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ON NON-REALIZATION RESULTS AND CONJECTURES OF N. KUHN

NGUYEN THE CUONG, GÉRALD GAUDENS, GEOFFREY POWELL, AND LIONEL SCHWARTZ

Abstract. We discuss two extensions of results conjectured by Nick Kuhn about the non-realization of unstable algebras as the mod p singular cohomology of a space, for p a prime. The first extends and refines earlier work of the second and fourth authors, using Lannes' mapping space theorem. The second (for the prime 2) is based on an analysis of the -1 and -2 columns of the Eilenberg-Moore spectral sequence, and of the associated extension.

In both cases, the statements and proofs use the relationship between the categories of unstable modules and functors between $\mathbb{F}_p$-vector spaces. The second result in particular exhibits the power of the functorial approach.

1. Introduction

Let p be a prime number, $\mathcal{U}$ denote the category of unstable modules and $\mathcal{K}$ the category of unstable algebras over the mod p Steenrod algebra $\mathcal{A}_p$ [Sch94]. The mod p singular cohomology of a space X is denoted H^*X.

In the first part of the paper, the topological spaces X considered are p-complete and connected. We assume that H^*X is of finite type (finite dimensional in each degree) and, moreover, that Jean Lannes' functor T_V acts nicely on H^*X, in the sense that $T_V(H^*X)$ is of finite type for all V. In order to apply Lannes' theory [Lan92] we also suppose the spaces considered are 1-connected and that $T_V(H^*X)$ is 1-connected for all V. For the current arguments, the connectivity hypothesis is not a significant restriction, since it is always possible to collapse the 1-skeleton. The finiteness hypotheses can be relaxed using methods of Fabien Morel, as explained by François-Xavier Dehon and Gérald Gaudens [DG03].

In an earlier paper[1], the second and fourth authors gave a proof of the following result, Nick Kuhn's realization conjecture [Kuh95b]:

Theorem 1.1. [GS13] *Let X be a space such that H^*X is finitely-generated as an $\mathcal{A}_p$-module, then H^*X is finite.*

This result is a consequence of the following stronger result, Kuhn's strong realization conjecture [Kuh95b], which uses the Krull filtration

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}$$

of the category $\mathcal{U}$ (see Section 2); in particular, $\mathcal{U}_0$ is the full subcategory of locally finite unstable modules.

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[1][GS13] was published two years late, due to an email lost by the second of the two authors.

Theorem 1.2. [GS13] *Let X be a space such that $H^*X \in \mathcal{U}_n$ for some $n \in \mathbb{N}$, then $H^*X \in \mathcal{U}_0$*

The aim of this paper is to extend these results in two directions. The first exploits the nilpotent filtration (see Section 3)

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots$$

of the category of unstable modules.

Here $\mathcal{N}il_1$ is the full subcategory of nilpotent unstable modules. For $p = 2$, an unstable module is nilpotent if the operator $\mathrm{Sq}_0 \colon x \mapsto \mathrm{Sq}^{|x|}x$ acts nilpotently on any element (a similar definition applies for p odd), hence, by the restriction axiom, a connected unstable algebra is nilpotent if and only if its augmentation ideal is a nilpotent unstable module.

Recall that an unstable module is reduced if it contains no non-trivial nilpotent submodule. The following result explains how unstable modules are built from (suspensions of) reduced unstable modules:

Proposition 1.3. [Sch94, Kuh95b] *An unstable module M has a natural, convergent decreasing filtration $\{\mathrm{nil}_s M\}_{s\geq 0}$ with $\mathrm{nil}_s M \in \mathcal{N}il_s$ and $\mathrm{nil}_s M/\mathrm{nil}_{s+1} M \cong \Sigma^s R_s M$, where $R_s M$ is a reduced unstable module.*

A reduced unstable module M is said to have degree $n \in \mathbb{N}$ (written $\deg(M) = n$) if $M \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, otherwise $\deg(M) = \infty$. Following Kuhn, define the profile function $w_M \colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ of an unstable module M by

$$w_M(i) := \deg(R_i M).$$

Recall that the module of indecomposable elements $QH^*X := \tilde{H}^*X/(\tilde{H}^*X)^2$, of the cohomology of a space X is an unstable module. Set $w_X := w_{H^*X}$ and $q_X := w_{QH^*X}$.

The following theorem provides a generalization of Theorem 1.2 and stresses the relationship between the profile functions w_X and q_X (which is examined in greater detail in Section 4). A version of the theorem was announced with a sketch proof in [GS12]; the current statement strengthens and unifies existing results.

Theorem 1.4. *Let X be a space such that $\tilde{H}^*X$ is nilpotent. The following conditions are equivalent:*

(1) $H^*X \in \mathcal{U}_0$;
(2) $w_X = 0$;
(3) $q_X = 0$;
(4) $w_X \leq \mathrm{Id}$;
(5) $q_X \leq \mathrm{Id}$;
(6) $w_X - \mathrm{Id}$ *is bounded.*

To see that this result implies Theorem 1.2, let X be a space such that $H^*(X) \in \mathcal{U}_n$. This condition implies easily (see Sections 2 and 3) that $w_{\Sigma X} - \mathrm{Id}$ is bounded, hence $H^*(\Sigma X)$ is locally finite, by Theorem 1.4, thus H^*X also.

Theorem 1.4 provides evidence for the following:

Conjecture 1. Let X be a space such that $\tilde{H}^*X$ is nilpotent. If q_X is bounded, then $H^*X \in \mathcal{U}_0$.

This should be compared with the following stronger conjecture, which is equivalent to the unbounded strong realization conjecture of [Kuh95b]:

Conjecture 2. Let X be a space such that $\tilde{H}^*X$ is nilpotent. If q_X takes finite values, then $H^*X \in \mathcal{U}_0$.

The second generalization concerns the first and second layers of the nilpotent filtration. The method of proof is of independent interest and can be applied in other situations; a generalization of this approach may lead to a proof of Conjecture 1.

The fact that the argument is based upon the Eilenberg-Moore spectral sequence for computing $H^*\Omega X$ from H^*X means that the restrictions upon the space X can be relaxed in the following theorem, in which the prime is taken to be 2.

Theorem 1.5. *Let X be a 1-connected space such that $\tilde{H}^*X$ is of finite type and nilpotent. If $\deg(R_1H^*X) = d \in \mathbb{N}$ then $\deg(R_2H^*X) \geq 2d$.*

The result proved is slightly stronger, giving a precise statement on the cup product on H^*X (see Remark 6.14).

The strategy of proof for Theorem 1.5 is different from that of the previous results and is related to that of [Sch98]. It depends on an analysis of the second stage of the Eilenberg-Moore filtration of $H^*\Omega X$ and using results on triviality and non-triviality of certain extension groups $\mathrm{Ext}^1_{\mathcal{F}}(-,-)$, where $\mathcal{F}$ is the category of functors on $\mathbb{F}_2$-vector spaces (see Section 2).

The paper is organized as follows. The Krull filtration is reviewed in Section 2 and the nilpotent filtration in Section 3; using this material, the profile functions are considered in Section 4. Theorem 1.4 is proved in Section 5 based on Lannes' theory, which is reviewed rapidly. Section 6 is devoted to the proof of Theorem 1.5, using the Eilenberg-Moore spectral sequence.

Remark 1.6. A first version of this work was made available by three of the authors in [NGS14]. The third-named author proposed the current approach in Section 6.

2. The Krull filtration on $\mathcal{U}$

Gabriel [Gab62] introduced the Krull filtration of an abelian category; for the category of unstable modules, $\mathcal{U}$, this gives the filtration

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}$$

by thick subcategories stable under colimits, which is described in [Sch94].

The category $\mathcal{U}_0$ is the largest thick sub-category generated by the simple objects and stable under colimits, which identifies with the subcategory of locally finite modules ($M \in \mathcal{U}$ is locally finite if $\mathcal{A}_p x$ is finite for any $x \in M$). The categories $\mathcal{U}_n$ are then defined recursively as follows. Having defined $\mathcal{U}_n$, form the quotient category $\mathcal{U}/\mathcal{U}_n$; then $\mathcal{U}_{n+1}$ is the pre-image under the canonical projection $\mathcal{U} \to \mathcal{U}/\mathcal{U}_n$ of the subcategory $(\mathcal{U}/\mathcal{U}_n)_0 \subset (\mathcal{U}/\mathcal{U}_n)$ (which is defined as above).

For M an unstable module and $n \in \mathbb{N}$, k_nM denotes the largest sub-module of M that is in $\mathcal{U}_n$ (see [Kuh13], for example).

Proposition 2.1. *For $M \in \mathcal{U}$, $M = \cup_n k_n M$.*

Example 2.2. For $k, n \in \mathbb{N}$, $\Sigma^k F(n) \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, where $F(n)$ is the free unstable module on a generator of degree n. (In the terminology of the introduction, $\deg F(n) = n$.)

Proposition 2.3. *(*[Sch94, 6.1.4] *and* [Kuh95b]*.) If M is a finitely-generated unstable module,*

(1) *there exists $d \in \mathbb{N}$ such that $M \in \mathcal{U}_d$;*
(2) *for each $n \in \mathbb{N}$, $R_n M$ is finitely-generated;*
(3) *the nilpotent filtration of M is finite (*$\mathrm{nil}_s M = 0$ *for $s \gg 0$).*

There is a characterization of the Krull filtration in terms of Lannes' T-functor. Recall that the functor T_V, for V an elementary abelian p-group, is left adjoint to $M \mapsto H^*BV \otimes M$; $T_{\mathbb{F}_p}$ is denoted simply T. Since $H^*B\mathbb{Z}/p$ splits in $\mathcal{U}$ as $\mathbb{F}_p \oplus \tilde{H}^*B\mathbb{Z}/p$, the functor T is naturally equivalent to $Id \oplus \bar{T}$, where $\bar{T}$ is left adjoint to $\tilde{H}^*B\mathbb{Z}/p \otimes -$.

Theorem 2.4. [Lan92, Sch94] *The functor T_V is exact and commutes with colimits; moreover there is a canonical isomorphism*

$$T_V(M_1 \otimes M_2) \cong T_V(M_1) \otimes T_V(M_2);$$

in particular, $T_V(\Sigma M) \cong \Sigma T_V(M)$.

Theorem 2.5. [Sch94, Kuh13] *The following are equivalent:*

(1) $M \in \mathcal{U}_n$,
(2) $\bar{T}^{n+1} M = 0$.

Corollary 2.6. *If $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$ then $M \otimes N \in \mathcal{U}_{m+n}$.*

There is also a combinatorial characterization of modules in $\mathcal{U}_n$ which are of finite type. This is stated here for $p = 2$; there are analogous results for odd primes. Denote by $\alpha(k)$ the sum of the digits in the 2-adic expansion of k.

Theorem 2.7. [Sch94, Sch06] *For M an unstable $\mathcal{A}_2$-module and $n \in \mathbb{N}$,*

(1) *if M is reduced, $M \in \mathcal{U}_n$ if and only if $M^j = 0$ for $\alpha(j) > n$;*
(2) *if M is finitely-generated, $M \in \mathcal{U}_n$ if and only if its Poincaré series $\Sigma_i a_i t^i$ has the following property: there exits $k \in \mathbb{N}$ such that, if $a_d \neq 0$, then $\alpha(d-i) \leq n$, for some $0 \leq i \leq k$.*

Let $\mathcal{F}$ be the category of functors from finite dimensional $\mathbb{F}_p$-vector spaces to $\mathbb{F}_p$-vector spaces. There is an exact functor [HLS93] $f\colon \mathcal{U} \longrightarrow \mathcal{F}$ defined by $f(M)(V) := \mathrm{Hom}_{\mathcal{U}}(M, H^*(BV))' = T_V(M)^0$ which induces an embedding of $\mathcal{U}/\mathcal{N}il_1$ in $\mathcal{F}$.

The functor $\bar{T}$ corresponds to the difference functor $\Delta : \mathcal{F} \to \mathcal{F}$ which is defined on $F \in \mathcal{F}$ by

$$\Delta(F)(V) := \mathrm{Ker}\big(F(V \oplus \mathbb{F}_p) \longrightarrow F(V)\big).$$

Namely, for $M \in \mathcal{U}$, $\Delta(fM) \cong f(\bar{T}M)$.

Let $\mathcal{F}_n \subset \mathcal{F}$ be the subcategory of polynomial functors of degree at most n, defined as the full subcategory of functors F such that $\Delta^{n+1}(F) = 0$. The polynomial degree of a functor F is written $\deg F \in \mathbb{N} \cup \{\infty\}$.

By Theorem 2.5, the following holds:

Proposition 2.8. *For $n \in \mathbb{N}$, the functor $f : \mathcal{U} \to \mathcal{F}$ restricts to an exact functor $f : \mathcal{U}_n \to \mathcal{F}_n$.*

Corollary 2.9. [Sch94] *For M a reduced module, the following are equivalent*

(1) *$M \in \mathcal{U}_d$;*
(2) *$deg(fM) \leq d$;*
(3) *$M^j = 0$ for all j such that $\alpha(j) > d$.*

3. The nilpotent filtration

The main results of the paper concern spaces X such that the positive degree elements of the cohomology H^*X are nilpotent; by the restriction axiom for unstable algebras this corresponds to $\tilde{H}^*X$ being nilpotent as an unstable module.

For $p = 2$, the following definition applies, where Sq_0 is the operator $x \mapsto \mathrm{Sq}^{|x|}(x)$.

Definition 3.1. An unstable $\mathcal{A}_2$-module M is nilpotent if, for any $x \in M$, there exists k such that $\mathrm{Sq}_0^k x = 0$.

The archetypal example of a nilpotent unstable module is a suspension and the following holds for all primes.

Proposition 3.2. [Sch94] *An unstable module is nilpotent if and only if it is the colimit of unstable modules which have a finite filtration whose quotients are suspensions.*

The full subcategory of nilpotent unstable modules is denoted $\mathcal{N}il_1 \subset \mathcal{U}$ and an unstable module is said to be reduced if it contains no non-trivial subobject which lies in $\mathcal{N}il_1$ (this is equivalent to containing no non-trivial suspension).

More generally the category $\mathcal{U}$ is filtered by thick subcategories $\mathcal{N}il_s$, $s \geq 0$, where $\mathcal{N}il_s$ is the smallest thick subcategory stable under colimits and containing all s-fold suspensions:

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots \ .$$

Proposition 3.3. [Sch94, Kuh95b, Kuh13] *The inclusion functor $\mathcal{N}il_s \hookrightarrow \mathcal{U}$ admits a right adjoint $\mathrm{nil}_s : \mathcal{U} \to \mathcal{N}il_s$ so that $M \in \mathcal{U}$ has a convergent decreasing filtration*

$$\ldots \subset \mathrm{nil}_{s+1}M \subset \mathrm{nil}_s M \subset \ldots \subset M$$

and $\mathrm{nil}_s M_s/\mathrm{nil}_{s+1}M \cong \Sigma^s R_s M$, where $R_s(M)$ is a reduced unstable module.

The convergence statement is a consequence of the fact that, for any unstable module M, $\mathrm{nil}_s M$ is $(s-1)$-connected.

Proposition 3.4. [Sch94, Kuh95b, Kuh13]

1. *The T-functor restricts to $T\colon \mathcal{N}il_s \to \mathcal{N}il_s$ and $T \circ \mathrm{nil}_s \cong \mathrm{nil}_s \circ T$.*
2. *The tensor product restricts to $\otimes\colon \mathcal{N}il_s \otimes \mathcal{N}il_t \to \mathcal{N}il_{s+t}$.*
3. *For M a finitely-generated unstable module, the nilpotent filtration is finite (i.e. $\mathrm{nil}_s M = 0$ for $s \gg 0$) and each $R_s M$ is finitely-generated.*
4. *An unstable module M lies in $\mathcal{U}_n$ if and only if $R_s M \in \mathcal{U}_n$ for all $s \in \mathbb{N}$.*

Proof. The result follows from the commutation of T with suspension and the respective definitions. For the final part, since T does not commute with projective limits in general, in addition the connectivity of elements of $\mathcal{N}il_s$ is used. □

Notation 3.5. For K an augmented unstable algebra with augmentation $\varepsilon\colon K \longrightarrow \mathbb{F}_p$, denote by $\bar{K}$ the augmentation ideal $\mathrm{ker}\varepsilon$ and $QK \in \mathcal{U}$ the module of indecomposables: $QK := \bar{K}/(\bar{K})^2$.

Proposition 3.6. [Sch94, Section 6.4] *For K an unstable algebra, $QK \in \mathcal{N}il_1$. If $p = 2$, QK is a suspension.*

The following result is applied in Section 6; for simplicity only the case $p = 2$ is considered.

Proposition 3.7. *Let $\psi\colon M \to N$ be a morphism of unstable $\mathcal{A}_2$-modules such that*

(1) *$M \in \mathcal{N}il_d$ and $N \in \mathcal{N}il_{d+1}$, for some $d \in \mathbb{N}$;*
(2) *$R_d M$ is finitely-generated.*

Then there exists a finitely-generated submodule $U \subset M$ such that

(1) *the restriction $\psi|_U$ is trivial;*
(2) *the monomorphism $R_d\psi : R_d U \to R_d M$ has nilpotent kernel (equivalently $fR_d\psi$ is an isomorphism).*

Proof. It is straightforward to reduce to the case where M is finitely-generated. Moreover, without loss of generality, we may assume that ψ is surjective.

For $k \gg 0$ (an explicit bound can be supplied), consider the following diagram

$$\begin{tikzcd} & M \arrow[r, two heads, "\psi"] \arrow[d, two heads] & N \\ \Sigma^d\Phi^k R_d M \arrow[ur, dashed, "\sigma_k"] \arrow[r, hook] & \Sigma^d R_d M & \end{tikzcd}$$

where Φ denotes the Frobenius functor and the bottom arrow is the canonical inclusion (recall that $R_d M$ is reduced). The dashed arrow denotes a choice of linear section σ_k (not $\mathcal{A}_2$-linear in general).

By hypothesis, M is finitely-generated, hence so is N; thus there exists $h \in \mathbb{N}$ such that $R_s N = 0$ if $s \notin [d+1, h]$. Moreover, since each $R_s N$ is finitely-generated, Proposition 2.9 implies that N is concentrated in degrees of the form $\ell + t$, $\ell \in [d+1, h]$ and $t \in \mathbb{N}$ such that $\alpha(t) \leq D$ for some $D \in \mathbb{N}$. (This argument is related to the second part of Theorem 2.7.)

Elements of $\Sigma^d\Phi^k R_d M$ lie in degrees of the form $d + 2^k v$, for $v \in \mathbb{N}$; hence a non-zero element in the image of $\psi\sigma_k$ lies in a degree of the form:

$$d + 2^k v = \ell + t,$$

so that $t = 2^k v - (\ell - d)$, where $\alpha(t) \leq D$ and $\ell - d \in [1, h-d]$, where the values of D and h are independent of k. If k is sufficiently large, this leads to a contradiction; thus, for $k \gg 0$, $\psi\sigma_k = 0$.

Finally, consider the submodule U of M generated by the image under σ_k of a (finite-dimensional) space of generators of the unstable module $\Sigma^d\Phi^k R_d M$, which is finitely-generated, since $R_d M$ is. The functor R_d preserves monomorphisms, hence $R_d U \subset R_d M$ and the conclusion follows. □

4. Kuhn's profile functions

The profile function of an unstable module is defined using ideas of Kuhn [Kuh95b]:

Definition 4.1. For $M \in \mathcal{U}$, the profile function $w_M\colon \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ is defined by:

$$w_M(i) := \deg f(R_i M) = \deg R_i M.$$

Remark 4.2. For M an unstable module, $w_M = 0$ if and only if M is locally finite (i.e. $M \in \mathcal{U}_0$).

Notation 4.3. For functions $f, g : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ write $f \leq g$ if $f(i) \leq g(i)$ for all $i \in \mathbb{N}$. The inclusion $\mathbb{N} \hookrightarrow \mathbb{N} \cup \{\infty\}$ is denoted Id.

The following operations on functions $\mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ are useful:

Definition 4.4. For functions $f, g : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$, define functions:

(1) $f \bullet g(i) := \sup_k \{f(k) + g(i-k)\}$;
(2) $\sup\{f, g\}(i) := \sup\{f(i), g(i)\}$ (likewise for arbitrary sets of functions);
(3) $\partial f(i) := \sup\{0, f(i) - 1\}$.
(4) $[f](i) := \sup\{f(j) | 0 \leq i \leq j\}$.

Proposition 4.5. *For M, N unstable modules,*

(1) $w_{M \otimes N} \leq w_M \bullet w_N$;
(2) $w_{\bar{T}M} = \partial w_M$.

For a short exact sequence of unstable modules, $0 \to M_1 \to M_2 \to M_3 \to 0$, the following hold

(1) $w_{M_1} \leq w_{M_2}$;
(2) $w_{M_2} \leq \sup\{w_{M_1}, w_{M_3}\}$;
(3) $w_{M_3} \leq [w_{M_2}]$.

Proof. For tensor products, the statement holds by the behaviour of the functor R_i with respect to tensor products [Kuh95b, Proposition 2.5]. The statement for the reduced T-functor is a consequence of the compatibility of T with the nilpotent filtration and the definition of polynomial degree (see Proposition 3.4 and [Sch94, Kuh95b]).

For the short exact sequence, the first two properties follow from the left exactness of the composite functor $f \circ R_i$ [Kuh07, Corollary 3.2] and the fact that $\mathcal{F}_n$ is thick.

For the final point, it suffices to show that $\Sigma^t R_t M_3$ lies in $\mathcal{U}_d$, where

$$d = [w_{M_2}](t) = \sup\{w_{M_2}(i) | 0 \leq i \leq t\}.$$

Now $\Sigma^t R_t M_3$ is a subquotient of $M_2/\mathrm{nil}_{t+1} M_2$ and the latter lies in $\mathcal{U}_d$, since each $\Sigma^i R_i M_2$ does, for $0 \leq i \leq t$, by definition of d. □

Remark 4.6. The final statement on w_{M_3} can be strengthened slightly; the current presentation is sufficient for current purposes.

Example 4.7. Let M, N be unstable modules such that $w_M \leq \mathrm{Id}$ and $w_N \leq \mathrm{Id}$, then $w_{M \otimes N} \leq \mathrm{Id}$. Hence, if $\mathbb{T}(M) := \bigoplus_i M^{\otimes i}$ denotes the tensor algebra on M, $w_{\mathbb{T}(M)} \leq Id$.

For K an unstable algebra, Proposition 4.5 immediately provides a bound for w_{QK} in terms of w_K; the following provides a converse.

Proposition 4.8. *Let K be an augmented unstable algebra such that $\bar{K} \in \mathcal{N}il_1$ (thus K is connected). Then*

$$w_K \leq \sup\{[w_{QK}]^{\bullet t} | t \in \mathbb{N}\}.$$

Proof. The result follows from an analysis of the short exact sequence

$$0 \to (\bar{K})^2 \to \bar{K} \to QK \to 0$$

together with the surjection $\bar{K} \otimes \bar{K} \twoheadrightarrow (\bar{K})^2$. Proposition 4.5 provides the inequalities $w_{(\bar{K})^2} \leq [w_{\bar{K} \otimes \bar{K}}] \leq [w_{\bar{K}}] \bullet [w_{\bar{K}}]$, so that $w_K \leq \sup\{w_{QK}, [w_{\bar{K}}] \bullet [w_{\bar{K}}]\}$.

The hypothesis that $\bar{K}$ is nilpotent implies that $w_{\bar{K}}(0) = 0 = w_{QK}(0)$. It follows that $w_{\bar{K}\otimes\bar{K}}(i)$ only depends on $w_{\bar{K}}(j)$ for $0 < j < i$. A similar statement holds for $[w_{\bar{K}}] \bullet [w_{\bar{K}}]$ in terms of $[w_{\bar{K}}]$.

The result follows by showing by induction on i that $[w_K](i) \leq [w_{QK}]^{\bullet t}(i)$ for $1 \leq t \leq i$. The cases $i \in \{0, 1\}$ are clear. For the inductive step, as above

$$[w_K](i) \leq \sup\{[w_{QK}](i), [w_{\bar{K}}] \bullet [w_{\bar{K}}](i)\}.$$

Now

$$\begin{aligned} [w_{\bar{K}}] \bullet [w_{\bar{K}}](i) &= \sup_j\{[w_{\bar{K}}](j) + [w_{\bar{K}}](i-j)\} \\ &\leq \sup_j\{[w_{QK}]^{\bullet t_1}(j) + [w_{QK}]^{\bullet t_2}(i-j)\} \end{aligned}$$

by the inductive hypothesis, where $1 \leq t_1 \leq j$ and $1 \leq t_2 \leq i - j$. One identifies $\sup_j\{[w_{QK}]^{\bullet t_1}(j) + [w_{QK}]^{\bullet t_2}(i-j)\} = [w_{QK}]^{\bullet(t_1+t_2)}(i)$, which concludes the inductive step, since $t_1 + t_2 \leq i$. □

The following is clear:

Corollary 4.9. *Let K be an augmented unstable algebra such that $\bar{K} \in \mathcal{N}il_1$, then $\bar{K}$ is locally finite if and only if QK is locally finite; equivalently $w_K = 0$ if and only if $w_{QK} = 0$.*

Corollary 4.10. *Let K be an augmented unstable algebra such that $\bar{K} \in \mathcal{N}il_1$, then $w_K \leq \mathrm{Id}$ if and only if $w_{QK} \leq \mathrm{Id}$.*

Proof. The hypothesis $w_K \leq \mathrm{Id}$ implies that $[w_K] \leq [\mathrm{Id}] = \mathrm{Id}$. The result follows from the observation that $\mathrm{Id} \bullet \mathrm{Id} = \mathrm{Id}$. □

Notation 4.11. For X a space, write $w_X := w_{H^*X}$ and $q_X := w_{QH^*X}$.

5. Proof of Theorem 1.4

We commence by a rapid review of Lannes' theory, which is the main ingredient in the proof of Theorem 1.4 and is also the reason for the restrictions imposed on the topological spaces considered.

Let X be a p-complete, 1-connected space and assume that TH^*X is of finite type and 1-connected. The evaluation map:

$$B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, X) \longrightarrow X$$

induces a map in cohomology $H^*X \longrightarrow H^*B\mathbb{Z}/p \otimes H^*\mathrm{map}(B\mathbb{Z}/p, X)$ and hence, by adjunction, $TH^*X \longrightarrow H^*\mathrm{map}(B\mathbb{Z}/p, X)$.

Theorem 5.1. [Lan92] *Under the above hypotheses, the natural map $TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X)$ is an isomorphism of unstable algebras.*

Notation 5.2. [Kuh95b] Denote by $\Delta(X)$ the homotopy cofibre of the map $X \longrightarrow \mathrm{map}(B\mathbb{Z}/p, X)$ induced by $B\mathbb{Z}/p \to *$.

Theorem 5.1 yields the following:

Proposition 5.3. *Under the above hypotheses on X, $H^*\Delta(X) \cong \bar{T}H^*X$, hence*

1. *$w_{\Delta(X)} = \partial w_X$;*
2. *if $H^*X \in \mathcal{U}_n$, then $H^*\Delta(X) \in \mathcal{U}_{n-1}$.*

The functor T induces $T : \mathcal{K} \to \mathcal{K}$, which commutes with the indecomposables functor for augmented unstable algebras [Sch94, Lemma 6.4.2]: namely, for K an augmented unstable algebra, there is a natural isomorphism $T(QK) \cong Q(TK)$. There is no analogous statement for $\bar{T}$; however, if Z is an H-space, there is a homotopy equivalence:

$$\mathrm{map}(B\mathbb{Z}/p, Z) \cong Z \times \mathrm{map}_*(B\mathbb{Z}/p, Z).$$

For example, this leads to:

Proposition 5.4. [CCS07] *For Z an H-space (also satisfying the global hypotheses)*

$$QH^*\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z) \cong \bar{T}^n QH^*Z.$$

Proof of Theorem 1.4. Proposition 3.4 implies that an unstable module M is locally finite if and only if $w_M = 0$, which implies the equivalence (1) $\Leftrightarrow$ (2). Corollary 4.9 gives the equivalence (2) $\Leftrightarrow$ (3) and Corollary 4.10 the equivalence (4) $\Leftrightarrow$ (5). The implications (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (6) are clear, hence it suffices to establish:

- (6) $\Rightarrow$ (2): if $w_X - \mathrm{Id}$ is bounded then $w_X = 0$.

For clarity, we first establish (4) $\Rightarrow$ (2) and then indicate how to deduce the more general result.

Suppose that there exists a space X (satisfying the global hypotheses) such that $0 \neq w_X \leq \mathrm{Id}$ and $\tilde{H}^*X \in \mathcal{N}il_1$; *reductio ad absurdam.* The first step is Kuhn's reduction [Kuh95b]. For $s = \inf\{i | w_X(i) \neq 0\}$, the space $Y := \Delta^{w_X(s)-1}X$ satisfies

$$\begin{aligned} w_Y(i) &= 0,\ i < s \\ w_Y(s) &= 1 \\ w_Y(i) &\leq i - s + 1,\ i > s. \end{aligned}$$

By collapsing down a low-dimensional skeleton and p-completing, we may assume that Y satisfies the global hypotheses and that $R_iH^*Y = 0$ for $0 < i < s$. In order to work with pointed mapping spaces, consider $Z := \Omega(\Sigma X)^\wedge_p$. By the Bott-Samelson theorem $H^*Z \cong \mathbb{T}(\tilde{H}^*X)$ as an unstable module, hence the global hypotheses are satisfied by Z. Moreover, by Proposition 4.5, one has $R_iH^*Z = 0$ for $0 < i < s$ and $R_iH^*Z \in \mathcal{U}_{i-s+1}$ for $i \geq s$. Thus, by Theorem 2.5 and the definition of the nilpotent filtration, $\bar{T}^nH^*Z$ is $(n + s - 2)$-connected.

It follows from [BK72] (in particular using Chapters 6 and 9 to show that there are no phantom maps) that $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is $(n + s - 2)$-connected.

By construction, R_sH^*Z is a reduced unstable module in $\mathcal{U}_1$, hence (by [Kuh95b, Proposition 0.6], for example) there exists a non-trivial morphism $R_sH^*Z \to F(1)$, where $F(1)$ is the free unstable module on a generator of degree 1. Composing with the canonical inclusion $F(1) \hookrightarrow \tilde{H}^*B\mathbb{Z}/p$ gives

$$\varphi_s^* : H^*Z \twoheadrightarrow H^*Z/\mathrm{nil}_{s+1}H^*Z \cong \mathbb{F}_2 \oplus \Sigma^s R_sH^*Z \to \mathbb{F}_2 \oplus \Sigma^s F(1) \subset \mathbb{F}_2 \oplus \Sigma^s \tilde{H}^*B\mathbb{Z}/p,$$

which is a morphism of unstable algebras, by compatibility of the nilpotent filtration with multiplicative structures.

By Lannes' theory [Lan92], the morphism φ_s^* can be realized as the cohomology of a map $\varphi_s : \Sigma^s B\mathbb{Z}/p \to Z$, by applying Theorem 5.1 together with the Hurewicz theorem, since $\mathrm{map}_*(B\mathbb{Z}/p, Z)$ is $(s - 1)$-connected.

Thus, consider the extension problem

$$\begin{tikzcd} \Sigma^s B\mathbb{Z}/p \arrow[r, "\varphi_s"] \arrow[d] & Z \\ \Sigma^{s-1}K(\mathbb{Z}/p,2), \arrow[ur, dotted] & \end{tikzcd}$$

where the vertical map is the $(s-1)$-fold suspension of the canonical map $\Sigma B\mathbb{Z}/p \to K(\mathbb{Z}/p,2)$.

Algebraically it is clear that no such extension can exist, since φ_s^* is non-trivial in positive degrees (by construction), whereas

$$\mathrm{Hom}_{\mathcal{U}}(\tilde{H}^*Z, \tilde{H}^*\Sigma^{s-1}K(\mathbb{Z}/p,2)) = 0$$

since $\tilde{H}^*Z \in \mathcal{N}il_s$ and $\tilde{H}^*\Sigma^{s-1}K(\mathbb{Z}/p,2)$ is the $(s-1)$-fold suspension of a reduced module ($H^*K(\mathbb{Z}/p,2)$ is reduced by Proposition 5.5 below).

However, obstruction theory shows that one can construct such a factorization, as follows. Recall that $K(\mathbb{Z}/p,2)$ can be built, starting from $\Sigma B\mathbb{Z}/p$, using Milnor's construction; in particular there is a filtration $* = C_0 \subset C_1 = \Sigma B\mathbb{Z}/p \subset C_2 \subset \ldots \subset \cup_n C_n = K(\mathbb{Z}/p,2)$ with associated (homotopy) cofibre sequences

$$\Sigma^{n-1}B\mathbb{Z}/p^{\wedge n} \to C_{n-1} \to C_n.$$

The associated obstructions to extending $\varphi_s\colon \Sigma^s B\mathbb{Z}/p \to Z$ lie in the pointed homotopy groups

$$[\Sigma^{n+s-2}(B\mathbb{Z}/p)^{\wedge n}, Z] = \pi_{n+s-2}\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z).$$

The groups $\pi_{n+s-2}\mathrm{map}_*B\mathbb{Z}/p^{\wedge n}, Z)$ are trivial, since $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is $(n+s-2)$-connected, as observed above. It follows that an extension exists, which is a contradiction to the existence of such a space X, completing the proof of (4) $\Rightarrow$ (2).

Finally consider (6) $\Rightarrow$ (2). Suppose that X is a space such that $w_X - \mathrm{Id} \leq t$ for some $t \in \mathbb{N}$; if $t = 0$ then the previous case applies, otherwise consider $\Delta^t X$. Proposition 5.3 implies that $w_{\Delta^t X} \leq \mathrm{Id}$, so, by the previous argument (collapsing a low-dimensional skeleton and p-completing as required), $w_{\Delta^t X} = 0$ so that $w_X \leq t$, again by Proposition 5.3. This implies that $H^*X \in \mathcal{U}_t$, by Proposition 3.4 and, without significant loss of generality, we may assume that $H^*X \notin \mathcal{U}_{t-1}$. Suppose that $t > 0$, then $\Delta^{t-1}X$ (again collapsing a low-dimensional skeleton and p-completing as required) satisfies $0 \neq w_{\Delta^{t-1}X} \leq \mathrm{Id}$, contradicting the previous result. Thus $t = 0$, as required. □

Proposition 5.5. *The unstable module $H^*K(\mathbb{Z}/p,2)$ is reduced.*

Proof. (This result is well known to the experts, and holds for $H^*K(\mathbb{Z}/p,n)$ for any $n \in \mathbb{N}$. As we do not know a convenient reference, we indicate a proof.)

The cohomology $H^*K(\mathbb{Z}/p,2)$ is isomorphic (with the usual notation) to

$$\mathbb{F}_p[x, \beta P^{I_h}\beta(x)] \otimes E(P^{I_\ell}\beta(x)) \cong \mathbb{F}_p[x, x_h; h \geq 1] \otimes E(y_\ell; \ell \geq 0)$$

with $|x| = 2$, $I_h = (p^{h-1}, p^{h-2}, \ldots, p, 1)$, $h \geq 1$, and $\beta(x) = y_0$; in particular, $\beta(y_h) = x_h$, $h \geq 1$. It is enough to show that, for any non zero element $z \in \tilde{H}^*K(\mathbb{Z}/p,2)$, there exists an operation θ such that $\theta(z) \in \mathbb{F}_p[x, x_h]$ and $\theta(z) \neq 0$.

Any element z can be written as a sum $\sum_{0\leq i\leq t}\sum_j P_{i,j}(x, x_h) \otimes L_{i,j}(y_\ell)$. If z has degree $2n$, and in such an expression there is a non-trivial term with exterior

part of degree 0, a straightforward application of the Cartan formula shows that $\theta = P^n$ suffices, since the reduced powers act trivially on the exterior generators.

For a general element, one reduces to such elements by applying operations which decrease (non-trivially) the length of the exterior factors that occur. Consider amongst the exterior factors a term $L_{i,j}$ of minimal length, and the minimal ℓ for which y_ℓ occurs in it. Let Q_i be the usual Milnor derivation at an odd prime. The element $[\beta, Q_i]$ is also a derivation; it acts trivially on the x_h, sends x to y_{i+1}, and y_ℓ to $x_{i-\ell+1}^{p^\ell}$. Using a standard lexicographic order argument, one can see that this operation does the job for i large enough. □

6. Using the Eilenberg-Moore spectral sequence

In this section, the prime p is taken to be 2 and the space X is 1-connected such that $\tilde{H}^*X$ is nilpotent and of finite type.

The object of this last section is to prove the following:

Theorem 6.1. *Let X be a space such that $\tilde{H}^*X$ is of finite type and is nilpotent. If $w_X(1) = d \in \mathbb{N}$ then $w_X(2) \geq 2d$.*

The interest of the result is to give some control on R_2H^*X, starting from information about R_1H^*X. See Remark 6.14 for a slightly refined version of this theorem.

Remark 6.2. This theorem is proved only for $p = 2$; the difficulties that occur in the odd primary case in [Sch98, Sch10] also arise here, but look more manageable.

The method was originally suggested by the following observation:

Proposition 6.3. *Let M be a connected, reduced unstable module such that $\deg(fM) = d \in \mathbb{N}$. If $d > 0$, then the unstable module $\mathbb{T}(M) = \bigoplus_i M^{\otimes i}$ does not carry the structure of an unstable algebra.*

Remark 6.4. This result is a special case of a general structure result for reduced unstable algebras. The idea, from the point of view of this paper, is that cup products of classes in M should appear in $M \otimes M$ whereas the restriction axiom for unstable algebras implies that cup squares occur in M. Thus, the triviality of the extension between M and $M \otimes M$ is incompatible with an unstable algebra structure.

The proof of Theorem 1.5 is based on the analysis of an $\mathrm{Ext}^1_{\mathcal{F}}$ group, playing off the following non-splitting result against a vanishing criterion.

Recall from Section 2 that $\mathcal{F}$ denotes the category of functors on $\mathbb{F}_2$-vector spaces; a functor of $\mathcal{F}$ is finite if it has a finite composition series. As usual, S^n denotes the nth symmetric power functor and Λ^n the nth exterior power functor.

Lemma 6.5. *For $F \in \mathcal{F}$ a non-constant finite functor, post-composition with the short exact sequence $0 \to S^1 \to S^2 \to \Lambda^2 \to 0$, where $S^1 \to S^2$ is the Frobenius $x \mapsto x^2$, induces an exact sequence*

$$0 \longrightarrow F \longrightarrow S^2(F) \longrightarrow \Lambda^2(F) \longrightarrow 0$$

which does not split.

Proof. Follows from [Kuh95a, Theorem 4.8], since $0 \longrightarrow S^1 \to S^2 \to \Lambda^2 \longrightarrow 0$ does not split. □

Notation 6.6. For $F \in \mathcal{F}$ a finite functor of polynomial degree d, set $\overline{F} := \mathrm{k}er\{F \twoheadrightarrow q_{d-1}F\}$, where $q_{d-1}F$ is the largest quotient of degree $\leq d-1$.

Lemma 6.5 will be played off against the vanishing result for $\mathrm{Ext}^1_{\mathcal{F}}$ in the following statement.

Lemma 6.7. *Let F be a finite functor of polynomial degree d, $G_{<d}$ of degree $< d$, $G_{<2d}$ of degree $< 2d$ and $G_{\leq d}$ of degree $\leq d$. Then*

(1) $\mathrm{Hom}_{\mathcal{F}}(\overline{F}, G_{<d}) = 0$;
(2) $\mathrm{Hom}_{\mathcal{F}}(\overline{F} \otimes \overline{F}, G_{<2d}) = 0 = \mathrm{Hom}_{\mathcal{F}}(\Lambda^2(\overline{F}), G_{<2d})$;
(3) $\mathrm{Ext}^1_{\mathcal{F}}(\overline{F} \otimes \overline{F}, G_{\leq d}) = 0$.

Proof. The first statement is a consequence of the definition of $\overline{F}$.

The second is similar and follows from the compatibility of the polynomial filtration of $\mathcal{F}$ with tensor products, which implies that $\overline{F} \otimes \overline{F}$ has no quotient of polynomial degree $< 2d$. The second equality follows from the fact that the functor $\Lambda^2(\overline{F})$ is a quotient of $\overline{F} \otimes \overline{F}$.

The result for Ext^1 is proved as follows. Using *dévissage* it is straightforward to reduced to the case where $G_{\leq d}$ is simple. Since a simple functor of polynomial degree n embeds in the nth tensor functor $T^n : V \mapsto V^{\otimes n}$, which is finite and has polynomial degree n, using the previous statement for Hom, it suffices to show that $\mathrm{Ext}^1_{\mathcal{F}}(\overline{F} \otimes \overline{F}, T^n) = 0$ for $n \leq d$. This follows by the standard methods introduced in [FLS94], exploiting the tensor product on the left hand side. □

Using these results in conjunction with the Eilenberg-Moore spectral sequence, we prove Theorem 1.5. For relevant details (and further references) on the Eilenberg-Moore spectral sequence computing $H^*\Omega X$ from H^*X, see [Sch94, Section 8.7]. Note that the hypothesis that X is simply-connected ensures strong convergence of the spectral sequence.

Recall that the (-2)-layer of the Eilenberg-Moore filtration $F_\infty^{-2,*}$ on $H^*\Omega X$ is an extension in unstable modules between the column $E_\infty^{-1,*}$ desuspended once and the column $E_\infty^{-2,*}$ desuspended twice:

$$0 \longrightarrow \Sigma^{-1}E_\infty^{-1,*} \longrightarrow F_\infty^{-2,*} \longrightarrow \Sigma^{-2}E_\infty^{-2,*} \longrightarrow 0.$$

The term $E_\infty^{-1,*}$ is a quotient of $QH^*X \cong E_2^{-1,*}$ by a submodule in $\mathcal{N}il_2$, by [Sch94, Theorem 8.7.1]. Similarly, $E_\infty^{-2,*}$ is a quotient of $\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$ (which belongs to $\mathcal{N}il_2$ by [Sch94, Theorem 6.1]), by a submodule in $\overline{\mathcal{N}il}_3$ by [Sch94, Proposition 8.7.7]. (Here $\overline{\mathcal{N}il}_3$ is generated by $\mathcal{N}il_3$ and $\mathcal{U}_0$ - see [Sch94, Section 6.2].)

Thus, applying the exact functor $f : \mathcal{U} \to \mathcal{F}$ yields the short exact sequence:

$$0 \to fR_1QH^*X \to fF_\infty^{-2,*} \to fR_2\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2) \to 0 \tag{6.1}$$

and, moreover, $fR_1QH^*X \cong fR_1\tilde{H}^*X$. This allows arguments to be carried out in the category of functors $\mathcal{F}$.

Notation 6.8. Write

$$\begin{aligned} F_1 &:= fR_1\tilde{H}^*X \cong f(\Sigma^{-1}E_\infty^{-1,*}) \\ F_2 &:= f(F_\infty^{-2,*}), \end{aligned}$$

so that $F_2/F_1 \cong fR_2\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$ by (6.1).

Remark 6.9. With this notation, the short exact sequence (6.1) represents a class $[F_2] \in \mathrm{Ext}^1_{\mathcal{F}}(F_2/F_1, F_1)$.

The compatibility of the cup product with the Eilenberg-Moore spectral sequence gives a morphism $S^2(F_1) \to F_2$ which gives a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & S^2(F_1) \arrow[r] \arrow[d] & \Lambda^2(F_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & F_2/F_1 \arrow[r] & 0
\end{tikzcd}$$

so that the right hand square is a pull-back.

By hypothesis, the functor F_1 is of degree d; the inclusion $\overline{F_1} \subset F_1$ induces inclusions $S^2(\overline{F_1}) \subset S^2(F_1)$ and $\Lambda^2(\overline{F_1}) \subset \Lambda^2(F_1)$ which fit into the following diagram of morphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \overline{F_1} \arrow[r] \arrow[d, hook] & S^2(\overline{F_1}) \arrow[r] \arrow[d] & \Lambda^2(\overline{F_1}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \Lambda^2(\overline{F_1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & S^2(F_1) \arrow[r] \arrow[d] & \Lambda^2(F_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & F_2/F_1 \arrow[r] & 0
\end{tikzcd} \tag{6.2}$$

in which the second row can be viewed either as the pushout of the top row or the pullback of the second. Moreover, the first and third rows are not split, by Lemma 6.5, hence the bottom row is not split.

Lemma 6.10. *If F_1 is non-constant the short exact sequence*

$$0 \to F_1 \to E \to \Lambda^2(\overline{F_1}) \to 0$$

does not split.

Proof. Consider the long exact sequence for $\mathrm{Ext}^*_{\mathcal{F}}$ induced by the defining short exact sequence $\overline{F_1} \to F_1 \to q_{d-1}F_1$, which gives the exact sequence:

$$\mathrm{Hom}(\Lambda^2(\overline{F_1}), q_{d-1}F_1) \to \mathrm{Ext}^1_{\mathcal{F}}(\Lambda^2(\overline{F_1}), \overline{F_1}) \to \mathrm{Ext}^1_{\mathcal{F}}(\Lambda^2(\overline{F_1}), F_1)$$

in which the first term is zero, by Lemma 6.7. Thus the second morphism is injective.

The top row in diagram (6.2) represents a non-trivial class in $\mathrm{Ext}^1(\Lambda^2(\overline{F_1}), \overline{F_1})$, by Lemma 6.5, hence pushes out to a non-split short exact sequence represented by a non-zero class in $\mathrm{Ext}^1_{\mathcal{F}}(\Lambda^2(\overline{F_1}), F_1)$. □

Proposition 6.11. *The morphism $\Lambda^2\overline{F_1} \to F_2/F_1$ does not factor over $\overline{F_1} \otimes \overline{F_1}$:*

$$\begin{tikzcd}
\Lambda^2\overline{F_1} \arrow[r] \arrow[d] & \overline{F_1} \otimes \overline{F_1} \arrow[dl, dotted, "\nexists"] \\
F_2/F_1.
\end{tikzcd}$$

Proof. The non-trivial extension of Lemma 6.10 is the image of the class $[F_2] \in \mathrm{Ext}^1_{\mathcal{F}}(F_2/F_1, F_1)$ under the morphism induced by $\Lambda^2(\bar{F}_1) \to F_2/F_1$. Hence there can be no factorization across the group $\mathrm{Ext}^1_{\mathcal{F}}(\overline{F} \otimes \overline{F}, F)$, which is trivial by Lemma 6.7. □

Proposition 6.12. *Assume that $\tilde{H}^*X$ is nilpotent and of finite type.*
If $deg(fR_1H^*X) = d \in \mathbb{N}$ and $deg(fR_2(H^*X)) < 2d$, then

1. *there is an inclusion $\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2) \cong F_2/F_1$ with cokernel of degree $< 2d$;*
2. *the morphism $\Lambda^2(\overline{F_1}) \to F_2/F_1$ factors across $\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$.*

Proof. The cup product of H^*X induces a morphism $\tilde{H}^*X \otimes \tilde{H}^*X \to \mathrm{nil}_2 H^*X$, since $\tilde{H}^*X \in \mathcal{N}il_1$, hence in $\mathcal{F}$

$$F_1 \otimes F_1 = fR_1H^*X \otimes fR_1H^*X \to fR_2H^*X.$$

Restricting to $\overline{F_1} \otimes \overline{F_1} \subset F_1 \otimes F_1$ this morphism is trivial, by Lemma 6.7, since $\deg fR_2H^*X < 2d$, by hypothesis.

Lift $\overline{F_1}$ to a submodule M of $\tilde{H}^*X$ (so that fR_1M corresponds to $\overline{F_1}$) and consider the restriction of the product. By construction this gives:

$$M \otimes M \to \mathrm{nil}_3 H^*X$$

with $M \otimes M \in \mathcal{N}il_2$. Moreover, it is easily checked that the finiteness hypothesis required to apply Proposition 3.7 is satisfied, hence there exists a finitely-generated submodule $U \subset M \subset \tilde{H}^*X$ such that $fR_1U = fR_1M = \overline{F_1}$ and the cup product restricts to a trivial map $U \otimes U \to \tilde{H}^*X$. (This is a slight extension of Proposition 3.7 using the fact that the choices in the proof can be taken to be compatible with the tensor product $M \otimes M$.)

The calculation of $\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$ using the bar construction then implies that there is an inclusion $\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$. Moreover, by the definition of $\overline{F_1} \subset F_1$, it is clear that the cokernel has degree $< 2d$.

Finally, again by Lemma 6.7, there is no non-trivial map from $\Lambda^2(\overline{F_1})$ to a functor of degree $< 2d$, which gives the factorization statement. □

Remark 6.13. The appeal to Proposition 3.7 can be avoided so that the argument is carried out entirely within $\mathcal{F}$.

Proof of Theorem 6.1. Suppose that $deg(fR_2H^*X) < 2d$, then Proposition 6.12 provides a factorization

$$\Lambda^2(\overline{F_1}) \to \overline{F_1} \otimes \overline{F_1} \to F_2/F_1.$$

This contradicts Proposition 6.11. □

Remark 6.14. The result proved is slightly stronger, without supposing $w_X(2) < 2d$. Namely the argument shows that the composite:

$$\overline{F_1} \otimes \overline{F_1} \subset F_1 \otimes F_1 \to fR_2H^*X,$$

where the second morphism is induced by the cup product of H^*X, is necessarily non-trivial.

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