius δ . Therefore, by the uniform boundedness principle, there is a constant C such that

$$\left\| \frac{k_{z+\Delta} - k_z}{\Delta} \right\| \leq C, \quad (0 < |\Delta| < \delta).$$

Let $g \in \mathcal{H}$ be a weak limit of this fraction as $\Delta \rightarrow 0$. Then, on one hand, for each $f \in \mathcal{H}$ we have

$$\langle f, g \rangle = \lim_{\Delta \rightarrow 0} \left\langle f, \frac{k_{z+\Delta} - k_z}{\Delta} \right\rangle = \lim_{\Delta \rightarrow 0} \frac{f(z+\Delta) - f(z)}{\Delta} = f'(z).$$

On the other hand,

$$\begin{aligned} g(\zeta) &= \langle g, k_\zeta \rangle = \lim_{\Delta \rightarrow 0} \left\langle \frac{k_{z+\Delta} - k_z}{\Delta}, k_\zeta \right\rangle \\ &= \lim_{\Delta \rightarrow 0} \frac{k_{z+\Delta}(\zeta) - k_z(\zeta)}{\Delta} = \frac{\partial k_z}{\partial \bar{z}}(\zeta). \end{aligned}$$

In short, $g = \partial k_z / \partial \bar{z}$.

In the light of relation (6.2), we define the notation

$$(6.3) \quad k_{z,n}^{\mathcal{H}} = \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n,$$

i.e. the kernel of evaluation functional of the n -th derivative at $z \in \Omega$. The relation (6.2) is rewritten as

$$(6.4) \quad f^{(n)}(z) = \langle f, k_{z,n}^{\mathcal{H}} \rangle_{\mathcal{H}}, \quad (f \in \mathcal{H}).$$

In the above formula, if we replace f by $k_{z,n}^{\mathcal{H}}$, we obtain

$$(6.5) \quad (k_{z,n}^{\mathcal{H}})^{(n)}(z) = \|k_{z,n}^{\mathcal{H}}\|_{\mathcal{H}}^2.$$

There are some other formulas for $k_{z,n}^{\mathcal{H}}$ and each has its merits and useful in the applications. We treat some of them below. For the space $\mathcal{H}(b)$, instead of $k_{z,n}^{\mathcal{H}(b)}$ we will write $k_{z,n}^b$. Our first formula for $k_{z,n}^b$ is based on the operator X_b .

Lemma 6.1. *We have*

$$k_{z,n}^b = n!(I - \bar{z}X_b^*)^{-(n+1)} X_b^{*n} k_0^b.$$

Proof. For each $w, z \in \mathbb{D}$,

$$((I - \bar{w}S)k_w)(z) = k_w(z) - \bar{w}(Sk_w)(z) = \frac{1}{1 - \bar{w}z} - \bar{w}z \frac{1}{1 - \bar{w}z} = 1 = k_0(z).$$

Hence, $(I - \bar{w}S)k_w = k_0$. But, for each $w \in \mathbb{D}$, the operator $I - \bar{w}S$ is invertible. Thus,

$$k_w = (I - \bar{w}S)^{-1} k_0$$

and, for each $f \in H^2(\mathbb{D})$, we have

$$f(w) = \langle (I - wS^*)^{-1} f, k_0 \rangle_2, \quad (w \in \mathbb{D}).$$

Thus, if $f \in \mathcal{H}(b)$, we can write

$$f(w) = \langle (I - wX_b)^{-1} f, k_0 \rangle_2, \quad (w \in \mathbb{D}).$$

But, $k_0^b = (I - T_b T_b^*) k_0$ and

$$\langle (I - wX_b)^{-1} f, k_0 \rangle_2 = \langle (I - wX_b)^{-1} f, (I - T_b T_b^*) k_0 \rangle_b.$$

Hence,

$$f(w) = \langle (I - wX_b)^{-1} f, k_0^b \rangle_b = \langle f, (I - \bar{w}X_b^*)^{-1} k_0^b \rangle_b, \quad (w \in \mathbb{D}).$$

Since the last relation is valid for every function $f \in \mathcal{H}(b)$, we conclude that $(I - \bar{w}X_b^*)^{-1} k_0^b$ is precisely the reproducing kernel k_w^b . Changing w to z , we obtain

$$k_z^b = (I - \bar{z}X_b^*)^{-1} k_0^b.$$

Hence, using the definition (6.3), we get

$$\begin{aligned} k_{z,n}^b &= \frac{\partial^n k_z^b}{\partial \bar{z}^n} \\ &= \frac{\partial^n}{\partial \bar{z}^n} \left((I - \bar{z} X_b^*)^{-1} k_0^b \right) \\ &= n! (I - \bar{z} X_b^*)^{-(n+1)} X_b^{*n} k_0^b. \end{aligned}$$

□

The formula for $k_z^b = k_{z,0}^b$ is

$$k_z^b(w) = \frac{1 - \overline{b(z)} b(w)}{1 - \bar{z} w}, \quad (z, w \in \mathbb{D}).$$

Using Leibnitz' rule, by straightforward computations, we obtain

$$(6.6) \quad k_{z,n}^b(w) = \frac{\partial^n k_z^b(w)}{\partial \bar{z}^n} = \frac{h_{z,n}^b(w)}{(1 - \bar{z} w)^{n+1}},$$

where $h_{z,n}^b$ is the function

$$(6.7) \quad h_{z,n}^b(w) = n! w^n - b(w) \sum_{j=0}^n \binom{n}{j} \overline{b^{(j)}(z)} (n-j)! w^{n-j} (1 - \bar{z} w)^j.$$

Lemma 6.2. *Let $z_0 \in \mathbb{D}$ with $b(z_0) \neq 0$. Then*

$$h_{z_0,n}^b(1/\bar{z}_0) = (h_{z_0,n}^b)'(1/\bar{z}_0) = \cdots = (h_{z_0,n}^b)^{(n)}(1/\bar{z}_0) = 0.$$

Proof. The functional equation (4.2) shows that b is analytic in a neighborhood of the point $1/\bar{z}_0$. (If $b(z_0) = 0$, then b has pole at $1/\bar{z}_0$ of the same order as the order of b at z_0 .) Therefore, the formula for $k_{z_0}^b(w)$ shows that this kernel is a meromorphic function on $|w| > 1$ with poles as described above and a possible pole at $1/\bar{z}_0$. However, again by (4.2), we have $b(z_0) \overline{b(1/\bar{z}_0)} = 1$ and thus the pole is removable. In short, $w \mapsto k_{z_0}^b(w)$ is analytic at $1/\bar{z}_0$. Therefore, the same is true for the application $w \mapsto k_{z_0}^b(w)$. Respecting this property, the representation (6.6) implies that $h_{z_0,n}^b$ must have a zero of order $n+1$ at $1/\bar{z}_0$. □

We finish this section by studying $k_{z,n}^B$, where B is a Blaschke product formed with zeros $(a_n)_{n \geq 1}$. It is easy to see that

$$(6.8) \quad h_j(z) = \left(\prod_{k=1}^{j-1} \frac{a_k - z}{1 - \bar{a}_k z} \right) \frac{(1 - |a_j|^2)^{1/2}}{1 - \bar{a}_j z}, \quad (j \geq 1),$$

is an orthonormal basis for $K_B = \mathcal{H}(B)$. Sometimes, we will write

$$(6.9) \quad h_j(z) = (1 - |a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1 - \bar{a}_j z}, \quad (j \geq 1),$$

where B_j is the finite product formed with the first j zeros.

Lemma 6.3. *Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$. Let $z \in \mathbb{D}$. Then*

$$k_{z,n}^B = \sum_{j=1}^{\infty} \overline{h_j^{(n)}(z)} h_j.$$

The series converges in $H^2(\mathbb{D})$ -norm.

Proof. Since $(h_j)_{j \geq 1}$ forms an orthonormal basis for K_B , there are coefficients c_j , $j \geq 1$, such that

$$k_{z,n}^B = \sum_{j=1}^{\infty} c_j h_j,$$

where the series converges in $H^2(\mathbb{D})$ -norm. Moreover, thanks to orthonormality, c_j is given by

$$c_j = \langle k_{z,n}^B, h_j \rangle_2.$$

But, the formula 6.2 immediately implies $\bar{c}_j = h_j^{(n)}(z)$. \square

For a Blaschke product $X_B = S^*|_{K_B}$ and $k_0^B = P_B 1$. Thus, $X_B^* = P_B S = M_B$ is the compressed shift on K_B . Therefore, by Lemma 6.1, we have

$$(6.10) \quad k_{z,n}^B = n!(I - \bar{z}M_B)^{-(n+1)} M_B^n P_B 1.$$

Lemma 6.4. *Let $z_0 \in \mathbb{D}$, and $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$. Assume that there are functions $f, g \in H^2(\mathbb{D})$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1} f(z) + B(z)g(z), \quad (z \in \mathbb{D}).$$

Then we have $P_B f = k_{z_0, N}^B / N!$.

Proof. We write the above equation for f and g as

$$S^N 1 = (1 - \bar{z}_0 S)^{N+1} f + Bg.$$

Since M_B is the compression of S , if we apply P_B to both sides, we obtain

$$M_B^N P_B 1 = (1 - \bar{z}_0 M_B)^{N+1} P_B f.$$

Thus,

$$P_B f = (1 - \bar{z}_0 M_B)^{-N-1} M_B^N P_B 1$$

and the result follows from (6.10). \square

7. AN INTERPOLATION PROBLEM

There is a close relation between the existence of derivatives of elements of $\mathcal{H}(b)$ at the boundary and the containment of $X_b^{*N} k_0^b$ to the range of $(I - \bar{\zeta}_0 X_b^*)^{N+1}$. This is fully explored in Theorem 8.2. But, to reach that general result, we need to pave the road by studying some special cases. We start doing this by considering Blaschke products. First, a technical lemma.

Lemma 7.1. *Let $S, (S_n)_{n \geq 1} \in \mathcal{L}(\mathcal{H})$ with the following properties:*

(i) *Each S_n is invertible.*

- (ii) S is injective.
- (iii) $S_n \rightarrow S$ in the norm topology.
- (iv) There is a constant M such that

$$\|S_n^{-1}S\| \leq M, \quad (n \geq 1).$$

Let $y \in \mathcal{H}$. Then $(S_n^{-1}y)_{n \geq 1}$ is a bounded sequence in \mathcal{H} if and only if $y \in \mathbb{R}(S)$. Moreover, if this holds, we actually have $S_n^{-1}y \rightarrow S^{-1}y$ in the weak topology.

Proof. Assume that $(S_n^{-1}y)_{n \geq 1}$ is a bounded sequence in \mathcal{H} . Hence, it has at least one weak limit point in \mathcal{H} . Let $x \in \mathcal{H}$ be a weak limit point of the sequence. Since $S_n \rightarrow S$ in the norm topology, we surely have (at least for a subsequence) $S_n S_n^{-1}y \rightarrow Sx$. Therefore, $y = Sx$, i.e. $y \in \mathbb{R}(S)$. But, since S is injective, the above argument shows that the sequence has precisely one weak point (if x' is another weak limit point, we would have $y = Sx = Sx'$). In other words, the whole sequence tends weakly to x .

To prove the other (easy) direction, assume that $y \in \mathbb{R}(S)$, i.e. $y = Sx$ for some $x \in \mathcal{H}$. Then

$$\|S_n^{-1}y\| \leq \|S_n^{-1}Sx\| \leq \|S_n^{-1}S\| \|x\| \leq M\|x\|, \quad (n \geq 1).$$

□

The following corollary is a realization of the preceding lemma. The assumption are adjusted to fit our application in the study of derivatives of $\mathcal{H}(b)$ functions.

Corollary 7.2. *Let $T_k \in \mathcal{L}(\mathcal{H})$, $\zeta_k \in \mathbb{T}$ and $\lambda_{k,n} \in \mathbb{D}$, for $n \geq 1$ and $1 \leq k \leq p$, with the following properties:*

- (i) Each T_k is a contraction.
- (ii) Each $I - \zeta_k T_k$ is one to one.
- (iii) $T_k T_{k'} = T_{k'} T_k$ for $k, k' \in \{1, \dots, p\}$.
- (iv) For each k , $\lambda_{k,n}$ tends nontangentially to ζ_k as $n \rightarrow \infty$.

Let $y \in \mathcal{H}$. Then the sequence

$$\left((I - \lambda_{1,n} T_1)^{-1} \dots (I - \lambda_{p,n} T_p)^{-1} y \right)_{n \geq 1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I - \zeta_1 T_1) \dots (I - \zeta_p T_p)$, in which case,

$$(I - \lambda_{1,n} T_1)^{-1} \dots (I - \lambda_{p,n} T_p)^{-1} y \rightarrow (I - \zeta_1 T_1)^{-1} \dots (I - \zeta_p T_p)^{-1} y$$

in the weak topology.

Proof. We apply Lemma 7.1 with

$$S_n = (I - \lambda_{1,n} T_1) \dots (I - \lambda_{p,n} T_p)$$

and

$$S = (I - \zeta_1 T_1) \dots (I - \zeta_p T_p).$$

The only nontrivial property is the boundedness of $S_n^{-1}S$. Since T_k are commuting, it is enough to verify that the sequence

$$\left((I - \lambda_{k,n}T_k)^{-1}(I - \zeta_k T_k) \right)_{n \geq 1}$$

is bounded. But,

$$\begin{aligned} \|(I - \lambda_{k,n}T_k)^{-1}(I - \zeta_k T_k)\| &= \|I + (\lambda_{k,n} - \zeta_k)(I - \lambda_{k,n}T_k)^{-1}T_k\| \\ &\leq 1 + |\lambda_{k,n} - \zeta_k| \|(I - \lambda_{k,n}T_k)^{-1}\| \\ &\leq 1 + |\lambda_{k,n} - \zeta_k| (1 - |\lambda_{k,n}|)^{-1} \\ &\leq 1 + M_k. \end{aligned}$$

The last estimation holds since $\lambda_{k,n}$ tends nontangentially to ζ_k . The result thus follows. \square

As a matter of fact, we even need a special case of Corollary 7.2 in which $T_1 = \dots = T_p$.

Corollary 7.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction, $\zeta \in \mathbb{T}$ and $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$, with the following properties:*

- (i) $I - \zeta T$ is one to one.
- (ii) λ_n tends nontangentially to ζ as $n \rightarrow \infty$.

Let $y \in \mathcal{H}$. Then the sequence

$$\left((I - \lambda_n T)^{-p} y \right)_{n \geq 1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I - \zeta T)^p$, in which case,

$$(I - \lambda_n T)^{-p} y \longrightarrow (I - \zeta T)^{-p} y$$

in the weak topology.

Now we are ready to establish the connection between the existence of boundary derivatives in K_B and an interpolation problem.

Theorem 7.4. *Let $\zeta \in \mathbb{T}$, and let $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$ such that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|^{2N+2}} \leq A.$$

Then there are functions $f, g \in H^2(\mathbb{D})$ such that

$$z^N = (1 - \bar{\zeta}z)^{N+1} f(z) + B(z)g(z), \quad (z \in \mathbb{D}),$$

with

$$\|f\|_2 \leq C,$$

where $C = C(N, A)$ is a constant.

Proof. According to Lemma 6.3,

$$k_{z,N}^B = \sum_{j=1}^{\infty} \overline{h_j^{(N)}(z)} h_j.$$

Hence,

$$(7.1) \quad \|k_{z,n}^B\|^2 = \sum_{j=1}^{\infty} |h_j^{(n)}(z)|^2.$$

We rewrite the formula in (6.8) for h_j as

$$h_j(z) = (1 - |a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1 - \bar{a}_j z}.$$

Hence, by Leibnitz's formula,

$$h_j^{(N)}(z) = (1 - |a_j|^2)^{1/2} \sum_{k=1}^N \binom{N}{k} B_{j-1}^{(N-k)}(z) \frac{k!(-\bar{a}_j)^k}{(1 - \bar{a}_j z)^{k+1}}.$$

Therefore, by Theorem 3.1 and denoting the constant $C(N, A)$ of this theorem by C ,

$$\begin{aligned} |h_j^{(N)}(r\zeta)| &\leq (1 - |a_j|^2)^{1/2} \sum_{k=1}^N \binom{N}{k} C \frac{k!}{|1 - \bar{a}_j r\zeta|^{k+1}} \\ &\leq C(1 - |a_j|^2)^{1/2} \sum_{k=1}^N \binom{N}{k} \frac{2^{k+1}k!}{|1 - \bar{a}_j \zeta|^{k+1}} \\ &\leq C(1 - |a_j|^2)^{1/2} \sum_{k=1}^N \binom{N}{k} \frac{2^{N+1}k!}{|1 - \bar{a}_j \zeta|^{N+1}} \\ &= \left(2^{N+1} C \sum_{k=1}^N k! \binom{N}{k} \right) \frac{(1 - |a_j|^2)^{1/2}}{|\zeta - a_j|^{N+1}}. \\ &= C' \frac{(1 - |a_j|^2)^{1/2}}{|\zeta - a_j|^{N+1}}. \end{aligned}$$

Considering (7.1), we conclude

$$(7.2) \quad \|k_{r\zeta,N}^B\|^2 \leq 2AC'^2, \quad (0 < r < 1).$$

The next step is to appeal to the formula (6.10) and Corollary 7.3. Since $\sigma_p(M_B) \subset \mathbb{D}$, the operator $I - \bar{\zeta}M_B$ is injective. Hence, with $T = M_B$, $p = N + 1$ and $y = M_B^N P_B 1$, we see that $M_B^N P_B 1$ belongs to the range of $(I - \bar{\zeta}M_B)^{N+1}$. This means that there is a function $f \in H^2(\mathbb{D})$ such that

$$M_B^N P_B 1 = (I - \bar{\zeta}M_B)^{N+1} f.$$

Since M_B is the compressed shift, we can rewrite the preceding identity as

$$P_B(z^N) = P_B((1 - \bar{\zeta}z)^{N+1}f).$$

Hence, $z^N - (1 - \bar{\zeta}z)^{N+1}f \perp K_B$, or equivalently $z^N - (1 - \bar{\zeta}z)^{N+1}f \in BH^2(\mathbb{D})$. Therefore, there is $g \in H^2(\mathbb{D})$ such that

$$z^N - (1 - \bar{\zeta}z)^{N+1}f = Bg.$$

Finally, Corollary 7.3 also says that

$$f = (I - \bar{\zeta}M_B)^{-N-1}M_B^N P_B 1 = \lim_{r \rightarrow 1} (I - r\bar{\zeta}M_B)^{-N-1}M_B^N P_B 1.$$

Hence, by (6.10)

$$f = (I - \bar{\zeta}M_B)^{-N-1}M_B^N P_B 1 = \frac{1}{N!} \lim_{r \rightarrow 1} k_{r\bar{\zeta}, N}^B,$$

and, by (7.2), the latter is uniformly bounded by a constant. \square

The above result was referees as an interpolation problem since the equation

$$z^n = (1 - \bar{\zeta}z)f(z) + B(z)g(z)$$

has a solution if and only if there is a function $f \in H^2(\mathbb{D})$ such that

$$f(a_n) = \frac{a_n}{(1 - \bar{\zeta}a_n)^{N+1}}, \quad (n \geq 1).$$

Since

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{(1 - \bar{\zeta}a_n)^{N+1}} \right|^2 (1 - |a_n|^2) < \infty,$$

if $(a_n)_{n \geq 1}$ was an interpolation sequence, then the function f trivially exists. The surprising feature of Theorem 7.4 is that it ensures a solution, even with an additional growth restriction, always exists.

Theorem 7.4, in a sense, is reversible. Indeed, this is the version that we need in the proof of Theorem 8.2.

Theorem 7.5. *Let $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$. Assume that there are functions $f, g \in H^2(\mathbb{D})$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1}f(z) + B(z)g(z), \quad (z \in \mathbb{D}),$$

with

$$\|f\|_2 \leq C \quad \text{and} \quad \left(1 - \frac{1}{2C^2}\right)^{1/2} \leq |z_0| < 1,$$

where $C > 1$ is a constant. Then there is a constant $A = A(N, C)$ such that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - \bar{a}_n z_0|^{2N+2}} \leq A.$$

Proof. Since we appeal to induction, the function f and g that appear in the N -th step will be denoted by f_N and g_N . Note that, by Lemma 6.4,

$$(7.3) \quad P_B f_N = \frac{k_{z_0, N}^B}{N!}.$$

Case $N = 0$: By Lemma 6.3, the

$$k_{z_0}^B = \sum_{j=1}^{\infty} \overline{h_j(z_0)} h_j.$$

Hence, by (7.3), our condition $\|f_0\|_2 \leq C$ translates as

$$\sum_{j=1}^{\infty} |h_j(z_0)|^2 \leq C^2.$$

We use (6.9), to requote this estimation as

$$(7.4) \quad \sum_{j=1}^{\infty} |B_{j-1}(z_0)|^2 \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \leq C^2.$$

We just need to get rid of $|B_{j-1}(z_0)|^2$ to establish the result. To do so, just note that since B_j is a subproduct of B , we have

$$P_{B_j} k_{z_0}^B(z) = k_{z_0}^{B_j}(z) = \frac{1 - \overline{B_j(z_0)} B_j(z)}{1 - \bar{z}_0 z}.$$

Hence,

$$\frac{1 - |B_j(z_0)|^2}{1 - |z_0|^2} = k_{z_0}^{B_j}(z_0) = \|k_{z_0}^{B_j}\|^2 \leq \|k_z^{B_j}\|^2 \leq C^2.$$

The restriction $1 - 1/2C^2 \leq |z_0|^2 < 1$ now implies $|B_j(z_0)|^2 \geq 1/2$. Therefore, from (7.4), we conclude

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \leq 2C^2.$$

This settles the case $N = 0$.

Case $N \geq 1$: Assume that the result holds for $N - 1$. Our assumption is that there are functions $f_N, g_N \in H^2(\mathbb{D})$ such that

$$(7.5) \quad z^N = (1 - \bar{z}_0 z)^{N+1} f_N(z) + B(z) g_N(z), \quad (z \in \mathbb{D}),$$

with $\|f_N\|_2 \leq C$. Write

$$1 - (1 - \bar{z}_0 z)^N = - \sum_{k=1}^N \binom{N}{k} (-\bar{z}_0)^k z^k.$$

Multiply by z^{N-1} to get

$$z^{N-1} = (1 - \bar{z}_0 z)^N z^{N-1} - \left(\sum_{k=1}^N \binom{N}{k} (-\bar{z}_0)^k z^{k-1} \right) z^N.$$

Plugging (7.5) gives

$$z^{N-1} = (1 - \bar{z}_0 z)^N f_{N-1}(z) + B(z)g_{N-1}(z), \quad (z \in \mathbb{D}),$$

where

$$f_{N-1}(z) = z^{N-1} - \left(\sum_{k=1}^N \binom{N}{k} (-\bar{z}_0)^k z^{k-1} \right) (1 - \bar{z}_0 z) f_N(z).$$

Hence,

$$\|f_{N-1}\|_2 \leq 1 + 2^{N+1}C.$$

This means that all required conditions are fulfilled and we can apply the induction for $N - 1$. Thus, there is a constant A such that

$$(7.6) \quad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - \bar{a}_n z_0|^{2N}} \leq A.$$

Theorem 3.1 now ensures that $B_j^{(k)}(z_0)$, $0 \leq k \leq 2N - 1$, exist and are uniformly bounded by a constant A' , where B_j is any subproduct of B .

If we take N times the derivative of both sides in (6.9), we obtain

$$h_j^{(N)}(z) = (1 - |a_j|^2)^{1/2} \sum_{k=0}^N \binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1 - \bar{a}_j z)^{N-k+1}}, \quad (j \geq 1).$$

We rewrite this as

$$(7.7) \quad \begin{aligned} & (1 - |a_j|^2)^{1/2} B_{j-1}(z) \frac{N!(-\bar{a}_j)^N}{(1 - \bar{a}_j z)^{N+1}} \\ &= h_j^{(N)}(z) - (1 - |a_j|^2)^{\frac{1}{2}} \sum_{k=1}^N \binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1 - \bar{a}_j z)^{N-k+1}}. \end{aligned}$$

As we saw above, for $1 \leq k \leq N$,

$$\begin{aligned} \left| \binom{N}{k} B_{j-1}^{(k)}(z_0) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1 - \bar{a}_j z)^{N-k+1}} \right| &\leq \frac{A'N!}{|1 - \bar{a}_j z_0|^{N-k+1}} \\ &\leq \frac{A'N!2^N}{|1 - \bar{a}_j z_0|^N}. \end{aligned}$$

Thus, the right side of (7.7) is majorized by

$$|h_j^{(N)}(z)| + A'N!2^N \frac{(1 - |a_j|^2)^{1/2}}{|1 - \bar{a}_j z_0|^N}.$$

The left side of (7.7) is minorized by

$$\frac{N!}{2^{N+1}} \frac{(1 - |a_j|^2)^{1/2}}{|1 - \bar{a}_j z_0|^{N+1}}$$

for zeros $|a_j| \geq 1/2$. Hence, for such j , we have

$$\frac{(1 - |a_j|^2)^{1/2}}{|1 - \bar{a}_j z_0|^{N+1}} \leq 2^{N+1} |h_j^{(N)}(z_0)| + A' N 2^{2N+1} \frac{(1 - |a_j|^2)^{1/2}}{|1 - \bar{a}_j z_0|^N}.$$

Hence, by Minkowski's inequality, Lemmas 6.3 and 6.4, and (7.6),

$$\begin{aligned} & \left(\sum_{|a_j| \geq 1/2} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^{2N+2}} \right)^{1/2} \\ & \leq 2^{N+1} \left(\sum_{j=1}^{\infty} |h_j^{(N)}(z_0)|^2 \right)^{1/2} + A' N 2^{2N+1} \left(\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^{2N}} \right)^{1/2} \\ & \leq 2^{N+1} \|k_{z_0, N}^B\| + A' A N 2^{2N+1} \\ & \leq 2^{N+1} N! \|P_B f_N\| + A' A N 2^{2N+1} \\ & \leq 2^{N+1} N! C + A' A N 2^{2N+1}. \end{aligned}$$

For zeros with $|a_j| < 1/2$, we have

$$\sum_{|a_j| < 1/2} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^{2N+2}} \leq 4 \sum_{|a_j| < 1/2} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^{2N}} \leq 4A.$$

Hence, the result follows. \square

8. DERIVATIVES OF $\mathcal{H}(b)$ FUNCTIONS

There is a close connection between the analytic continuation of b across a subarc of \mathbb{T} on one hand, and the analytic continuation of all function of $\mathcal{H}(b)$ across the same subarc on the other hand. In this section, we treat a similar result. While we are studying the derivative of elements in $\mathcal{H}(b)$, we are content with the existence of nontangential boundary values.

We begin with a simple lemma which is a simple exercise from calculus and is interesting in its own rights. We do not it in such a generality since in our application even the derivative of order $n + m + 1$ exists at all points. However, the proof for the general case is essentially the same.

Lemma 8.1. *Let I be an open interval, and let $a, b \in I$. Suppose that the function $h : I \rightarrow \mathbb{C}$ satisfies the following properties:*

- (i) h has $n + m$ continuous derivatives on the I .
- (ii) $h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$.
- (iii) $h(b) = h'(b) = \dots = h^{(n-1)}(b) = 0$.

Put

$$k(x) = \frac{h(x)}{(x - b)^n}, \quad (x \in I).$$

Then k is $m + 1$ times differentiable on I and, moreover,

$$k^{(m+1)}(x) = \int_0^1 \cdots \int_0^1 h^{(m+n+1)}(b + t_1 \cdots t_n(x - b)) v(t) dt_1 \cdots dt_n,$$

where $v(t) = t_1^{p_1} \cdots t_n^{p_n}$ is some monomial.

Proof. Since $h(b) = 0$, the fundamental theorem of calculus says

$$\begin{aligned} h(x) &= \int_0^1 \frac{d}{dt_1} \left(h(b + t_1(x - b)) \right) dt_1 \\ &= (x - b) \int_0^1 h'(b + t_1(x - b)) dt_1. \end{aligned}$$

Applying the same result to the function $x \mapsto h'(b + t_1(x - b))$ gives

$$h'(b + t_1(x - b)) = t_1(x - b) \int_0^1 h''(b + t_1 t_2(x - b)) dt_2.$$

Therefore,

$$h(x) = (x - b)^2 \int_0^1 \int_0^1 t_1 h''(b + t_1 t_2(x - b)) dt_1 dt_2.$$

Continuing this process n times gives

$$k(x) = \int_0^1 \cdots \int_0^1 t_1^{n-1} t_2^{n-2} \cdots t_{n-1} h^{(n)}(b + t_1 \cdots t_n(x - b)) dt_1 \cdots dt_n.$$

Write $m(t) = t_1^{n-1} t_2^{n-2} \cdots t_{n-1}$.

Since h has $n + m$ continuous derivatives on the I , and $h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$, the function k has $m + 1$ continuous derivatives on the I and

$$\begin{aligned} k^{(m+1)}(x) &= \int_0^1 \cdots \int_0^1 m(t) \frac{\partial^{m+1}}{\partial x^{m+1}} h^{(n)}(b + t_1 \cdots t_n(x - b)) dt_1 \cdots dt_n \\ &= \int_0^1 \cdots \int_0^1 v(t) h^{(m+n+1)}(b + t_1 \cdots t_n(x - b)) dt_1 \cdots dt_n, \end{aligned}$$

where $v(t) = t_1^{m+n} t_2^{m+n-1} \cdots t_{n-1}^{m+2} t_n^{m+1}$. \square

The following result gives a criterion for the existence of the derivatives for functions of $\mathcal{H}(b)$ and it generalizes the Ahern-Clark result.

Theorem 8.2 (Fricain–Mashreghi [14]). *Let b be a point in the unit ball of $H^\infty(\mathbb{D})$ with the canonical factorization (4.1), let $\zeta_0 \in \mathbb{T}$ and let N be a nonnegative integer. Then the following are equivalent.*

- (i) For every $f \in \mathcal{H}(b)$, the functions $f(z), f'(z), \dots, f^{(N)}(z)$ have finite limits as z tends radially to ζ_0 .
- (ii) For every $f \in \mathcal{H}(b)$, the function $|f^{(N)}(z)|$ remains bounded as z tends radially to ζ_0 .
- (iii) $\|k_{z,N}^b\|_b$ is bounded on the ray $z \in [0, \zeta_0]$.
- (iv) $X_b^{*N} k_0^b$ belongs to the range of $(I - \bar{\zeta}_0 X_b^*)^{N+1}$.

(v) We have

$$\sum_n \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0 - \zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{|\log |b(\zeta)||}{|\zeta_0 - \zeta|^{2N+2}} dm(\zeta) < \infty.$$

Proof. (i) \implies (ii): This is trivial.

(ii) \implies (iii): In the light of representation (6.4), this implication follows from the principle of uniform boundedness.

(iii) \implies (iv): By Lemma 6.1,

$$(8.1) \quad k_{z,N}^b = N!(I - \bar{z}X_b^*)^{-(N+1)} X_b^{*N} k_0^b.$$

Since $\sigma_p(X_b^*) \subset \mathbb{D}$, the operator $I - \bar{\zeta}_0 X_b^*$ is injective. By assumption, $(I - \bar{z}_n X_b^*)^{-(N+1)} X_b^{*N} k_0^b$ is uniformly bounded for any sequence $z_n \in \mathbb{D}$ tending radially to ζ_0 . Now, we apply Corollary 7.3 with $T = X_b^*$, $p = N + 1$ and $y = X_b^{*N} k_0^b$ to conclude that $X_b^{*N} k_0^b$ belongs to the range of $(I - \bar{\zeta}_0 X_b^*)^{N+1}$.

(iv) \implies (i): Using once more Corollary 7.3, we see that

$$(I - \bar{z}_n X_b^*)^{-(N+1)} X_b^{*N} k_0^b \longrightarrow (I - \bar{\zeta}_0 X_b^*)^{-(N+1)} X_b^{*N} k_0^b$$

in the weak topology, for any sequence $z_n \in \mathbb{D}$ tending radially to ζ . But, (8.1) says that the left side is precisely $k_{z_n,N}^b$. Hence, in the light of (6.4), for every function f in $\mathcal{H}(b)$, The N -th derivative $f^{(N)}(z)$ has a finite limit as z tends radially to ζ_0 .

The rest is by induction. We have

$$I - (I - \bar{\zeta}_0 X_b^*)^N = - \sum_{k=1}^N \binom{N}{k} (-\zeta_0)^k X_b^{*k}.$$

Multiply both sides by $X_b^{*(N-1)} k_0^b$ to get

$$X_b^{*(N-1)} k_0^b = (I - \bar{\zeta}_0 X_b^*)^N X_b^{*(N-1)} k_0^b - \sum_{k=1}^N \binom{N}{k} (-\zeta_0)^k X_b^{*(k-1)} X_b^{*N} k_0^b.$$

Hence, $X_b^{*(N-1)} k_0^b$ belongs to the range of $(I - \bar{\zeta}_0 X_b^*)^N$. The above argument applies with N replaced by $N - 1$. We continue this process N times. Therefore, for every function f in $\mathcal{H}(b)$, $f^{(j)}(z)$, $0 \leq j \leq N$, has a finite limit as z tends radially to ζ_0 .

(v) \implies (iii): Without loss of generality we assume that $\zeta_0 = 1$. By Theorem 4.1, the condition (v) implies that

$$\lim_{r \rightarrow 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R \rightarrow 1^+} b^{(j)}(R)$$

exist and are equal for $0 \leq j \leq 2N + 1$. Moreover, since b can have only a finite number of real zeros, we can take $\delta > 0$ such that the interval $[1 - \delta, 1)$ is free of zeros of b . Therefore, b has $2N + 1$ continuous bounded derivatives on $[1 - \delta, 1 + \delta]$. Now, fix r in the interval $(1 - \delta, 1)$.

We remind that by (6.6) and (6.7), $k_{r,N}^b(x) = h_{r,N}^b(x)/(1-rx)^{N+1}$, where

$$h_{r,N}^b(x) = N!x^N - b(x) \sum_{j=0}^N \binom{N}{j} \overline{b^{(j)}(r)} (N-j)! x^{N-j} (1-rx)^j.$$

Hence, $h_{r,N}^b$ has $2N+1$ continuous bounded derivatives on $(1-\delta, 1+\delta)$. Moreover, by Lemma 6.2, we have

$$h_{r,N}^b(1/r) = (h_{r,N}^b)'(1/r) = \dots = (h_{r,N}^b)^{(N)}(1/r) = 0.$$

We now apply Lemma 8.1 with $I = (1-\delta, 1+\delta)$, $a = 1$, $b = 1/r$, $n = N+1$, $m = N$, and $h = h_{r,N}^b$. Note that

$$\frac{h(x)}{(x-b)^n} = \frac{h_{r,N}^b(x)}{(x-1/r)^{N+1}} = (-r)^{N+1} k_{N,r}^b(x).$$

Thus, lemma says that $(-r)^{N+1} (k_{N,r}^b)^{(N)}(x)$ is equal to

$$\int_0^1 \dots \int_0^1 (h_{r,N}^b)^{(2N+1)} \left(\frac{1}{r} + t_1 \dots t_{N+1} \left(x - \frac{1}{r} \right) \right) v(t) dt_1 \dots dt_{N+1}.$$

Since there is an M such that

$$|(h_{r,N}^b)^{(2N+1)}(s)| \leq M, \quad (1-\delta < s < 1+\delta),$$

we deduce that

$$|(k_{r,N}^b)^{(N)}(x)| \leq M \delta^{-N-1}, \quad (1-\delta < x < 1).$$

In particular, $(k_{r,N}^b)^{(N)}(r)$ is bounded as $r \rightarrow 1^-$. But, according to (6.5),

$$\|k_{z,N}^b\|_b^2 = (k_{z,N}^b)^{(N)}(z).$$

Thus, $\|k_{r,N}^b\|_b$ remains bounded as $r \rightarrow 1^-$.

(iii) \implies (v): Again, without loss of generality, assume that $\zeta_0 = 1$. Fix $r \in (0, 1)$. Considering the canonical factorization of b , since b is in the unit ball of H^∞ , we have

$$\sum_n \frac{1 - |a_n|^2}{|1 - a_n r|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|r - \zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{|\log |b(\zeta)||}{|r - \zeta|^{2N+2}} dm(\zeta) < \infty.$$

For simplicity of formulas, denote the left side by Δ_r . According to Theorem 5.4, there is a sequence $(B_j)_{j \geq 1}$ of Blaschke products, with zeros $(a_{jk})_{k \geq 1}$, converging uniformly to b on compact subsets of \mathbb{D} and such that

$$\sum_{k=1}^{\infty} \frac{1 - |a_{jk}|^2}{|1 - r a_{jk}|^{2N+2}} \longrightarrow \Delta_r$$

as $j \rightarrow \infty$. Hence, the formulas (6.6) and (6.7) show that $k_{w,N}^{B_j}$ tends to $k_{w,N}^b$ uniformly on compact subsets of \mathbb{D} . In particular, we must have

$$\lim_{j \rightarrow \infty} (k_{w,N}^{B_j})^{(N)}(w) = (k_{w,N}^b)^{(N)}(w).$$

In the light of (6.5), we rewrite this identity as

$$\lim_{j \rightarrow \infty} \|k_{w,N}^{B_j}\|_2 = \|k_{w,N}^b\|_b.$$

The assumption (iii) implies that there is a $C > 0$ such that

$$\|k_{r,N}^b\|_b \leq C, \quad (0 < r < 1).$$

Therefore, there is an index j_r such that

$$\|k_{r,N}^{B_j}\|_2 \leq C + 1, \quad (j \geq j_r).$$

The formulas (6.6) and (6.7) also show that

$$(1 - rz)^{N+1} k_{r,N}^{B_j}(z) = N! z^N - B_j(z) g_j(z),$$

where $g_j \in H^2(\mathbb{D})$. Hence, it follows from Theorem 7.5 that there is a constant $A = A(C, N)$ (independent of r) such that

$$\sum_k \frac{1 - |a_{jk}|^2}{|1 - ra_{jk}|^{2N+2}} \leq A, \quad (j \geq j_r),$$

Letting $j \rightarrow \infty$, we obtain $\Delta_r \leq A$ for all $r \in (0, 1)$. Finally, we let $r \rightarrow 1^-$ to get the desired condition (v). \square

We also mention that Sarason has obtained another criterion in terms of the Clark measure σ_λ associated with b .

Theorem 8.3 (Sarason, [26]). *Let ζ_0 be a point of \mathbb{T} and let ℓ be a nonnegative integer. The following conditions are equivalent.*

- (i) *Each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have nontangential limits at ζ_0 .*
- (ii) *There is a point $\lambda \in \mathbb{T}$ such that*

$$(8.2) \quad \int_{\mathbb{T}} |e^{i\theta} - \zeta_0|^{-2\ell-2} d\sigma_\lambda(e^{i\theta}) < +\infty.$$

- (iii) *The last inequality holds for all $\lambda \in \mathbb{T} \setminus \{b(\zeta_0)\}$.*
- (iv) *There is a point $\lambda \in \mathbb{T}$ such that μ_λ has a point mass at ζ_0 and*

$$\int_{\mathbb{T} \setminus \{\zeta_0\}} |e^{i\theta} - \zeta_0|^{-2\ell} d\sigma_\lambda(e^{i\theta}) < \infty.$$

Recently, Bolotnikov and Kheifets [5] gave a third criterion (in some sense more algebraic) in terms of the Schwarz-Pick matrix. Recall that if b is a function in the unit ball of H^∞ , then the matrix $\mathbf{P}_\ell^\omega(z)$, which will be referred to as to a Schwarz-Pick matrix and defined by

$$\mathbf{P}_\ell^b(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |b(z)|^2}{1 - |z|^2} \right]_{i,j=0}^\ell,$$

is positive semidefinite for every $\ell \geq 0$ and $z \in \mathbb{D}$. We extend this notion to boundary points as follows: given a point $\zeta_0 \in \mathbb{T}$, the boundary Schwarz-Pick matrix is

$$\mathbf{P}_\ell^b(\zeta_0) = \lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} \mathbf{P}_\ell^b(z) \quad (\ell \geq 0),$$

provided this non tangential limit exists.

Theorem 8.4. *Let b be a point in the unit ball of H^∞ , let $\zeta_0 \in \mathbb{T}$ and let ℓ be a nonnegative integer. Assume that the boundary Schwarz-Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists. Then each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have nontangential limits at ζ_0 .*

Further it is shown in [5] that the boundary Schwarz-Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists if and only if

$$(8.3) \quad \lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} d_{b,\ell}(z) < +\infty,$$

where

$$d_{b,\ell}(z) := \frac{1}{(\ell!)^2} \frac{\partial^{2\ell}}{\partial z^\ell \partial \bar{z}^\ell} \frac{1 - |b(z)|^2}{1 - |z|^2}.$$

We should mention that it is not clear to show direct connections between conditions (8.2), (8.3) and condition (v) of Theorem 8.2.

9. PASSAGE TO THE UPPER HALF PLANE \mathbb{C}_+

Let \mathbb{C}_+ denote the upper half plane in the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space consisting of analytic functions f on \mathbb{C}_+ which satisfy the growth restriction

$$\|f\|_2 = \sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)|^2 dx \right)^{1/2} < \infty.$$

For each function $f \in H^2(\mathbb{C}_+)$ and for almost all $x_0 \in \mathbb{R}$,

$$f^*(x_0) = \lim_{t \rightarrow 0^+} f(x_0 + it)$$

exists. Moreover, we have $f^* \in L^2(\mathbb{R})$ and $\mathcal{F}f^* = 0$ on $(-\infty, 0)$, where \mathcal{F} is the Fourier-Plancherel transformation, and $\|f^*\|_2 = \|f\|_2$.

A particularly interesting class of subspaces of $H^2(\mathbb{C}_+)$ is the $\mathcal{H}(b)$ classes of the upper half plane. The definitions are similar to those for the open unit disc. However, for the sake of completeness, we mention them below. We also mention some facts without proof.

For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi(f) = P_+(\varphi f), \quad (f \in H^2(\mathbb{C}_+)),$$

where P_+ is the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $b \in H^\infty$, with $\|b\|_\infty \leq 1$, the de Branges-Rovnyak space $\mathcal{H}(b)$ consists of those $H^2(\mathbb{C}_+)$ functions which are in the range of the operator $(I - T_b T_b^*)^{1/2}$. As before, $\mathcal{H}(b)$ is a Hilbert space when equipped with the inner product

$$\langle (I - T_b T_b^*)^{1/2} f, (I - T_b T_b^*)^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker (I - T_\varphi T_{\bar{b}})^{1/2}$. For each $w \in \mathbb{C}_+$, the function

$$k_w^b(z) = \frac{i}{2\pi} \frac{1 - \overline{b(w)}b(z)}{z - \bar{w}}, \quad (z \in \mathbb{C}_+),$$

is the reproducing kernel of $\mathcal{H}(b)$, that is

$$(9.1) \quad f(w) = \langle f, k_w^b \rangle_b, \quad (f \in \mathcal{H}(b)).$$

In particular, with $f = k_w^b(z)$, we obtain

$$(9.2) \quad \|k_w^b\|_b^2 = k_w^b(w) = \frac{1}{2\pi} \frac{1 - |b(w)|^2}{2\Im w}, \quad (f \in \mathcal{H}(b)).$$

Let $\rho(t) = 1 - |b(t)|^2$, $t \in \mathbb{R}$, and let $L^2(\rho)$ stand for the usual Hilbert space of measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with $\|f\|_\rho < \infty$, where

$$\|f\|_\rho^2 = \int_{\mathbb{R}} |f(t)|^2 \rho(t) dt.$$

For each $w \in \mathbb{C}_+$, the Cauchy kernel k_w belongs to $L^2(\rho)$. Hence, we define $H^2(\rho)$ to be the span in $L^2(\rho)$ of the functions k_w ($w \in \mathbb{C}_+$). If g is a function in $L^2(\rho)$, then $g\rho$ is in $L^2(\mathbb{R})$, being the product of $g\rho^{1/2} \in L^2(\mathbb{R})$ and the bounded function $\rho^{1/2}$. Thus, we define the operator $C_\rho : L^2(\rho) \rightarrow H^2(\mathbb{C}_+)$ by

$$C_\rho(g) = P_+(g\rho).$$

Then C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ whose initial space equals to $H^2(\rho)$ and it is an isometry if and only if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$.

Write $k_w^b = k_w - \overline{b(w)}bk_w$. Since $T_{\bar{b}}k_w = \overline{b(w)}k_w$, we obtain

$$(9.3) \quad \begin{aligned} T_{\bar{b}}k_w^b &= \overline{b(w)} (k_w - P_+(|b|^2 k_w)) \\ &= \overline{b(w)} P_+((1 - |b|^2)k_w) \\ &= \overline{b(w)} P_+(\rho k_w) \\ &= \overline{b(w)} C_\rho(k_w). \end{aligned}$$

In Section 8, we have studied the boundary behavior of the derivatives of functions in $\mathcal{H}(b)$ spaces of the open unit disc \mathbb{D} . We mention some parts of Theorems 4.1 and 8.2, modified for the upper half plane \mathbb{C}_+ , that are needed below.

Theorem 9.1. *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$ and let $b = BI_\mu O_b$ be its canonical factorization, where*

$$B(z) = \prod_k e^{i\alpha_k} \frac{z - z_k}{z - \bar{z}_k}$$

is a Blaschke product, the singular inner function I_μ is given by

$$I_\mu(z) = \exp \left(iaz - \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) d\mu(t) \right)$$

with a positive singular measure μ and $a \geq 0$, and O_b is the outer function

$$O_b(z) = \exp \left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) \log |b(t)| dt \right).$$

Put

$$(9.4) \quad S_n(x) = \sum_{k=1}^{\infty} \frac{\Im z_k}{|x-z_k|^n} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x-t|^n} + \int_{\mathbb{R}} \frac{|\log |b(t)||}{|x-t|^n} dt.$$

Then, for $x \in \mathbb{R}$ and for $n \geq 0$, the following hold.

(i) If $S_{n+1}(x) < \infty$, then the limits

$$b^{(j)}(x) = \lim_{t \rightarrow 0^+} b^{(j)}(x+it), \quad (0 \leq j \leq n),$$

exist.

(ii) For every function $f \in \mathcal{H}(b)$, the limits

$$f^{(j)}(x) = \lim_{t \rightarrow 0^+} f^{(j)}(x+it), \quad (0 \leq j \leq n),$$

exist if and only if $S_{2n+2}(x) < \infty$.

Theorem 9.1 suggests to define

$$E_n(b) = \{x \in \mathbb{R} : S_n(x) < \infty\}.$$

The upper half plane version of Corollary 4.2 says that, in $x \in E_2(b)$, then the modulus of the angular derivative of b at a point x is given by

$$(9.5) \quad |b'(x)| = a + \sum_k \frac{2\Im z_k}{|x-z_k|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{|x-t|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{|\log |b(t)||}{|x-t|^2} dt.$$

10. INTEGRAL REPRESENTATIONS FOR DERIVATIVES

In this section, our goal is to prove an integral representation for the derivatives of elements of $\mathcal{H}(b)$. we start by finding such a formula at a point w in the upper half plane. Since w is away from the boundary, the representation is easy to establish. In order to get an integral representation for the n th derivative of f at point w for functions in the de-Branges-Rovnyak spaces, we need to introduce the following kernels

$$(10.1) \quad k_{w,n}^b(z) = -\frac{n!}{2\pi i} \cdot \frac{1 - b(z) \sum_{p=0}^n \frac{\overline{b^{(p)}(w)}}{p!} (z - \bar{w})^p}{(z - \bar{w})^{n+1}}, \quad (z \in \mathbb{C}_+),$$

and

$$(10.2) \quad k_{w,n}^p(t) = -\frac{n!}{2\pi i} \cdot \frac{\sum_{p=0}^n \frac{\overline{b^{(p)}(w)}}{p!} (t - \bar{w})^p}{(t - \bar{w})^{n+1}}, \quad (t \in \mathbb{R}).$$

For $n = 0$, we see that $k_{w,0}^b = k_w^b$ and $k_{w,0}^\rho = \overline{b(w)}k_w$. As a matter of fact, (10.1) is the upper half plane version of (6.6), and we have

$$f^{(n)}(w) = \langle f, k_{w,n}^b \rangle_b, \quad (f \in \mathcal{H}(b)).$$

In the following lemma, we obtain a more friendly representation for $f^{(n)}(w)$.

Lemma 10.1. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\bar{b}}f = C_\rho(g)$. Then, for all $w \in \mathbb{C}_+$ and for any integer $n \geq 0$, we have $k_{w,n}^b \in \mathcal{H}(b)$ and $k_{w,n}^\rho \in H^2(\rho)$ and*

$$(10.3) \quad f^{(n)}(w) = \int_{\mathbb{R}} f(t) \overline{k_{w,n}^b(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{w,n}^\rho(t)} dt.$$

Proof. According to (9.1), we have

$$f(w) = \langle f, k_w^b \rangle_b = \langle f, k_w^b \rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}k_w^b \rangle_{\bar{b}}.$$

Hence, by (9.3),

$$f(w) = \langle f, k_w^b \rangle_2 + b(w) \langle C_\rho(g), C_\rho(k_w) \rangle_{\bar{b}}.$$

Since C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$, with initial space equals to $H^2(\rho)$, we conclude that

$$f(w) = \langle f, k_w^b \rangle_2 + b(w) \langle g, k_w \rangle_\rho = \langle f, k_w^b \rangle_2 + \langle \rho g, k_w^\rho \rangle_2.$$

We rewrite this identity as

$$f(w) = \langle f, k_{w,0}^b \rangle_2 + \langle \rho g, k_{w,0}^\rho \rangle_2,$$

which is precisely the representation (10.3) for $n = 0$.

Now straightforward computations show that

$$\frac{\partial^n k_{w,0}^b}{\partial \bar{w}^n} = k_{w,n}^b \quad \text{and} \quad \frac{\partial^n k_{w,0}^\rho}{\partial \bar{w}^n} = k_{w,n}^\rho.$$

Since $k_{w,0}^b \in \mathcal{H}(b)$ and $k_{w,0}^\rho \in H^2(\rho)$, as we justified similarly in Section 6, we have $k_{w,n}^b \in \mathcal{H}(b)$ and $k_{w,n}^\rho \in H^2(\rho)$, $n \geq 0$. The representation (10.3) follows now by induction and by differentiating under the integral sign. \square

The next step is to show that (10.3) is still valid at the boundary points x_0 which satisfy $S_{2n+2}(x_0) < \infty$. To do so, we need the boundary analogues of the kernels (10.1) and (10.2). In fact, both formulas make sense if we simply replace w by x_0 and assume that $S_{n+1}(x_0) < \infty$. However, we see that, under the stronger condition $S_{2n+2}(x_0) < \infty$, $k_{x_0,n}^b$ is actually the kernel function in $\mathcal{H}(b)$ for the functional of the n -th derivative at x_0 .

Lemma 10.2. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let $n \geq 0$, and let $x_0 \in \mathbb{R}$. Assume that x_0 satisfies the condition $S_{2n+2}(x_0) < \infty$. Then $k_{x_0,n}^b \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, we have*

$$(10.4) \quad f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b.$$

Proof. According to Theorem 9.1, the condition $S_{2n+2}(x_0) < \infty$ guarantees that, for every function $f \in \mathcal{H}(b)$, $f^{(n)}(w)$ tends to $f^{(n)}(x_0)$, as w tends radially to x_0 . Therefore, an application of the uniform boundedness principle shows that the functional $f \mapsto f^{(n)}(x_0)$ is bounded on $\mathcal{H}(b)$. Hence, by Riesz' theorem, there exists $\varphi_{x_0,n} \in \mathcal{H}(b)$ such that

$$f^{(n)}(x_0) = \langle f, \varphi_{x_0,n} \rangle_b, \quad (f \in \mathcal{H}(b)).$$

Now, the formula

$$f^{(n)}(w) = \langle f, k_{w,n}^b \rangle_b, \quad (f \in \mathcal{H}(b)),$$

implies that $k_{w,n}^b$ tends weakly to $\varphi_{x_0,n}$, as w tends radially to x_0 . Thus, for $z \in \mathbb{C}_+$, we can write

$$\begin{aligned} \varphi_{x_0,n}(z) &= \langle \varphi_{x_0,n}, k_z^b \rangle_b \\ &= \lim_{t \rightarrow 0^+} \langle k_{x_0+it,n}^b, k_z^b \rangle_b \\ &= \lim_{t \rightarrow 0^+} k_{x_0+it,n}^b(z) \\ &= \lim_{t \rightarrow 0^+} -\frac{n!}{2\pi i} \frac{1 - b(z) \sum_{p=0}^n \frac{\overline{b^{(p)}(x_0+it)}}{p!} (z - x_0 + it)^p}{(z - x_0 + it)^{n+1}} \\ &= -\frac{n!}{2\pi i} \frac{1 - b(z) \sum_{p=0}^n \frac{\overline{b^{(p)}(x_0)}}{p!} (z - x_0)^p}{(z - x_0)^{n+1}} = k_{x_0,n}^b(z). \end{aligned}$$

Hence, $k_{x_0,n}^b \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, (10.4) holds. \square

The next result gives a (standard) Taylor formula at a point on the boundary.

Lemma 10.3. *Let h be a holomorphic function in the upper-half plane \mathbb{C}_+ , let $n \geq 0$, and let $x_0 \in \mathbb{R}$. Assume that $h^{(n)}$ has a radial limit at x_0 . Then $h, h', \dots, h^{(n-1)}$ have radial limits at x_0 and*

$$h(w) = \sum_{p=0}^n \frac{h^{(p)}(x_0)}{p!} (w - x_0)^p + (w - x_0)^n \varepsilon(w), \quad (w \in \mathbb{C}_+),$$

with $\lim_{t \rightarrow 0^+} \varepsilon(x_0 + it) = 0$.

Proof. The proof is by induction.

Case $n = 0$: This is trivial.

Case $n \geq 1$: Assumes that the property is true for $n - 1$. Applying the induction hypothesis to $g = h'$, we see that $g = h', g' = h^{(2)}, \dots, g^{(n-1)} = h^{(n)}$ have a radial limit at x_0 and

$$h'(w) = \sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (w - x_0)^p + (w - x_0)^{n-1} \varepsilon_1(w),$$

with $\lim_{t \rightarrow 0^+} \varepsilon_1(x_0 + it) = 0$.

Since h' has a radial limit at x_0 , we can define $h(x_0)$ by

$$h(x_0) = h(x_0 + it) - \int_{[x_0, x_0 + it]} h'(u) du.$$

By Cauchy's theorem, $h(x_0)$ is well-defined and its value does not depend on t . However, this freedom in choosing t shows that $h(x_0) = \lim_{t \rightarrow 0} h(x_0 + it)$. Then another application of Cauchy's theorem reveals that we can even write

$$h(w) = h(x_0) + \int_{\Gamma_w} h'(u) du,$$

for all $w \in \mathbb{C}_+$. The path Γ_w is from x_0 to w . To have the uniform continuity of h on the path, we assume that in the beginning Γ_w is a vertical segment starting at x_0 (the height of segment is not important), and then it goes to w via a rectifiable path in \mathbb{C}_+ . Hence, we have

$$\begin{aligned} h(w) &= h(x_0) + \int_{\Gamma_w} \left(\sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (u - x_0)^p + (u - x_0)^{n-1} \varepsilon_1(u) \right) du \\ &= \sum_{p=0}^n \frac{h^{(p)}(x_0)}{p!} (w - x_0)^p + \int_{\Gamma_w} (u - x_0)^{n-1} \varepsilon_1(u) du. \end{aligned}$$

The natural choice for ε is

$$\varepsilon(w) = \frac{1}{(w - x_0)^n} \int_{\Gamma_w} (u - x_0)^{n-1} \varepsilon_1(u) du, \quad (w \in \mathbb{C}_+).$$

Thus,

$$\varepsilon(x_0 + it) = \frac{1}{(it)^n} \int_{s=0}^t (is)^{n-1} \varepsilon_1(x_0 + is) ds.$$

Therefore, based on the induction hypothesis on ε_1 , we have

$$\begin{aligned} |\varepsilon(x_0 + it)| &\leq \frac{1}{t^n} \int_{s=0}^t s^{n-1} |\varepsilon_1(x_0 + is)| ds \\ &\leq \frac{1}{t} \int_{s=0}^t |\varepsilon_1(x_0 + is)| ds \longrightarrow 0 \end{aligned}$$

as $t \rightarrow 0^+$. □

If x_0 satisfies the condition $S_{2n+2}(x_0) < \infty$ we also have $k_{x_0, n}^\rho \in L^2(\rho)$. Indeed, according to (10.2), it suffices to prove that $(t - x_0)^{-j} \in L^2(\rho)$, for $1 \leq j \leq n + 1$. Since $\rho \leq 1$, it is enough to verify this fact in a neighborhood of x_0 , say $I_{x_0} = [x_0 - 1, x_0 + 1]$. But according to the condition $S_{2n+2}(x_0) < \infty$, we have

$$\int_{I_{x_0}} \frac{1 - |b(t)|^2}{|t - x_0|^{2j}} dt \leq 2 \int_{I_{x_0}} \frac{|\log |b(t)||}{|t - x_0|^{2j}} dt \leq 2 \int_{I_{x_0}} \frac{|\log |b(t)||}{|t - x_0|^{2n+2}} dt < \infty.$$

Theorem 10.4 (Fricain–Mashreghi [15]). *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let $n \geq 0$, let $f \in \mathcal{H}(b)$, and let $g \in H^2(\rho)$ be such that $T_{\bar{b}}f = C_\rho(g)$. Then, for every point $x_0 \in \mathbb{R}$ satisfying the condition $S_{2n+2}(x_0) < \infty$, we have*

$$(10.5) \quad f^{(n)}(x_0) = \int_{\mathbb{R}} f(t) \overline{k_{x_0,n}^b(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{x_0,n}^\rho(t)} dt.$$

Proof. Recall that the condition $S_{2n+2}(x_0) < \infty$ guarantees that $b^{(j)}(x_0)$ exists for $0 \leq j \leq 2n+1$. Moreover, Lemma 10.2 implies that $k_{x_0,p}^b \in \mathcal{H}(b)$, for $0 \leq p \leq n$.

Put

$$h_{x_0,n}(z) = \frac{b(z) - \sum_{p=0}^n \frac{b^{(p)}(x_0)}{p!} (z - x_0)^p}{(z - x_0)^{n+1}}, \quad (z \in \mathbb{C}_+).$$

Let us verify that $h_{x_0,n}$ satisfies

$$(10.6) \quad h_{x_0,n} = 2\pi i \sum_{p=0}^n \frac{b^{(n-p)}(x_0)}{(n-p)! p!} k_{x_0,p}^b.$$

To simplify a little bit the next computations, we put $a_p := \frac{b^{(p)}(x_0)}{p!}$, $0 \leq p \leq n$. According to (10.1), we have

$$\begin{aligned} & 2\pi i \sum_{p=0}^n a_{n-p} \frac{k_{x_0,p}^b(z)}{p!} \\ &= \sum_{p=0}^n a_{n-p} \left(b(z) \frac{\sum_{j=0}^p \bar{a}_j (z - x_0)^j - 1}{(z - x_0)^{p+1}} \right) \\ &= \frac{1}{(z - x_0)^{n+1}} \left[\sum_{p=0}^n a_{n-p} (z - x_0)^{n-p} \left(b(z) \sum_{j=0}^p \bar{a}_j (z - x_0)^j - 1 \right) \right] \\ &= \frac{1}{(z - x_0)^{n+1}} \left[b(z) \left(\sum_{p=0}^n \sum_{j=0}^p a_{n-p} \bar{a}_j (z - x_0)^{n-p+j} \right) - \sum_{k=0}^n a_k (z - x_0)^k \right]. \end{aligned}$$

Therefore, we see that (10.6) is equivalent to

$$(10.7) \quad \sum_{p=0}^n \sum_{j=0}^p a_{n-p} \bar{a}_j (z - x_0)^{n-p+j} = 1.$$

But, putting $j = \ell - n + p$, we obtain

$$\begin{aligned} \sum_{p=0}^n \sum_{j=0}^p a_{n-p} \bar{a}_j (z - x_0)^{n-p+j} &= \sum_{\ell=0}^n \left(\sum_{p=n-\ell}^n a_{n-p} \overline{a_{\ell-n+p}} \right) (z - x_0)^\ell \\ &= \sum_{\ell=0}^n \left(\sum_{q=0}^{\ell} a_{\ell-q} \bar{a}_q \right) (z - x_0)^\ell. \end{aligned}$$

Consequently, (10.7) is equivalent to

$$(10.8) \quad |a_0|^2 = |b(x_0)|^2 = 1 \quad (\text{for } \ell = 0),$$

and

$$(10.9) \quad \sum_{q=0}^{\ell} a_{\ell-q} \overline{a_q} = 0, \quad (1 \leq \ell \leq n).$$

To establish (10.8) and (10.8), put

$$\varphi(z) = 1 - b(z) \sum_{p=0}^n \overline{a_p} (z - x_0)^p, \quad (z \in \mathbb{C}_+).$$

Then φ is holomorphic in \mathbb{C}_+ and φ and its derivatives up to order $2n + 1$ have radial limits at x_0 . An application of Lemma 10.3 shows that we can write

$$\varphi(z) = \sum_{p=0}^n \frac{\varphi^{(p)}(x_0)}{p!} (z - x_0)^p + o((z - x_0)^n),$$

as z tends radially to x_0 . Assume that there exists $p \in \{0, \dots, n\}$ such that $\varphi^{(p)}(x_0) \neq 0$ and put

$$p_0 = \min\{0 \leq p \leq n : \varphi^{(p)}(x_0) \neq 0\}.$$

Hence, as $t \rightarrow 0^+$,

$$|k_{x_0, n}^b(x_0 + it)| \sim \frac{1}{2\pi} \frac{|\varphi^{(p_0)}(x_0)|}{p_0!} t^{p_0 - (n+1)},$$

which implies that $\lim_{t \rightarrow 0^+} |k_{x_0, n}^b(x_0 + it)| = \infty$. This is a contradiction with the fact that $k_{x_0, n}^b$ belongs to $\mathcal{H}(b)$ and has a finite radial limit at x_0 . Therefore, we necessarily have $\varphi^{(p)}(x_0) = 0$, $0 \leq p \leq n$. But,

$$\varphi(x_0) = 1 - b(x_0) \overline{b(x_0)} = 1 - |b(x_0)|^2,$$

and if we use Leibnitz rule to compute the derivative of φ , for $1 \leq \ell \leq n$, we get

$$\varphi^{(\ell)}(x_0) = - \sum_{p=0}^{\ell} \overline{a_p} \binom{\ell}{p} p! b^{(\ell-p)}(x_0) = -\ell! \sum_{p=0}^{\ell} \overline{a_p} a_{\ell-p}.$$

Thus, we established (10.9) and (10.8), which in return proves (10.6). According to Lemma 10.2, (10.6) implies $h_{x_0, n} \in \mathcal{H}(b)$.

Now, for almost all $t \in \mathbb{R}$, we have

$$\begin{aligned}
& \overline{b(t)} k_{x_0, n}^b(t) \\
&= -\frac{n!}{2\pi i} \cdot \frac{\overline{b(t)} - |b(t)|^2 \sum_{p=0}^n \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} \\
&= -\frac{n!}{2\pi i} \cdot (1 - |b(t)|^2) \frac{\sum_{p=0}^n \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} - \frac{n!}{2\pi i} \cdot \frac{\overline{b(t)} - \sum_{p=0}^n \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} \\
&= \rho(t) k_{x_0, n}^\rho(t) - \frac{n!}{2\pi i} \cdot \overline{h_{x_0, n}(t)}.
\end{aligned}$$

Since $h_{x_0, n} \in \mathcal{H}(b) \subset H^2(\mathbb{C}_+)$, we get that $P_+(\overline{b}k_{x_0, n}^b) = P_+(\rho k_{x_0, n}^\rho)$, which can be written as $T_{\overline{b}}k_{x_0, n}^b = C_\rho k_{x_0, n}^\rho$. It follows from Lemma 10.2 that

$$\begin{aligned}
f^{(n)}(x_0) &= \langle f, k_{x_0, n}^b \rangle_b \\
&= \langle f, k_{x_0, n}^b \rangle_2 + \langle T_{\overline{b}}f, T_{\overline{b}}k_{x_0, n}^b \rangle_{\overline{b}} \\
&= \langle f, k_{x_0, n}^b \rangle_2 + \langle g, k_{x_0, n}^\rho \rangle_\rho \\
&= \int_{\mathbb{R}} f(t) \overline{k_{x_0, n}^b(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{x_0, n}^\rho(t)} dt,
\end{aligned}$$

which proves the relation (10.5). □

If b is inner, then it is clear that the second integral in (10.5) is zero and we obtain the easier formula

$$(10.10) \quad f^{(n)}(w) = \int_{\mathbb{R}} f(t) \overline{k_{w, n}^b(t)} dt.$$

We now provide a variation of Theorem 10.4 which is more suitable for obtaining Bernstein-type inequalities. To do so, we need the new kernel

$$(10.11) \quad \mathfrak{K}_{z_0, n}^\rho(t) = \overline{b(z_0)} \frac{\sum_{p=0}^n \binom{n+1}{p+1} (-1)^p \overline{b^p(z_0)} b^p(t)}{(t - \overline{z_0})^{n+1}}, \quad (t \in \mathbb{R}),$$

which is well-defined for all $z_0 \in \mathbb{C}_+$. It also makes sense whenever $z_0 = x_0 \in E_1(b)$. We highlight that b^p is the p -power of b and should not be mistaken with the p -th derivative.

Corollary 10.5 (Baranov–Fricain–Mashreghi [6]). *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$, let $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, and let $n \geq 0$. Then $(k_{z_0}^b)^{n+1} \in H^2(\mathbb{C}_+)$ and $\mathfrak{K}_{z_0, n}^\rho \in L^2(\rho)$. Moreover, for every function $f \in \mathcal{H}(b)$, we have*

$$(10.12) \quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \left(\int_{\mathbb{R}} f(t) \overline{(k_{z_0}^b)^{n+1}(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{\mathfrak{K}_{z_0, n}^\rho(t)} dt \right),$$

where $g \in H^2(\rho)$ is such that $T_{\overline{b}}f = C_\rho g$.

Proof. Step 1: To show that the first integral in (10.12) is well-defined, we show that $(k_{z_0}^b)^{n+1} \in H^2(\mathbb{C}_+)$.

Let $a_p = b^{(p)}(z_0)/p!$. Fix $0 \leq j \leq n$. Then we rewrite (10.1) as

$$-\frac{2\pi i}{j!} k_{z_0, j}^b(z) = \frac{1 - \overline{b(z_0)}b(z)}{(z - \overline{z_0})^{j+1}} - b(z) \sum_{p=1}^j \frac{\overline{a_p}}{(z - \overline{z_0})^{j+1-p}}.$$

Hence, multiplying by $(1 - \overline{b(z_0)}b(z))^j$ and rearranging the terms, we obtain

$$(10.13) \quad \begin{aligned} (k_{z_0}^b(z))^{j+1} &= (1 - \overline{b(z_0)}b(z))^j \left(-\frac{2\pi i}{j!} k_{z_0, j}^b(z) \right) \\ &+ b(z) \sum_{p=1}^j \overline{a_{j+1-p}} (1 - \overline{b(z_0)}b(z))^{j-p} (k_{z_0}^b(z))^p. \end{aligned}$$

Since $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, according to Lemmas 10.1 and 10.2, the functions $k_{z_0}^b$ and $k_{z_0, j}^b$ ($1 \leq j \leq n$) belong to $\mathcal{H}(b)$. Hence, using the recurrence relation (10.13) and that $1 - \overline{b(z_0)}b(z) \in H^\infty(\mathbb{C}_+)$, we see immediately by induction that $(k_{z_0}^b)^{n+1} \in H^2(\mathbb{C}_+)$.

Step 2: To show that the second integral in (10.12) is well-defined, we show that $\mathfrak{R}_{z_0, n}^\rho \in L^2(\rho)$.

We have $= (t - \overline{z_0})^{-(n+1)}\varphi(t)$, with

$$|\mathfrak{R}_{z_0, n}^\rho(t)| \leq \frac{2^{n+1}}{|t - z_0|^{n+1}}, \quad (t \in \mathbb{R}).$$

Hence, it is sufficient to prove that $(t - z_0)^{-(n+1)} \in L^2(\rho)$. If $z_0 \in \mathbb{C}_+$, this fact is trivial and if $z_0 \in E_{2n+2}(b)$, the inequality $1 - x \leq |\log x|$, $x \in [0, 1]$, implies

$$\int_{\mathbb{R}} \frac{\rho(t)}{|t - z_0|^{2n+2}} dt \leq \int_{\mathbb{R}} \frac{1 - |b(t)|^2}{|t - z_0|^{2n+2}} dt \leq 2 \int_{\mathbb{R}} \frac{|\log |b(t)||}{|t - z_0|^{2n+2}} dt < \infty,$$

which is the required result.

Step 3: It remains to prove that (10.12) holds.

Let ψ be any element of $H^2(\mathbb{C}_+)$. According to (10.5), we have

$$\begin{aligned} f^{(n)}(z_0) &= \langle f, k_{z_0, n}^b \rangle_2 + \langle \rho g, k_{z_0, n}^\rho \rangle_2 \\ &= \langle f, k_{z_0, n}^b - b\psi \rangle_2 + \langle \overline{b}f, \psi \rangle_2 + \langle \rho g, k_{z_0, n}^\rho \rangle_2. \end{aligned}$$

But we have $T_{\overline{b}}f = C_\rho g$, which means that $\overline{b}f - \rho g \perp H^2(\mathbb{C}_+)$. Since $\psi \in H^2(\mathbb{C}_+)$, it follows that $\langle \overline{b}f, \psi \rangle_2 = \langle \rho g, \psi \rangle_2$. Hence, the identity

$$(10.14) \quad f^{(n)}(z_0) = \langle f, k_{z_0, n}^b - b\psi \rangle_2 + \langle \rho g, k_{z_0, n}^\rho + \psi \rangle_2$$

holds for each $\psi \in H^2(\mathbb{C}_+)$. A very specific ψ gives us the required representation. To find the appropriate ψ note that, on one hand, we have

$$\begin{aligned} & -\frac{2\pi i}{n!} k_{z_0, n}^b(t) - (k_{z_0}^b)^{n+1}(t) \\ &= \frac{1 - b(t) \sum_{p=0}^n \overline{a_p}(t - \overline{z_0})^p - (1 - \overline{b(z_0)}b(t))^{n+1}}{(t - \overline{z_0})^{n+1}} \\ &= \frac{1 - (1 - \overline{b(z_0)}b(t))^{n+1}}{(t - \overline{z_0})^{n+1}} - b(t) \frac{\sum_{p=0}^n \overline{a_p}(t - \overline{z_0})^p}{(t - \overline{z_0})^{n+1}} \\ &= b(t)\psi(t), \end{aligned}$$

where

$$\psi(t) = \frac{\sum_{p=1}^{n+1} (-1)^{p+1} \binom{n+1}{p} (\overline{b(z_0)})^p (b(t))^{p-1}}{(t - \overline{z_0})^{n+1}} - \frac{\sum_{p=0}^n \overline{a_p}(t - \overline{z_0})^p}{(t - \overline{z_0})^{n+1}}.$$

On the other hand, we easily see that

$$\begin{aligned} & -\frac{2\pi i}{n!} k_{z_0, n}^\rho(t) + \psi(t) \\ &= \frac{\sum_{p=1}^{n+1} (-1)^{p+1} \binom{n+1}{p} (\overline{b(z_0)})^p (b(t))^{p-1}}{(t - \overline{z_0})^{n+1}} \\ &= \overline{b(z_0)} \frac{\sum_{p=0}^n (-1)^p \binom{n+1}{p+1} (\overline{b(z_0)})^p (b(t))^p}{(t - \overline{z_0})^{n+1}} = \mathfrak{R}_{z_0, n}^\rho(t). \end{aligned}$$

Therefore, (10.12) follows immediately from (10.14). □

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