In defense of a non-newtonian economic analysis

Diana A. Filip, Cyrille Piatecki

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Diana Andrada Filip† & Cyrille Piatecki‡

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Abstract

The double-entry bookkeeping promoted by Luca Pacioli in the fifteenth century could be considered a strong argument in behalf of the multiplicative calculus which can be developed from the Grossman and Katz non-newtonian calculus concept. In order to emphasize this statement we present a brief history of the accountancy in its early time and we make the point of Ellerman's research concerning the double-entry bookkeeping.

Contents

1 Introduction

According to Kuhn (1962), a paradigm shift is a radical change of our way to understand the world such that no one inside the field of reflection where it occurs can ever refers to the corpus that was pregnant before the shift. But, for a paradigm shift to occur, the failure of the older paradigm

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†Babeş-Bolyai University of Cluj-Napoca, Romania and LEO, Orléans University, France — diana.filip@econ.ubbcluj.ro.

‡LEO, Orléans University, France — cyrille.piatecki@univ-orleans.fr

must be acknowledged by a scientific community. And to acknowledge that there are failures, a scientific community must not be locked-in a way to look at some phenomenons or in a way to analyse those phenomenons.

And in complete opposition to what some could think, it is through our model of the world that we try to understand it because nobody has never been able to collect data without a conception of what are the data.

Now, because our cerebral enginery has evolved in such a way that, at birth, we are only able to conceive elementary additions and substractions, all other operations being the object of a complex process of learning — see for example Lakoff & Núñez (2000) —, we are locked in an additive conception of our world which is rarely questioned.

2 The accounting locked-in

It is hard to imagined that the way we take record of our transactions could have been managed in a different way, better adapted to the description of the growth or decline of ours operations.

2.1 A brief accountancy history in its early time

It is acknowledge that human beings count since its emergence on this world. But a serious accountancy system like the double entry bookkeeping appears only in the 14th century Italy and not in the merchant civilisation of the middle east or in Greece or in Rome. According to Littleton (1933), there are 7 *key ingredients* which led to the creation of a double entry book keeping :

① *Existence of a private property system* : This is a mandatory ingredient because bookkeeping is concerned with the record properties, of transfer of properties and on property rights.

② *Accumulation of capital* : the growth of the human activity has been effective only when those with a know-how has been able to borrow resources to those who possessed them in such a way that commerce and credit ceased to be trivial.

③ *Commerce at a widespread level* : At a local level, small volume trading does not create a pressure to organise a strong system of bookkeeping, because a simple accountancy is perfectly sufficient for small quantities transactions.

④ *Existence inter-personal credit with an enforcement guarantee* : If all transactions are untied on the spot, there is no incentive to keep any record. The enforcement guarantee need as a precondition the existence of an authority strong enough to sanction the non-payment of the principal and the interest due to the loaners. As is shown by the early civil code as the Ur-Numu or the Hamuraby ones — nearly 2285 B.C. —, the pre-existence of the State, if not a mandatory condition is necessary to the realisation of this ingredient.

⑤ *Writting* : Obviously, if one cannot write, because human memory is too fallible, there is no way to record any thing. The oral transfer of information from people to people cannot offer any guaranties of authenticity.

⑥ *Money* : It is also a mandatory condition, because without a *common denominator*, bookkeeping is nearly impossible. In the contrary, with money and from the point of view of bookkeeping, transactions are no more that a set of monetary values.

⑦ *An arithmetic* : This is also mandatory because without the mastery of an arithmetic there is no way to compute the monetary details of the transactions.

As signaled by Alexander (2002), many of these factors where present long before the 14th century but either there where not present in the same time and place or they where present in an unstructured form or with a not strong enough pregnancy. Therefore, in what concern writing, if it is not a necessary condition for civilization — the gallic civilisation is acknowledge as a civilization even if it never acquired writing —, it was mandatory to begin historical times since History could not exist without records. But even if the beginnings of arithmetic are contemporary of the civilization, arithmetic understood as the systematic manipulation of numbers, was not a tool acquired on a sufficient scale until the middle-age to help to the development of a double entry bookkeeping.

More than that, the nearly universal use of roman numeral in the occidental world long after that the arabic numeral have been introduced, has been a severe restraining factor because of the non existence of the zero in the roman numeral systems. Yet, a neutral element, and zero is the neutral element of the addition, is mandatory to the development of a double entry bookkeeping as would be shown in section 2.2.

Nevertheless, in the ancient times, the problems encountered by mercantile or even States where alike with ours. For instance, because of tax collecting, governments had strong incentives to keep records of receipts and expenditures. As rich peoples often hired agents or used slaves to perform their operations, they needed to realize audits to verify the honesty and/or skill of their factotum. But the illiteracy and the cost of writing — ink and parchment — was so high, and the monetary systems so inconsistent, that only in the case where transactions were incredibly large, could we imagine to record them.

Nevertheless, because of he first prosperity times in the mankind history in the area between the Tiger and the Euphrates rivers there was a need to record transaction. This need was early codified. For instance in the Hammurabi code, it was required that an agent who was selling a good in name of his principal gave him a price quotation under seal, the failure to perform this obligation being an invalidation argument for the transaction. So, transactions were engraved in clay tablets which were safely kept until final outcome and the recycled for a new transaction.

Egyptians use for a while the same support until the introduction of the papyrus which permits easily to extend records. A profession of specialized scribes was organized. It developed an elaborate internal verification system whose honesty and credibility were enforced by royal audits which conduct to important penalty in case of irregularities.

Ancient greeks introduced two major innovations : first, they established public accountants [1] to impose the authority and control of the State; secondly, about 600 B.C., they introduced coined money which not only facilitated transactions but lighten bookkeeping operations by the introduction of a local common unit. Under those innovations, the banking system, which has existed since the old sumerian times, reached a level never reached, allowing change and loan operations and even cash transfers on a scale never reach until this time.

The romans developed the first system which recurrently maintain, at the level of the households their daily receipts and the expenses in *an ad-*

[1] 10 chosen by lot.

versaria or account book. Then, they aggregated monthly those statements in a cashbook known as *a codex accepti et expensi*. All those operations, developed because the assets and liabilities of the citizens where the basis of the taxation systems and eventually were used to determine civil rights.

Then roman accountants developed an elaborate system of checks and balances for governmental receipts and spending which was necessary to fixed and verify all the operations links to a conquerors nation. Later on, when the empire was well established an accounting system was mandatory to keep record of all level of fiscal operations. To coordinate all the public financial operation, they conceive the first annual budgets.

But at the time of the fall of the roman empire in 582, the necessary structures needed to teach how to write an account has vanished. One must know that romans use nearly the same educational system which was nearly universally used in Greece. Until 12, young upper class boys stayed in their families where they received an education where the emphasis was set on letters, music and a great proportion of elementary arithmetic — essentially taught to know how counting either with the help of their fingers or with the abacus[2]. Then, they were normally send in a school where they learned literature, grammar, some elements of logic, rhetoric and dialectics.

Only those who needed a deeper mathematical teaching, as the one who planed to become *agrimensor*, that is surveyors, used to learn geometry. If they planed to become architect they obeyed the Vitruve's advices to learn geometry, optics, arithmetic, astronomy and others fields as law, medicine, music, philosophy and history. Galen gave some nearby advices for student in medicine... The reason why mathematics where not high ranked in the roman education was essentially that it was useful essentially for liberal professions when the royal road in education conduced to public functions.

But it seems that according to the accounting standards, quickly described above, this mathematical education was in all ways sufficient.

During the *dark ages*, because to learn you need peace, there was a fall in eduction in Europe to the unique exception of the british islands which, because of their insolation, were protected against the quasi-universal chaos.

[2]It is not clear if the abacus is an roman invention or if it is a chinese one which come in occident as the silk cloth through the trade of the red sea and the coasting navigation along india.

Of course, the higher centers of learning where rare but we do have testimonies of the transmission of the roman educational system. Toward the 9th century, in nearly every monasteries schools were organized to permit the acquisition of the christian culture to those destined to the priesthood or to the cloistral life. But, because of the intended object of this eduction, the studies were limited to reading, writing and the study of the Bible. Only in very rare places like the cathedral of York, mathematics were taught[3].

Horrified by the low standard of education in continental Europe, Charlemagne, asked Alcuin to leave York to organize a court school with the posted object to permit at least to the clerk to interpret the Holy Bible correctly.

But at his death, with the return of war, the educational level drop against until Gerbert was elected Pope under the name of Sylvestre II in 999. Gerbert was himself a mathematician who discovered some interesting document, so he favored the study of Boethius, one of the rare roman who studied mathematics in the 5th century[4]. But, as Boethius was himself interested by the work of the surveyor, the mathematics studied had a more pronounced flavor of geometry than the needed arithmetics necessary to keep the accounts.

At the end of the dark ages, the english church, fighting against the pagan influence, decided that mathematics had a too much paganist flavor and mathematics education again vanished. Even Sylvestre II mortal remains undergo the costs. Because of his efforts in the transmission of

[3] In 732, Egbert was bishop and head teacher of the York school. He organized the studies in such a way to teach rhetoric, law — essentially canonic —, physics, arithmetic, geometry. As the date of Easter was changing according to the moon, some arithmetics where also taught.

[4] Until the beginning of the XXth century, Boethius has been credited to the first usage of the hindu-arabic numerals in Occident. But Smith & Karpinski (1911) have erased this credit, showing, by an erudite demonstration, that a man of his inquiring mind couldn't have let aside the least quantum of information that spreads out in the mediterranean world through commercial relations, but, as in his time, the zero has not yet been incorporated as a place holder with the other numbers, the arabic ciphers where not an improvement in the calculation effort in comparison with roman numeral. So there where no reason for Boethius to use them. In what concerns his *Geometry*, one of the three works which are attributed to Boethius, where one can found the complete hindu-arabic numerals including the zero, Smith & Karpinski (1911) have demonstrated that it has been belatedly falsified since, if Boethius have known such a system he would have used it mainly in his *Arithmetics*, that none of his disciples and successor never used it, and that the falsification of text was under complaint even in his own time.

the arabic science he obtains through his connection with the jews during his three years apprenticeship travel trough Catalogna and Cordoba, and, mainly, his introduction of the zero, a veil of sorcery stay associated with his name for sorcery until his exhumation in 1648 to verify if he was still inhabited by all the devils of inferno. Not the least reason of the sad post-mortem history of Gerbert of Aurillac — Sylvestre II —, is his role in revealing to the occidental intellectuals the, in that time relative new, hindu-arabic positional system of numerical notation and the fondamental innovation attach to it that is to say the zero — initialy zephyrum — who was no more than the arabic *shifr* which itself was no more than the indian *sunya* which stand for void[5]. According to Turner (1951), the roman numerals which are, by nature, additives[6] were even more easier to add or to subtract than the hindu-arabic ones. For instance, subtracting or adding 126 — CXXVI — to 378 — CCCLXXVIII — is a very simple calculation which is perform without knowing any table as shown here after[7] :

							C	X		
	CCC	LXX	VIII	(378)			CCC	LXX	VIII	(378)
-	C	XX	VI	(126)		+	C	XX	VI	(126)
	CC	L	V	(255)			D	(C)	(X)VI	(504)

Al least, since Cajori (1950), the idea that multiplication and division were truly hard problems if carry out with roman numeral has diffused among scholars. On that subject, reinforced by the fact that no multiplicative roman calculus has never been discovered in the document inherited from roman or middle-age time, the common point of view was that those operations were performed through the abacus. But, as shown by Anderson (1956), it's a profound mistake[8]. For instance, in the case of the multiplication of 17 — XVII — by 60 — CL —, one has :

[5]On the transmission to Gerbert of the hindu-arabic numeral, one can consult Zuccato (2005).

[6]That is to say : 1977 or MCMLXXVII is, literally written, in roman numerals notation as $1 \times 1000 + 9 \times 100 + 7 \times 10 + 1 \times 7$.

[7]X = VV, L = XXXXX and D = CCCCC.

[8]In his appendix, Anderson (1956) take a brag pleasure to carry out the multiplication of DCCXXIII by CCCLXIV reputed by Cajory as impossible to compute with roman numerals.

	XVII					
×	LX					
	D	CCL	L	L		
+		C	L		X	X
	D	CCCL	C	L	X	X
=	D	CCCCC			X	X
=	D	D			X	X
=	M				X	X

17
60
00
102
1020

One can observe that the main difference between the two operating mode is that with roman numeral the operations are carried out from left to right contrary to the ones carried out with hindu-arabic numerals.

In what concern the division, there are no essential differences between the two type of numeral. For instance, to compute 209 ÷ 11 that is — CCIX÷XI —, one has

C		C	I	X	X I	20	9	11
C		X			X	11		19
	LXXXX		I	X	IX	9	9	
	LXXXX		I	X			0	

But, as is not instantly apparent, division is in fact simpler with roman numerals than with hindu-arabic ones because it entitles the possibility to not find the exact number of time the divisor will go in the dividend at each step[9]. Anderson (1956) also shows how to calculate powers or how to extract square roots. This experimental archaeological explanation shows that, contrary to the general conviction, it could be conjectured that it is not because of its intrinsic complexity that roman numerals were finally derelicted.

In the same time, one can conjecture that there was a disenchantment between middle-age thinkers and roman numerals. If for scholars as Murray (1991), the original sin of roman numerals was to embalm the primitive

[9]One must note that, in all those cases, there is a flavor a heriticism in the notation used conjointly with roman numeral since the sign + appears for the first time in a book by Hodder in 1672, the sign × is a joint product of the algorithm by Napier in 1618 and ÷ is due to Rahn in 1659. On those subjects, one can consult Smith (1958) and Cajori (1909).

principles of addition and substraction in such a way to block the entry, into notation, of the multiplication and the division, a point of view which is dismissed by the former paragraphs, King (2001) sees a conjonction of many practical different causes. For example, the roman numeral were not fitted to accommodate great numbers because the letters used to construct a number were limited. As they used X for ten, C for one hundred and M for a thousand, they were also obliged to write a horizontal line through or above the numerals to raised the value of the number by a factor of 10 or of 1000. King (2001) displays a tentative by a certain Adriaen vander Gucht, realized in 1569, to construct a table of power of 10 from 1 to 29. But as one can observe, it couldn't be used because of the ambiguity here attached.

Number	Representation				
I	$10^0 = 1$	10^6	10^{12}	10^{18}	10^{24}
X	10^1	10^7	10^{13}	10^{19}	10^{25}
C	10^2	10^8	10^{14}	10^{20}	10^{26}
M	10^3	10^9	10^{15}	10^{21}	10^{27}
XM	10^4	10^{10}	10^{16}	10^{22}	10^{28}
CM	10^5	10^{11}	10^{17}	10^{23}	10^{29}

As shown by the anecdote retrieved from Suetonius' *Life of Twelve Caesars* by Kaplan (2000), the roman numerals limited number of signs was a true deficiency which could conduct to many legal resorts. When dying, Livie, August's wife and certainly one of the most powerful women of the ancient times, decided to bequest to Galba 50000000 sesterces, that is to say, in roman numeral notations $|\overline{\mathrm{D}}|$. But, the son of Livie, the emperor Tiberius, who doesn't like Galba, insisted that $|\overline{\mathrm{D}}|$ be read as 500000 because *quia notata non perscripta erat summa*[10].

This may seems a venal sin of roman numerals but, as shown by Murray (1991), as time goes by, with the help of the need for large numbers, roman numerals became truly impractical. For instance, Murray reports a 1649

[10] Because the sum was in notation, not written in full.

selling of the english crown lands for a amount of 1 423 710 pounds, 18 shillings and 6 pence which have been noted by the exchequer

l′	C	M	C	ħ	s′	d′
M	iiij	xxiij	vij	x	xviij	iij

Even for smaller numbers, during the 15th century the roman numerals became inappropriate as a numbering system as one can realize in looking to the 24 hour clock-face situated on the inside of the western wall of the duomo Santa Maria del Fiore in Florence.

A second problem was that scribes had introduced some variations which were very difficult to decipher as in some french manuscripts written between the 13th and the 15th century as :

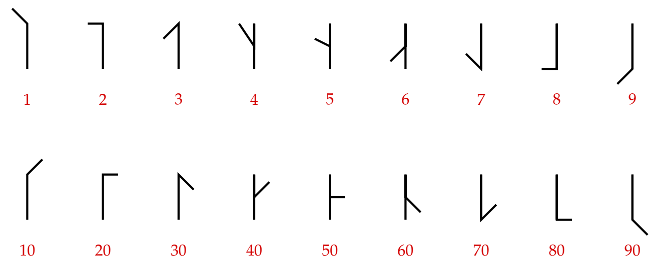
$$\begin{matrix}\mathrm{XX} & \\ & \bullet \\ \mathrm{IIII} & \end{matrix} = 81, \quad \mathrm{IX}^{\mathrm{XX}}\mathrm{III} = 183, \quad \mathrm{VII}^{\mathrm{XX}}\mathrm{VI} = 156$$

as is shown in Lehmann (1936). An other point which renders caduc the usage of the roman numeral was that, since the introduction of the hindu-arabic numerals, peoples used to mixed them with the use of roman numeral. For instance Bergner (1905) — cited by King (2001) — brings out some german inscriptions such as :

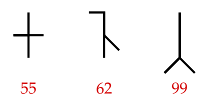
mcccc8	1408	stamp of an Augsburg religious dignitary
1• 4 • Lxiii	1463	on a gravestone from Salzburg
14XCIIII	1494	on the altar of St. Othmar in Naumburg
1 • V^{C} • V	1505	on a bell in Keila near Ziegenrück
1 • V^{C} • 6	1506	on a bell in Neustadt a. O.
15X5	1515	in Lauffen near Rottweill
MD.25	1525	in the Schlosskirche in Chemnitz

And this degenerated usage was not the only appanage of germans, since even the great french humanist Guillaume Budé use to note the year 1534 by M.5.34.

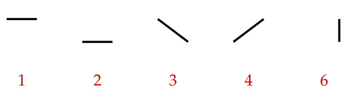
One couldn't say if this out of favoring has been the origin of the introduction of some to day forgotten numeral, but as shown by King (2001) that during the 10th century, John of Basingstoke, who was one of the first in england to master greek, is reputed to have introduced some new ciphers deemed of greek origin[11]. Those ciphers were very simple but they were not able to note numbers greater than 99 as is shown in the following table.



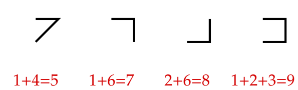
With such ciphers one can write very easily numbers as



Even if of limited range, this type of ciphers pleased at least the cistercian monks who designed some clever extensions. As shown by King (2001), cyphers of that type flourished in Europa. During the 13th century north of France, a clever vertical version of this type of cyphers, of whom design was particularly well fitted for arithmetic operations, reaches the academic mediums. Its particular structure not only permits operation up to 9999 but, thanks to its additive structure, made easy the four standard operations. From



one obtains



Each number is associated with a decimal value — unit, tenth, hundred, thousand — according to its position and symmetry along a vertical bar — *i.e.* :

[11] As a friend of Robert Grosseteste, chancellor of the University of Oxford and archdeacon of Leicester and later bishop of Lincoln, John of Basingstoke occupied a central position in the intellectual medium of the tenth century.

Even if no one has ever tried to compute with this ciphers, King (2001) has shown that they were perfectly apt to realize the four operations. But there was a major disfunction : it was impossible to define an algebra since if we define addition or multiplication as the internal operation there are numbers not representable with this numeral systems[12]. Be that as it may, this type of cyphers hardly has been of any used outside one 14th century astrolabe of Berselius and it's use for marking volumes on wine-barrels[13].

Before the end of the Middle-age, not only the numeral notation was deficient, so it was for the technology of the operations, at least in Europa, was deficient. Multiplication and division was hard tasks operated on hands or with an abacus.

Of course, not later than the 5th century, indian mathematicians imagined a positional system with a peculiar use of the zero since both, the positional system and the zero, were the basis of the sexagesimal system of the babylonian[14]. What make peculiar the indian numeral system is that, contrary to the 3th century b.c. babylonian numeral system, the zero was not just a mark to indicate a void position, but a true number.

The transmission through the arab mathematics to the west could only succeed in a civilization clash because, when the occident discovered the zero, it was under the monopoly of the christian doctrine which, at that time was completely devoted to the aristotelician philosophy for which the void associated with the zero was truly a non-concept.

First of all, as underlined by Seife (2000), zero brake the rubber band property of the integer numbers which come throughout multiplication and associativity. That is to say that, for every number, with the exception

[12] In 1953, then at *Bell Labs*, Claude Shannon constructed a mechanical calculator which computed in Roman numeral so as to demonstrate that, even if difficult, it was possible to compute this way.

[13] In the middle of the 16th century, Gerolamo Cardano proposed his own version of the ciphers in his *De subtilitate Libre XXI* of 1550 — see Cardano (1999[1666]).

[14] One may wonder why babylonians used a sexagesimal system. One possible explanation is that with such a system 1/2, 1/3 and 1/6 are integers which simplified astronomical operations.

of zero, the multiplication by any integer change the scale, as if the rubber band expanded. Secondly, if one adds two numbers and multiply the resulting number by any number we obtain the same result as if one has multiply the two primitive numbers by the same number and then adds the to resulting numbers. But with zero obeys its own rules : any number multiply by zero shrinks to zero and any number adds to zero stays the same. Thirdly, as the pythagorician's doctrine ascertained that every things that make sense in the universe had to be related to a neat proportion zero couldn't fit in this doctrine as any number divided by zero is infinite, to the notable exception of zero which, as is universally known, is undetermined.

So, to the notable exception of pythagoricians, if greek mathematics could accepts irrational and negatives numbers, avare of the sumerian sexagesimal numeral system which, as one has seen earlier, has a peculiar zero, they could not accept zero as a number on philosophical basis, not by ignorance.

And then, Zeno of Elea sets out his famous paradox of Achile and the tortoise. If Achile raced the tortoise that has a head start, even if he runs two time speeder than the lumbering tortoise, he will never catch up with the tortoise since each time he made half the distance between his position and the one of the tortoise that last one has increase the distance that Achille must make[15]. Because of the lack of a the concept — greeks have no word to name it —, it was impossible to cope with the paradoxe. Zeno conclusion was contrary to experiment : motion was impossible. Every thing is one and changeless[16]. Here enter Aristote who in front of Zeno paradoxe declares that there is no need of infinity which are could exist only in the mind of mathematicians without actual support. There was no infinity and no void, the earth was situated in the center of the universe surrounded by spheres in a pythagorian harmony. But as there were no infinity, the number of spheres was necessary finite, the last of them being the celestial vault. There was no such thing as beyond the celestial vault and the universe was contained in a nutshell.

[15]The solution will emerge from the development of mathematical analysis, with the introduction of the notion of limit by the indian mathematician Mādhava of Sañgamāgrama in the 15th century who was in all probability not aware of the problem, two centuries before that Cauchy strictly develops the analysis of the subject on the basis of the series convergence initialized by d'Alembert.

[16]More exactly, Zeno was a member of the eleatic school founded by Parmenides and his paradox was designed to support Parmenides arguments.

The reason why this conception of the universe lasts so long is that it incorporated a proof of god existence. As there was movements[17], something must be causing the motion and, by necessity, it could be only the prime mover — *i.e.* : god. As expressed by Seife (2000), as wrong as it is, the aristotelician was so successful that for a millennium it eclipsed all opposing views.

It causes many problems the most part of them being linked to the calendar since it was impossible to define a year zero. As the catholic church endorsed that philosophy, one should wait 1277 to see Étienne Tempier, bishop of Paris, condemn all doctrines contradicting god omnipotence which was the case for the aristotelician physics — see Piché (1999). All of a sudden, void was allowed since an omnipotent deity has not to follow any consequences of a human philosophy.

Unfortunatelly, Tempier condemnation was not the final blow of the aristotelician theory. The church will stay clinged to it for still some centuries, and even will reinvigorate this theory when attacked at the reform time.

Because of all this sad history, in those times, only a very small bunch of peoples where able to go further that counting on the hand fingers[18]. Then at that time, there has been a reversal in education. In the british islands, it stagnated. Of course, between the 12th and the 14th century, many great universities were created in all Europe and the teaching of the mathematics of the ancient rise — even new mathematics were developed and diffused as the one of Fibonnaci[19],[20] whose best promotor were Johannes de Sac-

[17]As the earth was at the center of the world, it couldn't move. So, the rotation of the celestial on itself should be explained.

[18]In Turner (1951), one can find a table showing how roman peoples were using their fingers to convey numbers from 1 to 99. More recently, Ifrah (1985[1981]) devotes a complete chapter to the history of fingers counting.

[19]His *Liber Abbaci, Practica Geometria* has been written in 1202. The success was so great that he realized a second edition publish in his living in 1228. Up to now, all copies of the 1202 have disappeared. For wha t concerns the 1228 edition only fourteen copies exists nowadays but only three, located in Italia, are complete or almost so and seven are mere fragments. It should wait 800 years to be translated in English — see Fibonacci & Sigler (2003[1202]).

[20]In any case, was Fibonacci known under this name in his life time. In the tradition of that time, he would have been known as Leonardo of Pisa. But, in the introduction of *Liber Abbaci*, he refers to himself as *filius Bonacci*, a name which was not the name of his father William — one knows also the name of a brother Bonaccingus, but nothing. Perhaps,

robosco — or John of HollyWood or John of Halifax — who learned and teach in Paris and wrote his *Tractatus de Sphaera* whose stayed one of the most studied astronomic book until the end of the Renaissance, the french Alexander of Villedieu's *Carmen of Algorism* written in 1240 and Jordanus Nemorarius — still known as Jordanus of Nemore — *Algorismus*, a more formal treatise, whose redaction date in the 13th century is unknown. In those days, the first translation in Latin of the *Elements* of Euclid was done, and at least in the 14th century, the mathematical curriculum included algorithmic, ptolemaic astronomy, perspective, proportion, measurement of surfaces and. . . fingers accounting which was a pre-requisite to the entry in the Universities.

What make Fibonnaci book so peculiar to his temporaries. The answer is astonishingly simple : it provided an original sum of knowledge linked to the economic of trade which, up to that innovation hardly at disposal even in the old technology of the roman numerals and, in a time where illiteracy was the rule, were, at the end, easy to memorize and manage.

Not only it was the first occidental exposition of the hindu-arabic numerals which diffuses outside the friaries, but it gives also the algorithm to compute the four operations and explains how to apply them to problems essentially linked to the mathematics of trade, freeing the accountants of any counting boards, contrary to what announced the title of the book since, until then, they were mandatory to compute any operation[21]. As such, he

was he proud to belong to the family of his mother. In 1838, the historian Guillaume Libri take the freedom to contract *filius Bonacci* in Fibonacci. Occasionally, he refers to himself as *Bigollo* which, in tuscan dialect signifies traveler and in others italian dialects blockhead. It is under this name that Pise decide to grant to him a salary of 20 lira. In fact this is a very complex subject since it seems that, Perizolo, notary of the Roman Empire, mentions Leonardo as Fibonacci in 1506 — for more details on that subject, see Drozdyuk & Drozdyuk (2010).

[21]That *Liber Abaci* should have been the gate through which practical algebra has been transmitted to Italy and then christian Europe has been convincingly recently disclaimed by Høyrup (2010). One can mention at least two early tentatives to work with hindu-arabic numeral : the first one was Rabi Abraham ben Meir Ibn Ezra who, in his 1148 *Sefer ha-Echad* — Book on Unity —, borrowed the Indian place-value system, but instead of using the traditional number signs, represented each with the first nine letters of the Hebrew alphabet — keeping the Indian sign for zero — and Thomas Le Brun yet known also under the name Qaid Brun, an englishman who was in turn secretary of William I of Sicily "The bad", and later a reformer of the exchequer under Henry II, who seems to have tried to introduced them in the usage of the exchequer but has ot been followed near by 1158 — here the conditional comes from the fact that we have find some

was immediately adopted by merchants who, unexpectedly, could, all of a sudden, acquire, in a time were knowledge was nearly a church monopoly, a modern technology which lowers the accounting costs.

For instance, even if it was not his own, Fibonacci introduced the multiplication *per jealousia* which made this operation more accessible to the most part of peoples who were able to read. For instance, if one want to multiply say 2345 by 467, as was exposed in the book, on can operate by associativity in the same way as with roman numeral — *i.e.* : one can note that 2345 = 2000 + 300 + 40 + 5 and 467 = 400 + 60 + 7.

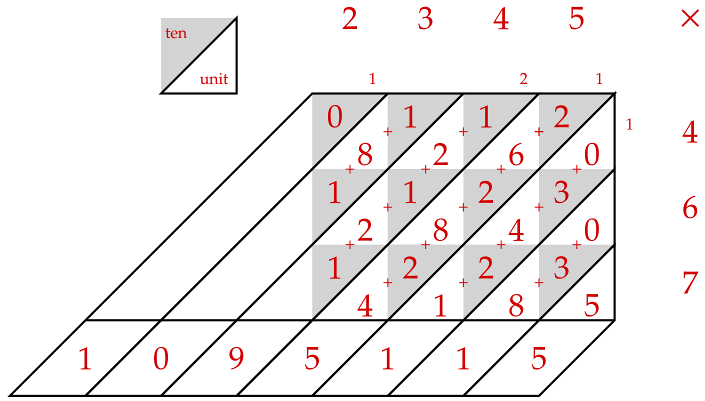
×	2000	300	40	5		
400	800000	120000	16000	2000	+	
60	120000	18000	2400	300	+	
7	14000	2100	280	35	+	
	934000 +	140100 +	18680 +	2335	=	1095115

but we note that that method imply huge numbers multiplications. The *jalousia method*[22] of 9 by 9. This time too, one begin by a double-entry table but in each cases one find the unit, the ten, the hundred, the thousand. . . then one operate as indicated in the following example.

---

recent sources on the subject attesting the existence of some documents which are not referenced. According to Smith & Karpinski (1911), in the same period one can find the first computations with hindu-arabic numerals in a german *Algorismus* of 1143. Fibanacci himself, in his introduction to *Liber Abaci*, tels us that, in what concerns hindu-arabic numbers, while in business, he went in all the places where they were studied and taught — *i.e.* : Egypt, Syria, Greece, Sicily, and Provence — and, according to the translation of Grimm (2002), he *pursued [his] study in depth and learned the give-and take of disputation*.

[22]According to the dictionary, the name comes from the *jalousie window*, a window compose of wooden louvers set in a frame. The louvers are locked together onto a track, so that they may be tilted open and shut in unison, to control airflow through the window. *Jalousies* are reputed to permit to people to see unnoticed.



In what concern the division, he introduced to the european public the *galley method* which is obviously of chinese origin — see Lay-Yong (1966)[23].

Suppose one wants to divide 7385 by 214. The method, which is here decomposed stage by stage, began as shown here under :

①

7385 |
214

As $2 \cdot 3 < 7 < 2 \cdot 4$, one write 3 in the right place, write 2 under the 7 and the rest in the difference between 7 and $2 \cdot 3$ above the seven and do the two same operations for the one crossing the numeral already used.

②

1
~~7~~385 | 3
~~2~~14

③

10
~~73~~85 | 3
~~21~~4

Now $3 \cdot 4 = 12$ but on the line there is only 8 so we must borrow a 1 to 10. The remainder is 9 and the remainder of 18 -12 = 6 so

[23] This method stays nearly for three centuries the lone method taught, since it is paper economizing and that the paper was first imported in occident in the 13th century.

④
9
~~1~~~~0~~2
~~7~~~~3~~~~8~~5 | 3
~~2~~~~1~~~~4~~

Now, as we have exhausted the use of 214 we rewrite it shifted by one position.

⑤
9
~~1~~~~0~~2
~~7~~~~3~~~~8~~5 | 3
~~2~~~~1~~~~4~~4
21

And one begins again.

⑤
1
~~9~~
~~1~~~~0~~6
~~7~~~~3~~~~8~~5 | 34
~~2~~~~1~~~~4~~4
~~2~~1

⑥
~~1~~
~~9~~2
~~1~~~~0~~~~6~~
~~7~~~~3~~~~8~~5 | 34
~~2~~~~1~~~~4~~4
~~2~~~~1~~

⑦
10
~~9~~~~8~~
~~1~~~~0~~~~6~~9
~~7~~~~3~~~~8~~~~5~~ | 34
~~2~~~~1~~~~4~~~~4~~
~~2~~~~1~~

At the end of the process, one find that there is 34 times 214 in 7385 with a reminder of 109. This method generate at least two remarks: first, it is not self-evident for a learner to understand that a number couldn't be written on the same line, but this is marginal. Secondly, there was no attempt to try to make any expansion of the remainder in decimal value as one uses to do nowadays. It was not in the spirit of the time and, even if it has been know in China, in Persia long ago before Fibonacci, even if it was developed in Europe in the 14th century writings of Immanuel ben Jacob Bonfils, it will be necessary to wait the 17th century and the *De Thiende* of the flemish mathematician Simon Stevin[24] so that the decimal development becomes

[24]To help the diffusion of his thinking, Steven had translated his work by himself.

an universal practice[25] — see Stevin (1935[1585]). But, from a practical point of view, that was perfectly justified. In a tradition inherited from the egyptian, it was the usage to write a non integer number as A.bcd in the form :

$$\frac{d}{10}\frac{c}{10}\frac{b}{10}A$$

where the denominator of each fraction was understood as the product of all denominator value that preceded it[26]. For instance, Fibonacci would have written the result of the former division $7385 \div 214$ as $\frac{109}{214}34$. Suppose now that this value be a monetary value. In his living time, the subdivision of monetary unit was complex[27] but with the Parma system of account Fibonacci would have written $\frac{2}{20}\frac{6}{12}34$ that is 34 *Lira* 6/12 of *Lira* and 2/240 of *Lira* or in actual money 34 *Lira* 6 *Soldi* and 2 *Denari*.

An other innovation of the *Liber Abaci*, is it advocacy for the use of negative numbers which Fibonacci interprets as debts or, in another work — *Flos*[28] —, throughout the deficit metaphor. Of course, even in that cas, Fibonacci was just a diffuser since negative numbers were of chinese and, later, indian origin. And even his interpretation was not original since, in the 7th century, the indian mathematician Brahmagupta has yet called them debts :

> *A debt cut off from nothingness becomes a credit;*
> *a credit cut off from nothingness becomes a debt.*

The true innovations of the *Liber Abacci* could be find in its numerous applications and nearly all applications, one can find in that book, are of

[25]But, one more time, we know now that in the middle of the 10th century, al-Uqlidisi of Damascus introduced place-value decimals to the right of the decimal point. Unfortunately, non one saw a particular reason to adopt it and this brillant idea sleeps for five hundred years before that arab scholars awake it and tree more centuries before Steven convinces europeans of its power — see Devlin (2010).

[26]It is clear that this way to write a number is of arabic origin since, under this structure, it is clearly written from left to right. One must also note that the fraction bar is a Fibonacci innovation.

[27]Until six subdivision as *Ducado*, *Lira*, *Grosso*, *Soldo*, *Piciolo* and *Denaro*, but there was no uniformity between towns. For instance, in Parma, the main money was the *Lira*, which was subdivided in 20 *Soldi* themselves subdivided in 12 *Denari*. It is notable that unit of weight and measure were even more complex.

[28]See Fibonacci (2010).

commercial and financial content. That is not to say that applications were not developed as a by product of the development of the Algebra and essentially from the work of Muhammad ibn Mūsá Al Khwārizmī from which Fibonacci transmitted essentially all the mathematical innovations but, the main difference is linked to the fact that, apart of his involvement in geometry, Khwārizmī's exemples are all essentially linked to legal problems of inheritance. Without any means to know that by himself, there are evidences, through the identity of some developed examples, that Fibonacci commercial and financial investigations are linked to those of earlier indian mathematicians as Āryabhata who, stradding on the 5th and the 6th century, developed some interest calculi in his astronomical book *Āryabhatīya* — see Clark (1930) —, Bhāskara whose commentaries on *Āryabhatīya*, written in the 7th century, include some problems related to partnership share divisions, and the relative pricing of commodities — Goetzmann (2004) — and Sridharacaryas who, in the 10th century, devote some of its 300 verse couplets *Trisastika*, to a few practical interest rate problems and a division of partnership problem — see Ramanujacharia & Kaye (1913). In *Liber Abacci* , one can even find a 9th century copy & past problem borrowed to the jain mathematician Mahavira's book, *Ganita Sara Sangraha*, where three merchants find a purse lying in the road. The first asserts that the discovery would make him twice as wealthy as the other two combined. The second claims his wealth would triple if he kept the purse, and the third claims his wealth would increase five fold[29].

But, two innovations denote a true financial innovative spirit. First, in *Liber Abacci*, one can find the first calculus of an equilibrium price as the result of *absence of arbitrage*.

Try to imagine how to determine an unknown price from a given quantity of merchandise when the price per unit is known[30] — *suppose a 100 rolls costs 40 lira, how much would five rolls cost ?* — without the apparatus

[29]In fact, nearly all, what in a near past, appeared to be Fibonacci innovations have been demonstrated borrowed to indian or arabic traditions. For instance, the celebrated Fibonacci sequence of the growth of a rabbit population has been borrowed to the *Chandahshastra — The Art of Prosody* — of the sanskrit grammarian Pingala, who was active somewhere between 450 and 200 BCE. Then, the indian mathematician Virahanka showed how the sequences arises in the analysis of metres with long and short syllabes. Finally, around 1150, the jain philosopher Hemachandra composed a text on the term value of this numbers — see Devlin (2010).

[30]It's the subject of the chapter 8.

links to the resolution of a first order equation with one unknown.

Nowadays, one would write :

$$x \cdot 100 = 40 \implies x = \frac{40}{100} \implies x \cdot 5 = \frac{40}{100} \cdot 5 = 2$$

a solution that, nowadays, nobody can find remarquable since first order equations are taught in the beginning of the secondary school. Leonardo present it under a different structure as the famous *rule of three* — *i.e.* :

$$\begin{array}{cc} 40 & 100 \\ ? & 5 \end{array} \qquad \frac{4 \cdot 5}{20}(=?)$$

As remarked by Goetzmann (2004), the rule of three is one of the oldest algebraic tools since it appears for the first time in the *Āryabhatīya*, and is extended and elaborated upon in *Bhaskara*s commentaries, in which he applies it to problems quite similar to those analyzed by Leonardo — see Sarma (2002). It's so simple that we can hardly imagine how were perform exchanges before it came of common knowledge by every traders. Of all evidence, in every transaction, one of the two party was certainly defrauded. As a proof for that case, one can advance that, if it has not been the case, Fibonacci should not have developed so many complex examples on that subject. Immediately, Fibonacci explains how to find the price ratio between two goods

> *It is proposed that 7 rolls of pepper are worth 4 bezants and 9 pounds of saffron are worth 11 bezants, and it is sought how much saffron will be had for 23 rolls of pepper*

One more time for the common run of people living in the XXIth millennium, there shouldn't be any difficulty with this operation : One find the unit price of the roll of pepper, the unit price for the pound of safran, then one calculate the price ratio and, after multiplication by the number of desired rolls, the mass is said, that is :

$$\left.\begin{array}{r} 7 \cdot p_p = 4 \\ 9 \cdot p_s = 11 \end{array}\right\} \implies \frac{p_p}{p_s} = \frac{4 \cdot 9}{11 \cdot 7}$$

that is to say that if one wants to exchange 23 rolls of pepper against saffron one will obtain :

$$\left(\frac{4 \cdot 9}{11 \cdot 7}\right) \cdot 23 = \frac{2}{11}\frac{8}{7}10 \approx 10.75324675\ldots$$

But, on more time it's only the application of the *rule of 5*, that Bhāskara and later Indian mathematicians developed for expressing price/quantity relationships across several goods — *i.e.* :

$$\begin{bmatrix} \text{Saffron} & \text{Bezants} & \text{Pepper} \\ ? & 4 & 7 \\ 9 & 11 & 23 \end{bmatrix}$$

Implicitely, Fibonacci teaches how, in that matrix, the balance is between the two diagonals

$$\begin{bmatrix} \text{Saffron} & \text{Bezants} & \text{Pepper} \\ ? & \textcircled{4} & 7 \\ \textcircled{9} & 11 & \textcircled{23} \end{bmatrix} \text{ and } \begin{bmatrix} \text{Saffron} & \text{Bezants} & \text{Pepper} \\ ? & 4 & \textcircled{7} \\ \textcircled{9} & \textcircled{11} & 23 \end{bmatrix}$$

and finally give the thing — ?.

Up to this point, Fibonacci also shows how to arbitrate between moneys because, in that time, even in Tuscany, even if cities have inherited of the late roman money division — *denari*, *soldi*, *lire*[31] —, the relative value and metallic composition of the various moneys varied considerably through time and across space[32]. On that subject, after 1252, the teachings of the *Liber Abacci*, will became unavoidable, due to the introduction of the golden florin which was the first money which could simultaneously serve as a unit of account and transaction on a 1:1 basis, since there were evidence, that it was never debased — see Velde (2000).

But, as bring out by Goetzmann (2004), and echoed by Rubinstein (2006), it's in the finance field that *Liber Abacci* shows all his genius in discussing four type of problems :

① How to split fairly the profit of a joint venture when contributions are unequal, are made at different points in time, and in different currencies or goods and in cases in which business partners borrow from each other

[31] Which finally remains in the english pounds, shilling and pence system.
[32] It's the reason why, he also give many exemples of minting and alloying of money.

② How to calculate the profits generated by a sequence of business trips in which profit and expense or withdrawal of capital occurs at each stop.

③ How to calculate the future values of investments made with banking houses.

④ The first use of present value analysis as a criterium to evaluate investments, including specifically the difference between annual and quarterly compound interest.

In what concerns the first point, on must know that in the 13th century italia, the business ventures were organized through the *commenda* contract[33] which stated how the *commender* — the partner who invested his funds — and the *traveler* — the partner who invested his labor — should divide the profits. If this last one, doesn't abound to the financement, the *commenda* was unilateral and the *commender* retained 3/4 of the benefits. If the traveller decide to abound to the financement, in which case the *commenda* contract was reputed bilateral, it was generally up to half of the *commender* abounding.

Then, according to the specific dispositions of the contract, the study of notarial archives from cities as Barcelona, Genoa, Venice, Amalfi, Marseilles, and Pisa[34] reveals that, generally, any profit was usually divided 1/2-1/2 while the *commendator* bore 2/3 of any loss and the *tractator* 1/3. But variations of this dispositions could be encountered. For instance, in Dubrovnik, the share 3/4-1/4 of the unilateral *commenda* was not a rule at all.

Here, Fibonacci innovates in explaining how to fairly — that is according to the initial contribution — divide the profit when there is more than one *commender* — in those time, they constitute a *societas*[35].

In what concern the travelling merchant, here is how Fibonacci states his basic problem :

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[33]Prior to Pryor (1977), it was common to place the origin of this contract in the muslim qīrad contract. Since then, the Jewish *'isqa* contract and the roman *societas* appear to be two other sources of the *commenda* contract.

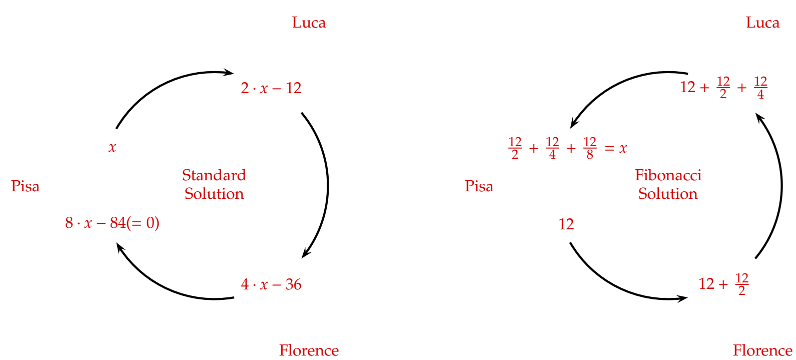
[34]According to Fibonacci & Sigler (2003[1202]), the *Constitutum Usus* of 1156 is the the earliest surviving municipal document specifying the conditions of the *commenda* contract.

[35]For the specific operations, one can consults one more time Goetzmann (2004) or Fibonacci & Sigler (2003[1202]).

A certain man proceeded to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and spent 12 denari. Then he returned to Pisa, doubled his money and it is proposed that he had nothing left. It is sought how much he had at the beginning.

Here, one can find the solution original under the restreint that words have the same meaning for us than for Fibonacci, that is to say that one must understand that, in returning in Pisa the man spent also 12 denari and that the money double before the expense.

Here is how it may be reasonable to solve the problem by a forward argument. If X is the initial capital, one has for an initial capital of x, a capital of $2 \cdot (x - 12)$ in Lucca, of $2 \cdot ((2 \cdot x - 24) - 12) = 4 \cdot x - 36$ in Florence and of $2 \cdot ((4 \cdot x - 36) - 12) = 8 \cdot x - 84$ at his return in Pisa, which is finally equal to 0, process which give birth to the forward dynamic scheme



As finally, he has spent all his wealth, one must solve the first order equation $8 \cdot x - 84 = 0$ which give $x = 10.5$. This is how Fibonacci could have simply solve his problem.

But as the sign of a great spirit, he find the solution in following a complete new path, which could certainly be identified as the first backward argument in the history of thought: he actualizes.

He states that, since its capital double at each stop, its initial capital is 1/2 the capital he owns in Lucca, which is 1/4 the capital he owns in Florence, which is 1/8 the capital he owns at is return in Pisa. In terms of the initial capital. That is to say that a denaro spent in his third stay could not be evaluated on the same basis as a dinaro spent in his second and in his first stay. So, 12 dinari spent in his final stay is worth only 1/8 of 12 dinari owned at the departure of the trip, 12 dinari spent in is penultimate stay ar worth 1/4 of the dinary owned at the departure and, 12 dinari spent in his first stay are worth 1/2 of the initial dinari. That is to say that, one could evaluate a flux of dinari at different dates at the departure time. This give :

$$\text{Total Expense} = \frac{12}{2} + \frac{12}{4} + \frac{12}{8} = 10.5 = \text{Initial Capital}$$

One must also remark that, at least by implication, Fibonacci's solution is linked to an anticipatory conscience of double entry bookkeeping since his solution discriminates clearly between capital and expenses in the sequences of accounts

Pisa

Ressources	Expenses
x	

Lucca

Ressources	Expenses
$2 \cdot x$	12
$2 \cdot x - 12$	

Florence

Ressources	Expenses
$2 \cdot (2 \cdot x - 12)$	12
$4 \cdot x - 36$	

Pisa

Ressources	Expenses
$2 \cdot (4 \cdot x - 36)$	12
$8 \cdot x - 84$	

One must note that as simple that those accounts may seem, it was a formidable achievement to write it, since the successives ressources incorporate negative numbers which he has just introduced in an earlier chapter. Now, in an exercice of experimental archeology one can try to reconstruct his argumentation. As one can interpret negative ressource values as vanished earning opportunities, they must slither from the Asset column to be written in the Liabilities column as shown hereunder.

Pisa → Lucca → Florence → Pisa

Pisa

Assets	Liabilities
x	

Lucca

Assets	Liabilities
$2 \cdot x$	12
$2 \cdot x - 12$	

Florence

Assets	Liabilities
$4 \cdot x$	Lucca : $2 \cdot 12$
	Pisa : 12
$4 \cdot x$	36

Pisa

Assets	Liabilities
$8 \cdot x$	Lucca : $4 \cdot 12$
	Florence : $2 \cdot 12$
	Pisa : 12
$8 \cdot x$	84

Under this presentation, it is self-evident that the last Pisa's book is written in terms of accumulated assets, not in terms of initial assets. So Fibonacci must simply have divided all terms by 8 to finally obtain the return to Pisa book in terms of initial assets, that is :

Pisa

Assets	Liabilities	
x	Lucca :	$\frac{1}{2} \cdot 12$
	Florence :	$\frac{1}{4} \cdot 12$
	Pisa :	$\frac{1}{8} \cdot 12$
x		10.5

What a *tour de force* ! For the first time, since human beings tried to maintain the balance sequences of his commercial operations realized in distinct places, Fibonacci, from scratch, shows how to relativize all the entries and express them in initial value entitling the comparison of various complex flux of income. And more than that, to realize this *tour de force*, his mind, at least tacitely, should have been aware of the possibility to keep all the entries by the distinction between income and expenses — *i.e.*: A double-entry bookkeeping operations which, as one will explain later in this paper, will be truly available only two centuries later.

There is no evidence to state if Fibonacci has derived is invention of actualization, but it is more than a conjecture as can be seen from the problems which follow the travelling merchant problem. They are a set of sophisticated banking problems such as :

> *A man placed 100 pounds at a certain [banking] house for 4 denari per pound per month interest and he took back each year a payment of 30 pounds. One must compute in each year the 30 pounds reduction of capital and the profit on the said 30 pounds. It is sought how many years, months, days and hours he will hold money in the house.*

It is worth the effort to follow, helped by the modern apparatus, the effort of Fibonacci to find the solution of his problem. According to the traveller's one, one must reason as follow. First of all, one must find the interest rate and, according to the decomposition *denari*, *soldi*, *lire,* as a lire — a pound — is worth 240 denari, and the return of the investment is 400 denari per month for 12 month, that is to say 4800 denari or 20 lira/year, the rate of return is .2. Now, one has :

$$0 \qquad 100 - \left(\frac{30}{1.2} + \frac{30}{1.2^2} + \frac{30}{1.2^3} + \frac{30}{1.2^4} + \frac{30}{1.2^5} + \frac{30}{1.2^6}\right) = 0.234697$$

$$1 \qquad 1.2 \cdot 100 - 30 = 90$$

$$2 \qquad 1.2 \cdot 90 - 30 = 78$$

$$3 \qquad 1.2 \cdot 78 - 30 = 63.6$$

$$4 \qquad 1.2 \cdot 63.6 - 30 = 46.32$$

$$5 \qquad 1.2 \cdot 46.32 - 30 = 25.584$$

$$6 \qquad 1.2 \cdot 25.584 - 30 = 0.7008$$

So there is a balance. And, as this balance is lower than 30, one knows that the account cannot stay open one more year. So one must reason in days because month are not regular unit. But to reason in days, one must also suppose that, all things equal, all the magnitudes are time homogeneous, that is to say that the man's daily expanses are equal to the 1/365th of his yearly expenses and that the daily interest rate is also simply 1/365th of the yearly one.

This carries to the new table :

$$0 \qquad 0.708 - \left(\frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^2} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^3} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^4} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^5} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^6} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^7} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^8}\right) = 0.641613$$

$$1 \qquad \left(1+\frac{2}{365}\right) \cdot 0.708 - \frac{30}{365} = 0.629688$$

$$2 \qquad \left(1+\frac{2}{365}\right) \cdot 0.629688 - \frac{30}{365} = 0.550946$$

$$3 \qquad \left(1+\frac{2}{365}\right) \cdot 0.550946 - \frac{30}{365} = 0.471773$$

$$4 \qquad \left(1+\frac{2}{365}\right) \cdot 0.471773 - \frac{30}{365} = 0.392167$$

$$5 \qquad \left(1+\frac{2}{365}\right) \cdot 0.392167 - \frac{30}{365} = 0.312124$$

$$6 \qquad \left(1+\frac{2}{365}\right) \cdot 0.312124 - \frac{30}{365} = 0.231642$$

$$7 \qquad \left(1+\frac{2}{365}\right) \cdot 0.231642 - \frac{30}{365} = 0.15072$$

$$8 \qquad \left(1+\frac{2}{365}\right) \cdot 0.15072 - \frac{30}{365} = 0.0693538$$

So, one has found that after 6 years and 8 hours the account will be nearly void — Fibonacci goes further since he displays the exact answer wich is 6 years, 8 days and $\frac{1}{2}\frac{3}{9}5$ hours in his own notation but one renounces to go to this stage not to tire the reader. But what a fantastic operation. It outperform any performed computation before centuries. And from this problem, Fibonacci constructed 11 other examples from which the following has been extracted because it's has been viewed as the founding problem of the modern finance since for the first time in history, not only the present value criterium is applied to discriminate between two apparently identical payment which differs by their sequence. It's the celebrated *On s soldier receiving three hundred bezants for his fief*.

The Fibonacci's story invented to expose this *sui generis* problem is the story of a soldier of whom the King want to reward his service record in granting him an annuity of 300 bezants/year, paid in quarterly installments of 75 bezants. Then the King alters the payment schedule to an annual year-end payment of 300 bezants. Knowing that the soldier is able to earn 2 bezants for one hundred invested bezants, Fibonacci asked if the situation of the soldier is better in the first or in the second schedule.

It's clearly a simple present value problem which can be analyzed in the first year of its payment. In the case where the grant is served in one shot, as the mensual rate of return that is accessible to the soldier est of 2%, the present value is :

$$V_{300} = \frac{300}{1.02^{12}} = 236.548$$

and in the case of the quarterly payment of 75 bezants, one find :

$$V_{75} = 75 + \frac{75}{1.02^4} + \frac{75}{1.02^8} + \frac{75}{1.02^{12}} = 267.437$$

So after alteration of the schedule, it comes that, in present value, the pension has been altered 30.8892 of bezants: The soldier should have been better off, if the king shouldn't have altered the payment schedule. As a by product of this accomplishment, and only as a by product, Fibonacci developed the rabbit population growth problem and as a natural consequence, the first analysis of a geometric sequence of number.

It could be tempting to minimize the stupendous financial accomplishment for Fibonacci pretending there is no news under the sun since capitalisation is an operation which was used before the Hammurabi code that is 18th b.c.. It's a very simple operation which amount to compound the due interests at the expiration date. And, as shown by Goetzmann (2004), one can also be astonished by the incredible connection between the *Liber Abacci* and the babylonian problem exposed in the tablet 8528 conserved in the Berlin museum and published by Neugebauer (1935).

TABLET 8528

If I lent one mina of silver at the rate of 12 shekels (a shekel is equal to 1/60 of a mina) per year, and I received in repayment, one talent (60 minas) and 4 minas. How long did the money accumulate?

LIBER ABACCI

A certain man gave one denaro at interest so that in five years he must receive double the denari, and in another five he must have double two of the denari and thus forever from 5 to 5 years the capital and interest are doubled. It is sought how many denari from this one denaro he must have in 100 years.

If capitalization is necessarily a by-product of the loan operation, as demonstrated by the nearly three thousand rolled years between the first operation of capitalisation and the actualization by Fibonacci, that last operation is linked to far more complex mental schemes.

But, as necessity has a value of law, this great innovation couldn't stay a dead letter. And one knows that it has not been the case. Certainly Fibonacci was a first class teacher who never spared himself in diffusing the new way to calculate without a mechanical support[36] or to promote his new financial instruments. One can support this assertion by three remarks: First, during his life time, precisely in 1228, there has been a second edition of the book, which was an exceptional event for such a book, secondly his fame arrived early to the ears of the Fredrick II[37], emperor of the Holy Roman Empire, whose court mathematician — John of Palermo —

[36]It would be tempting to write here with a pencil and a sheet of paper, but this wouldn't represent Fibonacci's actuality.

[37]It seems that he has read by himself Fibonacci's book.

was charged to ask three problems to him[38], two of them been either later expended in his *Liber Quadratorum* — see Fibonacci & Sigler (1987) — or incorporated in his *Flos* — currently untranslated, at least from the latin transcription of Fibonacci & Boncompagni (1857-1862) — both published in 1225. Third, in his old age, in 1241, Fibonacci receive a pension of Pisa which acknowledged equally his effort in the two distinct field of the eduction of the citizens and dedicated service[39]. That is to say that, during his time life, Fibonacci helped his town in performing fructuous financial operations.

But the effort of a *lone poor mathematician* to transmit a new way to cope with number would have been a waste, if it shouldn't have been relayed by some other means because, first the editing technology of that time was so expensive that only the happy few could have access to books.

A book — initially named a *Codex* — was a technological progress on the *Volumen* — a scroll of papyrus — because it was far more durable than the later. It was design on the same structure that the assembly of wooden and wax tablets used for drafts and provisional texts. But they were also far more costly since the pages were made of parchment, that is to say sheep hide, even if parchment were apt to support reverse writings, support to be fold and sewed in registers.

But, more than that, books were hand written and, since this was very costly time consuming operation, extraordinary costly. For example, Dittmar (2011) yield that in 1383, in England, a scribe was commissioned to write a single service book for the bishop of Westminster. For this work, the scribe was was paid £4, a sum equivalent to 208 days' wages for a

[38]The first problem is a problem looking like those resolved in *Liber Abaci*. The second was to find a rational number r such that both $r^2 - 5$and $r^2 + 5$ are rational squares for which, Fibonacci established that if $r^2 - n$ and $r^2 + n$ are rational squares, then there is a right triangle with rational sides, hypotenuse $2 \cdot r$, and area n, which conduct to the solution $n = 5$. In what concern the third problem — find a root of the cubic equation $x^3 + 2 \cdot x^2 + 10 \cdot x = 20$ —, according to Brown & Brunson (2008), it was borrowed to Omar Khayyams *Al-jabr* — see Kasir (1931-1972). Fibonacci proved that the solution of this cubic equation was neither an integer, nor a rational, nor a number of any of the forms from *Book X* of Euclid's *Elements*. So he decide to approximate it. Unfortunately, he failed to give a good approximation since the number he gives was neither a truncation nor a rounding up of the actual root — for more on that subject see Brown & Brunson (2008) who try to explained why he made this mistake.

[39]See H. Lhüneburg post at `http://www.mathematik.unikl.de/~luene/miszellen/Fibonacci.html`

skilled craftsman knowing that the costs of illustration, binding, and paper were listed separately and that he performed the transcription part time and completed the project over the course of two years. During the manuscript era, books were sufficiently rare and valuable that they were used as collateral on substantial cash loans extended to scholars by Oxford and Cambridge universities — Bell (1937).

Initially produced in the *scriptoria* of the friary, the most part of the codex was devoted to religious books. But, in the 12th-13th century, slowly, things began to change. In the secular world, the extraction of the scholar education from the cathedral and monastic schools to the Universities from Bologna (1088), to Siena (1240), while passing through La Sorbone (1150), Oxford (1167), Palencia (1208), Cambridge (1209), Salamanca (1218), Padua (1218), Montpellier (1220), Naples (1224) and Toulouse (1229), created and expanded markets for law, mathematical, medical... book began to develop due to a small literacy increase, causing the opening of scriptoria in towns but this movement stayed marginal — less that 25 % of the manuscript of this time are devoted to non religious subjects. So, a book like *Liber Abaci* could be written only for a very small audience which was yet ready to receive his message because, to accept to be involved in such a purchase one must have been persuaded that it was worth the expense. So the book was written necessarily for rich merchant and more precisely for those who were called to rule the city.

In what concern the replacement of the roman numerals by the hindu-arabic numerals, the audience was certainly not too difficult to convince, because with the development of exchanges with the arabic world, it went to be secure that the later outshine the former. After all, as recorded by Fibonacci himself, his father, who has been send in Bugia[40] to supervise the pisean commerce with this tunisian town, was persuaded that his son could, to some purpose, find some interest in learning the arabian way of computing. There is no reason to doubt that he was alone of his kind, even if he was the lone to have a son which became able to convert it in a scientific revolution.

[40]Now Bejaia in Algeria. In those times, Bugia was one of the main intellectual centers of the arabic world of equal fame with Sevilla and Toledo. Strikingly, the last crusade which ended under the walls of Tunis after the passing of Louis IX of France by a treatise between Charles d'Anjou and Tunis's emir which bestowed, among other things, to christian the right to free trade in the Tunisia territory. But in that time, Fibonacci was dead since 40 years.

Secondly, toscan merchants involved with mediterranean trade were certainly already ready to cope with the hindu-arabic numeral. Because, if from 1095 to 1291, the crusades failed to hit their main objective, which was, in the time terminology, to free Jerusalem of the faithless, despite the recurrent conflicts which will kept on going between christians and muslims, they will have, as unintended consequences, to secure the wealth of merchant italian cities. Their wealth began by the provision of ships to transport many of the crusaders to the Middle East and set up with the trade perfumes, spices as saffron, jewels, silk, dyes, tapestries, ivory and other products which the Europeans gradually came to value[41].

The croisades influenced also the intellectual development of Europe. Above all they liberalized the minds of the crusaders which were in contact with the leading science from which they had so much to learn. In particular, certain great spirits began to untie from the aristotelian tradition endorsed by the catholic church. One can easily conjecture that, in that manner, the ground was set for merchants to endorse hindu-arabic numerals.

And Fibonacci has been capable to convince his pisean fellow citizens, the inhabitants of the other main toscan and venetian cities and, finally, the world. But this will take more than a century and at the end Pisa would have been absorbed in the florencean *contado*[42]. The more astonishing is that, at least for a contemporaneous literate man, *Liber Abacci* was a big and difficult book whose messages could only be assimilated by peoples ready to set aside the ancient way of doing calculation

The instruments of the diffusion of the Fibonacci's canons, were multiples : in a first time, the *Liber Abaci* litigated for himself toward the educated people and after the approval of the Emperor in 1228, the official start was given to try to transmit the new corpus to merchants.

Apparently, Leonardo was the first to teach and practice his method since, according to the State archives of Pisa for 1233-1241 — see Bonaini (1858) —, among other reasons adduced to the payment by the city of a 20 lira pension, one specifies that he taught the new method with discretion and wisdom.

[41]On that subject and, more precisely, on the role and luck of Pisa in the mediterranean trade, one could consult Tangheroni (2002) and Tangheroni (2003).

[42]During the Middle-Ages, in the north of Italy, the *contado* was the territory under control and dependant of a major city. Pisa, short of its maritime power, will loose its independence in 1406.

If he should have act alone, there would have been no revolution because, in those times, education was restricted to a minimum which was humanities skewed, and whose access was reserved to a few number of peoples. At the primary level, even in Italy, one can say that no things that really matter have changed since, in 789, Charlemagne has enacted that every cathedral and monastery should open schools, reserved to a male assistance, where they could learn grammar, rhetoric, logic, latin, astronomy, philosophy and mathematics — essentially arithmetic and geometry. Essentially, young peoples learned to read and write in latin et vernacular languages — see Durant (1950).

By slow degrees, and for at least four centuries, the first system of commercial private and/or public schools was organized in Tuscany. According to Ulivi (2002), as soon as 1265, a man named Pietro, who should have been a direct pupil of the maestro, appears as witness in a bill of sale. This Pietro had a son and both of them were *maestro d'abaco*. They certainly taught in private school. During the last quarter of the 13th century, one can testify the existence of some communal abaco schools, the oldest testimony coming from the municipality of San Geminiano who hired a Michele in 1279. In 1282, Bologne institutes its own school. In the first rotuli of Bologna University, in 1384,1388 and 1407, one find that a Antonio Bonini Biliotti, a university professor, should lecture in arithmetics, and geometry at the pre-universitary level, a decision which will stay in vitality at least until the middle of the 16th century. In the 15th century, Scipione del Ferro, who was the first to use the imaginary numbers[43], would be one of the *Maestro del Abaco*. A similar situation appears to have been set in Perugia where in 1389 and 1396, it is attested that, near by the secondary school professor of grammar, there was a professor of geometry and abaco. Interestingly,

[43] According to Cardan, del Ferro find the universal solution to second order equations. But et never published it since, in that time the university mathematical pulpit was regularly put in competition, according to a rituel where the resolution of arithmetic problems was mendatory. Dying, he bequests his solution to his son-in-law, Hannibal de Nave, and one of his students Antonio Fior. This last one in competition for a university vacancy with Nicolo Fontana — *Tartaglia* that is to say the stammerer — was unwise in asking this alter one to resolve too much third order equations. Tartaglia guessed the existence of this universal solution and finally found it. Then Cardan, called to care the agonising Tartaglia, convinced him to transmit to him his knowledge. Tartaglia accepted under the strict condition that Cardan kept the solution for himself. Fortunately for us, Cardan doesn't respect his parole, but beeing one of the best physician of his time he really cure Tartaglia who recover his health and the controversy began.

as it can be deduced from Ulivi (2002), the municipality had deliberated a first time in 1277 on the opportunity to create a school but the decision was gained only after the third deliberation in 1285. During their life time, the Verona schools will see the teaching of Tartaglia and of Francesco Feliciano de Lazisio both in the 16th century. It's only in 1399 that Pisa hired two professor for the creation of a local school.

One must also notice that small commercial towns picked out a public school system contrary to the great ones, as Florence or Venezia which opted for a private one. According to Black (2004), Florence never followed the practice of other toscan towns which propounded academic subventions to their citizens. This certainly explain the differential treatment of the professors which was dependent of the fortune of the cities where they taught.

In Venezia, one dispose of testimonies that atteste the presence of *Abaco Schools* since 1305 and, as time goes by, the number of students of those schools could reach a little less of 150. The most famous one was the *Scuola di Rialto*, established in 1408, where instruction covered logic, natural philosophy, theology, astronomy and mathematic, and the teaching was of such good level that it was a kind of foundation courses for being a candidate to the University. And its least title of glory was to have had Luca Pacioli as pupil. Around 1360-62, Leonardo da Vinci received a formation in his birth town just before being send in Florence to enter in Andrea Verrochio's studio. If one doesn't know with precision the birthday date of Piero della Francesca, one knows that, before 1430, he was a pupil of a scuola del abaco.

According to the memories of his father Bernardo, in 1480-1481, Niccolò Machiavelli studied also in such a school — see Machiavelli (1954[$\backsim$ 1488]).

In what concern Florence, for which we have testimonies since 1283, the reputation of the dispense teaching was so high that the city trained the most part of the *maestri del Abaco* who teached in Toscany and Venetia for at least tree centuries. Between, the end of the 13th century and the first third of the 16th century, nearly sixty maestri have been counted which operated in twenty schools[44]. For the world education standards of those times, Toscany appears truly outstanding. According to the chronicler Giovani Villani — see Villani (1906[1341]) —, quoted in Davis (1965), that

[44] Ulivi (2002) gives a very elaborate description of the implantation of the schools in the diverse districts of the town.

approximatively 10 % of Florence's population, that is to say between 8000 and 10000 boys and girls were being taught to read. One fourth of the boys — between 1000 and 1200 — decide to study in one of the six scuola del abaca[45]. The social basis of recruitment was broad from patriciate to smal shopkeepers. And, as noted by Goldthwaite (1972), they learned their lessons well :

> *anyone who is familiar with the complexity of the arithmetic problems of their monetary systems, international exchange rates and other accounting and business practices and who has tested the accuracy of the calculations in their commercial recors can vouch the effectiveness of the training Florentines received in* scuola d'abbaco.

One must also note that the black plague which, in 1348, spread from Genoa to Rome invading Toscany, ravaging Florence to finally reach the papal states, only slowed down the development of the merchant educational system[46]. On the content, duration and organization of the studies in the scuoli, we are pretty well documented, but one throw back to Goldthwaite (1972), Ulivi (2002) and Black (2007). One must shortly underline that those schools were perceived as high school accessible with the prerequisite of the attendance to a primary cycle where children learned to read and to write. In average, the school days began at 10 or 11 years and last two days but according to the competence and abilities of the child it could variegate accordingly.

After the school, they enter as novice in the trade but not only because swiftly others corporations find that the formation dispensed in the scuoli could represent a good prequisit. Even if geometry had a very small part in the teaching of the scuoli, this part was sufficient and largely outperform what was taught on that subject in other places all over Europa. For instance, Zervas (1975) argues convincingly that the design of the florentine baptistery door of Santa Maria del Fiore by Andrea Pisano owe much to the abaco teachings.

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[45]The others, either came in one of the fourth latin school to study grammar and logic — nearly one eight —, in rethoric school, other in canon law school — certainly the cathedral school, and the last part, was taught in notarial schools. This crumbling of the formation was certainly linked to the fact that, in 1321, Florence was fighting to found a university.

[46]During the black plague the most part of the great cities had lost more than 40 % of their inhabitants. Florence lost more of the third of its population in the first six month, and between 45 % and 75 % in the first year.

Nevertheless for a life expectancy of less than 25 years to enter in the trade at the age of thirteen years after two years of formations was the most than can be demanded in those times.

To help them in their teachings, some of the maestri departed from Fibonacci and had draft new books.

As more and more clergymen became mathematical educated, helped by the fact that Dominicans, who were more involved in education than older orders, attracted more and more poor young peoples eager to become monks, the mathematical education raised at all levels even if the used methods of teaching conducts more to rote learning than true understanding[47].

But, during all those times, the far most advanced mathematical teaching was done by the trade guilds. The apprenticeship lasted seven years : a master of a trade then taught to an apprentice all he thought he should know.

Of course, those studies were purely practical and no apprentice could have learned more that what artisans and merchants could teach classes had he wanted to. Architect and builders but also merchants and traders and in the big cities the early forms of money landers use to learn geometry and arithmetics.

So from the trades, because they imposed the study of mathematics to their practitioners, came the conditions for a rapid advance in accounting technology which will be achieved during the Renaissance.

We must underline also that, if in 976 the *Codex Albelendsis seu Vigilanus* was the first document to use arabic numeral, it take nearly 300 years before Leonardo Fibonacci, in his *Liber Abbaci*, advocated their universal use in replacement of the roman numerals[48]. But, and this could explain why trades used them so early, the spreading of their use come also as a by product of teh development of the oriental trades generated by the crusades and perhaps from the play of cards, whose origin is uncertain but likely situated in orient or in the middle-east. In the primitive literature on the subject, it is postulated that they were introduced in Europe by the arabs.

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[47]Pupils were obliged to learn the question they could ask to the teacher and to learn also the answers.

[48]The importance of the Fibonacci advocacy was early recognized by the italian cities. As a proof, he receives a permanent income from Pisa, certainly to teach arithmetics.

From playing card with arabic numeral to bookkeeping with the same medium the step was natural for merchants who rapidly could not use two counting device, and the rise of the commercial relation of cities like Venezia or Genova was a good incentive to develop a new accounting system.

This new accounting system was finally explained and advocated by Luca Pacioli whose fame is shown in its enigmatic portrait[49]. As a Renaissance man, Pacioli accepted the interrelatedness of all the subject under study in his time : religion, business, military science, mathematics, medicine, art, music, law, language. He finally acquired a high level of knowledge in all those fields but he cherished the most those which exhibited harmony and balance as mathematics and accounting — see Alexander (2002).

If Pacioli is often credited to be the father of double entry bookkeeping, he never claimed for himself its invention. The most part of the authors who write on this subject argued that double entry bookkeeping was a common practice since the 13th century italian cities but there are rare exceptions as Kats (1930) who convincingly argued that it was a common practice in Rome long before this time or Lauwers & Willekens (1994) who refers to Colt (1844) whose thesis is that italians pick up their knowledge of double entry bookkeeping at Alexandria, Constantinople or some other eastern cities[50].

But many accounting historians, as Roover (1955), do not accept double registration of a transaction, on time in credit and one time in debit, as a sufficient condition to qualify the accounting system under study as a double entry system. For instance, de Roover insists on the fact that all transactions be twice recorded. *"This principle involves the existence of an integrated system of accounts, both real and nominal, so that the books will balance in the end, record changes in the owner's equity and permit the determination of*

[49]For an analysis of the portrait painted by Jacopo de Barbari in 1495 which is exposed in many internet sites, it is worth to read McKinnon (1993).

[50]Recent investigations tend to confirm this hypothesis. For instance, Albraiki (1994) proves that at the beginning of the Mamluk period between 1250 and 1517, double entry book keepin was already in use in Egypt and Syria. More than that, the old *Cairo geniza collection* — *geniza* meaning burial —, a collection of more than 200000 fragments found in the Ben Ezra synagogue during its restoration in 1890, contains a fragment dated from 1080 in the form of a journal and a four page account dated from 1134 listing both credit and debits — see Scorgie (1994).

profit and loss".

With this in mind, according to Roover (1955), the oldest discovered record of a compete double-entry system is the one of the *Messari* — the *treasurers* — accounts of the city of Genoa in 1340 because not only it contains debits and credits journalised in a bilateral form, but each transaction is recorded twice in the ledger.

Pacioli himself credited the first description of the system to Benedetto Cotrugli, a Raguze merchant, who has written a book *Delai Mercatura et del Mercante Perfetto* — *Of Trading and the Perfect Trader* — which has been published a century later, but of which he was familiar.

The *Summa de Arithmetica, Geometria, Proportioni et Proportionality* — *Everything about Arithmetic, Geometry and Proportion* — has been written as a digest and guide to existing mathematical knowledge and bookkeeping was only one of the five topics covered. The presence of bookkeeping in this master piece and the fact that 37 short chapters entitled *De Computis and Scripturis,* were devoted to its study, acknowledged the fact that for Pacioli it was a major and perfectly legitimate mathematical subject for his time, a true algebraic application.

But, not only for Pacioli was it a major subject. The proof come from the fact that it has not circulated as manuscript copies but as printed copies — as soon as it has been finished, it has been directly printed in 1494 from the Paganino de Paganini printing house. Yet, even if the fact to print drive the reproduction price to a low level in comparison to the manual copy, in that times, printing could be achieved only to a high cost, and as the invention was only in its thirteen decade, it must have been a recognized urgency, to print such a book when so much manuscript where waiting to be printed. In all the cases, the book is an *incunabulum* that is to say a immeasurable value book printed when printing was still in its cradle.

More than that, in choosing to print his master piece in Venice where he came to accomplish this project which is demonstrated by the fact that since 1486, he visited main courts and lectured mathematics at various Italian universities such as Perugia, Florence, Rome and Napels where he taught also military science when he occupied no recorded function in Venice, Pacioli was certainly aware of the fact that he could discuss with some masters in accounting in the case where it happen to be necessary, and that it was the only place in the world where he was covered by some author rights since they have been invented by the Serenissima in 1474, even if it was only for the venetian states and for a 10 year duration.

Even if this copyright give no protection against the copy by hands, there is no known hand copy of the *Summa*. It could seems strange because there was an abundance of scribes, whose work will last until the end of the 16th century because the strong resistance from some bibliophiles who preferred to possess an handwritten unique manuscript. According to Sangster, Stoner & McCarthy (2007), the lack of pirated copies of the *Summa* could be explained in part by the presence of diagrams and marginal notes which could make the copy relatively unattractive.

And, as Padova was in the venetian states, and because in Padova the University was independent of the Papacy, liberal fields were taught opening a large market for the book which nevertheless has been written mainly for the merchants[51],[52].

The commercial success of the book was such that it was printed two times the same year and that his publisher Paganino de Paganini signed again Pacioli for two others books which were published in 1509[53]. In 1523, the son of Paganini published a new edition of the book.

In what concern the number of printed copies of the *Summa* a first estimation of 300 printed copies by Antinori (1980) has been disallowed by Sangster et al. (2007) on the basis that it does not take into account the size of the print-runs of the late 15th century. On this basis, in their opinion it would be reasonable to infer that at least 500 copies were printed. But other factors, indicate that a greater number of copy could have been printed.

First of all some pages have been independently printed. It was the case in 1502 and in 1509 certainly, in the case of the first date, avoid the expiration of the 10-year copyright and in the case of the second one to take advantage of a 15 year copyright witch have been granted to Pacioli himself.

After some very convincing arguments, partially coming from the fact that the editor financed the publication of the 1523 second edition, Sangster et al. (2007) arrive to the conclusion that more than 1000 and perhaps up to 2000 copies of the *Summa* where sell.

[51]Sangster et al. (2007) argue that because the book has no worked examples, he has been written for merchants who need not them.

[52]

[53]*De divina proportione* which was controversial in the sense where the third part is a translation in italian of the treatise on the five regular solid written by Pierro della Francesca and the translation of Euclid's *Elements*.

2.2 The accountancy group

In accountancy an double-entry account can be defined as an ordered pair of number $(d, c) \in \mathbb{Z}^2$ where c is the credit and d is a debit.

$$(d, c) = \begin{array}{c|c} \text{Debit} & \text{Credit} \\ \hline d & c \end{array}$$

In a newtonian accountancy, the gap between c and d is the balance or, in a more economic mood, the profit. In a newtonian world, we can add two accounts in such a way that if (d_1, c_1) is the first account and (d_2, c_2) is the second, we have : $(d_1, c_2) + (d_2, c_2) = (d_1 + d_2, c_1 + c_2)$. We can also add three accounts in such a way that if the first is (d_1, c_1), the second (d_2, c_2), and the third (d_3, c_3) we will have $((d_1, c_1) + (d_2, c_2)) + (d_3, c_3) = (d_1, c_1) + ((d_2, c_2) + (d_3, c_3))$. We can also remark that as $(0, 0) \in \mathbb{Z}^2$, $(d_1, c_1) + (0, 0) = (d_1, c_1)$. In other terms, in accountancy, $(0, 0)$ a neutral element.

Since if $(d_1, c_1) \in \mathbb{Z}^2$, $-(d_1, c_1) \in \mathbb{Z}^2$, in accountancy each accountancy has obviously an inverse because $(d_1, c_1) + [-(d_1, c_1)] = (0, 0)$. And last but not least, the order in which one adds the accounts has no consequences because $(d_1, c_1) + (d_2, c_2) = (d_2, c_2) + (d_1, c_1)$.

In the mathematical terminology, an euclidian accountancy is associative, it possess a neutral element and each account has an inverse and the addition of two accounts is commutative. In short, an accountancy is what mathematicians call an *additive abelian group*.

This as been noticed by Ellerman (1986) and incidentally by Lim (1966). Ellerman call it the *Pacioli group*[54]. As he remarks, a century earlier, the great Arthur Cayley — voir Cayley (1984[1896]) — has noticed by himself that if mathematicians do not seriously looked to accountancy as a mathematical object it is because of its apparent simplicity[55],[56]. Cayley was also the first to linked double-entry book-keeping to the euclidian ratios :

[54] Ellerman use the notation $[c \| d]$ because it seems that it has been suggested by Pacioli himself.

[55] Cayley has been a lawyer for 14 years.

[56] Augustus DeMorgan is the only other great mathematician who was interested in accountancy — see DeMorgan (1869).

The Principles of Book-keeping by Double Entry constitutes a theory which is mathematically by no means uninteresting: it is in fact Euclid's theory of ratios an absolutely perfect one, and it is only its extreme simplicity which prevents it from being a interesting as it would be otherwise — Cayley (1984[1896])

But each mathematical presentation of group theory display conjointly an additive and a multiplicative group. So why is there no non-newtonian accountancy based on the multiplicative one. Even there is only one likely answer to this question which is to say that our brain is delivered only to compute additions and substractions, all other operations being acquired which give a great advantage to addition over multiplication, one must study the possibility of a multiplicative accountancy on the basis that the conception of the modern accountancy system as been a very time consuming process and that nothing could have prevent this system to have a distinct look from the one it finally takes.

If we call multiplicative accountancy an **accountancy*, it must operate with ordered pairs $(d, c) \in \mathbb{N}^\star \times \mathbb{N}^\star$. In this non-newtonian system the ratio between debit and credit — *i.e.* : d/c — becomes the profit. We can multiply two accounts in such a way that if (d_1, c_1) is the first account and (d_2, c_2) is the second, we will have $(d_1, c_1) \times (d_2, c_2) = (d_1 d_2, c_1 c_2)$. We can also multiply three accounts (d_1, c_1), (d_2, c_2) and (d_3, c_3) and find that this is an associative operation — *i.e.* : $((d_1, c_1) \times (d_2, c_2)) \times (d_3, c_3) = (d_1, c_1) \times ((d_2, c_2) \times (d_3, c_3))$. The *accountancy is also commutative since $(d_1, c_1) \times (d_2, c_2) = (d_2, c_2) \times (d_1, c_1)$.The account $(1, 1) \in \mathbb{N}^\star \times \mathbb{N}^\star$ plays the role of the neutral element for the *accountancy since $(d, c) \times (1, 1) = (d, c)$. Each account (d, c) has an inverse since $(d, c) \times (d^{-1}, c^{-1}) = (1, 1)$.

So as the newtonian accountancy is an abelian group, the non-newtonian *accountancy is also an abelian group — Ellerman (2010) call it a *Pacioli multiplicative group* by contrast with the newtonian accountancy which is a *Pacioli additive group*.

3 Conclusion

Taking into account that it could be shown that ratios better compare two positive quantities than differences, which has been discussed by many Renaissance scholars including Galileo — Grossman & Katz (1972) — and that from this it follows that growth phenomenon are better described by the multiplicative point of view than by the additive one, one can consider that the ⋆accountancy approach could at least been used to express balance sheets in the analysis of the economic growth. It's what we have tried to do in Filip & Piatecki (2011).

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