



**HAL**  
open science

## Bell polynomials in combinatorial Hopf algebras

Ammar Aboud, Jean-Paul Bultel, Ali Chouria, Jean-Gabriel Luque, Olivier Mallet

► **To cite this version:**

Amarr Aboud, Jean-Paul Bultel, Ali Chouria, Jean-Gabriel Luque, Olivier Mallet. Bell polynomials in combinatorial Hopf algebras. 2015. hal-00945543v2

**HAL Id: hal-00945543**

**<https://hal.science/hal-00945543v2>**

Preprint submitted on 8 Apr 2015 (v2), last revised 21 Jan 2016 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Bell polynomials in combinatorial Hopf algebras

Ammar Aboud\*, Jean-Paul Bultel†, Ali Chouria‡  
Jean-Gabriel Luque§ and Olivier Mallet¶

April 3, 2015

**Keywords:** Bell polynomials, Symmetric functions, Hopf algebras, Faà di Bruno algebra, Lagrange inversion, Word symmetric functions, Set partitions.

## Abstract

Partial multivariate Bell polynomials have been defined by E.T. Bell in 1934. These polynomials have numerous applications in Combinatorics, Analysis, Algebra, Probabilities *etc.* Many of the formulæ on Bell polynomials involve combinatorial objects (set partitions, set partitions into lists, permutations *etc.*). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed with these objects. In this paper we investigate the connexions between Bell polynomials and several combinatorial Hopf algebras: the Hopf algebra of symmetric functions, the Faà di Bruno algebra, the Hopf algebra of word symmetric functions *etc.* We show that Bell polynomials can be defined in all these algebras and we give analogues of classical results. To this aim, we construct and study a family of combinatorial Hopf algebras whose bases are indexed by colored set partitions.

## Abstract

Partial multivariate Bell polynomials have been defined by E.T. Bell in 1934. These polynomials have numerous applications in Combinatorics, Analysis, Algebra, Probabilities *etc.* Many of the formulæ on Bell polynomials involve combinatorial objects (set partitions, set partitions into lists, permutations *etc.*). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed with these objects. In this paper we investigate the connexions between Bell polynomials and several combinatorial Hopf algebras: the Hopf algebra of symmetric functions, the Faà di Bruno algebra, the Hopf algebra of word symmetric functions *etc.* We show that Bell polynomials can be defined in all these algebras and we give analogues of classical results. To this aim, we construct and study a family of combinatorial Hopf algebras whose bases are indexed by colored set partitions.

---

\*aboudam@gmail.com. USTHB, Faculty of Mathematics, Po. Box 32 El Alia 16111 Algiers, Algeria.

†jean-paul.bultel@univ-rouen.fr. Laboratoire LITIS - EA 4108, Université de Rouen, Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex

‡ali.chourial@univ-rouen.fr. Laboratoire LITIS - EA 4108, Université de Rouen, Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex

§jean-gabriel.luque@univ-rouen.fr. Laboratoire LITIS - EA 4108, Université de Rouen, Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex

¶olivier.mallet@univ-rouen.fr. Laboratoire LITIS - EA 4108, Université de Rouen, Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex

# 1 Introduction

Partial multivariate Bell polynomials (Bell polynomials for short) have been defined by E.T. Bell in [3] in 1934. But their name is due to Riordan [23] which studied the Faà di Bruno formula [10, 11] allowing one to write the  $n$ th derivative of a composition  $f \circ g$  in terms of the derivatives of  $f$  and  $g$  [22]. The applications of Bell polynomials in Combinatorics, Analysis, Algebra, Probabilities *etc.* are so numerous that it should be very long to detail them in the paper. Let us give only a few seminal examples.

- The main applications to Probabilities follow from the fact that the  $n$ th moment of a probability distribution is a complete Bell polynomial of the cumulants.
- Partial Bell polynomials are linked to the Lagrange inversion. This follows from the Faà di Bruno formula.
- Many combinatorial formulæ on Bell polynomials involve classical combinatorial numbers like Stirling numbers, Lah numbers *etc.*

The Faà di Bruno formula and many combinatorial identities can be found in [6]. The PhD thesis of M. Mihoubi [20] contains a rather complete survey of the applications of these polynomials together with numerous formulæ.

Some of the simplest formulæ are related to the enumeration of combinatorial objects (set partitions, set partitions into lists, permutations *etc.*). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed by these objects. Combinatorial Hopf algebras are graded bibebras with bases indexed by combinatorial objects such that the product and the coproduct have some compatibilities. The graduation implies that the Hopf structure is equivalent to the fact that the coproduct is a morphism for the product.

In fact, most of the identities on Bell polynomials can be obtained by manipulating generating functions and are closely related to some other identities occurring in literature. Typically, the relation between the complete Bell polynomials  $A_n(a_1, a_2, \dots)$  and the variables  $a_1, a_2, \dots$  is very closely related to the Newton Formula which links the generating functions of complete symmetric functions  $h_n$  (Cauchy series) to those of the power sums  $p_n$ . The symmetric functions form a commutative algebra  $Sym$  freely generated by the complete functions  $h_n$  or the power sum functions  $p_n$ . So, specializing the variable  $a_n$  to some numbers is equivalent to specializing the power sum functions  $p_n$ . More soundly, the algebra  $Sym$  can be endowed with coproducts conferring to it a structure of Hopf algebra. For instance, the coproduct for which the power sums are primitive turns  $Sym$  into a self-dual Hopf algebra. The coproduct can be translated in terms of generating functions by a product of two Cauchy series. This kind of manipulations appears also in the context of Bell polynomials, for instance when computing the complete Bell polynomials of the sum of two sequences of variables  $a_1 + b_1, a_2 + b_2, \dots$ . Another coproduct turns  $Sym$  into a non-cocommutative Hopf algebra called the Faà di Bruno algebra which is related to the Lagrange inversion. Finally, the coproduct such that the power sums are group like can be related also to a few other formulæ on Bell polynomials. The aim of Section 2 is to investigate these connexions.

Section 3 is devoted to the study of the Hopf algebras of colored set partitions. After having introduced this family of algebras, we give some special cases which can be found in the literature. The main application shows how to factorize the specialization  $Sym \rightarrow \mathbb{C}$  through this algebra. This explains that we can recover some identities on Bell polynomials when the variables are specialized to combinatorial numbers from analogous identities in some combinatorial Hopf algebras. We show that the algebra **WSym** of word symmetric functions has an important role for this construction. Finally, in Section 4, we give a few analogues of complete and partial Bell polynomials in **WSym**,  $\Pi QSym = \mathbf{WSym}^*$  and  $\mathbb{C}\langle A \rangle$  and investigate their main properties.

## 2 Bell polynomials and symmetric functions

The aim of this section is to restate some known results in terms of symmetric functions and virtual alphabets. We give also a few new results that are difficult to prove without the help of symmetric functions.

### 2.1 Definitions and basic properties

The (complete) *Bell polynomials* [6, p133] are usually defined on an infinite set of commuting variables  $\{a_1, a_2, \dots\}$  by the following generating function:

$$\sum_{n \geq 0} A_n(a_1, \dots, a_p, \dots) \frac{t^n}{n!} = \exp\left(\sum_{m \geq 1} a_m \frac{t^m}{m!}\right) \quad (1)$$

and the *partial Bell polynomials* are defined by

$$\sum_{n \geq 0} B_{n,k}(a_1, \dots, a_p, \dots) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{m \geq 1} a_m \frac{t^m}{m!}\right)^k. \quad (2)$$

So, one has

$$A_n(a_1, a_2, \dots) = \sum_{k=1}^n B_{n,k}(a_1, a_2, \dots), \forall n \geq 1 \text{ and } A_0(a_1, a_2, \dots) = 1. \quad (3)$$

Let  $S_{n,k}$  denote the Stirling number of the second kind which counts the number of ways to partition a set of  $n$  objects into  $k$  nonempty subsets. The following identity holds

$$B_{n,k}(1, 1, \dots) = S_{n,k}. \quad (4)$$

Note also that  $A_n(x, x, \dots) = \sum_{k=0}^n S_{n,k} x^k$  is the classical univariate Bell polynomial denoted by  $\phi_n(x)$  in [3]. Several other identities involve combinatorial numbers. For instance, one has

$$B_{n,k}(1!, 2!, 3!, \dots) = \binom{n-1}{k-1} \frac{n!}{k!}, \text{ Unsigned Lah numbers } \underline{\text{A105278}} \text{ in [24]}, \quad (5)$$

$$B_{n,k}(1, 2, 3, \dots) = \binom{n}{k} k^{n-k}, \text{ Idempotent numbers } \underline{\text{A059297}} \text{ in [24]}, \quad (6)$$

$$B_{n,k}(0!, 1!, 2!, \dots) = |s_{n,k}|, \text{ Stirling numbers of the first kind } \underline{\text{A048994}} \text{ in [24]}. \quad (7)$$

We can also find many other examples in [3, 6, 19, 27, 21].

**Remark 2.1.** Without loss of generality, when needed, we will suppose  $a_1 = 1$  in the remainder of the paper. Indeed, if  $a_1 \neq 0$ , then the generating function gives

$$B_{n,k}(a_1, \dots, a_p, \dots) = a_1^k B_{n,k}\left(1, \frac{a_2}{a_1}, \dots, \frac{a_p}{a_1}\right) \quad (8)$$

and when  $a_1 = 0$ ,

$$B_{n,k}(0, a_2, \dots, a_p, \dots) = \begin{cases} 0 & \text{if } n < k \\ \frac{n!}{(n-k)!} B_{n,k}(a_2, \dots, a_p, \dots) & \text{if } n \geq k. \end{cases} \quad (9)$$

## 2.2 Bell polynomials as symmetric functions

The algebra of symmetric functions [18, 17] is isomorphic to its polynomial realization  $Sym(\mathbb{X})$  on an infinite set  $\mathbb{X} = \{x_1, x_2, \dots\}$  of commuting variables, so the algebra  $Sym(\mathbb{X})$  is defined as the set of polynomials invariant under permutation of the variables. As an algebra,  $Sym(\mathbb{X})$  is freely generated by the power sum symmetric functions  $p_n(\mathbb{X})$ , defined by  $p_n(\mathbb{X}) = \sum_{i \geq 1} x_i^n$  or the complete symmetric functions  $h_n$ , where  $h_n$  is the sum of all the monomials of total degree  $n$  in the variables  $x_1, x_2, \dots$ . The generating function for the  $h_n$ , called *Cauchy formula*, is

$$\sigma_t(\mathbb{X}) = \sum_{n \geq 0} h_n(\mathbb{X}) t^n = \prod_{i \geq 1} (1 - x_i t)^{-1}. \quad (10)$$

The relationship between the two families  $(p_n)_{n \in \mathbb{N}}$  and  $(h_n)_{n \in \mathbb{N}}$  is described in terms of generating series by the *Newton formula*:

$$\sigma_t(\mathbb{X}) = \exp\left\{\sum_{n \geq 1} p_n(\mathbb{X}) \frac{t^n}{n}\right\}. \quad (11)$$

Notice that  $Sym$  is the free commutative algebra generated by  $p_1, p_2, \dots$  *i.e.*  $Sym = \mathbb{C}[p_1, p_2, \dots]$  and  $Sym(\mathbb{X}) = \mathbb{C}[p_1(\mathbb{X}), p_2(\mathbb{X}), \dots]$  when  $\mathbb{X}$  is an infinite alphabet without relations on the variables. As a consequence of the Newton Formula (11), it is also the free commutative algebra generated by  $h_1, h_2, \dots$ . The freeness of the algebra provides a mechanism of specialization: that is, for any sequence of commutative scalars  $u = (u_n)_{n \in \mathbb{N}}$ , there is a morphism of algebra  $\phi_u$  sending each  $p_n$  to  $u_n$  (resp. sending  $h_n$  to a certain  $v_n$  which can be deduced from  $u$ ). These morphisms are manipulated *as if* there exists an underlying alphabet (so called *virtual alphabet*)  $\mathbb{X}_u$  such that  $p_n(\mathbb{X}_u) = u_n$  (resp.  $h_n(\mathbb{X}_u) = v_n$ ). The interest of such a vision is that one defines operations on sequences and symmetric functions by manipulating alphabets.

The bases of  $Sym$  are indexed by the *partitions*  $\lambda \vdash n$  of all the integers  $n$ . A partition  $\lambda$  of  $n$  is a finite noncreasing sequence of positive integers  $(\lambda_1 \geq \lambda_2 \geq \dots)$  such that  $\sum_i \lambda_i = n$ . The *multiplicity*  $m_j(\lambda)$  of  $j$  in  $\lambda \vdash n$  is the number of parts of  $\lambda$  equal to  $j$ , the *length*  $\ell(\lambda)$  of  $\lambda$  is its number of parts, and  $n$  is the *weight*  $|\lambda|$  of  $\lambda$ . We set also  $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$ . Let  $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k}$  for any partition  $\lambda = [\lambda_1, \dots, \lambda_k]$ .

Given two alphabets  $\mathbb{X}$  and  $\mathbb{Y}$ , we also define (see *e.g.* [17]) the alphabet  $\mathbb{X} + \mathbb{Y}$  by:

$$p_n(\mathbb{X} + \mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y}) \quad (12)$$

and the alphabet  $\alpha\mathbb{X}$  (resp  $\mathbb{X}\mathbb{Y}$ ), for  $\alpha \in \mathbb{C}$  by:

$$p_n(\alpha\mathbb{X}) = \alpha p_n(\mathbb{X}) \quad (\text{resp } p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})). \quad (13)$$

In terms of Cauchy functions, these transforms imply

$$\sigma_t(\mathbb{X} + \mathbb{Y}) = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y}) \quad (14)$$

and

$$\sigma_t(\mathbb{X}\mathbb{Y}) = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda(\mathbb{X}) p_\lambda(\mathbb{Y}) t^{|\lambda|}. \quad (15)$$

In fact the  $\sigma_t(\mathbb{X}\mathbb{Y})$  encodes the kernel of the scalar product defined by  $\langle p_\lambda, c_\mu \rangle = \delta_{\lambda, \mu}$  with  $c_\lambda = \frac{p_\lambda}{z_\lambda}$ . Notice that  $c_n = \frac{p_n}{n}$  and

$$Sym = \mathbb{C}[c_1, c_2, \dots]. \quad (16)$$

From (1) and (11), we obtain

**Proposition 2.2.**  $h_n = \frac{1}{n!} A_n(1!c_1, 2!c_2, \dots)$ .

Conversely, Equality (16) implies that the morphism  $\phi_a$  sending each  $c_i$  to  $\frac{a_i}{i!}$  is well defined for any sequence of numbers  $a = (a_i)_{i \in \mathbb{N} \setminus \{0\}}$  and  $\phi_a(h_n) = \frac{1}{n!} A_n(a_1, a_2, \dots)$ . Let us define also  $h_n^{(k)}(\mathbb{X}) = [\alpha^k] h_n(\alpha \mathbb{X})$ . From (11) and (13) we have

$$h_n^{(k)} = \sum_{\lambda = [\lambda_1, \dots, \lambda_k] \vdash n} c_\lambda = [t^n] \frac{1}{k!} \left( \sum_{i \geq 1} c_i t^i \right)^k$$

and so, all works as if we use a special (virtual) alphabet  $\mathbb{X}^{(a)}$  satisfying  $c_n(\mathbb{X}^{(a)}) = n! a_n$ . More precisely:

**Proposition 2.3.**

$$\phi_a(h_n^{(k)}) = h_n^{(k)}(\mathbb{X}^{(a)}) = \frac{1}{n!} B_{n,k}(a_1, \dots, a_k, \dots). \quad (17)$$

**Example 2.4.** Let  $\mathbf{1}$  be the virtual alphabet defined by  $c_n(\mathbf{1}) = \frac{1}{n}$  for each  $n \in \mathbb{N}$ . In this case the Newton Formula gives  $h_n(\mathbf{1}) = 1$ . Hence  $A_n(0!, 1!, 2!, \dots, (m-1)!, \dots) = n!$  and  $B_{n,k}(0!, 1!, 2!, \dots, (m-1)!, \dots) = n! [\alpha^k] [t^n] \left( \frac{1}{1-t} \right)^\alpha = s_{n,k}$ , the Stirling number of the first kind.

**Example 2.5.** A more complicated example is treated in [1, 16] where  $a_i = i^{i-1}$ . In this case, the specialization gives  $\sigma_t(\alpha \mathbb{X}^{(a)}) = \exp\{-\alpha W(-t)\}$  where  $W(t) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{t^n}{n!}$  is the Lambert  $W$  function satisfying  $W(t) \exp\{W(t)\} = t$  (see *e.g.* [5]). Hence,  $\sigma_t(\alpha \mathbb{X}^{(a)}) = \left( \frac{W(-t)}{-t} \right)^\alpha$ . But the expansion of the series  $\left( \frac{W(t)}{t} \right)^\alpha$  is known to be:

$$\left( \frac{W(t)}{t} \right)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \alpha(\alpha+n)^{n-1} (-t)^n. \quad (18)$$

Hence, we obtain  $B_{n,k}(1, 2, 3^2, \dots, m^{m-1}, \dots) = \binom{n-1}{k-1} n^{n-k}$ . Note that the expansion of  $W(t)$  and (18) are usually obtained by the use of the Lagrange inversion.

**Example 2.6.** With these notations we have  $B_{n,k}(a_1 + b_1, \dots) = \frac{1}{n!} h_n^{(k)}(\mathbb{X}^{(a)} + \mathbb{X}^{(b)})$ , and classical properties of Bell polynomials can be deduced from symmetric functions through this formalism. For instance, the equalities  $c_n(\mathbb{X}^{(a)} + \mathbb{X}^{(b)}) = c_n(\mathbb{X}^{(a)}) + c_n(\mathbb{X}^{(b)})$  and  $h_n(\mathbb{X}^{(a)} + \mathbb{X}^{(b)}) = \sum_{i+j=n} h_i(\mathbb{X}^{(a)}) h_j(\mathbb{X}^{(b)})$  give

$$A_n(a_1 + b_1, \dots) = \sum_{i+j=n} \binom{n}{i} A_i(a_1, a_2, \dots) A_j(b_1, b_2, \dots)$$

and

$$B_{n,k}(a_1 + b_1, \dots) = \sum_{r+s=k} \sum_{i+j=n} \binom{n}{i} B_{i,r}(a_1, a_2, \dots) B_{j,s}(b_1, b_2, \dots).$$

**Example 2.7.** Another example is given by

$$A_n(1a_1 b_1, 2a_2 b_2, \dots, ma_m b_m, \dots) = n! \sum_{\lambda \vdash n} \det \left| \frac{A_{\lambda_i - i + j}(a_1, a_2, \dots)}{(\lambda_i - i + j)!} \right| \times \det \left| \frac{A_{\lambda_i - i + j}(b_1, b_2, \dots)}{(\lambda_i - i + j)!} \right|,$$

using the convention  $A_{-n} = 0$  for  $n > 0$ . This formula is an emanation of the Jacobi-Trudi formula and is derived from the Cauchy kernel (15), remarking that  $c_n(\mathbb{X}^{(a)} \mathbb{X}^{(b)}) = n c_n(\mathbb{X}^{(a)}) c_n(\mathbb{X}^{(b)})$  and

$$h_n(\mathbb{X}^{(a)} \mathbb{X}^{(b)}) = \sum_{\lambda \vdash n} s_\lambda(\mathbb{X}^{(a)}) s_\lambda(\mathbb{X}^{(b)}) = \sum_{\lambda \vdash n} \det \left| h_{\lambda_i - i + j}^{(a)}(\mathbb{X}^a) \right| \det \left| h_{\lambda_i - i + j}^{(b)}(\mathbb{X}^b) \right|,$$

where  $s_\lambda = \det \left| h_{\lambda_i - i + j}^{(a)} \right|$  is a Schur function (see *e.g.*[18]).

### 2.3 Other interpretations

First we focus on the identity (2) and we interpret it as the Cauchy function  $\sigma_t(k\hat{\mathbb{X}}^{(a)})$  where  $\hat{\mathbb{X}}^{(a)}$  is the virtual alphabet such that  $h_{i-1}(\hat{\mathbb{X}}^{(a)}) = \frac{a_i}{i!}$ . This means that we consider the morphism  $\hat{\phi}_a : \text{Sym} \rightarrow \mathbb{C}$  sending  $h_i$  to  $\frac{a_{i+1}}{(i+1)!}$ . We suppose that  $a_1 = 1$  otherwise we use (8) and (9). With these notations we have

**Proposition 2.8.**

$$B_{n,k}(a_1, a_2, \dots) = \frac{n!}{k!} h_{n-k}(k\hat{\mathbb{X}}^{(a)}). \quad (19)$$

**Example 2.9.** If  $a_i = i$  we have  $h_i(\hat{\mathbb{X}}^{(a)}) = \frac{1}{i!}$  and so  $\sigma_t(k\hat{\mathbb{X}}^{(a)}) = \exp(kt)$ . Hence, we recover the classical result

$$B_{n,k}(1, 2, \dots, m, \dots) = \binom{n}{k} k^{n-k}.$$

From  $h_n(\mathbb{X} + \mathbb{Y}) = \sum_{i+j=n} h_i(\mathbb{X})h_j(\mathbb{Y})$  we deduce two classical identities:

$$\binom{k_1 + k_2}{k_1} B_{n, k_1 + k_2}(a_1, a_2, \dots) = \sum_{i=0}^n \binom{n}{i} B_{i, k_1}(a_1, a_2, \dots) B_{n-i, k_2}(a_1, a_2, \dots) \quad (20)$$

and

$$\binom{n}{k} B_{n-k, k} \left( a_1 b_1, \dots, \frac{1}{m+1} \sum_{i=1}^m \binom{m+1}{i} a_i b_{m+1-i}, \dots \right) = \sum_{i=k}^{n-k} \binom{n}{i} B_{i, k}(a_1, a_2, \dots) B_{i, k}(b_1, b_2, \dots). \quad (21)$$

Indeed, formula (20) is obtained by setting  $\mathbb{X} = k_1 \hat{\mathbb{X}}^{(a)}$  and  $\mathbb{Y} = k_2 \hat{\mathbb{X}}^{(a)}$ . Formula (21) is called convolution formula for Bell polynomials (see *e.g.* [20]) and is obtained by setting  $\mathbb{X} = \hat{\mathbb{X}}^{(a)}$  and  $\mathbb{Y} = \hat{\mathbb{X}}^{(b)}$  in the left hand side and  $\mathbb{X} = k\hat{\mathbb{X}}^{(a)}$  and  $\mathbb{Y} = k\hat{\mathbb{X}}^{(b)}$  in the right hand side.

**Example 2.10.** The partial Bell polynomials are known to be involved in interesting identities on binomial functions. Let us first recall that a binomial sequence is a family of functions  $(f_n)_{n \in \mathbb{N}}$  satisfying  $f_0(x) = 1$  and

$$f_n(a+b) = \sum_{k=0}^n \binom{n}{k} f_k(a) f_{n-k}(b), \quad (22)$$

for all  $a, b \in \mathbb{C}$  and  $n \in \mathbb{N}$ . Setting  $h_n(\mathbb{A}) := \frac{f_n(a)}{n!}$  and  $h_n(\mathbb{B}) := \frac{f_n(b)}{n!}$ , with these notations  $f_n(ka) = n! h_n(k\mathbb{A})$ . Hence,

$$B_{n,k}(1, \dots, i f_{i-1}(a), \dots) = \frac{n!}{k!} h_{n-k}(k\mathbb{A}) = \binom{n}{k} f_{n-k}(ka). \quad (23)$$

Notice that from (20),  $f_n(k) = \begin{cases} \binom{n}{k}^{-1} B_{n,k}(a_1, a_2, \dots) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$ , is binomial and we obtain

$$\binom{n}{k_1 k_2}^{-1} B_{n, k_1}(1, \dots, i \binom{i-1}{k}^{-1} B_{i-1, k_2}(a_1, a_2, \dots), \dots) = \binom{n-k_1}{k_1 k_2}^{-1} B_{n-k_1, k_1 k_2}(a_1, a_2, \dots). \quad (24)$$

Several related identities are compiled in [20].

**Example 2.11.** Taking the coefficient of  $t^{n-k-1}$  in the left hand side and the right hand side of the equality  $\frac{d}{dt}\sigma_t((k+1)\mathbb{X}) = (k+1)\left(\frac{d}{dt}\sigma_t(\mathbb{X})\right)\sigma_t(k\mathbb{X})$ , we obtain

$$(n-k)h_{n-k}((k+1)\mathbb{X}) = (k+1)\sum_{i=1}^{n-k} ih_i(\mathbb{X})h_{n-i-k}(\mathbb{X})$$

and we recover the identity (see *e.g.* [7]):

$$B_{n,k}(a_1, a_2, \dots) = \frac{1}{n-k} \sum_{i=1}^{n-k} \binom{n}{i} \left[ (k+1) - \frac{n+1}{i+1} \right] (i+1)a_i B_{n-i,k}(a_1, a_2, \dots). \quad (25)$$

**Example 2.12.** Let  $(a_n)_{n>0}$  and  $(b_n)_{n>0}$  be two sequences of numbers such that  $a_1 = b_1 = 1$  and  $d_n = n! \sum_{\lambda \vdash n-1} \det \left| \frac{a_{\lambda_i - i + j + 1}}{(\lambda_i + j + 1)!} \right| \det \left| \frac{b_{\lambda_i - i + j + 1}}{(\lambda_i + j + 1)!} \right|$  with the convention  $a_{-n} = b_{-n} = 0$  if  $n \geq 0$ . The Cauchy kernel and the orthogonality of Schur functions give

$$B_{n,k}(d_1, d_2, \dots) = \frac{n!}{k!} \sum_{\lambda \vdash n-k} (k_1! k_2!)^{\ell(\lambda)} \det \left| \frac{B_{\lambda_i - i + j + k_1, k_1}(a_1, a_2, \dots)}{(\lambda_i - i + j + k_1)!} \right| \det \left| \frac{B_{\lambda_i - i + j + k_1, k_1}(b_1, b_2, \dots)}{(\lambda_i - i + j + k_1)!} \right|,$$

for any  $k_1 k_2 = k$ . Indeed, it suffices to use the fact that

$$h_n(k\mathbb{X}^{(a)}\mathbb{X}^{(b)}) = \sum_{\lambda \vdash n} s_\lambda(k_1\mathbb{X}^{(a)})s_\lambda(k_2\mathbb{X}^{(b)}).$$

The sum  $\mathbb{X} + \mathbb{Y}$  and the product  $\mathbb{X}\mathbb{Y}$  of alphabets are two examples of coproducts endowing  $Sym$  with a structure of Hopf algebra. The sum of alphabets encodes the coproduct  $\Delta$  for which the power sums are of type Lie (*i.e.*  $\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \sim p_n(\mathbb{X} + \mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y})$  by identifying  $f \otimes g$  with  $f(\mathbb{X})g(\mathbb{Y})$ ) whilst the product of alphabets encodes the coproduct  $\Delta'$  for which the power sums are group like (*i.e.*  $\Delta'(p_n) = p_n \otimes p_n \sim p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})$ ).

The algebra of symmetric functions can be endowed with another coproduct that confers a structure of Hopf algebra: this is the Faà di Bruno algebra [8, 15]. This algebra is rather important since it is related to the Lagrange-Bürmann formula. The Bell polynomials appear also in this context. As a consequence, one can define a new operation on alphabets corresponding to the composition of Cauchy generating functions. Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two alphabets and set  $f(t) = t\sigma_t(\mathbb{X})$  and  $g(t) = t\sigma_t(\mathbb{Y})$ . The composition  $\mathbb{X} \circ \mathbb{Y}$  is defined by  $\sigma_t(\mathbb{X} \circ \mathbb{Y}) = \frac{1}{t}f \circ g(t)$ . The relationship with Bell polynomials can be established by observing that we have

$$\frac{1}{t}f \circ g = \sum_{n \geq 0} \left( \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} h_{k-1}(\mathbb{X}) B_{n+1,k}(1, 2!h_1(\mathbb{Y}), 3!h_2(\mathbb{Y}), \dots) \right) t^n.$$

Equivalently,  $h_n(\mathbb{X} \circ \mathbb{Y}) = \sum_{k=0}^n \frac{(k+1)!}{(n+1)!} h_k(\mathbb{X}) B_{n+1,k+1}(1, 2!h_1(\mathbb{Y}), 3!h_2(\mathbb{Y}), \dots)$ .

The antipode of the Faà di Bruno algebra is also described in terms of alphabets as the operation which associates to each alphabet  $\mathbb{X}$  the alphabet  $\mathbb{X}^{(-1)}$  satisfying  $\sigma_t(\mathbb{X} \circ \mathbb{X}^{(-1)}) = 1$ . More explicitly one has

$$h_n(\mathbb{X}^{(-1)}) = \frac{n!}{(2n+1)!(n+1)} B_{2n+1,n}(1, -2!e_1(\mathbb{X}), 3!e_2(\mathbb{X}), \dots), \quad (26)$$

where  $e_n(\mathbb{X})$  is the elementary symmetric function defined by  $\sum_n e_n(\mathbb{X})t^n = \frac{1}{\sigma_{-t}(\mathbb{X})}$ .



**Example 2.13.** Let  $\omega(t) = t\sigma_t(\mathbb{X})$ . The Lagrange inversion consists in finding an alphabet  $\mathbb{X}'$  such that  $\phi(t) = \sigma_t(\mathbb{X}')$ . According to (26), it suffices to set  $\mathbb{X}' = -\mathbb{X}^{(-1)}$ . Let  $F(t) = \sigma_t(\mathbb{Y})$ . When stated in terms of alphabets, the Lagrange-Bürmann formula reads

$$F(\omega(t)) = 1 + \sum_{n \geq 1} \frac{d^{n-1}}{du^{n-1}} [\sigma'_u(\mathbb{Y})\sigma_u(-n\mathbb{X}^{(-1)})] \Big|_{u=0} \frac{t^n}{n!}.$$

In other words one has,  $h_{n-k}(-n\mathbb{X}^{(-1)}) = \frac{(k-1)!}{(n-1)!} B_{n,k}(1, 2!h_1(\mathbb{X}), 3!h_3(\mathbb{X}), \dots)$ . So we recover a result due to Sadek Bouroubi and Moncef Abbas [1]:

$$B_{n,k}(1, h_1(2\mathbb{X}), \dots, (m-1)!h_{m-1}(m\mathbb{X}), \dots) = \frac{(n-1)!}{(k-1)!} h_{n-k}(n\mathbb{X}).$$

### 3 Hopf algebras of colored set partitions

#### 3.1 Colored partitions and Bell polynomials

Let  $a = (a_m)_{m \geq 1}$  be a sequence of nonnegative integers. A *colored set partition* associated to the sequence  $a$  is a set of couples

$$\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$$

such that  $\pi = \{\pi_1, \dots, \pi_k\}$  is a partition of  $\{1, \dots, n\}$  for some  $n \in \mathbb{N}$  and  $1 \leq i_\ell \leq a_{\#\pi_\ell}$  for each  $1 \leq \ell \leq k$ . The integer  $n$  is the *size* of  $\Pi$ . We will write  $|\Pi| = n$ ,  $\Pi \models n$  and  $\Pi \Rightarrow \pi$ . We will denote by  $\mathcal{CP}_n(a)$  the set of colored partitions of size  $n$  associated to the sequence  $a$ . Notice that these sets are finite. We will set also  $\mathcal{CP}(a) = \bigcup_n \mathcal{CP}_n(a)$ . We endow  $\mathcal{CP}$  with the additional statistic  $\#\Pi$  and set  $\mathcal{CP}_{n,k}(a) = \{\Pi \in \mathcal{CP}_n(a) : \#\Pi = k\}$ .

**Example 3.1.** Consider the sequence whose first terms are  $a = (1, 2, 3, \dots)$ . The colored partitions of size 3 associated to  $a$  are

$$\begin{aligned} \mathcal{CP}_3(a) = & \{ \{ \{ \{1, 2, 3\}, 1 \} \}, \{ \{ \{1, 2, 3\}, 2 \} \}, \{ \{ \{1, 2, 3\}, 3 \} \}, \{ \{ \{1, 2\}, 1 \}, \{ \{3\}, 1 \} \}, \\ & \{ \{ \{1, 2\}, 2 \}, \{ \{3\}, 1 \} \}, \{ \{ \{1, 3\}, 1 \}, \{ \{2\}, 1 \} \}, \{ \{ \{1, 3\}, 2 \}, \{ \{2\}, 1 \} \}, \\ & \{ \{ \{2, 3\}, 1 \}, \{ \{1\}, 1 \} \}, \{ \{ \{2, 3\}, 2 \}, \{ \{1\}, 1 \} \}, \{ \{ \{1\}, 1 \}, \{ \{2\}, 1 \}, \{ \{3\}, 1 \} \} \}. \end{aligned}$$

The colored partitions of size 3 and cardinality 2 are

$$\begin{aligned} \mathcal{CP}_{3,2}(a) = & \{ \{ \{ \{1, 2\}, 1 \}, \{ \{3\}, 1 \} \}, \{ \{ \{1, 2\}, 2 \}, \{ \{3\}, 1 \} \}, \{ \{ \{1, 3\}, 1 \}, \{ \{2\}, 1 \} \}, \\ & \{ \{ \{1, 3\}, 2 \}, \{ \{2\}, 1 \} \}, \{ \{ \{2, 3\}, 1 \}, \{ \{1\}, 1 \} \}, \{ \{ \{2, 3\}, 2 \}, \{ \{1\}, 1 \} \} \}. \end{aligned}$$

It is well known that the number of colored set partitions of size  $n$  for a given sequence  $a = (a_n)_n$  is equal to the evaluation of the complete Bell polynomial  $A_n(a_1, \dots, a_m, \dots)$  and that the number of colored set partitions of size  $n$  and cardinality  $k$  is given by the evaluation of the partial Bell polynomial  $B_n(a_1, a_2, \dots, a_m, \dots)$ . That is

$$\#\mathcal{CP}_n(a) = A_n(a_1, a_2, \dots) \text{ and } \#\mathcal{CP}_{n,k}(a) = B_{n,k}(a_1, a_2, \dots).$$

Now, let  $\Pi = \{[\pi_1, i_1], \dots, [\pi_k, i_k]\}$  be a set such that the  $\pi_j$  are finite sets of nonnegative integers such that no integer belongs to more than one  $\pi_j$ , and  $1 \leq i_j \leq a_{\#\pi_j}$  for each  $j$ . Then, the *standardized*  $\text{std}(\Pi)$  of  $\Pi$  is well defined as the unique colored set partition obtained by replacing the  $i$ th smallest integer in the  $\pi_j$  by  $i$ .

**Example 3.2.** For instance:

$$\begin{aligned} \text{std}(\{ \{ \{1, 4, 7\}, 1 \}, \{ \{3, 8\}, 1 \}, \{ \{5\}, 3 \}, \{ \{10\}, 1 \} \}) = \\ \{ \{ \{1, 3, 5\}, 1 \}, \{ \{2, 6\}, 1 \}, \{ \{4\}, 3 \}, \{ \{7\}, 1 \} \} \end{aligned}$$

We define two binary operations  $\uplus : \mathcal{CP}_{n,k}(a) \otimes \mathcal{CP}_{n',k'}(a) \longrightarrow \mathcal{CP}_{n+n',k+k'}(a)$ ,

$$\Pi \uplus \Pi' = \Pi \cup \Pi'[n],$$

where  $\Pi'[n]$  means that we add  $n$  to each integer occurring in the sets of  $\Pi'$  and  $\uplus : \mathcal{CP}_{n,k} \otimes \mathcal{CP}_{n',k'} \longrightarrow \mathcal{P}(\mathcal{CP}_{n+n',k+k'})$  by

$$\Pi \uplus \Pi' = \{\hat{\Pi} \cup \hat{\Pi}' \in \mathcal{CP}_{n+n',k+k'}(a) : \text{std}(\hat{\Pi}) = \Pi \text{ and } \text{std}(\hat{\Pi}') = \Pi'\}.$$

**Example 3.3.** We have

$$\{\{1, 3\}, 5\}, \{2\}, 3\} \uplus \{\{1\}, 2\}, \{2, 3\}, 4\} = \{\{1, 3\}, 5\}, \{2\}, 3\}, \{4\}, 2\}, \{5, 6\}, 4\},$$

and

$$\begin{aligned} & \{\{1\}, 5\}, \{2\}, 3\} \uplus \{\{1, 2\}, 2\} = \{\{\{1\}, 5\}, \{2\}, 3\}, \{3, 4\}, 2\}, \\ & \{\{1\}, 5\}, \{3\}, 3\}, \{2, 4\}, 2\}, \{\{1\}, 5\}, \{4\}, 3\}, \{2, 3\}, 2\}, \\ & \{\{2\}, 5\}, \{3\}, 3\}, \{1, 4\}, 2\}, \{\{2\}, 5\}, \{4\}, 3\}, \{1, 3\}, 2\}, \{\{3\}, 5\}, \{4\}, 3\}, \{1, 2\}, 2\}. \end{aligned}$$

The operator  $\uplus$  provides an algorithm which computes all the colored partitions

$$\mathcal{CP}_{n,k}(a) = \bigcup_{i_1 + \dots + i_k = n} \bigcup_{j_1 = 1}^{a_{i_1}} \dots \bigcup_{j_k = 1}^{a_{i_k}} \{\{1, \dots, i_1\}, j_1\} \uplus \dots \uplus \{\{1, \dots, i_k\}, j_k\}. \quad (27)$$

Nevertheless each colored partition is generated more than once using this process. For a triple  $(\Pi, \Pi', \Pi'')$  we will denote by  $\alpha_{\Pi', \Pi''}^{\Pi}$  the number of pairs of disjoint subsets  $(\hat{\Pi}', \hat{\Pi}'')$  of  $\Pi$  such that  $\hat{\Pi}' \cup \hat{\Pi}'' = \Pi$ ,  $\text{std}(\hat{\Pi}') = \Pi'$  and  $\text{std}(\hat{\Pi}'') = \Pi''$ .

**Remark 3.4.** Notice that for  $a = \mathbf{1} = (1, 1, \dots)$  (i.e. the ordinary set partitions), there is an alternative simple way to construct efficiently the set  $\mathcal{CP}_n(\mathbf{1})$ . It suffices to use the induction

$$\begin{aligned} \mathcal{CP}_0(\mathbf{1}) &= \{\emptyset\}, \\ \mathcal{CP}_{n+1}(\mathbf{1}) &= \{\pi \cup \{n+1\} : \pi \in \mathcal{CP}_n(\mathbf{1})\} \cup \{(\pi \setminus \{e\}) \\ & \quad \cup \{e\} \cup \{n+1\} : \pi \in \mathcal{CP}_n(\mathbf{1}), e \in \pi\}. \end{aligned} \quad (28)$$

Applying this recurrence the set partitions of  $\mathcal{CP}_{n+1}(\mathbf{1})$  are each obtained exactly once from the set partitions of  $\mathcal{CP}_n(\mathbf{1})$ .

## 3.2 The Hopf algebras $\mathbf{CWSym}(a)$ and $\mathbf{CHQSym}(a)$

Let  $\mathbf{CWSym}(a)$  ( $\mathbf{CWSym}$  for short when there is no ambiguity) be the algebra defined by its basis  $(\Phi_{\Pi})_{\Pi \in \mathcal{CP}(a)}$  indexed by colored set partitions associated to the sequence  $a = (a_m)_{m \geq 1}$  and the product

$$\Phi_{\Pi} \Phi_{\Pi'} = \Phi_{\Pi \uplus \Pi'}. \quad (29)$$

**Example 3.5.** One has

$$\Phi_{\{\{1,3,5\},3\},\{2,4\},1\}} \Phi_{\{\{1,2,5\},4\},\{3\},1\},\{4\},2\}} = \Phi_{\{\{1,3,5\},3\},\{2,4\},1\},\{6,7,10\},4\},\{8\},1\},\{9\},2\}}.$$

Let  $\mathbf{CWSym}_n$  be the subspace generated by the elements  $\Phi_{\Pi}$  such that  $\Pi \vDash n$ .

For each  $n$  we consider an infinite alphabet  $\mathbb{A}_n$  of noncommuting variables and we suppose  $\mathbb{A}_n \cap \mathbb{A}_m = \emptyset$  when  $n \neq m$ . For each colored set partition  $\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$ , we construct a polynomial  $\Phi_{\Pi}(\mathbb{A}_1, \mathbb{A}_2, \dots) \in \mathbb{C}\langle \bigcup_n \mathbb{A}_n \rangle$

$$\Phi_{\Pi}(\mathbb{A}_1, \mathbb{A}_2, \dots) := \sum_{\mathbf{w} = \mathbf{a}_1 \dots \mathbf{a}_n} \mathbf{w}, \quad (30)$$

where the sum is over the words  $\mathbf{w} = \mathbf{a}_1 \dots \mathbf{a}_n$  satisfying

- For each  $1 \leq \ell \leq k$ ,  $\mathbf{a}_j \in \mathbb{A}_{i_\ell}$  if and only if  $j \in \pi_\ell$ .
- If  $j_1, j_2 \in \pi_\ell$  then  $\mathbf{a}_{j_1} = \mathbf{a}_{j_2}$ .

**Example 3.6.**

$$\Phi_{\{\{\{1,3\},3\},\{\{2\},1\},\{\{4\},3\}\}}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \sum_{\substack{\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{A}_3 \\ \mathbf{b} \in \mathbb{A}_1}} \mathbf{a}_1 \mathbf{b} \mathbf{a}_1 \mathbf{a}_2.$$

**Proposition 3.7.** *The family  $\Phi(a) := (\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots))_{\Pi \in \mathcal{CP}(a)}$  spans a subalgebra of  $\mathbb{C}\langle \bigcup_n \mathbb{A}_n \rangle$  which is isomorphic to  $\mathbf{CWSym}(a)$ .*

*Proof.* First remark that  $\text{span}(\Phi(a))$  is stable under concatenation. Indeed,

$$\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots) \Phi_{\Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \Phi_{\Pi \uplus \Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots).$$

Furthermore, this shows that  $\text{span}(\Phi(a))$  is homomorphic to  $\mathbf{CWSym}(a)$  and that an explicit (onto) morphism is given by  $\Phi_\Pi \longrightarrow \Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots)$ . Observing that the family  $\Phi(a)$  is linearly independent, the fact that the algebra  $\mathbf{CWSym}(a)$  is graded in finite dimension implies the result. ■

We turn  $\mathbf{CWSym}$  into a Hopf algebra by considering the coproduct

$$\Delta(\Phi_\Pi) = \sum_{\substack{\hat{\pi}_1 \cup \hat{\pi}_2 = \Pi \\ \hat{\pi}_1 \cap \hat{\pi}_2 = \emptyset}} \Phi_{\text{std}(\hat{\pi}_1)} \otimes \Phi_{\text{std}(\hat{\pi}_2)} = \sum_{\Pi_1, \Pi_2} \alpha_{\Pi_1, \Pi_2}^\Pi \Phi_{\Pi_1} \otimes \Phi_{\Pi_2}. \quad (31)$$

Indeed,  $\mathbf{CWSym}$  splits as a direct sum of finite dimension spaces

$$\mathbf{CWSym} = \bigoplus_n \mathbf{CWSym}_n.$$

This defines a natural graduation on  $\mathbf{CWSym}$ . Hence, since it is a connected algebra, it suffices to verify that it is a bialgebra. More precisely:

$$\begin{aligned} \Delta(\Phi_\Pi \Phi_{\Pi'}) &= \Delta(\Phi_{\Pi \uplus \Pi'}) \\ &= \sum_{\substack{\hat{\pi}_1 \cup \hat{\pi}_2 = \Pi, \hat{\pi}'_1 \cup \hat{\pi}'_2 = \Pi' \\ \hat{\pi}_1 \cap \hat{\pi}_2 = \emptyset, \hat{\pi}'_1 \cap \hat{\pi}'_2 = \emptyset}} \Phi_{\text{std}(\hat{\pi}_1) \uplus \text{std}(\hat{\pi}'_1)} \otimes \Phi_{\text{std}(\hat{\pi}_2) \uplus \text{std}(\hat{\pi}'_2)} \\ &= \Delta(\Phi_\Pi) \Delta(\Phi_{\Pi'}). \end{aligned}$$

Notice that  $\Delta$  is cocommutative.

**Example 3.8.** For instance,

$$\begin{aligned} \Delta(\Phi_{\{\{\{1,3\},5\},\{\{2\},3\}\}}) &= \Phi_{\{\{\{1,3\},5\},\{\{2\},3\}\}} \otimes 1 + \Phi_{\{\{1,2\},5\}} \otimes \Phi_{\{\{1\},3\}} + \\ &\quad \Phi_{\{\{1\},3\}} \otimes \Phi_{\{\{1,2\},5\}} + 1 \otimes \Phi_{\{\{1,3\},5\},\{\{2\},3\}}. \end{aligned}$$

The graded dual  $\mathbf{CHQSym}(a)$  ( $\mathbf{CHQSym}$  for short when there is no ambiguity) of  $\mathbf{CWSym}$  is the Hopf algebra generated as a space by the dual basis  $(\Psi_\Pi)_{\Pi \in \mathcal{CP}(a)}$  of  $(\Phi_\Pi)_{\Pi \in \mathcal{CP}(a)}$ . Its product and its coproduct are given by

$$\Psi_{\Pi'} \Psi_{\Pi''} = \sum_{\Pi \in \Pi' \uplus \Pi''} \alpha_{\Pi', \Pi''}^\Pi \Psi_\Pi \quad \text{and} \quad \Delta(\Psi_\Pi) = \sum_{\Pi' \uplus \Pi'' = \Pi} \Psi_{\Pi'} \otimes \Psi_{\Pi''}.$$

**Example 3.9.** For instance, one has

$$\begin{aligned} \Psi_{\{\{1,2\},3\}} \Psi_{\{\{1\},4\},\{2\},1\}} &= \Psi_{\{\{1,2\},3\},\{3\},4\},\{4\},1\}} + \Psi_{\{\{1,3\},3\},\{2\},4\},\{4\},1\}} \\ &+ \Psi_{\{\{1,4\},3\},\{2\},4\},\{3\},1\}} + \Psi_{\{\{2,3\},3\},\{1\},4\},\{4\},1\}} \\ &+ \Psi_{\{\{2,4\},3\},\{1\},4\},\{3\},1\}} + \Psi_{\{\{3,4\},3\},\{1\},4\},\{2\},1\}} \end{aligned}$$

and

$$\begin{aligned} \Delta(\Psi_{\{\{1,3\},3\},\{2\},4\},\{4\},1\}}) &= 1 \otimes \Psi_{\{\{1,3\},3\},\{2\},4\},\{4\},1\}} + \Psi_{\{\{1,3\},3\},\{2\},4\}} \otimes \Psi_{\{\{1\},1\}} \\ &+ \Psi_{\{\{1,3\},3\},\{2\},4\},\{4\},1\}} \otimes 1. \end{aligned}$$

### 3.3 Special cases

In this section, we investigate a few interesting special cases of the construction.

#### 3.3.1 Word symmetric functions

The most prominent example follows from the specialization  $a_n = 1$  for each  $n$ . In this case, the Hopf algebra  $\mathbf{CWSym}$  is isomorphic to  $\mathbf{WSym}$ , the Hopf algebra of word symmetric functions. Let us recall briefly its construction. The algebra of word symmetric functions is a way to construct a noncommutative analogue of the algebra  $Sym$ . Its bases are indexed by set partitions. After the seminal paper [26], this algebra was investigated in [?, 4, 13] as well as an abstract algebra as in its realization with noncommutative variables. Its name comes from its realization as a subalgebra of  $\mathbb{C}\langle \mathbb{A} \rangle$  where  $\mathbb{A} = \{a_1, \dots, a_n, \dots\}$  is an infinite alphabet.

Consider the family  $\Phi := \{\Phi_\pi\}_\pi$  whose elements are indexed by set partitions of  $\{1, \dots, n\}$ . The algebra  $\mathbf{WSym}$  is formally generated by  $\Phi$  using the shifted concatenation product:  $\Phi_\pi \Phi_{\pi'} = \Phi_{\pi\pi'[n]}$  where  $\pi$  and  $\pi'$  are set partitions of  $\{1, \dots, n\}$  and  $\{1, \dots, m\}$ , respectively, and  $\pi'[n]$  means that we add  $n$  to each integer occurring in  $\pi'$ . The polynomial realization  $\mathbf{WSym}(\mathbb{A}) \subset \mathbb{C}\langle \mathbb{A} \rangle$  is defined by  $\Phi_\pi(\mathbb{A}) = \sum_{\mathbf{w}} \mathbf{w}$  where the sum is over the words  $\mathbf{w} = a_1 \cdots a_n$  where  $i, j \in \pi_\ell$  implies  $a_i = a_j$ , if  $\pi = \{\pi_1, \dots, \pi_k\}$  is a set partition of  $\{1, \dots, n\}$ .

**Example 3.10.** For instance, one has  $\Phi_{\{\{1,4\}\{2,5,6\}\{3,7\}\}}(\mathbb{A}) = \sum_{a,b,c \in \mathbb{A}} \text{abcabbc}$ .

Although the construction of  $\mathbf{WSym}(\mathbb{A})$ , the polynomial realization of  $\mathbf{WSym}$ , seems to be close to  $Sym(\mathbb{X})$ , the structures of the two algebras are quite different since the Hopf algebra  $\mathbf{WSym}$  is not autodual. Surprisingly, the graded dual  $\Pi\text{QSym} := \mathbf{WSym}^*$  of  $\mathbf{WSym}$  admits a realization in the same subspace ( $\mathbf{WSym}(\mathbb{A})$ ) of  $\mathbb{C}\langle \mathbb{A} \rangle$  but for the shuffle product.

With no surprise, we notice the following fact:

**Proposition 3.11.**

- The algebras  $\mathbf{CWSym}(1, 1, \dots)$ ,  $\mathbf{WSym}$  and  $\mathbf{WSym}(\mathbb{A})$  are isomorphic.
- The algebras  $\text{CPIQSym}(1, 1, \dots)$ ,  $\Pi\text{QSym}$  and  $(\mathbf{WSym}(\mathbb{A}), \sqcup)$  are isomorphic.

In the rest of the paper, when there is no ambiguity, we will identify the algebras  $\mathbf{WSym}$  and  $\mathbf{WSym}(\mathbb{A})$ .

The word analogue of the basis  $(c_\lambda)_\lambda$  of  $Sym$  is the dual basis  $(\Psi_\pi)_\pi$  of  $(\Phi_\pi)_\pi$ .

Other bases are known, for example, the word monomial functions defined by  $\Phi_\pi = \sum_{\pi \leq \pi'} M_{\pi'}$ , where  $\pi \leq \pi'$  indicates that  $\pi$  is finer than  $\pi'$ , i.e., that each block of  $\pi'$  is a union of blocks of  $\pi$ .

**Example 3.12.** For instance,

$$\begin{aligned} \Phi_{\{\{1,4\}\{2,5,6\}\{3,7\}\}} &= M_{\{\{1,4\}\{2,5,6\}\{3,7\}\}} + M_{\{\{1,2,4,5,6\}\{3,7\}\}} + M_{\{\{1,3,4,7\}\{2,5,6\}\}} \\ &+ M_{\{\{1,4\}\{2,3,5,6,7\}\}} + M_{\{\{1,2,3,4,5,6,7\}\}}. \end{aligned}$$

From the definition of the  $M_\pi$ , we deduce that the polynomial representation of the word monomial functions is given by  $M_\pi(\mathbb{A}) = \sum_{\mathbf{w}} \mathbf{w}$  where the sum is over the words  $\mathbf{w} = \mathbf{a}_1 \cdots \mathbf{a}_n$  where  $i, j \in \pi_\ell$  if and only if  $\mathbf{a}_i = \mathbf{a}_j$ , where  $\pi = \{\pi_1, \dots, \pi_k\}$  is a set partition of  $\{1, \dots, n\}$ .

**Example 3.13.**  $M_{\{\{1,4\}\{2,5,6\}\{3,7\}\}}(\mathbb{A}) = \sum_{\substack{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{A} \\ \mathbf{a} \neq \mathbf{b}, \mathbf{a} \neq \mathbf{c}, \mathbf{b} \neq \mathbf{c}}} \mathbf{a}\mathbf{b}\mathbf{c}\mathbf{a}\mathbf{b}\mathbf{c}$ .

The analogue of complete symmetric functions is the basis  $(S_\pi)_\pi$  of  $\Pi\text{QSym}$  which is the dual of the basis  $(M_\pi)_\pi$  of  $\mathbf{WSym}$ .

The algebra  $\Pi\text{QSym}$  is also realized in the space  $\mathbf{WSym}(\mathbb{A})$ : it is the subalgebra of  $(\mathbb{C}\langle \mathbb{A} \rangle, \sqcup)$  generated by  $\Psi_\pi(\mathbb{A}) = \pi! \Phi_\pi(\mathbb{A})$  where  $\pi! = \#\pi_1! \cdots \#\pi_k!$  for  $\pi = \{\pi_1, \dots, \pi_k\}$ . Indeed, the linear map  $\Psi_\pi \rightarrow \Phi_\pi(\mathbb{A})$  is a bijection sending  $\Psi_{\pi_1} \Psi_{\pi_2}$  to

$$\begin{aligned} \sum_{\substack{\pi = \pi'_1 \cup \pi'_2, \pi'_1 \cap \pi'_2 = \emptyset \\ \pi_1 = \text{std}(\pi'_1), \pi_2 = \text{std}(\pi'_2)}} \Psi_\pi(\mathbb{A}) &= \pi_1! \pi_2! \sum_{\substack{\pi = \pi'_1 \cup \pi'_2, \pi'_1 \cap \pi'_2 = \emptyset \\ \pi_1 = \text{std}(\pi'_1), \pi_2 = \text{std}(\pi'_2)}} \Phi_\pi(\mathbb{A}) \\ &= \pi_1! \pi_2! \Phi_{\pi_1}(\mathbb{A}) \sqcup \Phi_{\pi_2}(\mathbb{A}) = \Psi_{\pi_1}(\mathbb{A}) \sqcup \Psi_{\pi_2}(\mathbb{A}). \end{aligned}$$

With these notations the image of  $S_\pi$  is  $S_\pi(\mathbb{A}) = \sum_{\pi' \leq \pi} \Psi_{\pi'}(\mathbb{A})$ . For our realization, the duality bracket  $\langle | \rangle$  implements the scalar product  $\langle | \rangle$  on the space  $\mathbf{WSym}(\mathbb{A})$  for which  $\langle S_{\pi_1}(\mathbb{A}) | M_{\pi_2}(\mathbb{A}) \rangle = \langle \Phi_{\pi_1}(\mathbb{A}) | \Psi_{\pi_2}(\mathbb{A}) \rangle = \delta_{\pi_1, \pi_2}$ .

Since the subalgebra of  $(\mathbf{WSym}(\mathbb{A}), \sqcup)$  generated by the complete functions  $S_{\{\{1, \dots, n\}\}}(\mathbb{A})$  is isomorphic to  $Sym$ , we define  $\sigma_t^W(\mathbb{A})$  and  $\phi_t^W(\mathbb{A})$  by  $\sigma_t^W(\mathbb{A}) = \sum_{n \geq 0} S_{\{\{1, \dots, n\}\}}(\mathbb{A}) t^n$  and  $\phi_t^W(\mathbb{A}) = \sum_{n \geq 1} \Psi_{\{\{1, \dots, n\}\}}(\mathbb{A}) t^{n-1}$ . These series are linked by the equality

$$\sigma_t^W(\mathbb{A}) = \exp_{\sqcup}(\phi_t^W(\mathbb{A})), \quad (32)$$

where  $\exp_{\sqcup}$  is the exponential in  $(\mathbf{WSym}(\mathbb{A}), \sqcup)$ . Furthermore, the coproduct of  $\mathbf{WSym}$  consists in identifying the algebra  $\mathbf{WSym} \otimes \mathbf{WSym}$  with  $\mathbf{WSym}(\mathbb{A} + \mathbb{B})$ , where  $\mathbb{A}$  and  $\mathbb{B}$  are two alphabets such that the letters of  $\mathbb{A}$  commute with those of  $\mathbb{B}$ . Hence, one has  $\sigma_t^W(\mathbb{A} + \mathbb{B}) = \sigma_t^W(\mathbb{A}) \sqcup \sigma_t^W(\mathbb{B})$ . In particular, we define the multiplication of an alphabet  $\mathbb{A}$  by a constant  $k \in \mathbb{N}$  by

$$\sigma_t^W(k\mathbb{A}) = \sum_{n \geq 0} S_{\{\{1, \dots, n\}\}}(k\mathbb{A}) t^n = \sigma_t^W(\mathbb{A})^k.$$

Nevertheless, the notion of specialization is subtler to define than in  $Sym$ . Indeed, the knowledge of the complete functions  $S_{\{\{1, \dots, n\}\}}(\mathbb{A})$  does not allow us to recover all the polynomials using uniquely the algebraic operations. In [2], we made an attempt to define virtual alphabets by reconstituting the whole algebra using the action of an operad. Although the general mechanism remains to be defined, the case where each complete function  $S_{\{\{1, \dots, n\}\}}(\mathbb{A})$  is specialized to a sum of words of length  $n$  can be understood via this construction. More precisely, we consider the family of multilinear  $k$ -ary operators  $\sqcup_{\Pi}$  indexed by set compositions (a set composition is a sequence  $[\pi_1, \dots, \pi_k]$  of subsets of  $\{1, \dots, n\}$  such that  $\{\pi_1, \dots, \pi_k\}$  is a set partition of  $\{1, \dots, n\}$ ) acting on words by  $\sqcup_{[\pi_1, \dots, \pi_k]}(a_1^1 \cdots a_{n_1}^1, \dots, a_1^k \cdots a_{n_k}^k) = b_1 \cdots b_n$  with  $b_{i_\ell}^p = a_\ell^p$  if  $\pi_p = \{i_1^p < \cdots < i_{n_p}^p\}$  and  $\sqcup_{[\pi_1, \dots, \pi_k]}(a_1^1 \cdots a_{n_1}^1, \dots, a_1^k \cdots a_{n_k}^k) = 0$  if  $\#\pi_p \neq n_p$  for some  $1 \leq p \leq k$ .

Let  $P = (P_n)_{n \geq 1}$  be a family of a homogeneous word polynomials such that  $\deg(P_n) = n$  for each  $n$ . We set  $S_{\{\{1, \dots, n\}\}}[\mathbb{A}^{(P)}] = P_n$  and

$$S_{\{\pi_1, \dots, \pi_k\}}[\mathbb{A}^{(P)}] = \sqcup_{[\pi_1, \dots, \pi_k]}(S_{\{\{1, \dots, \#\pi_1\}\}}[\mathbb{A}^{(P)}], \dots, S_{\{\{1, \dots, \#\pi_k\}\}}[\mathbb{A}^{(P)}]).$$

The space  $\mathbf{WSym}[\mathbb{A}^{(P)}]$  generated by the polynomials  $S_{\{\pi_1, \dots, \pi_k\}}[\mathbb{A}^{(P)}]$  and endowed with the two products  $\cdot$  and  $\sqcup$  is homomorphic to the double algebra  $(\mathbf{WSym}(\mathbb{A}), \cdot, \sqcup)$ . Indeed, let  $\pi =$

$\{\pi_1, \dots, \pi_k\} \vDash n$  and  $\pi' = \{\pi'_1, \dots, \pi'_{k'}\} \vDash n'$  be two set partitions, one has

$$\begin{aligned} S_\pi [\mathbb{A}^{(P)}] \cdot S_{\pi'} [\mathbb{A}^{(P)}] &= \sqcup_{[\{1, \dots, n\}, \{n+1, \dots, n+n'\}]} (S_\pi [\mathbb{A}^{(P)}], S_{\pi'} [\mathbb{A}^{(P)}]) \\ &= \sqcup_{[\pi_1, \dots, \pi_k, \pi'_1[n], \dots, \pi'_{k'}[n]]} (S_{\{1, \dots, \#\pi_1\}} [\mathbb{A}^{(P)}], \dots, S_{\{1, \dots, \#\pi_k\}} [\mathbb{A}^{(P)}], \\ &\quad S_{\{1, \dots, \#\pi'_1\}} [\mathbb{A}^{(P)}], \dots, S_{\{1, \dots, \#\pi'_{k'}\}} [\mathbb{A}^{(P)}]) \\ &= S_{\pi \uplus \pi'} [\mathbb{A}^{(P)}] \end{aligned}$$

and

$$\begin{aligned} S_\pi [\mathbb{A}^{(P)}] \sqcup S_{\pi'} [\mathbb{A}^{(P)}] &= \sum_{I \cup J = \{1, \dots, n+n'\}, I \cap J = \emptyset} \sqcup_{[I, J]} (S_\pi [\mathbb{A}^{(P)}], S_{\pi'} [\mathbb{A}^{(P)}]) \\ &= \sum \sqcup_{[\pi''_1, \dots, \pi''_{k+k'}]} (S_{\{1, \dots, \#\pi_1\}} [\mathbb{A}^{(P)}], \dots, S_{\{1, \dots, \#\pi_k\}} [\mathbb{A}^{(P)}], \\ &\quad S_{\{1, \dots, \#\pi'_1\}} [\mathbb{A}^{(P)}], \dots, S_{\{1, \dots, \#\pi'_{k'}\}} [\mathbb{A}^{(P)}]) \end{aligned}$$

where the second sum is over the partitions  $\{\pi''_1, \dots, \pi''_{k+k'}\} \in \pi \uplus \pi'$  satisfying, for each  $k+1 \leq i \leq k+k'$ ,  $\text{std}(\{\pi''_1, \dots, \pi''_k\}) = \pi$ ,  $\text{std}(\{\pi''_{k+1}, \dots, \pi''_{k+k'}\}) = \pi'$ ,  $\#\pi''_i = \pi_i$ . Hence,

$$S_\pi [\mathbb{A}^{(P)}] \sqcup S_{\pi'} [\mathbb{A}^{(P)}] = \sum_{\pi'' \in \pi \uplus \pi'} S_{\pi''} [\mathbb{A}^{(P)}].$$

In other words, we consider the elements of  $\mathbf{WSym} [\mathbb{A}^{(P)}]$  as word polynomials in the virtual alphabet  $\mathbb{A}^{(P)}$  specializing the elements of  $\mathbf{WSym}(\mathbb{A})$ .

### 3.3.2 Biword symmetric functions

The bi-indexed word algebra  $\mathbf{BWSym}$  was defined in [2]. We recall its definition here: the bases of  $\mathbf{BWSym}$  are indexed by set partitions into lists, which can be constructed from a set partition by ordering each block. We will denote by  $\mathcal{PL}_n$  the set of the set partitions of  $\{1, \dots, n\}$  into lists.

**Example 3.14.**  $\{[1, 2, 3], [4, 5]\}$  and  $\{[3, 1, 2], [5, 4]\}$  are two distinct set partitions into lists of the set  $\{1, 2, 3, 4, 5\}$ .

The number of set partitions into lists of an  $n$ -element set (or set partitions into lists of size  $n$ ) is given by Sloane's sequence [A000262](#) [24]. The first values are

$$1, 1, 3, 13, 73, 501, 4051, \dots$$

If  $\hat{\Pi}$  is a set partition into lists of  $\{1, \dots, n\}$ , we will write  $\hat{\Pi} \Vdash n$ . Set  $\hat{\Pi} \uplus \hat{\Pi}' = \hat{\Pi} \cup \{[l_1+n, \dots, l_k+n] : [l_1, \dots, l_k] \in \hat{\Pi}'\} \Vdash n+n'$ . Let  $\hat{\Pi}' \subset \hat{\Pi} \Vdash n$ , since the integers appearing in  $\hat{\Pi}'$  are all distinct, the standardized  $\text{std}(\hat{\Pi}')$  of  $\hat{\Pi}'$  is well defined as the unique set partition into lists obtained by replacing the  $i$ th smallest integer in  $\hat{\Pi}$  by  $i$ . For example,  $\text{std}(\{[5, 2], [3, 10], [6, 8]\}) = \{[3, 1], [2, 6], [4, 5]\}$ .

The Hopf algebra  $\mathbf{BWSym}$  is formally defined by its basis  $(\Phi_{\hat{\Pi}})$  where the  $\hat{\Pi}$  are set partitions into lists, its product  $\Phi_{\hat{\Pi}} \Phi_{\hat{\Pi}'} = \Phi_{\hat{\Pi} \uplus \hat{\Pi}'}$  and its coproduct

$$\Delta(\Phi_{\hat{\Pi}}) = \sum^{\wedge} \Phi_{\text{std}(\hat{\Pi}')} \otimes \Phi_{\text{std}(\hat{\Pi}'')}, \quad (33)$$

where the  $\sum^{\wedge}$  means that the sum is over the  $(\hat{\Pi}', \hat{\Pi}'')$  such that  $\hat{\Pi}' \cup \hat{\Pi}'' = \hat{\Pi}$  and  $\hat{\Pi}' \cap \hat{\Pi}'' = \emptyset$ .

The product of the graded dual  $\mathbf{BIIQSym}$  of  $\mathbf{BWSym}$  is completely described in the dual basis  $(\Psi_{\hat{\Pi}})_{\hat{\Pi}}$  of  $(\Phi_{\hat{\Pi}})_{\hat{\Pi}}$  by

$$\Psi_{\hat{\Pi}_1} \Psi_{\hat{\Pi}_2} = \sum_{\substack{\hat{\Pi} = \hat{\Pi}'_1 \cup \hat{\Pi}'_2, \hat{\Pi}'_1 \cap \hat{\Pi}'_2 \\ \text{std}(\hat{\Pi}'_1) = \hat{\Pi}_1, \text{std}(\hat{\Pi}'_2) = \hat{\Pi}_2}} \Psi_{\hat{\Pi}}.$$

Now consider a bijection  $\iota_n$  from  $\{1, \dots, n\}$  to the symmetric group  $\mathfrak{S}_n$ . The linear map  $\kappa : \mathcal{CP}(1!, 2!, 3!, \dots) \rightarrow \mathcal{CL}$  sending

$$\{[\{i_1^1, \dots, i_{n_1}^1\}, m_1], \dots, [\{i_1^k, \dots, i_{n_k}^k\}, m_1]\} \in \mathcal{CP}_n(1!, 2!, 3!, \dots),$$

with  $i_1^j \leq \dots \leq i_{n_j}^j$ , to

$$\{[i_{\iota_{n_1}(1)}^1, \dots, i_{\iota_{n_1}(n_1)}^1], \dots, [i_{\iota_{n_\ell}(1)}^\ell, \dots, i_{\iota_{n_\ell}(n_\ell)}^\ell]\}$$

is a bijection. Hence, a fast checking shows that the linear map sending  $\Psi^\Pi$  to  $\Psi^{\kappa(\Pi)}$  is an isomorphism. So, we have

**Proposition 3.15.**

- The Hopf algebras  $\mathbf{CWSym}(1!, 2!, 3!, \dots)$  and  $\mathbf{BWSym}$  are isomorphic.
- The Hopf algebras  $\mathbf{CIQSym}(1!, 2!, 3!, \dots)$  and  $\mathbf{BIQSym}$  are isomorphic.

### 3.3.3 Word symmetric functions of level 2

We consider the algebra  $\mathbf{WSym}_{(2)}$  which is spanned by the  $\Phi_\Pi$  where  $\Pi$  is a set partition of level 2, that is, a partition of a partition  $\pi$  of  $\{1, \dots, n\}$  for some  $n$ . The product of this algebra is given by  $\Phi_\Pi \Phi_{\Pi'} = \Phi_{\Pi \cup \Pi' [n]}$  where  $\Pi' [n] = \{e [n] : e \in \Pi'\}$ . The dimensions of this algebra are given by the exponential generating function

$$\sum_i b_i^{(2)} \frac{t^i}{i!} = \exp(\exp(\exp(t) - 1) - 1).$$

The first values are

$$1, 3, 12, 60, 358, 2471, 19302, 167894, 1606137, \dots$$

see sequence [A000258](#) of [24].

The coproduct is defined by  $\Delta(\Phi_\Pi) = \sum_{\substack{\Pi' \cup \Pi'' = \Pi \\ \Pi' \cap \Pi'' = \emptyset}} \Phi_{\text{std}(\Pi')} \otimes \Phi_{\text{std}(\Pi'')}$  where, if  $\Pi$  is a partition of a partition of  $\{i_1, \dots, i_k\}$ ,  $\text{std}(\Pi)$  denotes the standardized of  $\Pi$ , that is the partition of partition of  $\{1, \dots, k\}$  obtained by substituting each occurrence of  $i_j$  by  $j$  in  $\Pi$ . The coproduct being co-commutative, the dual algebra  $\Pi\text{QSym}_{(2)} := \mathbf{WSym}_{(2)}^*$  is commutative. The algebra  $\Pi\text{QSym}_{(2)}$  is spanned by a basis  $(\Psi_\Pi)_\Pi$  satisfying  $\Psi_\Pi \Psi_{\Pi'} = \sum_{\Pi''} C_{\Pi, \Pi'}^{\Pi''} \Psi_{\Pi''}$  where  $C_{\Pi, \Pi'}^{\Pi''}$  is the number of ways to write  $\Pi'' = A \cup B$  with  $A \cap B = \emptyset$ ,  $\text{std}(A) = \Pi$  and  $\text{std}(B) = \Pi'$ .

Let  $b_n$  be the  $n$ th Bell number  $A_n(1, 1, \dots)$ . Considering a bijection from  $\{1, \dots, b_n\}$  to the set of the set partitions of  $\{1, \dots, n\}$  for each  $n$ , we obtain, in the same way as in the previous subsection, the following result

**Proposition 3.16.**

- The Hopf algebras  $\mathbf{CWSym}(b_1, b_2, b_3, \dots)$  and  $\mathbf{WSym}_{(2)}$  are isomorphic.
- The Hopf algebras  $\mathbf{CIQSym}(b_1, b_2, b_3, \dots)$  and  $\Pi\text{QSym}_{(2)}$  are isomorphic.

### 3.3.4 Cycle word symmetric functions

We consider the Grossman-Larson Hopf algebra of heap-ordered trees  $\mathfrak{S}\mathbf{Sym}$  [12]. The combinatorics of this algebra has been extensively investigated in [13]. This Hopf algebra is spanned by the  $\Phi_\sigma$  where  $\sigma$  is a permutation. We identify each permutation with the set of its cycles (for example, the permutation 321 is  $\{(13), (2)\}$ ). The product in this algebra is given by  $\Phi_\sigma \Phi_\tau = \Phi_{\sigma \cup \tau [n]}$ , where

$n$  is the size of the permutation  $\sigma$  and  $\tau[n] = \{(i_1 + n, i_2 + n, \dots, i_k + n) \mid (i_1, \dots, i_k) \in \tau\}$ . The coproduct is given by

$$\Delta(\Phi_\sigma) = \sum \Phi_{\text{std}(\sigma|_I)} \otimes \Phi_{\text{std}(\sigma|_J)}, \quad (34)$$

where the sum is over the partitions of  $\{1, \dots, n\}$  into 2 sets  $I$  and  $J$  such that the action of  $\sigma$  lets the sets  $I$  and  $J$  globally invariant,  $\sigma|_I$  denotes the restriction of the permutation  $\sigma$  to the set  $I$  and  $\text{std}(\sigma|_I)$  is the permutation obtained from  $\sigma|_I$  by replacing the  $i$ th smallest label by  $i$  in  $\sigma|_I$ .

**Example 3.17.**

$$\Delta(\Phi_{3241}) = \Phi_{3241} \otimes 1 + \Phi_1 \otimes \Phi_{231} + \Phi_{231} \otimes \Phi_1 + 1 \otimes \Phi_{3241}.$$

The basis  $(\Phi_\sigma)$  and its dual basis  $(\Psi_\sigma)$  are respectively denoted by  $(S^\sigma)$  and  $(M_\sigma)$  in [13]. The Hopf algebra  $\mathfrak{S}\mathbf{Sym}$  is not commutative but it is cocommutative, so it is not autodial and not isomorphic to the Hopf algebra of free quasi-symmetric functions.

Let  $\iota_n$  be a bijection from the set of the cycles of  $\mathfrak{S}_n$  to  $\{1, \dots, (n-1)!\}$ . We define the bijection  $\kappa : \mathfrak{S}_n \leftrightarrow \mathcal{CP}(0!, 1!, 2!, \dots)$  by

$$\kappa(\sigma) = \{[\text{support}(c_1), \iota_{\#\text{support}(\text{std}(c_1))}(c_1)], \dots, [\text{support}(c_k), \iota_{\#\text{support}(\text{std}(c_k))}(c_k)]\},$$

if  $\sigma = c_1 \dots c_k$  is the decomposition of  $\sigma$  into cycles and  $\text{support}(c)$  denotes the support of the cycle  $c$ , *i.e.* the set of the elements which are permuted by the cycle.

**Example 3.18.** For instance, set

$$\iota_1(1) = 1, \quad \iota_3(231) = 2, \quad \text{and} \quad \iota_3(312) = 1.$$

One has

$$\kappa(32415867) = \{[\{2\}, 1], [\{1, 3, 4\}, 2], [\{5\}, 1], [\{6, 7, 8\}, 1]\}.$$

The linear map  $K : \mathfrak{S}\mathbf{Sym} \rightarrow \mathbf{CWSym}(0!, 1!, 2!, \dots)$  sending  $\Phi_\sigma$  to  $\Phi_{\kappa(\sigma)}$  is an isomorphism of algebra. Indeed, it is straightforward to see that it is a bijection and furthermore  $\kappa(\sigma \cup \tau[n]) = \kappa(\sigma) \uplus \kappa(\tau)$ . Moreover, if  $\sigma \in \mathfrak{S}_n$  is a permutation and  $\{I, J\}$  is a partition of  $\{1, \dots, n\}$  into two subsets such that the action of  $\sigma$  lets  $I$  and  $J$  globally invariant, we check that  $\kappa(\sigma) = \Pi_1 \cup \Pi_2$  with  $\Pi_1 \cap \Pi_2 = \emptyset$ ,  $\text{std}(\Pi_1) = \kappa(\text{std}(\sigma|_I))$  and  $\text{std}(\Pi_2) = \kappa(\text{std}(\sigma|_J))$ . Conversely, if  $\kappa(\sigma) = \Pi_1 \cup \Pi_2$  with  $\Pi_1 \cap \Pi_2 = \emptyset$  then there exists a partition  $\{I, J\}$  of  $\{1, \dots, n\}$  into two subsets such that the action of  $\sigma$  lets  $I$  and  $J$  globally invariant and  $\text{std}(\Pi_1) = \kappa(\text{std}(\sigma|_I))$  and  $\text{std}(\Pi_2) = \kappa(\text{std}(\sigma|_J))$ .

In other words,

$$\Delta(\Phi_{\kappa(\sigma)}) = \sum \Phi_{\kappa(\text{std}(\sigma|_I))} \otimes \Phi_{\kappa(\text{std}(\sigma|_J))},$$

where the sum is over the partitions of  $\{1, \dots, n\}$  into 2 sets  $I$  and  $J$  such that the action of  $\sigma$  lets the sets  $I$  and  $J$  globally invariant. Hence  $K$  is a morphism of cogebras and, as for the previous examples, one has

**Proposition 3.19.**

- The Hopf algebras  $\mathbf{CWSym}(0!, 1!, 2!, \dots)$  and  $\mathfrak{S}\mathbf{Sym}$  are isomorphic.
- The Hopf algebras  $\mathbf{CIQSym}(0!, 1!, 2!, \dots)$  and  $\mathfrak{S}\mathbf{Sym}^*$  are isomorphic.

### 3.3.5 Miscellaneous subalgebras of the Hopf algebra of endofunctions

We denote by  $\text{End}$  the combinatorial class of endofunctions (an endofunction of size  $n \in \mathbb{N}$  is a function from  $\{1, \dots, n\}$  to itself). Given a function  $f$  from a finite subset  $A$  of  $\mathbb{N}$  to itself, we denote by  $\text{std}(f)$  the endofunction  $\phi \circ f \circ \phi^{-1}$ , where  $\phi$  is the unique increasing bijection from  $A$  to  $\{1, 2, \dots, \text{card}(A)\}$ . Given a function  $g$  from a finite subset  $B$  of  $\mathbb{N}$  (disjoint from  $A$ ) to itself, we



denote by  $f \cup g$  the function from  $A \cup B$  to itself whose  $f$  and  $g$  are respectively the restrictions to  $A$  and  $B$ . Finally, given two endofunctions  $f$  and  $g$ , respectively of size  $n$  and  $m$ , we denote by  $f \bullet g$  the endofunction  $f \cup \tilde{g}$ , where  $\tilde{g}$  is the unique function from  $\{n+1, n+2, \dots, n+m\}$  to itself such that  $\text{std}(\tilde{g}) = g$ .

Now, let  $\text{EQSym}$  be the Hopf algebra of endofunctions [13]. This Hopf algebra is defined by its basis  $(\Psi_f)$  indexed by endofunctions, the product

$$\Psi_f \Psi_g = \sum_{\text{std}(\tilde{f})=f, \text{std}(\tilde{g})=g, \tilde{f} \cup \tilde{g} \in \text{End}} \Psi_{\tilde{f} \cup \tilde{g}} \quad (35)$$

and the coproduct

$$\Delta(\Psi_h) = \sum_{f \bullet g = h} \Psi_f \otimes \Psi_g. \quad (36)$$

This algebra is commutative but not cocommutative. We denote by  $\mathbf{ESym} := \text{EQSym}^*$  its graded dual, and by  $(\Phi_f)$  the basis of  $\mathbf{ESym}$  dual to  $(\Psi_f)$ . The bases  $(\Phi_\sigma)$  and  $(\Psi_\sigma)$  are respectively denoted by  $(S^\sigma)$  and  $(M_\sigma)$  in [13]. The product and the coproduct in  $\mathbf{ESym}$  are respectively given by

$$\Phi_f \Phi_g = \Phi_{f \bullet g} \quad (37)$$

and

$$\Delta(\Phi_h) = \sum_{f \cup g = h} \Phi_{\text{std}(f)} \otimes \Phi_{\text{std}(g)}. \quad (38)$$

*Remark :* The  $\Psi_f$ , where  $f$  is a bijective endofunction, span a Hopf subalgebra of  $\text{EQSym}$  obviously isomorphic to  $\mathfrak{S}\text{QSym} := \mathbf{ESym}^*$ , that is isomorphic to  $\text{CPIQSym}(0!, 1!, 2!, \dots)$  from (3.3.4).

As suggested by [13], we investigate a few other Hopf subalgebras of  $\text{EQSym}$ .

- The Hopf algebra of idempotent endofunctions is isomorphic to the Hopf algebra  $\text{CPIQSym}(1, 2, 3, \dots)$ . The explicit isomorphism sends  $\Psi_f$  to  $\Psi_{\phi(f)}$ , where for any idempotent endofunction  $f$  of size  $n$ ,

$$\phi(f) = \left\{ \left[ f^{-1}(i), \text{card}(\{j \in f^{-1}(i) \mid j \leq i\}) \right] \mid 1 \leq i \leq n, f^{-1}(i) \neq \emptyset \right\}. \quad (39)$$

- The Hopf algebra of involutive endofunctions is isomorphic to

$$\text{CPIQSym}(1, 1, 0, \dots, 0, \dots) \hookrightarrow \text{PIQSym}.$$

Namely, it is a Hopf subalgebra of  $\mathfrak{S}\text{QSym}$ , and the natural isomorphism from  $\mathfrak{S}\text{QSym}$  to  $\text{CPIQSym}(0!, 1!, 2!, \dots)$  sends it to the sub algebra  $\text{CPIQSym}(1, 1, 0, \dots, 0, \dots)$ .

- In the same way the endofunctions such that  $f^3 = \text{Id}$  generate a Hopf subalgebra of  $\mathfrak{S}\text{QSym} \hookrightarrow \text{EQSym}$  which is isomorphic to the Hopf algebra  $\text{CPIQSym}(1, 0, 2, 0, \dots, 0, \dots)$ .
- More generally, the endofunctions such that  $f^p = \text{Id}$  generate a Hopf subalgebra of  $\mathfrak{S}\text{QSym} \hookrightarrow \text{EQSym}$  isomorphic to  $\text{CPIQSym}(\tau(p))$  where  $\tau(p)_i = (i-1)!$  if  $i \mid p$  and  $\tau(p)_i = 0$  otherwise.

### 3.4 About specializations

The aim of this section is to show how the specialization  $c_n \longrightarrow \frac{a_n}{n!}$  factorizes through  $\text{PIQSym}$  and  $\text{CPIQSym}$ .

Notice first that the algebra  $\text{Sym}$  is isomorphic to the subalgebra of  $\text{PIQSym}$  generated by the family  $(\Psi_{\{1, \dots, n\}})_{n \in \mathbb{N}}$ ; the explicit isomorphism  $\alpha$  sends  $c_n$  to  $\Psi_{\{1, \dots, n\}}$ . The image of  $h_n$  is  $S_{\{1, \dots, n\}}$  and the image of  $c_\lambda = \frac{1}{\lambda!} c_{\lambda_1} \cdots c_{\lambda_k}$  is  $\sum_{\pi \models \lambda} \Psi_\pi$  where  $\pi \models \lambda$  means that  $\pi = \{\pi_1, \dots, \pi_k\}$  is a set partition such that  $\#\pi_1 = \lambda_1, \dots, \#\pi_k = \lambda_k$  and  $\lambda! = \frac{\lambda_1 \cdots \lambda_k}{z_\lambda} = \prod_i \mathbf{m}_i(\lambda)!$

where  $\mathbf{m}_i(\lambda)$  denotes the multiplicity of  $i$  in  $\lambda$ . Indeed,  $c_\lambda$  is mapped to  $\frac{1}{\lambda!} \Psi_{\{1, \dots, \lambda_1\}} \cdots \Psi_{\{1, \dots, \lambda_k\}}$  and  $\Psi_{\{1, \dots, \lambda_1\}} \cdots \Psi_{\{1, \dots, \lambda_k\}} = \lambda! \sum_{\pi \models \lambda} \Psi_\pi$ .

Now the linear map  $\beta_a : \Pi\text{QSym} \rightarrow \text{C}\Pi\text{QSym}(a)$  sending each  $\Psi_\pi$  to the element  $\sum_{\Pi \ni \pi} \Psi_\Pi$  is

a morphism of algebra and the subalgebra  $\widetilde{\Pi\text{QSym}} := \beta_a(\Pi\text{QSym})$  is isomorphic to  $\Pi\text{QSym}$  if and only if  $a \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ .

Let  $\gamma_a : \text{C}\Pi\text{QSym}(a) \rightarrow \mathbb{C}$  be the linear map sending  $\Psi_\Pi$  to  $\frac{1}{|\Pi|!}$ . We have

$$\gamma_a(\Psi_{\Pi_1} \Psi_{\Pi_2}) = \sum_{\substack{\Pi = \Pi'_1 \cup \Pi'_2, \Pi'_1 \cap \Pi'_2 = \emptyset \\ \text{std}(\Pi'_1) = \Pi_1, \text{std}(\Pi'_2) = \Pi_2}} \gamma_a(\Psi_\Pi).$$

From any subset  $A$  of  $\{1, 2, \dots, |\Pi_1| + |\Pi_2|\}$  of cardinality  $n$ , one has  $\text{std}(\Pi'_1) = \Pi_1$ ,  $\text{std}(\Pi'_2) = \Pi_2$  and  $\Pi'_1 \cap \Pi'_2 = \emptyset$ , where  $\Pi'_1$  is obtained from  $\Pi_1$  by replacing each label  $i$  by the  $i$ th smallest element of  $A$ , and  $\Pi'_2$  is obtained from  $\Pi_2$  by replacing each label  $i$  by the  $i$ th smallest element of  $\{1, \dots, |\Pi_1| + |\Pi_2|\} \setminus A$ . Since there are  $\binom{|\Pi_1| + |\Pi_2|}{|\Pi_1|}$  ways to construct  $A$ , one has

$$\gamma_a(\Psi_{\Pi_1} \Psi_{\Pi_2}) = \frac{1}{(|\Pi_1| + |\Pi_2|)!} \binom{|\Pi_1| + |\Pi_2|}{|\Pi_1|} = \frac{1}{|\Pi_1|! |\Pi_2|!} = \gamma_a(\Psi_{\Pi_1}) \gamma_a(\Psi_{\Pi_2}).$$

In other words,  $\gamma_a$  is a morphism of algebra. Furthermore, the restriction  $\hat{\gamma}_a$  of  $\gamma_a$  to  $\widetilde{\Pi\text{QSym}}$  is a morphism of algebra that sends  $\beta_a(\Psi_{\{1, \dots, n\}})$  to  $\frac{\alpha_n}{n!}$ . It follows that if  $f \in \text{Sym}$ , then one has

$$f(\mathbb{X}^{(a)}) = \tilde{\gamma}_a(\beta_a(\alpha(f))). \quad (40)$$

The following theorem summarizes the section:

**Theorem 3.20.** *The diagram*

$$\begin{array}{ccccc} \text{C}\Pi\text{QSym}(a) & \longleftarrow & \widetilde{\Pi\text{QSym}} & \xleftarrow{\beta_a} & \Pi\text{QSym} \\ & \searrow \gamma_a & \downarrow \hat{\gamma}_a & & \uparrow \alpha \\ & & \mathbb{C} = \text{Sym}[\mathbb{X}^{(a)}] & \longleftarrow & \text{Sym} \end{array}$$

is commutative.

## 4 Word Bell polynomials

### 4.1 Bell polynomials in $\Pi\text{QSym}$

Since  $\text{Sym}$  is isomorphic to the subalgebra of  $\Pi\text{QSym}$  generated by the elements  $\Psi_{\{1, \dots, n\}}$ , we can compute  $A_n(\Psi_{\{1\}}, \Psi_{\{1,2\}}, \dots, \Psi_{\{1, \dots, m\}}, \dots)$ . From (32), we have

$$A_n(1! \Psi_{\{1\}}, 2! \Psi_{\{1,2\}}, \dots, m! \Psi_{\{1, \dots, m\}}, \dots) = n! S_{\{1, \dots, n\}} = n! \sum_{\pi \models n} \Psi_\pi. \quad (41)$$

Notice that, from the previous section, the image of the Bell polynomial  $A_n(\Psi_{\{1\}}, \Psi_{\{1,2\}}, \dots, \Psi_{\{1, \dots, m\}}, \dots)$  by the morphism  $\gamma$  sending  $\Psi_{\{1, \dots, n\}}$  to  $\frac{1}{n!}$  is  $\gamma(A_n(1! \Psi_{\{1\}}, 2! \Psi_{\{1,2\}}, \dots, m! \Psi_{\{1, \dots, m\}}, \dots)) = b_n = A_n(1, 1, \dots)$ .

In the same way, we have

$$B_{n,k}(1! \Psi_{\{1\}}, 2! \Psi_{\{1,2\}}, \dots, m! \Psi_{\{1, \dots, m\}}, \dots) = n! \sum_{\substack{\pi \models n \\ \# \pi = k}} \Psi_\pi. \quad (42)$$

If  $(F_n)_n$  is a homogeneous family of elements of  $\Pi\text{QSym}$ , such that  $|F_n| = n$ , we define

$$\mathcal{A}_n(F_1, F_2, \dots) = \frac{1}{n!} A_n(1!F_1, 2!F_2, \dots, m!F_m, \dots) \quad (43)$$

and

$$\mathcal{B}_{n,k}(F_1, F_2, \dots) = \frac{1}{n!} B_{n,k}(1!F_1, 2!F_2, \dots, m!F_m, \dots). \quad (44)$$

Considering the map  $\beta_a \circ \alpha$  as a specialization of  $Sym$ , the following identities hold in  $\text{CPIQSym}(a)$ :

$$\mathcal{A}_n \left( \sum_{1 \leq i \leq a_1} \Psi_{\{\{\{1\}, i\}\}}, \sum_{1 \leq i \leq a_2} \Psi_{\{\{\{1,2\}, i\}\}}, \dots, \sum_{1 \leq i \leq a_m} \Psi_{\{\{\{1, \dots, m\}, i\}\}}, \dots \right) = \sum_{\Pi = n} \Psi_{\Pi}$$

and

$$\mathcal{B}_{n,k} \left( \sum_{1 \leq i \leq a_1} \Psi_{\{\{\{1\}, i\}\}}, \sum_{1 \leq i \leq a_2} \Psi_{\{\{\{1,2\}, i\}\}}, \dots, \sum_{1 \leq i \leq a_m} \Psi_{\{\{\{1, \dots, m\}, i\}\}}, \dots \right) = \sum_{\substack{\Pi = n \\ \#\Pi = k}} \Psi_{\Pi}.$$

**Example 4.1.** In  $\text{BPIQSym} \sim \text{CPIQSym}(1!, 2!, \dots)$ , we have

$$\mathcal{B}_{n,k} \left( \Psi_{\{\{1\}\}}, \Psi_{\{\{1,2\}\}} + \Psi_{\{\{2,1\}\}}, \dots, \sum_{\sigma \in \mathfrak{S}_m} \Psi_{\{\{\sigma\}\}}, \dots \right) = \sum_{\substack{\hat{\Pi} = n \\ \#\hat{\Pi} = k}} \Psi_{\hat{\Pi}},$$

where the sum on the right is over the set partitions of  $\{1, \dots, n\}$  into  $k$  lists. Considering the morphism sending  $\Psi_{\{\{\sigma_1, \dots, \sigma_n\}\}}$  to  $\frac{1}{n!}$ , Theorem 33 allows us to recover  $B_{n,k}(1!, 2!, 3!, \dots) = L_{n,k}$ , the number of set partitions of  $\{1, \dots, n\}$  into  $k$  lists.

**Example 4.2.** In  $\text{PIQSym}_{(2)} \sim \text{CPIQSym}(b_1, b_2, \dots)$ , we have

$$\mathcal{B}_{n,k} \left( \Psi_{\{\{\{1\}\}\}}, \Psi_{\{\{\{1,2\}\}\}} + \Psi_{\{\{\{1\}, \{2\}\}\}}, \dots, \sum_{\pi \in \mathfrak{S}_m} \Psi_{\{\{\pi\}\}}, \dots \right) = \sum_{\substack{\Pi \text{ partition of } \pi = n \\ \#\hat{\Pi} = k}} \Psi_{\Pi},$$

where the sum on the right is over the set partitions of  $\{1, \dots, n\}$  of level 2 into  $k$  blocks. Considering the morphism sending  $\Psi_{\{\{\pi\}\}}$  to  $\frac{1}{n!}$  for  $\pi \in \mathfrak{S}_n$ , Theorem 33 allows us to recover  $B_{n,k}(b_1, b_2, b_3, \dots) = S_{n,k}^{(2)}$ , the number of set partitions into  $k$  sets of a partition of  $\{1, \dots, n\}$ .

**Example 4.3.** In  $\mathfrak{S}\text{Sym}^* \sim \text{CPIQSym}(0!, 1!, 2!, \dots)$ , we have

$$\mathcal{B}_{n,k} \left( \Psi_{[1]}, \Psi_{[2,1]}, \Psi_{[2,3,1]} + \Psi_{[3,1,2]}, \dots, \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ is a cycle}}} \Psi_{\{\{\sigma\}\}}, \dots \right) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ has } k \text{ cycles}}} \Psi_{\sigma},$$

where the sum on the right is over the permutations of size  $n$  having  $k$  cycles. Considering the morphism sending  $\Psi_{\sigma}$  to  $\frac{1}{n!}$  for  $\sigma \in \mathfrak{S}_n$ , Theorem 33 allows us to recover  $B_{n,k}(0!, 1!, 2!, \dots) = s_{n,k}$ , the number of permutations of  $\mathfrak{S}_n$  having exactly  $k$  cycles.

**Example 4.4.** In the Hopf algebra of idempotent endofunctions, we have

$$\mathcal{B}_{n,k} \left( \Psi_{f_{1,1}}, \Psi_{f_{2,1}} + \Psi_{f_{2,2}}, \Psi_{f_{3,1}} + \Psi_{f_{3,2}} + \Psi_{f_{3,3}}, \dots, \sum_{i=1}^n \Psi_{f_{n,i}}, \dots \right) = \sum_{|f|=n, \text{card}(f(\{1, \dots, n\}))=k} \Psi_f,$$

where for  $i \geq j \geq 1$ ,  $f_{i,j}$  is the constant endofunction of size  $i$  and of image  $\{j\}$ . Here, the sum on the right is over idempotent endofunctions  $f$  of size  $n$  such that the cardinality of the image of  $f$  is  $k$ . Considering the morphism sending  $\Psi_f$  to  $\frac{1}{n!}$  for  $|f| = n$ , Theorem 33 allows us to recover that  $B_{n,k}(1, 2, 3, \dots)$  is the number of these idempotent endofunctions, that is the idempotent number  $\binom{n}{k} k^{n-k}$  [14, 25].

## 4.2 Bell polynomials in $\mathbf{WSym}$

Recursive descriptions of Bell polynomials are given in [9]. In this section we rewrite this result and other ones related to these polynomials in the Hopf algebra of word symmetric functions  $\mathbf{WSym}$ . We define the operator  $\partial$  acting linearly on the left on  $\mathbf{WSym}$  by

$$1\partial = 0 \text{ and } \Phi_{\{\pi_1, \dots, \pi_k\}} \partial = \sum_{i=1}^k \Phi_{(\{\pi_1, \dots, \pi_k\} \setminus \pi_i) \cup \{\pi_i \cup \{n+1\}\}}.$$

In fact, the operator  $\partial$  acts on  $\Phi_\pi$  almost as the multiplication of  $M_{\{\{1\}\}}$  on  $M_\pi$ . More precisely :

**Proposition 4.5.** *We have:*

$$\partial = \phi^{-1} \circ \mu \circ \phi - \mu,$$

where  $\phi$  is the linear operator satisfying  $M_\pi \phi = \Phi_\pi$  and  $\mu$  is the multiplication by  $\Phi_{\{\{1\}\}}$ .

**Example 4.6.** For instance, one has

$$\begin{aligned} \Phi_{\{\{1,3\}, \{2,4\}\}} \partial &= \Phi_{\{\{1,3\}, \{2,4\}\}} (\phi^{-1} \mu \phi - \mu) \\ &= M_{\{\{1,3\}, \{2,4\}\}} \mu \phi - \Phi_{\{\{1,3\}, \{2,4\}, \{5\}\}} \\ &= (M_{\{\{1,3,5\}, \{2,4\}\}} + M_{\{\{1,3\}, \{2,4,5\}\}} + M_{\{\{1,3\}, \{2,4\}, \{5\}\}}) \phi \\ &\quad - \Phi_{\{\{1,3\}, \{2,4\}, \{5\}\}} \\ &= \Phi_{\{\{1,3,5\}, \{2,4\}\}} + \Phi_{\{\{1,3\}, \{2,4,5\}\}} + \Phi_{\{\{1,3\}, \{2,4\}, \{5\}\}} \\ &\quad - \Phi_{\{\{1,3\}, \{2,4\}, \{5\}\}} \\ &= \Phi_{\{\{1,3,5\}, \{2,4\}\}} + \Phi_{\{\{1,3\}, \{2,4,5\}\}}. \end{aligned}$$

Following Remark 17, we define recursively the elements  $\mathfrak{A}_n$  of  $\mathbf{WSym}$  as

$$\mathfrak{A}_0 = 1, \quad \mathfrak{A}_{n+1} = \mathfrak{A}_n (\Phi_{\{\{1\}\}} + \partial). \quad (45)$$

So we have

$$\mathfrak{A}_n = 1 (\Phi_{\{\{1\}\}} + \partial)^n. \quad (46)$$

Easily, one shows that  $\mathfrak{A}_n$  provides an analogue of complete Bell polynomials in  $\mathbf{WSym}$ .

**Proposition 4.7.**

$$\mathfrak{A}_n = \sum_{\pi \models n} \Phi_\pi.$$

Noticing that the multiplication by  $\Phi_{\{\{1\}\}}$  adds one part to each partition, we give the following analogue for partial Bell polynomials.

**Proposition 4.8.** *If we set*

$$\mathfrak{B}_{n,k} = [t^k] 1 (t \Phi_{\{\{1\}\}} + \partial)^n, \quad (47)$$

then we have  $\mathfrak{B}_{n,k} = \sum_{\substack{\pi \models n \\ \#\pi = k}} \Phi_\pi$ .

**Example 4.9.** We have

$$\begin{aligned} 1(t\Phi_{\{1\}} + \partial)^4 &= t^4\Phi_{\{1\},\{2\},\{3\},\{4\}} + t^3(\Phi_{\{1,2\},\{3\},\{4\}} + \Phi_{\{1,3\},\{2\},\{4\}} \\ &+ \Phi_{\{1,4\},\{2,3\},\{4\}} + \Phi_{\{1,4\},\{2\},\{3\}} + \Phi_{\{1\},\{2,4\},\{3\}} + \Phi_{\{1\},\{2\},\{3,4\}}) + \\ &t^2(\Phi_{\{1,3,4\},\{2\}} + \Phi_{\{1,2,3\},\{4\}} + \Phi_{\{1,2,4\},\{3\}} + \Phi_{\{1,2\},\{3,4\}} + \Phi_{\{1,3\},\{2,4\}} \\ &+ \Phi_{\{1,4\},\{2,3\}} + \Phi_{\{1\},\{2,3,4\}}) + t\Phi_{\{1\},\{3\},\{4\},\{2\}}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathfrak{B}_{4,2} &= \Phi_{\{1,3,4\},\{2\}} + \Phi_{\{1,2,3\},\{4\}} + \Phi_{\{1,2,4\},\{3\}} + \Phi_{\{1,2\},\{3,4\}} \\ &+ \Phi_{\{1,3\},\{2,4\}} + \Phi_{\{1,4\},\{2,3\}} + \Phi_{\{1\},\{2,3,4\}}. \end{aligned}$$

### 4.3 Bell polynomials in $\mathbb{C}\langle\mathbb{A}\rangle$

Both  $\mathbf{WSym}$  and  $\Pi\mathbf{QSym}$  admit word polynomial realizations in a subspace  $\mathbf{WSym}(\mathbb{A})$  of the free associative algebra  $\mathbb{C}\langle\mathbb{A}\rangle$  over an infinite alphabet  $\mathbb{A}$ . When endowed with the concatenation product,  $\mathbf{WSym}(\mathbb{A})$  is isomorphic to  $\mathbf{WSym}$  and when endowed with the shuffle product, it is isomorphic to  $\Pi\mathbf{QSym}$ . Alternatively to the definitions of partial Bell numbers in  $\Pi\mathbf{QSym}$  (44) and in  $\mathbf{WSym}$ , we set, for any sequence of polynomials  $(F_i)_{i \in \mathbb{N}}$  in  $\mathbb{C}\langle\mathbb{A}\rangle$ :

$$\sum_{n \geq 0} \mathbf{B}_{n,k}(F_1, \dots, F_m, \dots) t^n = \frac{1}{k!} \left( \sum_i F_i t^i \right)^{\sqcup k} \quad (48)$$

and

$$\mathbf{A}_n(F_1, \dots, F_m, \dots) = \sum_{k \geq 1} \mathbf{B}_{n,k}(F_1, \dots, F_m, \dots). \quad (49)$$

This definition generalizes (44) and (47) in the following sense:

**Proposition 4.10.** *We have*

$$\mathbf{B}_{n,k}(\Psi_{\{1\}}(\mathbb{A}), \dots, \Psi_{\{1, \dots, m\}}(\mathbb{A}), \dots) = \mathcal{B}_{n,k}(\Psi_{\{1\}}(\mathbb{A}), \dots, \Psi_{\{1, \dots, m\}}(\mathbb{A}), \dots)$$

and

$$\mathbf{B}_{n,k}(\Phi_{\{1\}}(\mathbb{A}), \dots, \Phi_{\{1, \dots, m\}}(\mathbb{A}), \dots) = \mathfrak{B}_{n,k}(\mathbb{A}).$$

*Proof.* The two identities follow from

$$\Psi_{\pi_1}(\mathbb{A}) \sqcup \Psi_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi = \pi'_1 \cup \pi'_2, \pi'_1 \cap \pi'_2 = \emptyset \\ \text{std}(\pi'_1) = \pi_1, \text{std}(\pi'_2) = \pi_2}} \Psi_{\pi}(\mathbb{A}).$$

■

Equality (48) allows us to show more general properties. For instance, let  $\mathbb{A}'$  and  $\mathbb{A}''$  be two disjoint subalphabets of  $\mathbb{A}$  and set

$$S_n^{\mathbb{A}'}(\mathbb{A}'') = S_{\{1\}}(\mathbb{A}') \sqcup S_{\{1, \dots, n-1\}}(\mathbb{A}'').$$

Remarking that

$$\begin{aligned} \sum_n \mathbf{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''), \dots, S_m^{\mathbb{A}'}(\mathbb{A}''), \dots) t^n &= \\ t^k S_{\{1\}, \dots, \{k\}}(\mathbb{A}') \sqcup \left( \sum_{n \geq 0} S_{\{1, \dots, n\}}(\mathbb{A}'') t^n \right)^{\sqcup k} &= t^k S_{\{1\}, \dots, \{k\}}(\mathbb{A}') \sqcup \sigma_t^W(k\mathbb{A}''), \end{aligned}$$

we obtain a word analogue of (19):

**Proposition 4.11.**

$$\mathbb{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''), \dots, S_m^{\mathbb{A}'}(\mathbb{A}''), \dots) = S_{\{\{1\}, \dots, \{k\}\}}(\mathbb{A}') \sqcup S_{\{\{1, \dots, n-k\}\}}(k\mathbb{A}'').$$

For simplicity, let us write  $\mathbb{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'') := \mathbb{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''), \dots, S_m^{\mathbb{A}'}(\mathbb{A}''), \dots)$ .

Let  $k = k_1 + k_2$ . From

$$S_{\{\{1\}, \dots, \{k_1\}\}}(\mathbb{A}') \sqcup S_{\{\{1\}, \dots, \{k_2\}\}}(\mathbb{A}') = \binom{k}{k_1} S_{\{\{1\}, \dots, \{k\}\}}(\mathbb{A}')$$

and

$$S_{\{\{1, \dots, n-k\}\}}(k\mathbb{A}'') = \sum_{i+j=n-k} S_{\{\{1, \dots, i\}\}}(k_1\mathbb{A}'') \sqcup S_{\{\{1, \dots, i\}\}}(k_2\mathbb{A}''),$$

we deduce an analogue of (20):

**Corollary 4.12.** *Let  $k = k_1 + k_2$  be three nonnegative integers. We have*

$$\binom{k}{k_1} \mathbb{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'') = \sum_{i=0}^n \mathbb{B}_{i,k_1}^{\mathbb{A}'}(\mathbb{A}'') \sqcup \mathbb{B}_{n-i,k_2}^{\mathbb{A}'}(\mathbb{A}''). \quad (50)$$

**Example 4.13.** Consider a family of functions  $(f_k)_k$  such that  $f_k : \mathbb{N} \rightarrow \mathbb{C}\langle \mathbb{A} \rangle$  satisfying

$$f_0 = 1 \text{ and } f_n(\alpha + \beta) = \sum_{n=i+j} f_i(\alpha) \sqcup f_j(\beta). \quad (51)$$

From (48), we obtain

$$\mathbb{B}_{n,k}(f_0(a), \dots, f_{m-1}(a), \dots) t^n = \frac{1}{k!} \sum_{i_1 + \dots + i_k = n-k} f_{i_1}(a) \sqcup \dots \sqcup f_{i_k}(a).$$

Hence, iterating (51), we deduce

$$\mathbb{B}_{n,k}(f_0(a), \dots, f_{m-1}(a), \dots) = \frac{1}{k!} f_{n-k}(ka).$$

Set  $f_n(k) = k! \mathbb{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'')$  and  $f_0(k) = 1$ . By (50), the family  $(f_n)_{n \in \mathbb{N}}$  satisfies (51). Hence we obtain an analogue of (24):

$$k_1! \mathbb{B}_{n,k_1}(1, \dots, k_2! \mathbb{B}_{m-1,k_2}^{\mathbb{A}'}(\mathbb{A}''), \dots) = (k_1 k_2)! \mathbb{B}_{n-k_1, k_1 k_2}^{\mathbb{A}'}(\mathbb{A}'').$$

Suppose now  $\mathbb{A}'' = \mathbb{A}_1'' + \mathbb{A}_2''$ .

By  $S_{\{\{1, \dots, n\}\}}(\mathbb{A}'') = \sum_{i=0}^n S_{\{\{1, \dots, i\}\}}(\mathbb{A}_1'') \sqcup S_{\{\{1, \dots, n-i\}\}}(\mathbb{A}_2'')$ , Proposition (44) allows us to write a word analogue of the convolution formula for Bell polynomials (21).

**Corollary 4.14.**

$$S_{\{\{1\}, \dots, \{k\}\}}(\mathbb{A}') \sqcup \mathbb{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'') = \sum_{i=0}^n \mathbb{B}_{i,k}^{\mathbb{A}'}(\mathbb{A}_1'') \sqcup \mathbb{B}_{n-i,k}^{\mathbb{A}'}(\mathbb{A}_2''). \quad (52)$$

Let  $k_1$  and  $k_2$  be two positive integers. We have

$$\begin{aligned} & \sum_n \mathbb{B}_{n,k_1}(\mathbb{B}_{k_2,k_2}^{\mathbb{A}'}(\mathbb{A}''), \dots, \mathbb{B}_{k_2+m-1,k_2}^{\mathbb{A}'}(\mathbb{A}''), \dots) t^n = \\ & \frac{1}{k_1!} \left( \sum_{m \geq 1} \mathbb{B}_{k_2+m-1,k_2}^{\mathbb{A}'}(\mathbb{A}'') t^m \right)^{\sqcup k_1} = t^{k_1} S_{\{\{1\}, \dots, \{k_2 k_1\}\}}(\mathbb{A}') \sqcup \sigma_t^W(k_1 k_2 \mathbb{A}''). \end{aligned}$$

Hence,

**Proposition 4.15.**

$$\mathbb{B}_{n,k_1}(\mathbb{B}_{k_2,k_2}^{\mathbb{A}'}(\mathbb{A}''), \dots, \mathbb{B}_{k_2+m-1,k_2}^{\mathbb{A}'}(\mathbb{A}''), \dots) = \mathbb{B}_{n-k_1+k_1 k_2, k_1 k_2}^{\mathbb{A}'}(\mathbb{A}''). \quad (53)$$

## 4.4 Specialization again

In [2] we have shown that one can construct a double algebra which is homomorphic to  $(\mathbf{WSym}(\mathbb{A}), \cdot, \sqcup)$ . This is a general construction which is an attempt to define properly the concept of virtual alphabet for  $\mathbf{WSym}$ . In our context the construction is simpler, let us recall briefly it.

Let  $F = (F_\pi(\mathbb{A}))_\pi$  be a basis of  $\mathbf{WSym}(\mathbb{A})$ . We will say that  $F$  is *shuffle-compatible* if

$$F_{\{\pi_1, \dots, \pi_k\}}(\mathbb{A}) = \sqcup_{[\pi_1, \dots, \pi_k]} (F_{\{\{1, \dots, \#\pi_1\}\}}(\mathbb{A}), \dots, F_{\{\{1, \dots, \#\pi_k\}\}}(\mathbb{A})).$$

Hence, one has

$$F_{\pi_1}(\mathbb{A}) \sqcup F_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi = \pi'_1 \cup \pi'_2, \pi'_1 \cap \pi'_2 = \emptyset \\ \text{std}(\pi'_1) = \pi_1, \text{std}(\pi'_2) = \pi_2}} F_\pi \text{ and } F_{\pi_1}(\mathbb{A}) \cdot F_{\pi_2}(\mathbb{A}) = F_{\pi_1 \uplus \pi_2}(\mathbb{A}).$$

**Example 4.16.** The bases  $(S_\pi(\mathbb{A}))_\pi$ ,  $(\Phi_\pi(\mathbb{A}))_\pi$  and  $(\Psi_\pi(\mathbb{A}))_\pi$  are shuffle-compatible but not the basis  $(M_\pi(\mathbb{A}))_\pi$ .

Straightforwardly, one has:

**Claim 4.17.** Let  $(F_\pi(\mathbb{A}))_\pi$  be a shuffle-compatible basis of  $\mathbf{WSym}(\mathbb{A})$ . Let  $\mathbb{B}$  be another alphabet and let  $P = (P_k)_{k>0}$  be a family of noncommutative polynomials of  $\mathbb{C}\langle \mathbb{B} \rangle$  such that  $\deg P_k = k$ . Then, the space spanned by the polynomials  $F_{\{\pi_1, \dots, \pi_k\}}[\mathbb{A}_F^{(P)}] := \sqcup_{[\pi_1, \dots, \pi_k]} (P_{\#\pi_1}, \dots, P_{\#\pi_k})$  is stable under concatenation and shuffle product in  $\mathbb{C}\langle \mathbb{B} \rangle$ . So it is a double algebra which is homomorphic to  $(\mathbf{WSym}(\mathbb{A}), \cdot, \sqcup)$ . We will call  $\mathbf{WSym}[\mathbb{A}_F^{(P)}]$  this double algebra and  $f[\mathbb{A}_F^{(P)}]$  will denote the image of an element  $f \in \mathbf{WSym}(\mathbb{A})$  by the morphism  $\mathbf{WSym}(\mathbb{A}) \rightarrow \mathbf{WSym}[\mathbb{A}_F^{(P)}]$  sending  $F_{\{\pi_1, \dots, \pi_k\}}$  to  $F_{\{\pi_1, \dots, \pi_k\}}[\mathbb{A}_F^{(P)}]$ .

With these notations, one has

$$\mathbf{B}_{n,k}(P_1, \dots, P_m, \dots) = \mathfrak{B}_{n,k}[\mathbb{A}_\Phi^{(P)}].$$

**Example 4.18.** We define a specialization by setting

$$\Phi_{\{\{1, \dots, n\}\}}[\mathfrak{S}] = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma_1 = 1}} \mathbf{b}_{\sigma[1]} \dots \mathbf{b}_{\sigma[n]},$$

where the letters  $\mathbf{b}_i$  belong to an alphabet  $\mathbb{B}$ . Let  $\sigma \in \mathfrak{S}_n$  be a permutation and  $\sigma = c_1 \circ \dots \circ c_k$  its decomposition into cycles. Each cycle  $c^{(i)}$  is denoted by a sequence of integers  $(n_1^{(i)}, \dots, n_{\ell_i}^{(i)})$  such that  $n_1^{(i)} = \min\{n_1^{(i)}, \dots, n_{\ell_i}^{(i)}\}$ . Let  $\widetilde{c}^{(i)} \in \mathfrak{S}_{\ell_i}$  be the permutation which is the standardized of the sequence  $n_1^{(i)} \dots n_{\ell_i}^{(i)}$ . The cycle support of  $\sigma$  is the partition

$$\text{support}(\sigma) = \{\{n_1^{(1)}, \dots, n_{\ell^{(1)}}^{(1)}\}, \dots, \{n_1^{(k)}, \dots, n_{\ell^{(k)}}^{(k)}\}\}.$$

We define  $w[c^{(i)}] = \mathbf{b}_{\widetilde{c}^{(i)}[1]} \dots \mathbf{b}_{\widetilde{c}^{(i)}[\ell^{(i)}]}$  and  $w[\sigma] = \sqcup_{[\pi_1, \dots, \pi_k]} (w[c^{(1)}], \dots, w[c^{(k)}])$  where  $\pi_i = \{n_1^{(i)}, \dots, n_{\ell^{(i)}}^{(i)}\}$  for each  $1 \leq i \leq k$ .

For instance, if  $\sigma = 312654 = (132)(46)(5)$  we have  $w[(132)] = \mathbf{b}_1 \mathbf{b}_3 \mathbf{b}_2$ ,  $w[(46)] = \mathbf{b}_1 \mathbf{b}_2$ ,  $w[(5)] = \mathbf{b}_1$  and  $w[\sigma] = \mathbf{b}_1 \mathbf{b}_3 \mathbf{b}_2 \mathbf{b}_1 \mathbf{b}_1 \mathbf{b}_2$ .

So, we have

$$\mathbf{B}_{n,k}(\Phi_{\{\{1\}\}}[\mathfrak{S}], \dots, \Phi_{\{\{1, \dots, m\}\}}[\mathfrak{S}], \dots) = \mathfrak{B}_{n,k}[\mathfrak{S}] = \sum_{\substack{\pi \models n \\ \#\pi = k}} \Phi_\pi[\mathfrak{S}] = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \#\text{support}(\sigma) = k}} w[\sigma].$$

For instance

$$\begin{aligned}
& \mathbf{B}_{4,2}(\mathbf{b}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 + \mathbf{b}_1\mathbf{b}_3\mathbf{b}_2, \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4 + \mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_4 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_4\mathbf{b}_3 + \mathbf{b}_1\mathbf{b}_3\mathbf{b}_4\mathbf{b}_2 + \mathbf{b}_1\mathbf{b}_4\mathbf{b}_2\mathbf{b}_3 + \mathbf{b}_1\mathbf{b}_4\mathbf{b}_3\mathbf{b}_2, \dots) \\
&= \Phi_{\{\{1\}, \{2,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{2\}, \{1,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{3\}, \{1,2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{4\}, \{1,2,3\}\}}[\mathfrak{S}] + \Phi_{\{\{1,2\}, \{3,4\}\}}[\mathfrak{S}] + \\
&\Phi_{\{\{1,3\}, \{2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{1,4\}, \{2,3\}\}}[\mathfrak{S}] \\
&= 2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 + 2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_3\mathbf{b}_2 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_1\mathbf{b}_3 + \mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_1 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_1 + \mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_1 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_1\mathbf{b}_2 + 2\mathbf{b}_1\mathbf{b}_1\mathbf{b}_2\mathbf{b}_2.
\end{aligned}$$

Notice that the sum of the coefficients of the words occurring in the expansion of  $\mathfrak{B}_{n,k}[\mathfrak{S}]$  is equal to the Stirling number  $s_{n,k}$ . Hence, this specialization gives another word analogue of formula (7).

## 5 Conclusion

In brief, we have shown that many identities on Bell polynomials come from algebraic and cogebral structures. The algebra of symmetric functions provides an interesting frame for the study of the Bell polynomials. Indeed, the mechanism of specialization makes it possible to reinterpret some classical formulæ. Three coproducts  $p_n \rightarrow p_n(\mathbb{X}) + p_n(\mathbb{Y})$ ,  $p_n \rightarrow p_n(\mathbb{X})p_n(\mathbb{Y})$  and  $h_n \rightarrow h_n(\mathbb{X} \circ \mathbb{Y})$  have been investigated in this context. Each of them makes it possible to prove and understand a lot of formulæ.

The mechanism of specialization has been described in other combinatorial Hopf algebras in order to obtain some equalities on word Bell polynomials with no multiplicities by manipulating directly the combinatorial objects. These formulæ project on known identities. Analogues of the first coproduct and its consequences on word Bell polynomials have also been studied. It remains to investigate word analogues of the other two coproducts. In particular, it should be interesting to define a word Faà di Bruno algebra.

## Acknowledgments

This paper is partially supported by the PHC MAGHREB project IThèM and the ANR project CARMA.

## References

- [1] Sadek Bouroubi and Moncef Abbas. New identities for Bell's polynomials. New approaches. In *Rostock. Math. Kolloq*, volume 61, pages 49–55, 2006.
- [2] Jean-Paul Bultel, Ali Chouria, Jean-Gabriel Luque, and Olivier Mallet. Word symmetric functions and the Redfield-Pólya theorem. *FPSAC'2013, DMTCS Proceedings*, (01):563–574, 2013.
- [3] Eric Temple Bell. Exponential polynomials. *Annals of Mathematics*, pages 258–277, 1934.
- [4] Nantel Bergeron, Christophe Reutenauer, Mercedes Rosas, and Mike Zabrocki. Invariants and coinvariants of the symmetric groups in noncommuting variables. *Canad. J. Math.*, 60(2):266–296, 2008.
- [5] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert  $W$  function. *Advances in Computational Mathematics*, 5:329–359, 1996.
- [6] Louis Comtet. *Advanced Combinatorics: The art of finite and infinite expansions*. Springer, 1974.
- [7] Djurdje Cvijović. New identities for the partial Bell polynomials. *Applied Mathematics Letters*, 24(9):1544–1547, 2011.



- [8] Peter Doubilet. A Hopf algebra arising from the lattice of partitions of a set. *Journal of Algebra*, 28(1):127–132, 1974.
- [9] Kurusch Ebrahimi-Fard, Alexander Lundervold, and Dominique Manchon. Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras. *International Journal of Algebra and Computation*, 24(05):671–705, 2014.
- [10] Francesco Faà di Bruno. Sullo sviluppo delle funzioni. *Annali di Scienze Matematiche e Fisiche*, 6:479–480, 1855.
- [11] Francesco Faà di Bruno. Note sur une nouvelle formule de calcul différentiel. *Quarterly J. Pure Appl. Math.*, 1:359–360, 1857.
- [12] Robert Grossman and Richard G Larson. Hopf-algebraic structure of families of trees. *Journal of Algebra*, 126(1):184–210, 1989.
- [13] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Commutative combinatorial Hopf algebras. *Journal of Algebraic Combinatorics*, 28(1):65–95, 2008.
- [14] Bernhard Harris and Lowell Schoenfeld. The number of idempotent elements in symmetric semigroups. *Journal of Combinatorial Theory*, 3(2):122–135, 1967.
- [15] SA Joni and Gian-Carlo Rota. Coalgebras and bialgebras in combinatorics. *Stud. Appl. Math.*, 61(2):93–139, 1979.
- [16] S. Khelifa and Y. Cherruault. New results for the Adomian method. *Kybernetes.*, 29(n3):332–355, 2000.
- [17] A. Lascoux. *Symmetric Functions and Combinatorial Operators on Polynomials*. CBMS Regional Conference Series in Mathematics, American Mathematical Society, 2003.
- [18] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 1998.
- [19] Miloud Mihoubi. Bell polynomials and binomial type sequences. *Discrete Mathematics*, 308(12):2450–2459, 2008.
- [20] Miloud Mihoubi. Polynômes multivariés de Bell et polynômes de type binomial. *Thèse de Doctorat, L’Université des Sciences et de la Technologie Houari Boumediene*, 2008.
- [21] Miloud Mihoubi. Partial Bell polynomials and inverse relations. *J. Integer Seq.*, 13(4), 2010.
- [22] John Riordan. Derivatives of composite functions. *Bulletin of American Mathematical Society*, 52:664–667, 1946.
- [23] John Riordan. *An Introduction to Combinatorial Analysis*. John Wiley & Sons, New York, 1958 ; Princeton University Press, Princeton, NJ, 1980.
- [24] N. J. A. Sloane. The on-line encyclopedia of integer sequences, 2003.
- [25] Melvin Tainiter. Generating functions on idempotent semigroups with application to combinatorial analysis. *Journal of Combinatorial Theory*, 5(3):273–288, 1968.
- [26] Margarete C Wolf. Symmetric functions of non-commutative elements. *Duke Mathematical Journal*, 2(4):626–637, 1936.
- [27] Weiping Wang and Tianming Wang. General identities on Bell polynomials. *Computers and Mathematics with Applications*, 58(1):104–118, 2009.