An overview on the non-newtonian calculus and its potential applications to economics

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Abstract. Until now, non-newtonian calculus, multiplicative calculus in particular, has been presented as a curiosity and is nearly ignored for the social scientists field. In this paper, after a brief presentation of this calculus, we try to show how it could be used to re-explore from another perspective classical economic theory, more particularly the economic growth and in the maximum likelihood method from statistics.

§1 Introduction

The development of the occidental science has been an incredible success through the introduction of the newtonian calculus. One could say that until the XXth century when discontinuous methods became available, without the derivative nothing of great could have been accomplished.

In the social science, for a very long time, the derivative has been the center of nearly all the analysis. But in an unexpected way, it appears that we can develop another approach which seems to threat more realistically growth phenomenon which are involved in the model of economic growth.

The change of paradigm is to consider that the variations are more naturally taken into account if the deviations are measured by ratios instead of differences. But even if Galileo has discussed briefly such an opportunity, it is not until 1972 that Grossman and Katz [11] imagine a non-newtonian calculus.

In the last twenty years, a number of reviews of the non-Newtonian calculus have appeared, scattered in various journals, primarily of pedagogical scope.

We think that the works of Bashirov Bashirov et al. [1] and Özyapici et al. [12] has open a new era where non-newtonian calculus, from a mere curiosity, will become an operational calculus. We have shown elsewhere Filip and Piatecki [9] how calculus has been a locked-in and how Grossman and Katz [11], surfing on Volterra [16], has begun to unlock it. There have been

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two main reasons to this state of fact: the first one is that non-mathematician researchers are very conservative in what concern the mathematical instruments they use and don't understand why they should explore other kinds of mathematics, since the classical methods are well fited for what they expect. The second one is that, in the particular case of the non-newtonian calculus, as there is a gateway between calculus and non-newtonian calculus, they doesn't find any reason to adopt another type of calculus.

Nevertheless, inside any social science modelization, there is always an equilibrium problem, based on a balance equation. Since the end of the 6^{th} century, following Luca Paccioli, book accounting are universally based on an additive algebra. But, as Ellerman [7] has clearly proved, one could also defined a multiplicative bookkeeping. Now, if in the standard accounting system resources are kept through positive numbers and deficit through negative ones, giving to zero the balance role in an additive system, in a multiplicative bookkeeping, the balance is given by one, resources by numbers greater than one, deficit by numbers lower than one, in a multiplicative system.

So, if one adopts a multiplicative bookkeeping system, multiplicative calculus becomes the only way to coherently manage models, since in multiplicative calculus, as in multiplicative accounting system, one plays the role devoted to zero in the standard system.

§2 Glimpses on the non-newtonian calculus

2.1 Definition of the **derivative*

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a positive function. The **derivative* of the function f is given by:

$$f^\star(t) = \lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}}$$

In fact, we have :

$$\begin{aligned} f^\star(t) &= \lim_{h\to 0}\left(\tfrac{f(t+h)}{f(t)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(1+\tfrac{f(t+h)-f(t)}{f(t)}\right)^{\left(\frac{f(t)}{f(t+h)-f(t)}\cdot\frac{f(t+h)-f(t)}{h}\cdot\frac{1}{f(t)}\right)} \\ &= \lim_{h\to 0}\left(\left(1+\tfrac{f(t+h)-f(t)}{f(t)}\right)^{\left(\frac{f(t)}{f(t+h)-f(t)}\right)}\right)^{\left(\frac{f(t+h)-f(t)}{h}\cdot\frac{1}{f(t)}\right)} \\ &= e^{\frac{\dot{f}(t)}{f(t)}} = e^{(\ln\circ f)'(t)} \end{aligned}$$

for $(\ln\circ f)(t) = \ln(f(t))$. If the second order derivative of f at t exists, then by substitution, one will find that :

$$f^{\star\star}(t) = e^{(\ln\circ f^\star)'(t)} = e^{(\ln\circ f)''(t)}$$

We can see that $(\ln\circ f)''$ exists because $f''(t)$ exists. If we repeat n times this procedure, we can conclude that if $f(t)$ is a positive function and $f^{(n)}(t)$ exists,

$$f^{\star(n)}(t) = e^{(\ln \circ f)^{(n)}(t)}$$

We must note that this formula includes the case $n = 0$ as well because :

$$f(t) = e^{(\ln \circ f)(t)}$$

It must signaled that in multiplicative calculus **differentiability* imply continuity but continuity doesn't imply **differentiability*. Secondly the **derivative* of a positive constant function is 1 and to fix some results one has the Table 1.

In Stanley [15] is stated that if a function $f(t)$ is *differentiable* at t, then is also **differentiable* at t, and the converse affirmation is true in the following terms: if a positive function $f(t)$ is **differentiable* at t, and if $f^*(t) \neq 0$, then it is also *differentiable* at t.
We have:

$$f^*(t) = e^{\frac{f'(t)}{f(t)}}$$
$$f'(t) = f(t) \ln f^*(t)$$
$$\ln(f^*(t)) = \ln'(f(t))$$

The commutative diagram presented in Figure 1.

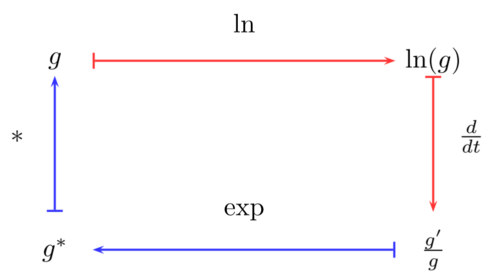

Figure 1: RELATIONSHIP OF g' AND g^*

2.2 Interpretation of the **derivative*

In order to give an interpretation of **derivative*, let consider f as being a function of two variables x and y. If c is a value in the range of the function f, then the equation $f(x, y) = c$ describes a curve lying on the plane $z = c$ called the trace of the graph of f in the plane $z = c$. If this trace is projected onto the xy-plane, the resulting curve in the xy-plane is called

As we has already signaled, for a complete account of the product calculus we must consult Bashirov et al. [1] but it may be useful to begin by Stanley [15].

Stanley [15].

For more details see Filip and Curt [8].

a level curve. Note that if the function f is an utility function, then a level curve is called an indifference curve.

The slope of the line that is tangent to the level curve $f(x,y)=c$ at a particular point is given by the derivative $y'(x)$. This derivative is the rate of change of y with respect to x on the level curve and hence is approximately the amount by which the y coordinate of a point on the level curve changes when the x coordinate is increased by 1. Since

$$y'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\approx \frac{\Delta y}{\Delta x}$$

then

$$\Delta y \approx y'(x)\ \Delta x$$

which gives us the change in y needed to compensate a small change Δx in x so that the value of the function f remain unchanged.

Recall the **derivative* formula

$$f^\star(x)=\lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}$$

and consider $\Delta x = x_2 - x_1$ the variation of the independent variable, and $\Delta^* y = \frac{y_2}{y_1} = \frac{y(x_2)}{y(x_1)}$ the variation of the independent variable. It follows that

$$y^*(x)=\lim_{\Delta x\to 0}\left(\Delta^* y\right)^{\frac{1}{\Delta x}}$$

and we consider that

$$y^\star(x)\approx \left(\Delta^\star y\right)^{\frac{1}{\Delta x}}.$$

By taking the Δx power of this expression we obtain

$$\Delta^\star y \approx \left[y^\star(x)\right]^{\Delta x}$$

$$\Delta^\star y \approx e^{\frac{y'(x)}{y(x)}\Delta x}$$

which gives

$$y_2 \approx y_1 e^{\frac{y'(x)}{y(x)}\Delta x}.$$

We can reach a similar result if we analyse the **Euler's method* for solving a differential equation, as it can be seen in Campbell [4]. Let consider the first-order differential equation with initial conditions (the Cauchy problem)

$$\begin{cases} y'(t)=f(t,y) \\ y(t_0)=y_0 \end{cases}$$

Using the limit definition

$$y'(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

and considering small Δt, we can write

$$y(t+\Delta t) \approx y(t) + y'(t)\Delta t$$

This approximation naturally induces the idea of the *Euler's method* which uses an iterative process in order to produce approximate values for the unknown function $f(t)$ as follows:

$$\begin{cases} t_{n+1} = t_n + \Delta t \\ a_{n+1} = a_n + f(t_n, a_n)\Delta t \end{cases}$$

for $n = 0, 1, 2, ...$. A two-variable discrete dynamical system was obtained.

If we consider now the **calculus*, the approximation which uses the **derivative* has the form:

$$y(t+\Delta t) \approx y(t) \left(y^\star(t)\right)^{\Delta t}$$

and, by using the relation which gives us the relation between the **derivative* and the *derivative*, i.e.

$$y^\star(t) = e^{\frac{y'(t)}{y(t)}} = e^{\frac{f(t,y(t))}{y(t)}}$$

the **approximation* will be

$$y(t+\Delta t) \approx y(t) e^{\frac{f(t,y(t))}{y(t)}\Delta t}$$

which implies the two-variable dynamical system

$$\begin{cases} t_{n+1} = t_n + \Delta t \\ a_{n+1} = a_n e^{\frac{f(t_n, a_n)}{a_n}\Delta t} \end{cases}$$

One can remark that the *Euler's method* gives the exact solutions only in the case of linear functions, which implies that differential equations of the form $y'(t) = m$ can be exactly solved. In Campbell [4] is stated that **Euler's method* gives exact solution to the differential equations of the form:

$$\begin{cases} y'(t) = ky \\ y(0) = y_0 \end{cases}$$

over the time interval $0 \leq t \leq T$, where k is a constant, which is the case of exponential functions. By choosing a number of N iterations and set $\Delta t = \frac{T}{N}$ and set initial conditions $t_0 = 0$ and $a_0 = y_0$, then iterate as follows:

$$\begin{cases} t_{n+1} = t_n + \Delta t \\ a_{n+1} = a_n e^{\frac{k y_n}{y_n}\Delta t} = a_n e^{k\Delta t} \end{cases}$$

It can be seen that these iterations are repeated multiplications by the constant $e^{k\Delta t}$. This implies that the closed form of this recurrence relation is

$$a_N = a_0\left(e^{k\Delta t}\right)^N = y_0 e^{kT}$$

which is the exact solution.

The connection between *Euler's method* and **Euler's method* can be obtained by using *Taylor series* of the function e^x $\left(e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...\right)$

$$\begin{aligned} a_{n+1} &= a_n e^{\frac{f(t_n,a_n)}{a_n}\Delta t} \\ &= a_n\left[1 + \frac{f(t_n,a_n)}{a_n}\Delta t + \frac{1}{2!}\left(\frac{f(t_n,a_n)}{a_n}\Delta t\right)^2 + ...\right] \\ &= a_n + f(t_n,a_n)\Delta t + \frac{1}{2!}\frac{[f(t_n,a_n)]^2}{a_n}(\Delta t)^2 + ... \end{aligned}$$

from which we can see that *Euler's method* is simply a truncation of **Euler's method.* In addition, when $a_n > 0$ we get an answer which is greater than the answer from the *Euler's method* and when $a_n < 0$ the answer is less than the Euler's method answer. This is connected by so-called the concavity problem. This problem appears as well if we study the *linear approximation* versus the *exponential approximation*, as can be seen in Campbell [4].

2.3 Linear approximation versus exponential approximation

As we have shown in Filip and Piatecki [9] and it can be found as well in Stanley [15], linear functions of the form $x(t) = at + b$ has constant *derivative* and $x(t+1) = x(t) + x'(t)$. In addition, if any two differential functions $x(t)$ and $y(t)$ differ by a constant, then $x'(t) = y'(t)$. Furthermore, a function $x(t)$ differentiable at a point $t = a$ has a *linear approximation* L near a defined as follows:

$$L(t) = x(a) + x'(t)(t - a)$$

On the other side, exponential functions of the form $x(t) = ab^t$, with $b > 0$, has constant **derivative* and $x(t+1) = x^\star(t)\,x(t)$. Moreover, the following two properties are true for all positive differential functions:

1. If one positive function is a constant multiple of another, i.e. $x(t) = c\,y(t)$, they have the same **derivative.* This is the central idea of the *multiplicative rate of change*: if one function is a constant multiple of another, their *multiplicative rate of change* is the same (as the **derivative* quantifies the *multiplicative rate of change*) that is, the two functions change by the same factor over any given interval $[t, t + \Delta t]$. In other words, for any functions $x(t)$ and $y(t) = c\,x(t)$, if we have $x(t + \Delta t) = G\,x(t)$ for some G, then $y(t + \Delta t) = G\,y(t)$ for the same factor G (G is the *multiplicative rate of change*).

2. Any positive function $x(t)$ differentiable at a point $t = a$ has an *exponential approximation* E near a defined as follows:

Stanley [15]

$$E(t) = x(a)x^{\star}(a)^{(t-a)}.$$

In Stanley [15], it is stated that the graph of the *exponential approximation* E "hugs" the graph of the function $x(t)$ and furthermore, the graph of E is a curve that coincides with the graph of x at a point a in these three ways:

- its value: $E(a) = x(a)$
- its **derivative*: $E^{\star}(a) = x^{\star}(a)$
- its *additive derivative* (its slope): $E'(a) = x'(a)$

In order to illustrate *linear* and *exponential approximation* in Stanley [15]aut]Stanley, D. is taken the function $x(t) = \frac{1}{t}$ and a point $a = t = 2$. Using the rules for **derivative* presented in Table 1 we have

$$x(t) = \frac{1}{t},\ x'(t) = -\frac{1}{t^2},\ x^{\star} = e^{-\frac{1}{t}}$$
$$x(2) = \frac{1}{2},\ x'(t) = -\frac{1}{4},\ x^{\star} = e^{-\frac{1}{2}}$$

Linear approximation at $t = 2$ is

$$L(t) = x(2) + x'(2)(t-2) = \frac{1}{2} - \frac{1}{4}(t-2) = 1 - \frac{1}{4}t$$

Exponential approximation at $t = 2$ is

$$E(t) = x(2)\, x^{\star}(2)^{(t-2)} = \frac{1}{2}e^{-\frac{1}{2}(t-2)} = \frac{1}{2}e^{-\frac{t}{2}+1}$$

For this example, the exponential approximation is a closer match that the linear one, as it can be seen in the Figure 2.

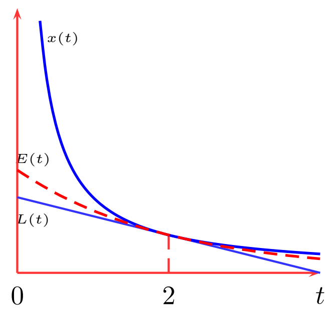

Figure 2: LINEAR VERSUS EXPONENTIAL APPROXIMATION TO $x(t) = \frac{1}{t}$ AT THE POINT $t = 2$

The problem is that the **derivative* is an exponential function which is a convex function and it cannot approximate closely any concave function. For instance (see Stanley [15]aut]Stanley, D.) the exponential approximation to concave function $\sin t$ at $t = \frac{\pi}{6}$ is far worse than the linear one, for the function $1 - \sin t$ its exponential approximation is fairly good and has about the same degree of closeness of match as the linear one.

In Stanley [15]aut]Stanley, D. the **mean value theorem* versus the *mean value theorem*, the **Rolle's theorem* versus the *Rolle's theorem* and **Taylor products* versus *Taylor series* are presented.

For the function $x(t) = \frac{1}{1+e^{-t}}$ at $t = 1$ second order polynomial and exponential approximations are computed and better results are obtained in the exponential approximation case.

Recall the function $x(t) = \sin t$ and $a = \frac{\pi}{6}$. Computing $E(t)$, $E_2(t)$, $E_3(t)$, $E_4(t)$ and $E_5(t)$ — Coco [5] — where

$$
\begin{aligned}
E(t) &= x(a)\left[x^\star(a)\right]^{t-a} = \left(\sin\frac{\pi}{6}\right)\left[e^{\cot\frac{\pi}{6}}\right] = \frac{1}{2}e^{\sqrt{3}(t-\frac{\pi}{6})}\\
E_2(t) &= x(a)\left[x^\star(a)\right]^{t-a}\left[x^{\star\star}(a)\right]^{\frac{(t-a)^2}{2!}} = E(t)\left[e^{-\frac{1}{\sin^2\frac{\pi}{6}}}\right]^{\frac{1}{2}(t-\frac{\pi}{6})^2} =\\
&= E(t)\, e^{-2(t-\frac{\pi}{6})^2} = \frac{1}{2}e^{\sqrt{3}(t-\frac{\pi}{6})}e^{-2(t-\frac{\pi}{6})^2}\\
E_3(t) &= x(a)\left[x^\star(a)\right]^{t-a}\left[x^{\star\star}(a)\right]^{\frac{(t-a)^2}{2!}}\left[x^{\star(3)}(a)\right]^{\frac{(t-a)^3}{3!}} =\\
&= E_2(t)\ \left(e^{8\sqrt{3}}\right)^{\frac{1}{6}(t-\frac{\pi}{6})^3}\\
E_4(t) &= E_3(t)\ \left(e^{-80}\right)^{\frac{1}{24}(t-\frac{\pi}{6})^4}\\
E_5(t) &= E_4(t)\ \left(e^{352\sqrt{3}}\right)^{\frac{1}{120}(t-\frac{\pi}{6})^5}
\end{aligned}
$$

and illustrating by graphics we can see that by increasing the order of the exponential approximation we obtain increasingly better results.

One can remark that the multiplicative calculus can be extended as follows: if φ is a bijective function define §*derivative* and §*integral* by — Coco [5]:

$$f^\S = \varphi\left(\varphi^{-1}\circ f\right)'(x)$$

(we can see that $f^\star(x) = e^{(\ln\circ|f|)'(x)}$)

$$\int_a^b f(x)d^\S x = \varphi\left(\int(\varphi^{-1}\circ f)(x)dx\right)$$

(we can see that $\int_a^b f(x)^{dx} = e^{\int_a^b(\ln\circ|f|)(x)dx}$).

2.4 Geometric calculus and bigeometric calculus

As we have mentioned, the *multiplicative calculus* or **calculus* was called by Grossman and Katz [11] *geometric calculus* in order to emphasize that changes in function arguments are measured by differences, while changes in values are measured by ratios. The entire theory works only for positive functions. This is why it is mandatory not to work with standard (additive) accounting since, in case of deficit, we should have negative numbers, while with multiplicative accounting, all numbers are greater than zero. In addition, a positive function which is continuous and has the same geometric change over any two intervals of equal classical extent (i.e. $\frac{x(t_2)}{x(t_1)} = \frac{x(t_4)}{x(t_3)}$ if $t_2 - t_1 = t_4 - t_3$ and $t_1 \leq t_2$, $t_3 \leq t_4$) is called *geometrically-uniform* function.

The name *geometric calculus* comes from at least two reasons. The first one is the property of a *geometrically-uniform* function that each arithmetic progression of arguments implies geometric progression of corresponding values. The second one must be started by recalling the definition of the geometric average of n positive numbers $v_1, v_2, ..., v_n$, that is the positive number $(v_1 v_2 ... v_n)^{\frac{1}{n}}$. In Grossman and Katz [11] and Grossman [10], is explained how a simply algebraic identity let them to construct the *geometric calculus* as follows:

Consider any n points (a_1, v_1),(a_2, v_2)...,(a_n, v_n), where $a_1 < a_2 < ... < a_n$ and $a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1} = k$. Thus, $k(n-1) = a_n - a_1$. Connect the n points by line segments, of which are $n-1$. The arithmetic average of their classical slopes equals the classical slope of the line segment containing the end points (a_1, v_1) and (a_n, v_n), that is:

$$\frac{\frac{v_2 - v_1}{k} + \frac{v_3 - v_2}{k} + ... + \frac{v_n - v_{n-1}}{k}}{n-1} = \frac{v_n - v_1}{a_n - a_1}$$

If we assume that the v_i values are positive, the following identity is obvious:

$$\left[\left(\frac{v_2}{v_1} \right)^{\frac{1}{k}} \left(\frac{v_3}{v_2} \right)^{\frac{1}{k}} ... \left(\frac{v_n}{v_{n-1}} \right)^{\frac{1}{k}} \right]^{\frac{1}{n-1}} = \left(\frac{v_n}{v_1} \right)^{\frac{1}{a_n - a_1}}$$

The left side of this equality is the geometric average of the $n-1$ numbers $\left(\frac{v_i}{v_{i-1}} \right)^{\frac{1}{k}}$ which they imagined to be the slopes of a new kind.

Speaking about the representation of a *geometrically-uniform* function x on a *semi-log paper* (logarithmically scaled on the ordinate axis), the result is a straight line whose classical slope equals the natural logarithm of the *geometric slope* of x.

One could ask what happens if both changes in arguments and values are measured by ratios? The theory developed by Grossman and Katz [11] and Grossman [10] works for functions with positive arguments and positive values and it is called *bigeometric calculus.*

A positive function which is continuous and has the same geometric change over any two positive intervals of equal geometric extent (i.e. $\frac{x(t_2)}{x(t_1)} = \frac{x(t_4)}{x(t_3)}$ if $\frac{t_2}{t_1} = \frac{t_4}{t_3}$ and $0 < t_1 \leq t_2$, $0 < t_3 \leq t_4$) is called *bigeometrically-uniform* function. The characteristic of a *bigeometrically-uniform* function is that each geometric progression of arguments implies geometric progression of values.

Speaking about the representation of a *bigeometrically-uniform* function x on a *log-log paper* (logarithmically scaled on both abscise and ordinate axis), the result is a straight line whose classical slope equals the natural logarithm of the *bigeometric slope* of x.

In the *bigeometric calculus*, the *bigeometric derivative* is defined as being

$$\lim_{t \to a} \left[\frac{x(t)}{x(a)} \right]^{\frac{1}{\ln t - \ln a}}$$

and, by making an analogy with the *geometric derivative* or **derivative* we propose the notation $x^{\star}(a^{\star})$ for the *bigeometric derivative* of the function f on a.

It can be proved that $x'(a)$ and $x^{\star}(a^{\star})$ coexist and the relationship between them is

$$x^{\star}(a^{\star}) = e^{\frac{ax'(a)}{x(a)}}.$$

The expression $\frac{ax'(a)}{x(a)}$ is called by economists *the elasticity* of x at a, while Grossman and Katz [11] and Grossman [10] called $x^{\star}(a^{\star})$ *the resiliency* of x at a. We can see that the elasticity equals the natural logarithm of resiliency.

For this work we stop here with reviewing *non-newtonian calculi*, even if in Grossman and Katz [11] other kind of calculi are presented and developed.

§3 Applications and further research directions

3.1 *calculus or geometric calculus

3.1.1 Galilean ratios

In the book by Grossman and Katz [11] is presented a problem which comes from Renaissance during Galileo, when two estimates, i.e. 10 and 1000 , were proposed for the price of a horse and the question was which estimate, if any, deviates less from the true value of 100? The first approach was to measure the deviations by differences and because of this, the estimate of 10 appears to be closer to the true value. Galileo proposed that the deviation should be measured by ratios and from this, both estimates deviated equally form the true value. From this story we can state as it was shown in Filip and Piatecki [9] that growth phenomenon could be better modelled by using **derivative*.

3.1.2 Personal finance application

In the same book by Grossman and Katz [11], a more sophisticated problem is presented in order to make a comparison between *classical calculus* and **calculus*: at time t_1 a person invested $x(t_1)$ dollars with a promoter who guarantees that at a certain subsequent of time t_2, the value of the investment would be $x(t_2)$ dollars. In event that the investor should desire to withdraw at any other time $t \in (t_1, t_2)$, it was agreed that the value of the investment increases continuously and uniformly. The question is how much, i.e. $x(t)$, would the investor be entitled to at time t? Grossman and Katz [11] presented two solutions which are both reasonable.

The first one is based on the assumption that the function x increases by equal amounts in equal times. This implies (because of the continuous increasing of the investment) that the growth has to be linear and, by considering the equation of a straight line which pass through the points $(t_1, x(t_1))$ and $(t_2, x(t_2))$ which is

$$\frac{x(t) - x(t_1)}{t - t_1} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

we have

$$x(t) = x(t_1) + \frac{x(t_2) - x(t_1)}{t_2 - t_1}(t - t_1).$$

Notice that the ratio $\frac{x(t_2)-x(t_1)}{t_2-t_1}$ represents the classical slope of the straight line which pass through the points $(t_1, x(t_1))$ and $(t_2, x(t_2))$.

The second solution is based on the assumption that if the value of the investment increases uniformly we may reasonable consider that it increases by equal percents in equal times. From this we can prove that the growth has to be exponential, i.e.

$$\left[\frac{x(t)}{x(t_1)}\right]^{\frac{1}{t-t_1}} = \left[\frac{x(t_2)}{x(t_1)}\right]^{\frac{1}{t_2-t_1}}$$

which gives

$$x(t) = x(t_1)\left\{\left[\frac{x(t_2)}{x(t_1)}\right]^{\frac{1}{t_2-t_1}}\right\}^{t-t_1}$$

The expression $\left[\frac{x(t_2)}{x(t_1)}\right]^{\frac{1}{t_2-t_1}}$ is, when $t_2 \to t_1$, the **derivative* $x^\star(t_1)$.

3.1.3 Relative derivative

Grossman and Katz [11] spoke about the relative change of a positive function x over the interval $[t_1, t_2]$, i.e. $\frac{x(t_2)-x(t_1)}{x(t_1)}$ which equalized by 1 gives the geometric change of the function x over the interval $[t_1, t_2]$, i.e. $\frac{x(t_2)}{x(t_1)}$. The relative slope of a *geometrically-uniform* function is its relative change over any interval of classical extent 1, from where if the interval is $[t_1, t_2]$ we have $\left[\frac{x(t_2)}{x(t_1)}\right]^{\frac{1}{t_2-t_1}}$ as relative slope. The example given by the mentioned authors is one from securities analysis which is called *compound growth rate*. They supposed that if one paid 64 dollars for a share of stock, three years later the price of the share will be 216 dollars. The compound (annual) growth rate of the price over the time interval $[0, 3]$ is given by:

$$\left(\frac{216}{64}\right)^{\frac{1}{3-0}} - 1 = 0.5 = 50\%$$

The significance of this result is that if an original investment of 64 dollars increases 50% in each of the three years, the final value would be 216 dollars, which is true if we compute directly $64 + \frac{64}{2} + \frac{96}{2} + \frac{144}{2} = 216$.

They introduced the notion of *relative derivative* of a positive function x as being the limit:

$$\lim_{t\to a}\left[\left(\frac{x(t)}{x(a)}\right)^{\frac{1}{t-a}} - 1\right]$$

if it exists and is greater than -1. They also state that the *relative derivative* of x at a coexists with, and equals 1 less than, the *geometric derivative* (**derivative*) of x at a.

3.1.4 Non-newtonian Euler's formula for homogeneous function

As an example of application of the non-newtonian calculus, we have shown, in the paper Filip and Piatecki [9], what happen to the celebrated Euler's formula for homogeneous functions. This approach was also developed by Córdova-Lepe [6]. But it's results was given for another definition of multiplicative derivative for positive functions :

$$Qf(x_0)=\lim_{h\to 1}\left(\frac{f(x_0 h)}{f(x_0)}\right)^{\frac{1}{\ln(h)}},\quad f:]0,\infty[\to]0,\infty[,\quad x_0\in]0,\infty[,\quad \text{if this limit exists.}$$

If we take an homogeneous function of degree r, which means $f(\mu x_1,\ldots,\mu x_n)=\mu^r f(x_1,\ldots,x_n)$ we have

$$rf(x_1,\ldots,x_n)=x_1 f'_{x_1}(x_1,\ldots,x_n)+\cdots+x_n f'_{x_n}(x_1,\ldots,x_n)$$

which is equivalent with

$$r=\frac{x_1 f'_{x_1}(x_1,\ldots,x_n)}{f(x_1,\ldots,x_n)}+\cdots+\frac{x_n f'_{x_n}(x_1,\ldots,x_n)}{f(x_1,\ldots,x_n)}$$

and by exponentiation, we obtain :

$$e^r=\left(e^{\frac{f'_{x_1}}{f}}\right)^{x_1}\cdots\left(e^{\frac{f'_{x_n}}{f}}\right)^{x_n}$$

We introduce the **partial derivative* of f as being

$$f^\star_{x_1}=e^{\frac{f'_{x_1}}{f}}$$

by making a parallel with the definition of the **derivative* of f:

$$f^\star=e^{\frac{f'}{f}}$$

We obtain the non-newtonian Euler's formula for homogeneous functions:

$$\left(f^\star_{x_1}\right)^{x_1}\cdots\left(f^\star_{x_n}\right)^{x_n}=e^r$$

In the case of the Cobb-Douglas function, we have :

$$F(K,L)=AK^\alpha L^\beta$$

the non-newtonian Euler's formula is :

$$\left(e^{\frac{\alpha}{K}}\right)^K\left(e^{\frac{\beta}{L}}\right)^L=e^r\quad\Longleftrightarrow\quad e^{\alpha+\beta}=e^r$$

in such a way that if there are constant return to scale (the function is homogeneous of degree one), $\alpha+\beta=1=r$.

3.1.5 Growth model

The neoclassical growth model. In the paper Filip and Piatecki [9] we have considered the Solow-Swan exogenous growth model and we have try to reconstruct the model.

If there is a model know by all macro-economists, it is certainly this one, but it is necessary to remember how it works. We have :

$$\begin{cases} I(t) = S(t) & \text{Equilibrium of the good \& services market} \\ S(t) = sY(t) & \text{Saving function} \\ Y(t) = F(K(t), L(t)) & \text{Production function homogeneous of first degree} \\ \dot{K}(t) = I(t) - \delta K(t) & \text{The increase in the capital is equal to the} \\ & \text{investment less the obsolescence} \\ \dot{L}(t)/L(t) = n & \text{The constant of growth of the labor force} \end{cases}$$

for I the investment, S the savings, Y the production, K the capital, L the labor force, δ the rate of obsolescence of the capital and n the constant rate of growth of the labor force. We know that if we define $k(t)$ as the capital by head — *i.e.* : $k(t) = K(t)/L(t)$ —, we will find that his accumulation is given by the celebrated ordinary differential equation :

$$\dot{k}(t) = sf(k(t)) - (n+\delta)k(t), \quad k(0) = k_0$$

which can be further developed in postulating for instance that the production function is a Cobb-Douglas one $Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}$. This gives Bernoulli first order ODE :

$$\dot{k}(t) = sAk^{\alpha}(t) - (n+\delta)k(t), \quad k(0) = k_0$$

whose solution is given by :

$$k(t) = \left[\left[k(0)^{1-\alpha} - \frac{s}{\delta+n}\right] e^{-(1-\alpha)(\delta+n)t} + \frac{s}{\delta+n}\right]^{\frac{1}{1-\alpha}}$$

Since the production fonction is homogeneous of degree one, one knows, by one of the most used Euler's theorems, that if the representative enterprise remunerate its factors to the marginal product ($F_K = r/p$ and $F_L = w/p$), the product is completely affected between the two type of costs, *i.e.*:

$$Y = F_K K + F_L L = \frac{r}{p}K + \frac{w}{p}L$$

It is know that this conduct to the fact that the profit by head $\pi = \Pi/L$ could be written :

$$\pi = pf(k) - rk - w$$

which on its turn implies for a maximising firm that $f'(k) = r/p$ and that $\pi = 0$ or equivalently

$$f(k) = \frac{r}{p}k + w$$

All those information are subsumed in the Figure 3. We must also stress that in the long term when all capital adjustment has been done, that is to say that, in equilibrium, we have :

$$\dot{k}(t) = 0 \iff sf(\overline{k}) = (n+\delta)\overline{k}$$

We must apologize to present here what is elementary text book economics, but we think that, in order to contrast our results with the standard approach this is mandatory.

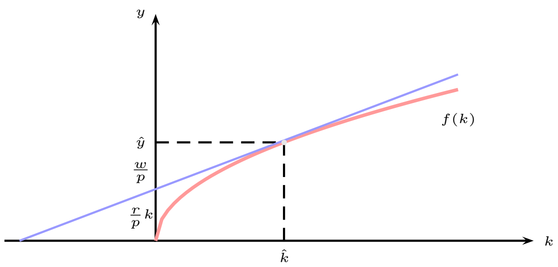

Figure 3: THE REPARTITION OF THE PRODUCT IN THE SOLOW-SWAN MODEL

which says simply that savings $sf(\overline{k})$ must equal the capital taken into account the constant growth of the labor force and to replace the obsolete one. In the steady state, output per worker is constant but total output increases at the rate n than the labor force. One must notice that obviously we obtain the same results if we decide to use the **derivative* to calculate marginal productivity.

This is for the newtonian Solow-Swan model. The problem one face now is to construct the non-newtonian equivalent approach.

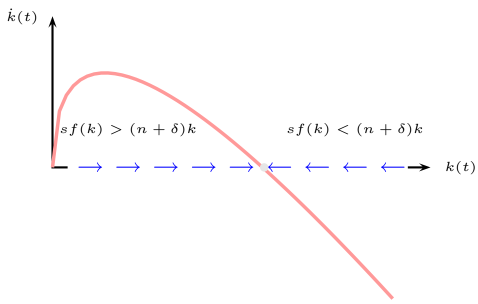

Figure 4: PHASE DIAGRAM OF CAPITAL ACCUMULATION in the Solow model

In the Figure 4, we can see that if savings are greater than $(n+\delta)k$, the capital by workers will tend to rise and, vice versa, if savings are lower than $(n+\delta)k$, it will tend to fall.

Now suppose that with Phelps [13] — see also Burmeister and Dobell [3] — , we want to see if, in the stationary state, there is a better saving rate among all the feasible saving rates. That is to say, we are searching for a saving rate which maximize the consumption by worker in the steady state, *i.e.* :

$$\hat{s} = \text{argmax}_{\{s\}} \{c = (1-s)f(k) | sf(k) = (n+\delta)k\}$$

From the constraint it comes that $s = (n+\delta)k/f(k)$ in such a way that the consumption by worker is given by :

$$c = f(k) - (n+\delta)k \implies f'(k) = (n+\delta) \implies \hat{s} = \frac{f'(k)k}{f(k)}$$

In other word, we learned that, if we ignore initial conditions and simply chose the collective saving rate associated with the steady growth path we must equal it to the share of the profit $f'(k)k$ in the product.

The non-newtonian Solow-Swan model. In order to show how non-newtonian calculus can be applied to keep Solow-Swan results for the exogenous growth, we are obliged to adapt the equations of the model in such a way that the rate of growth of the capital-labor ratio be described exactly by the same equation that the celebrated one. As it will be shown, only the differential equations must be changed.

In what concern the increase of the capital the non-newtonian equation must be the following:

$$K^{\star}(t) = \frac{I(t)}{K(t)^{\delta}}$$

As weird as can look this equation it is no more than the rate of growth of the capital written in terms of logarithm as it is shown by the fact that

$$K^{\star} = e^{\frac{\dot{K}}{K}} = \frac{I}{K^{\delta}} \iff \dot{\widehat{\ln K}} = \frac{\dot{K}}{K} = \ln I - \delta \ln K$$

This equation is, in fact the equation of the capital increase in the newtonian model, but which is not written in level.

In what concern the population growth, the Malthus equation becomes

$$L^{\star} = e^{n}$$

which, in fact, is nothing more than the standard equation, since according to the rule of the **calculus* :

$$L^{\star} = e^{\frac{\dot{L}}{L}} = e^{n}$$

So now, the non-newtonian Solow-Swan model is:

$$\begin{cases} I(t) = S(t) & \text{Equilibrium of the good \& services market} \\ S(t) = sY(t) & \text{Saving function} \\ Y(t) = F(K(t), L(t)) & \text{Production function homogeneous of first degree} \\ K^{\star}(t) = \dfrac{I(t)}{K(t)^{\delta}} & \text{The increase in the capital equals the ratio} \\ & \text{between investment and obsolescence} \\ L^{\star}(t) = e^{n} & \text{Growth of the labor force} \end{cases}$$

With some simple algebraic manipulations we can show that from this model we can derived the Solow-Swan capital-labor ratio growth equation.

$$k^\star = \left(\frac{K}{L}\right)^\star = \frac{K^\star}{L^\star} = \frac{e^{\frac{\dot K}{K}}}{L^\star} = \frac{e^{\frac{I(t)-\delta K(t)}{K(t)}}}{e^n} =$$
$$= \frac{e^{\frac{\frac{I(t)}{L(t)}-\delta\frac{K(t)}{L(t)}}{\frac{K(t)}{L(t)}}}}{e^n} = \frac{e^{\frac{\frac{I(t)}{K(t)}-\delta k}{k}}}{e^n} = \frac{e^{\frac{sf-\delta k}{k}}}{e^n} =$$
$$= e^{\frac{sf-\delta k-nk}{k}} = e^{\frac{sf-(n+\delta)k}{k}} = e^{\frac{\dot k}{k}}$$
$$\Rightarrow \quad \dot k = sf-(n+\delta)k$$

We can see that by re-writing the model by considering the **derivative* in the differential equations the classical model is conserved.

3.1.6 *Maximum likelihood method* versus the **maximum likelihood method*

With the help of the **derivative*, we are able to avoid the logarithmization of likelihood function in classical statistics. In the classical method, the random characteristic X, which has the PDF $f(x;\lambda_1,\lambda_2,...,\lambda_s)$ depending on unknown the parameters $\lambda_1,\lambda_2,...,\lambda_s$, is investigated. Let consider $X_1,X_2,...,X_n$ a sample of n independent and identically distributed (iid) observations which follow the same probability law as the random characteristic X. It follows that the PDF of the random vector $(X_1,X_2,...,X_n)$ is the function called the likelihood given by expression:

$$L(x_1,x_2,...,x_n;\lambda_1,\lambda_2,...,\lambda_s) = \prod_{k=1}^{n} f(x;\lambda_1,\lambda_2,...,\lambda_s)$$

The estimators $\hat\lambda_i = \hat\lambda_i(X_1,X_2,...,X_n)$ are called maximum likelihood estimators of the parameters $\lambda_1,\lambda_2,...,\lambda_s$ if they realize the likelihood maximum.

The maximum likelihood estimators are determined by solving the equations system:

$$\frac{\partial L(x_1,x_2,...,x_n;\lambda_1,\lambda_2,...,\lambda_s)}{\partial\lambda_i} = 0,\ i=\overline{1,s}$$

which is usually replaced by the system (easier to solve):

$$\frac{\partial \ln L(x_1,x_2,...,x_n;\lambda_1,\lambda_2,...,\lambda_s)}{\partial\lambda_i} = 0,\ i=\overline{1,s}$$

obtained by considering the optimization problem for the natural logarithm of the likelihood

$$\ln L(x_1,x_2,...,x_n;\lambda_1,\lambda_2,...,\lambda_s) = \sum_{k=1}^{n} \ln f(x_k;\lambda_1,\lambda_2,...,\lambda_s)$$

This calculus artifice is due to the calculus complexity or impossibility to calculate the partial derivatives of the likelihood.

Thanks to the product **derivative* rule, i.e. $(fg)^\star(x) = f^\star(x)g^\star(x)$, we are now able to avoid this problem, if we **differentiate* the likelihood function

$$L(x_1, x_2, ..., x_n; \lambda_1, \lambda_2, ..., \lambda_s) = f(x_1; \lambda_1, \lambda_2, ..., \lambda_s)...f(x_n; \lambda_1, \lambda_2, ..., \lambda_s)$$

as follows:

$$\begin{aligned} L^{\star}_{\lambda_i}(x_1, x_2, ..., x_n; \lambda_1, \lambda_2, ..., \lambda_s) &= f^{\star}_{\lambda_i}(x_1; \lambda_1, \lambda_2, ..., \lambda_s)...f^{\star}_{\lambda_i}(x_n; \lambda_1, \lambda_2, ..., \lambda_s) \\ &= e^{\frac{f'_{\lambda_i}(x_1;\lambda_1,\lambda_2,...,\lambda_s)}{f(x_1;\lambda_1,\lambda_2,...,\lambda_s)}} ... e^{\frac{f'_{\lambda_i}(x_n;\lambda_1,\lambda_2,...,\lambda_s)}{f(x_n;\lambda_1,\lambda_2,...,\lambda_s)}} \\ &= e^{\frac{f'_{\lambda_i}(x_1;\lambda_1,\lambda_2,...,\lambda_s)}{f(x_1;\lambda_1,\lambda_2,...,\lambda_s)} + ... + \frac{f'_{\lambda_i}(x_n;\lambda_1,\lambda_2,...,\lambda_s)}{f(x_n;\lambda_1,\lambda_2,...,\lambda_s)}} \\ &= 1 \\ &= e^0 \end{aligned}$$

which conduct to the maximum likelihood system:

$$\frac{f'_{\lambda_i}(x_1; \lambda_1, \lambda_2, ..., \lambda_s)}{f(x_1; \lambda_1, \lambda_2, ..., \lambda_s)} + ... + \frac{f'_{\lambda_i}(x_n; \lambda_1, \lambda_2, ..., \lambda_s)}{f(x_n; \lambda_1, \lambda_2, ..., \lambda_s)} = 0, \ i = \overline{1, s}$$

equivalent with the system obtained in the classical maximum likelihood method. Matched with the **Newton-Raphson* method developped by Özyapici et al. [12], which is known to be faster than the standard *Newton-Raphson* method, we expect that the implementation of the proposed **maximum likelihood* method improve the speed and the accuracy of the search for the maximum likelihood estimator.

3.2 Bigeometric calculus

3.2.1 Resiliency

As we have shown a relationship between the elasticity of a function x at a and the resiliency of x at a is the following:

$$\textit{elasticity} = \ln(\textit{resiliency})$$

where

$$\textit{elasticity} = \frac{ax'(a)}{x(a)} \text{ and } \textit{resiliency} = x^{\star}(a^{\star}) = e^{\frac{ax'(a)}{x(a)}}.$$

3.2.2 Suggested applications of resiliency

Grossman and Katz [11] and Grossman [10] supposed that:

- psychophysicists could find some interests in *bigeometric calculus* because of one of their basic laws which can be stated as follows: *"the resiliency of the stimulus-sensation function is constant"*, and they continued: *"that constant is determined by the nature of the stimulus"*.

- biologists could use the *bigeometric calculus* for a fundamental law of growth as follows: *"if f is the function relating the size of one organ to the size of any other given organ in the same body at the same instant, then, within certain time limits, the resiliency of f is constant"*.

- physicists who preferred not to settle on specific units of time and distance could assert that the *bigeometric speed* of an object falling freely to the earth is constant.

3.2.3 Bigeometric derivative

In Grossman [10] is studied the case of a bipositive function when each of its points (x, y) is changed to (px, qy), where p and q are positive constants, so a *change of scales* has been made in the function. It could be proved that the *bigeometric derivative* is independent of the scales used for function arguments and values and this fact is considered to be as a point of interest for scientists who wish to express laws in scale-free form.

§4 Conclusion

For nearly thirty years, non-newtonian calculus stays confidential because of the conservative position of social scientists and perhaps a lack of information on the fact that it could help to acquired new insight on classical subjects, or solve directly some problems which could only reached by approximations.

Until now, there is a lot of papers which describe this calculus as Stanley [15], Spivey [14], Bashirov et al. [1], Bashirov et al. [2], but economic applications are rare and doesn't get away from the simple justification of the technics.

In this paper, we have tried to show that it has some very interesting potentials that we have begin to study.

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Derivative Rules		Examples	
		$f(t)$	$f^\star(t)$
	$(cx)^\star(t) = (x)^\star(t)$	C	1
Product	$(xy)^\star(t) = f^\star(t)y^\star(t)$	$Ce^{\alpha t}$	e^{α}
Quotient	$\left(\frac{x}{y}\right)^\star(t) = \frac{f^\star(t)}{y^\star(t)}$		
		$Ce^{\sin(t)}$	$e^{\cos(t)}$
		$C\alpha^t$	α
	$(f^y)^\star(t) = f^\star(t)^{y(t)} f(t)^{y'(t)}$	Ct	$e^{\frac{1}{t}}$
Chain rule	$(x \circ y)^\star(t) = f^\star(y(t))^{y'(t)}$	$\alpha t + \beta$	$e^{\frac{\alpha}{\alpha t+\beta}}$
Sum rule	$(x+y)^\star(t) = f^\star(t)^{\frac{f(t)}{f(t)+y(t)}} y^\star(t)^{\frac{y(t)}{f(t)+y(t)}}$	Ct^{α}	$e^{\frac{\alpha}{t}}$
		$C\ln(t)$	$e^{\frac{1}{t\ln(t)}}$
		$C\ln(f(t))$	$[f^\star(t)]^{\frac{1}{\ln(f(t))}}$
		$C\sin(t)$	$e^{\cot(t)}$
		$C\cos(t)$	$e^{\tan(t)}$

Table 1: Some examples of product calculus