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# On an extremal problem of Garcia and Ross

I. Chalendar\*, E. Fricain† and D. Timotin‡

## Abstract

We show the equivalence of two extremal problems on Hardy spaces, thus answering a question posed by Garcia and Ross. The proof uses a slight generalization of complex symmetric operators.

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*Key words and phrases:* Extremal problem, Hankel operator, complex symmetric operator.

## 1 Introduction

In [4] Garcia and Ross discuss a nonlinear extremal problem for functions in the Hardy space and its relation to a well studied linear extremal problem. Specifically, let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disc in the complex plane and  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  the unit circle. For  $p > 0$ , let  $H^p$  denote the classical Hardy space on  $\mathbb{D}$  (identified, as usual, with a closed subspace of  $L^p = L^p(\mathbb{T})$ ). For fixed  $\psi \in L^\infty$ , the following nonlinear extremal problem is considered in [4]:

$$\Gamma(\psi) := \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) f(z)^2 dz \right|. \quad (1)$$

This is closely related to the well known classical linear extremal problem

$$\Lambda(\psi) := \sup_{\substack{F \in H^1 \\ \|F\|_1=1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) F(z) dz \right|; \quad (2)$$

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it is noted in [4] that we always have  $\Gamma(\psi) \leq \Lambda(\psi)$ , and it is proved that in some particular cases, including the case of rational  $\psi$ , we have equality. We show in this short note that equality actually holds for all  $\psi \in L^\infty$ , thus answering an open question stated in [4].

The two problems can be reformulated in terms of operators on a Hilbert space. Denote by  $P_+$  the projection in  $L^2$  onto  $H^2$  and by  $P_-$  the projection onto  $H_-^2 := L^2 \ominus H^2$ . The Hankel operator of symbol  $\psi$  is  $\mathfrak{H}_\psi : H^2 \rightarrow H_-^2$ , defined by  $\mathfrak{H}_\psi f = P_- \psi f$ .

By changing the variable  $z = e^{it}$  and denoting  $\zeta(t) = e^{it}$ , we have

$$\Gamma(\psi) = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t) \psi(e^{it}) f(e^{it})^2 dt \right| = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} |\langle \psi f, \bar{\zeta} \bar{f} \rangle| = \sup_{\substack{f \in H^2 \\ \|f\|_2=1}} |\langle \mathfrak{H}_\psi f, \bar{\zeta} \bar{f} \rangle|. \quad (3)$$

On the other hand, any function  $F \in H^1$  may be written as  $F = fg$  with  $f, g \in H^2$  and  $\|f\|_2 = \|g\|_2 = \|F\|_1$ . Therefore we get

$$\Lambda(\psi) = \sup_{\substack{f, g \in H^2 \\ \|f\|_2=\|g\|_2=1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t) \psi(e^{it}) f(e^{it}) g(e^{it}) dt \right| = \sup_{\substack{f, g \in H^2 \\ \|f\|_2=\|g\|_2=1}} |\langle \psi f, \bar{\zeta} \bar{g} \rangle| = \|\mathfrak{H}_\psi\|. \quad (4)$$

Both problems (1) and (2) are thus rephrased in terms of Hankel operators. A convenient reference for these, including all results that we shall use below, is [9].

## 2 Complex symmetric operators and their relatives

In [2, 3] the authors introduce the notion of *complex symmetric* operator on a Hilbert space, which has since found several applications; in particular, complex symmetric operators are used in [4] to prove the equivalence, in a particular case, of the two extremal problems. We need an extension of some of these facts to operators acting between two different spaces.

Suppose then that  $\mathcal{X}, \mathcal{Y}$  are two Hilbert spaces. Define  $\mathfrak{c} : \mathcal{X} \rightarrow \mathcal{Y}$  to be an *antiunitary* operator if it is a conjugate linear surjective map which satisfies  $\langle \mathfrak{c}x, \mathfrak{c}x' \rangle = \langle x', x \rangle$  for all  $x, x' \in \mathcal{X}$ . It is then immediate that  $\mathfrak{c}^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$  is also an antiunitary operator. A *conjugation* is an antiunitary operator which acts on the same space and is equal to its inverse. If  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ , we say that  $T$  is  *$\mathfrak{c}$ -symmetric* if  $T = \mathfrak{c}T^*\mathfrak{c}$ . If  $T \in \mathcal{L}(\mathcal{X})$  and there exists a conjugation  $C$  such that  $T$  is  $C$ -symmetric, then one says that  $T$  is *complex symmetric*; this is the class considered in [2, 3].

In order to go from complex symmetric to  $\mathfrak{c}$ -symmetric operators, the main tool is the following lemma.

**Lemma 2.1.** *If  $\mathfrak{c} : \mathcal{X} \rightarrow \mathcal{Y}$  is an antiunitary operator, then there exists a unitary operator  $V : \mathcal{X} \rightarrow \mathcal{Y}$  (not uniquely defined) such that  $C = V^*\mathfrak{c}$  is a conjugation on  $\mathcal{X}$ . If such a  $V$  is*

fixed, then the map  $T \mapsto V^*T$  is a bijection between  $\mathfrak{c}$ -symmetric operators and  $C$ -symmetric operators.

*Proof.* Take an orthonormal basis  $(e_n)$  in  $\mathcal{X}$ , and define  $V$  to be the unitary operator which maps  $e_n$  into  $\mathfrak{c}e_n$ . Then it is easily seen that  $C = V^*\mathfrak{c}$  is precisely the conjugation on  $\mathcal{X}$  associated with the basis  $(e_n)$ .

Now, if  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ , then

$$\begin{aligned} T = \mathfrak{c}T^*\mathfrak{c} &\Leftrightarrow V^*T = V^*\mathfrak{c}T^*\mathfrak{c} \Leftrightarrow V^*T = V^*\mathfrak{c}T^*VV^*\mathfrak{c} \\ &\Leftrightarrow V^*T = C(V^*T)^*C, \end{aligned}$$

which proves the second part of the lemma.  $\square$

As a consequence, we obtain the result that interests us, namely the analogue of Theorem 1 in [4] (which deals with the complex symmetric case).

**Lemma 2.2.** *Suppose  $\mathfrak{c} : \mathcal{X} \rightarrow \mathcal{Y}$  is an antiunitary operator and  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is  $\mathfrak{c}$ -symmetric. Then:*

- (i)  $\|T\| = \sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x \rangle|$ .
- (ii) *The supremum in (i) is attained if and only if  $T$  attains its norm (or, equivalently, if  $\|T\|$  is an eigenvalue for  $|T|$ .) In this case  $Tx = \omega\|T\|\mathfrak{c}x$  for some unimodular constant  $\omega$ .*

*Proof.* Suppose that  $V$  is the unitary operator and  $C$  is the conjugation given by Lemma 2.1; thus  $T' := V^*T$  is  $C$ -symmetric. Theorem 1 in [4] says then that  $\|T'\| = \sup_{\|x\|=1} |\langle T'x, Cx \rangle|$ . Since  $\|T\| = \|T'\|$  and

$$\sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x \rangle| = \sup_{\|x\|=1} |\langle V^*Tx, V^*\mathfrak{c}x \rangle| = \sup_{\|x\|=1} |\langle T'x, Cx \rangle|,$$

the first assertion is proved.

For the second, it is immediate by Schwarz's inequality that, if  $\|x\| = 1$ , then  $\|T\| = |\langle Tx, \mathfrak{c}x \rangle|$  if and only if  $Tx = \omega\|T\|\mathfrak{c}x$  for some unimodular constant  $\omega$ . But it is a general fact (for any operator  $T$ ) that  $T$  attains its norm if and only if  $\|T\|$  is an eigenvalue of  $|T|$ , given that  $\|T\| = \||T|\|$  and  $\||T|\| = \sup_{\|x\|=1} \langle |T|x, x \rangle$ .  $\square$

It might be of independent interest to state, as a corollary, the corresponding version of Theorem 2 in [1], characterizing the spectrum of the modulus of a  $\mathfrak{c}$ -symmetric operator in terms of what Garcia [1] calls an approximate antilinear eigenvalue problem.

**Proposition 2.3.** *Let  $T$  be a bounded  $\mathfrak{c}$ -symmetric operator and  $\lambda \geq 0$ . Then*

- (i)  $\lambda$  belongs to the spectrum of  $|T|$  if and only if there exists a sequence of unit vectors  $(f_n)_n$  such that  $\lim_{n \rightarrow \infty} \|(T - \lambda \mathfrak{c})f_n\| = 0$ .
- (ii)  $\lambda$  is a singular value of  $T$  if and only if  $Tf = \lambda \mathfrak{c}f$  has a nonzero solution  $f$ .

### 3 Main result

We can now prove the equivalence of the two problems (1) and (2) in the general case.

**Theorem 3.1.** *For any  $\psi \in L^\infty$  we have  $\Gamma(\psi) = \Lambda(\psi)$ .*

*Proof.* We intend to apply Lemma 2.2 to the following situation:  $\mathcal{X} = H^2$ ,  $\mathcal{Y} = H_-^2$ ,  $T = \mathfrak{H}_\psi$  and  $\mathfrak{c} : H^2 \rightarrow H_-^2$  defined by  $\mathfrak{c}f = \bar{\zeta}f$ . It is easy to see that  $\mathfrak{c}$  is antiunitary. Note that  $\mathfrak{c}^{-1} : H_-^2 \rightarrow H^2$  is given formally by the same formula as  $\mathfrak{c}$ . To be more accurate, we will define  $\mathfrak{C} : L^2 \rightarrow L^2$  by  $\mathfrak{C}f = \bar{\zeta}f$ . Then  $\mathfrak{c} = \mathfrak{C}|H^2 = P_- \mathfrak{C}|H^2$  and  $\mathfrak{c}^{-1} = \mathfrak{C}|H_-^2 = P_+ \mathfrak{C}|H_-^2$ . Moreover, we have  $\mathfrak{C}P_+ = P_- \mathfrak{C}$ .

Then  $\mathfrak{H}_\psi$  is  $\mathfrak{c}$ -symmetric:  $\mathfrak{H}_\psi^* : H_-^2 \rightarrow H^2$  acts by the formula  $\mathfrak{H}_\psi^* g = P_+ \bar{\psi} g$ , so

$$\begin{aligned} (\mathfrak{c}\mathfrak{H}_\psi^* \mathfrak{c})(f) &= (\mathfrak{c}\mathfrak{H}_\psi^*)(\bar{\zeta}f) = \mathfrak{c}(P_+ \bar{\psi} \bar{\zeta}f) = \mathfrak{C}P_+(\bar{\psi} \bar{\zeta}f) \\ &= P_- \mathfrak{C}(\bar{\psi} \bar{\zeta}f) = P_-(\bar{\zeta} \psi \zeta f) = P_-(\psi f) = \mathfrak{H}_\psi f. \end{aligned}$$

We may apply Lemma 2.2 (i), which gives:

$$\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \mathfrak{H}_\psi f, \mathfrak{c}f \rangle| = \sup_{\|f\|=1} |\langle P_-(\psi f), \bar{\zeta}f \rangle|.$$

Since  $\mathfrak{c}f = \bar{\zeta}f \in H_-^2$ , there is no need of  $P_-$  in the last scalar product, and therefore, by (3),

$$\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \psi f, \bar{\zeta}f \rangle| = \Gamma(\psi).$$

Since  $\|\mathfrak{H}_\psi\| = \Lambda(\psi)$  by (4), the theorem is proved.  $\square$

Also, from the second part of Lemma 2.2 it follows that the existence of an extremal function (a function that realizes  $\Gamma(\psi)$ ) is equivalent to the fact that the Hankel operator attains its norm. This happens, for instance, if  $\mathfrak{H}_\psi$  is compact, which is equivalent, via Hartman's theorem [6], to  $\psi \in H^\infty + C(\mathbb{T})$ , where  $C(\mathbb{T})$  denotes the algebra of continuous functions on  $\mathbb{T}$ .

Note that in [4] the solution to the extremal problem is related to truncated Toeplitz operators. These are operators on  $K_\Theta = H^2 \ominus \Theta H^2$  defined, for  $\phi \in H^\infty$ , by the formula

$$A_\phi^\Theta(f) = P_\Theta \phi f, \quad f \in K_\Theta,$$

where  $P_\Theta$  is the orthogonal projection onto  $K_\Theta$ . More precisely, it is shown in [4] that, if there is an inner function  $\Theta$  such that  $\psi\Theta \in H^\infty$ , then

$$\Lambda(\psi) = \Gamma(\psi) = \|A_{\psi\Theta}^\Theta\|.$$

The relation with Theorem 3.1 above is made by the following observation. Consider the orthogonal decompositions  $H^2 = K_\Theta \oplus \Theta H^2$  and  $H_-^2 = \bar{\Theta} K_\Theta \oplus \bar{\Theta} H_-^2$ . With respect to them, the only nonzero entry of the matrix of  $\mathfrak{H}_\psi$  is in the upper left corner, and it is equal to  $A_{\psi\Theta} : K_\Theta \rightarrow K_\Theta$  followed by multiplication with  $\bar{\Theta}$ . Consequently, in this case most of the results for the Hankel operators can be translated in terms of the truncated Toeplitz operator. Moreover, this is an *analytic* truncated Toeplitz operator, that is, one whose symbol is in  $H^\infty$ . Their theory is significantly simpler than in the case of general truncated Toeplitz operators, since we may apply Sarason's interpolation arguments.

## 4 Final remarks

This section has no claim of novelty; its purpose is to put some other results in [4] in a more general context.

**4.1** First, note that it is immediate that  $\Gamma(\psi) \leq \|\psi\|_\infty$ . Obviously  $\Gamma(\psi)$  depends only on the antianalytic part of  $\psi$ . Using the equivalence of (1) and (2), and Nehari's theorem [8], it follows that for each  $\psi \in L^\infty$  there exists  $\hat{\psi}$  such that  $\psi - \hat{\psi} \in H^\infty$  and  $\|\hat{\psi}\|_\infty = \Gamma(\psi)$ . In the context of truncated Toeplitz operators used in [4],  $\hat{\psi}$  corresponds to what is called therein a *norm attaining symbol*.

**4.2** In case an extremal function exists (equivalently, when the Hankel operator attains its norm) one can say more. With the previous notations, suppose  $g \in H^2$  is an extremal function with  $\|g\|_2 = 1$ ; thus  $\|\mathfrak{H}_{\hat{\psi}}g\|_2 = \|\mathfrak{H}_{\hat{\psi}}\|$ . The sequence of inequalities

$$\|\hat{\psi}\|_\infty = \|\mathfrak{H}_{\hat{\psi}}\| = \|\mathfrak{H}_{\hat{\psi}}g\|_2 = \|P_-(\hat{\psi}g)\|_2 \leq \|\hat{\psi}g\|_2 \leq \|\hat{\psi}\|_\infty \|g\|_2 = \|\hat{\psi}\|_\infty$$

imply that  $\|P_-(\hat{\psi}g)\|_2 = \|\hat{\psi}g\|_2 = \|\hat{\psi}\|_\infty$ . It follows then, first that  $\hat{\psi}$  has constant modulus, and secondly that  $\hat{\psi}g \in H_-^2$  and thus  $\hat{\psi}g_o \in H_-^2$ , where  $g_o$  is the outer part of  $g$ . Then  $\|\mathfrak{H}_{\hat{\psi}}g_o\|_2 = \|\hat{\psi}g_o\|_2 = \|\hat{\psi}\|_\infty = \|\mathfrak{H}_{\hat{\psi}}\|$ . Therefore, in case an extremal function exists, one can choose it outer, and thus not having zeros in  $\mathbb{D}$ . This recaptures the result of [7] quoted in [4], which says that if the symbol is continuous then there exists an extremal function which is nonzero on  $\mathbb{D}$  (as noted above, a Hankel operator with continuous symbol is compact and thus attains its norm).

**4.3** In case there exists an inner function  $\Theta$  such that  $\phi = \hat{\psi}\Theta \in H^\infty$ , then the result can be strengthened. With the above notations, we have then  $\bar{\Theta}\phi g_o \in H_-^2$ , which implies  $\phi g_o \in K_\Theta$ ; thus  $\phi$  is a scalar multiple of the inner part of a function in  $K_\Theta$ . This is essentially noticed in the remarks after [4, Theorem 2].

**4.4** Finally, let us note that, in case there exists no extremal function, norm attaining symbols might not have constant modulus. An example appears in [5, Ch. IV, Example 4.2]. Namely, suppose  $\Theta$  is an inner function that does not extend analytically across the unit circle in the neighborhood of 1, while  $f$  is a nonconstant invertible outer function with  $\|f\|_\infty = 1$  that has modulus 1 on an arc of  $\mathbb{T}$  around 1. Then the only norm attaining symbol for the Hankel operator  $\mathfrak{H}_{\bar{\Theta}f}$  is  $\bar{\Theta}f$ , which has not constant modulus.

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