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Abstract

The following note discuss two ways to extend conjectures of N. Kuhn about the non realization of certain unstable algebras as the mod p singular cohomology of spaces. The first case (established for all primes p) extends and refines preceding work the second and the third author [GS13]. The second one (only for the prime 2) discuss another approach, not depending on Lannes' mapping space theorem and relies on an analysis of the columns -1 and -2 of the Eilenberg-Moore spectral sequence, and the triviality of the associated extension in the category of unstable modules. In both case the statements and proofs will use the links between the categories of unstable modules and functors. In both cases it would be possible to work entirely within the category of unstable modules, however it is much more cleaner and efficient to use both categories, thus this note is propaganda for this point of view.

1 Introduction

Let p be a prime number, in all the sequel H^*X will denote the the mod p singular cohomology of the topological space X. All spaces X will be supposed p-complete and connected. We will denote as usual by $\mathcal{U}$ the abelian category of unstable modules and by $\mathcal{K}$ the category unstable algebras [S94]. One will assume moreover that the cohomology groups are finite dimensional in any degree and that the functor T_V of Jean Lannes acts nicely on H^*X in the sense that $T_V(H^*X)$ is also of finite dimension in any degree. As well, in order to apply a theorem of Lannes we will suppose the spaces 1-connected, as well as $T_V(H^*X$. For the problem we will consider this is not a significant restriction because it will be possible to collapse down the 1-skeleton. The dimensional condition could be relaxed using methods of Fabien Morel, as explained by François-Xavier Dehon and Gaudens. In a preceding paper [GS13] the second and the third author gave a proof of the conjecture of Nick Kuhn which follows

Theorem 1.1 *[G. Gaudens, L. Schwartz] Let X be a space such that H^*X is finitely generated as an $\mathcal{A}_p$-module. Then H^*X is finite.*

This result is as a consequence of the following stronger result also conjectured by Kuhn, and proved in the same reference, which uses the Krull filtration

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}$$

on the category $\mathcal{U}$ (see section 2) which is defined for any abelian category [Gab62].

Theorem 1.2 *[G. Gaudens, L. Schwartz] Let X, If $H^*X \in \mathcal{U}_n$ for some n, then $H^*X \in \mathcal{U}_0$*

The aim of this article is to extend the results in two directions, and to give details on 1.5 announced with ashort proof in [GS12] (note that [GS13] has been published two years in late, due to an email lost by the third author).

The first direction is that more can be said about the behavior of $\tilde{H}^*X$ with respect to its nilpotent filtration. To describe the statements one needs to introduce a function associated to an unstable module M and its nilpotent filtration. Above we introduced spaces with nilpotent cohomology. The restriction axiom for unstable algebras allows to express this in term of the action of the Steenrod algebra. The nilpotency condition (for $p = 2$) it is equivalent to require that the operation $Sq_0\colon x \mapsto Sq^{|x|}x$ acts nilpotently on any element. It makes it possible to extend the definition of nilpotency to any unstable module and also to define higher order of nilpotency. The subcategory $\mathcal{N}il_s$, $s \geq 0$, is the smallest thick subcategory stable under colimits and containing all s-suspensions of unstable modules.

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots$$

By the very definition *any $M \in \mathcal{N}il_s$ is $(s-1)$-connected.*.

Proposition 1.3 *[S94], [K95] Any M has a convergent decreasing filtration $\{M_s\}_{s\geq 0}$ with $M_s/M_{s+1} \cong \Sigma^s R_s(M)$ where $R_s(M)$ is a reduced unstable module, i.e. which does not contain a non trivial suspension.*

For a reduced unstable module M say that $\deg(M) = n$ if and only if $M \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$. Following N. Kuhn, to an unstable module M, let us associate to M the function $w_M : \mathbb{N} \to \mathbb{N} \cup \infty$ by:

$$w_M(i) = \deg(R_i(M))$$

Set $w_X = w_{H^*X}$.
Recall the indecomposable elements of the cohomology:

$$QH^*X = \tilde{H}^*X/(\tilde{H}^*X)^2$$

Set $q_X = w_{QH^*X}$.
One gets:

Theorem 1.4 (Gaudens, Schwartz) *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function w_X either is equal to 0 or $w_X - Id$ takes arbitrary large values, may be ∞.*

Theorem 1.5 (Nguyen T. Cuong, Gaudens, Schwartz) *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function q_X either is equal to 0 or $q_X - Id$ takes at least one positive (non zero) value.*

These are evidences (there are others) for:

Conjecture 1.1 (G. Gaudens) *Let X be a space so that any element in $\tilde{H}^*X$ is nilpotent. If $QH^*X \in \mathcal{U}_n$, then $QH^*X \in \mathcal{U}_0$.*

Conjecture 1.2 (G. Gaudens) *Let X be a space so that any element in $\tilde{H}^*X$ is nilpotent. If $QH^*X \notin \mathcal{U}_0$, then for some integer i, deg$(R_i(QH^*X) = +\infty$.*

Below is the second generalization. It concerns only the first and the second layers of the nilpotent filtration. Hopefully the method (comments about it are made below) extends to more general cases and may lead to a proof of the preceding conjectures.

Theorem 1.6 (Nguyen T. Cuong, Gaudens, Schwartz) *Let X be a space so that any element in $\tilde{H}^*X$ is nilpotent. Assume that $R_1H^*X \in \mathcal{U}_d$ then $R_2H^*X \in \mathcal{U}_{2d}$.*

The proof of this theorem will be slightly different from the proof of the preceding ones and be closer from the one in [S94]. It depends on an analysis of the second step of the Eilenberg-Moore filtration on $H^*\Omega X$, and computing certain $\mathrm{E}xt^1_{\mathcal{U}}(-,-)$.
The Krull and nilpotent filtrations will be shortly described in the first sections, for details we refer to [S94], and to [K95] and also [K13].
This work was partially supported by the VIASM, and the program ARCUS Vietnam of "Région Ile de France and Minisistère des Affaires Etrangères".

2 The Krull filtration on $\mathcal{U}$

The category of unstable modules $\mathcal{U}$, as any abelian category, has a Krull filtration. The filtration is by thick subcategories stable under colimits

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}$$

This is described in [S94], $\mathcal{U}_0$ is the largest thick sub-category generated by simple objects and stable under colimits. It is the subcategory of locally finite modules, $M \in \mathcal{U}$ is locally finite if the span over $\mathcal{A}_p$ of any $x \in M$ is finite. Having defined by induction $\mathcal{U}_n$ one defines $\mathcal{U}_{n+1}$ as follows. One introduces the quotient category $\mathcal{U}/\mathcal{U}_n$ whose objects are the same of those of $\mathcal{U}$ but where morphisms in $\mathcal{U}$ that have kernel and cokernel in $\mathcal{U}_0$ are formally inverted. Then $(\mathcal{U}/\mathcal{U}_n)_0$ is defined as above and $\mathcal{U}_{n+1}$ is the pre-image of this subcategory in $\mathcal{U}$ *via* the canonical projection functor, see [Gab62] for details. One has:

Theorem 2.1 *Let $M \in \mathcal{U}$ and $K_n(M)$ be the largest sub-object of M that is in $\mathcal{U}_n$, then*

$$M = \cup_n K_n(M)$$

Here is basic example:

- $\Sigma^k F(n) \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, the unstable modules $F(n)$ are the canonical generators of $\mathcal{U}$, generated in degree n by ι_n and $\mathbb{F}_p$-basis $P^I\iota_n$, I an admissible multi-index of excess less than n;

Proposition 2.2 *If M is a finitely generated unstable module, there exist d such that $M \in \mathcal{U}_d$, the nilpotent filtration is finite, and for any d $R_d(M)$ is finitely generated.*

See in particular [S94] 6.1.4 and [K95].
There is a characterization of the filtration in terms of the functor $\bar{T}$ introduced by Jean Lannes [S94]. The functor functor T_V, V elementary abelian p-group,is left adjoint to $M \mapsto H^*BV \otimes M$. If $V = \mathbb{F}_p$ denote it by T. As $H^*B\mathbb{Z}/p$ splits up, in $\mathcal{U}$, as $\mathbb{F}_p \oplus \tilde{H}^*B\mathbb{Z}/p$ the functor T is naturally equivalent to $Id \oplus \bar{T}$.
Below are the main properties of T_V

Theorem 2.3 *[L92], [S94] The functor T_V commutes with colimits (as a left adjoint). It is exact. Moreover there is a canonical isomorphism*

$$T_V(M_1 \otimes M_2) \cong T_V(M_1) \otimes T_V(M_2)$$

If $M_1 = \Sigma\mathbb{F}_p$ it writes as $T_V(\Sigma M) \cong \Sigma T_V(M)$.

Below is the characterization of the Krull filtration

Theorem 2.4 *The following two conditions are equivalent:*

- $M \in \mathcal{U}_n$,
- $\bar{T}^{n+1}(M) = \{0\}$.

Corollary 2.5 *If $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$ then $M \otimes N \in \mathcal{U}_{m+n}$*

There is also a characterization of objects in $\mathcal{U}_n$ of combinatorial nature (as soon as they are of finite dimension in any degree) for $p = 2$. As usual denote by $\alpha(k)$ is the number of 1 in the 2-adic expansion of k:

Theorem 2.6 *[S94] (see also [S06]) Let R be a reduced module, the $R \in \mathcal{U}_d$ if and only if:*

- $\bar{T}^{d+1}(R) \cong \{0\}$,
- *R is trivial in degrees k such that $\alpha(k) > d$.*

A similar condition holds for any prime p, but this result will be used in the last section which is done only at the prime 2.
In [S94] one refers in particular to chapter 5: 5.5.5, 5.5.7 and chapter 6: 6.1.4, 6.2.4, 6.2.5, 6.2.6. More generally one has:

Theorem 2.7 *[S06] A finitely generated unstable $\mathcal{A}_2$-module M is in $\mathcal{U}_n$ if and only if its Poincaré series $\Sigma_n a_n t^n$ has the following property. There exits an integer k so that the only non zero coefficients a_d satisfy $\alpha(d-i) \leq n$, for some $0 \leq i \leq k$.*

There are analogous results for $p > 2$.
Let $\mathcal{F}$ be the category of functors from finite dimensional $\mathbb{F}_p$-vector spaces to all vector spaces. Define a functor, [HLS93], $f : \mathcal{U} \to \mathcal{F}$ by

$$f(M)(V) = \mathrm{H}om_{\mathcal{U}}(M, H^*(BV))^* = T_V(M)^0$$

Let $\mathcal{F}_n$ be the sub-category of polynomial functors of degree less than n. It is defined as follows. Let $F \in \mathcal{F}$, let $\Delta(F) \in \mathcal{F}$ defined by

$$\Delta(F)(V) = \mathrm{Ker}(F(V \oplus \mathbb{F}_p) \to F(V))$$

Then by definition $F \in \mathcal{F}_n$ if and only if $\Delta^{n+1}(F) = 0$. As an an example $V \mapsto V^{\otimes n}$ is in $\mathcal{F}_n$. The following holds for any $M \in \mathcal{U}$

$$\Delta(f(M)) \cong f(\bar{T}(M)) \cong T_V^0(M)$$

Thus, the diagram below commutes:

$$\begin{array}{ccccccccc}
\mathcal{U}_0 & \dots & \mathcal{U}_{n-1} & \hookrightarrow & \mathcal{U}_n & \hookrightarrow & \mathcal{U} \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\
\mathcal{F}_0 & \dots & \mathcal{F}_{n-1} & \hookrightarrow & \mathcal{F}_n & \hookrightarrow & \mathcal{F}
\end{array}$$

Theorem 2.6 can be rewritten as:

Theorem 2.8 *[S94] Let R be a reduced module, the $R \in \mathcal{U}_d$ if and only if:*

- *$deg(f(R)) \leq d$,*
- *$\bar{T}^{d+1}(R) \cong \{0\}$,*
- *R is trivial in degrees k such that $\alpha(k) > d$.*

3 The nilpotent filtration and Kuhn's profile functions

Above one considered spaces so that any element in $\tilde{H}^*X$ is nilpotent and one introduced the terminology "nilpotent" for the cohomology. The restriction axiom allows to express this in term of the action of the Steenrod algebra. More precisely (for $p = 2$) it is equivalent to ask that the operation $Sq_0 \colon x \mapsto Sq^{|x|}x$ is "nilpotent" on any element. This makes it possible to extend this definition to any unstable module.

Definition 3.1 *One says that an unstable module M is nilpotent if for any $x \in M$ there exists k such that $Sq_0^k x = 0$.*

In particular an unstable module is 0-connected. A suspension is nilpotent. In fact:

Proposition 3.2 *An unstable module M is nilpotent if and only if it is the colimit of unstable modules which have a finite filtration whose quotients are suspensions.*

This allows to extend the definition for $p > 2$.
More generally one can define a filtration on $\mathcal{U}$. It is filtered by subcategories $\mathcal{N}il_s$, $s \geq 0$, $\mathcal{N}il_s$ is the smallest thick subcategory stable under colimits and containing all s-suspensions.

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots$$

By the very definition *any $M \in \mathcal{N}il_s$ is $(s-1)$-connected.*.

Proposition 3.3 *Any M has a convergent decreasing filtration $\{M_s\}_{s \geq 0}$ with $M_s/M_{s+1} \cong \Sigma^s R_s(M)$ where $R_s(M)$ is a reduced unstable module, i.e. does not contain a non trivial suspension.*

Only the second part of the proposition needs a small argument see [S94]. The following results are easy consequences of the commutation of T with suspension, the definition, and of **??**. The last part needs a small amount of care because T does not commutes with limits.

Proposition 3.4

- *Let M be a finitely generated unstable module, then the nilpotent filtration is finite (i.e. if k is large enough M_k is trivial), moreover all $R_i(M)$ are finitely generated;*
- *if $M \in \mathcal{N}il_m$, $N \in \mathcal{N}il_n$ then $M \otimes N \in \mathcal{N}il_{m+n}$;*
- *if $M \in \mathcal{N}il_m$ then $T(M) \in \mathcal{N}il_m$, in fact T preserves the nilpotent filtration,*
- *$M \in \mathcal{U}_n$ if and only if for any s $f(R_s(M)) \in \mathcal{F}_n$.*

Concerning the second claim If $M \in \mathcal{N}il_s$ the other condition is satisfied because of 2.4. On the other hand let M_s be the largest submodule of M in $\mathcal{N}il_s$. Consider the short exact sequence

$$\{0\} \to M_s \to M \to M/M_s \to \{0\}$$

Apply $\bar{T}^{n+1}$, the right hand side term cancels by hypothesis. As $\bar{T}^{n+1}(M_s)$ is $(s-1)$-connected, $\bar{T}^{n+1}(M)$ is also $(s-1)$-connected. But this holds for any s, the result follows.

Let $\varepsilon \colon K \to \mathbb{F}_p$ be an augmented unstable algebra. QK the quotient of indecomposable elements of K: $QK = \mathrm{k}er(\varepsilon)/\mathrm{k}er(\varepsilon)^2$.

Proposition 3.5 *The unstable module of indecomposable elements of an augmented unstable algebra K is in $\mathcal{N}il_1$. If $p = 2$ this is even a suspension.*

Following Kuhn, for an unstable module M, one defines a function $w_M : \mathbb{N} \to \mathbb{N} \cup \infty$ by

$$w_M(i) = \deg f(R_i(M))$$

The following lemma easily follows from the definitions, of the main properties of T, and of 2.4, see also [K95]

Lemma 3.6 *One has*

$$w_{M\otimes N}(i) = sup_k\ (w_M(k) + w_N(i-k))$$

Corollary 3.7 *Let M be such that $w_M \leq Id$: $w_M(i) \leq i$ for any i. Let $\mathbb{T}(M)$ be the the tensor algebra on M, the function $w_{\mathbb{T}(M)}$ has the same property.*

Proposition 3.8 *Let K be a connected unstable algebra, such that $\tilde{K} \in \mathcal{N}il_1$. If $q_K \leq Id$ then $w_K \leq Id$.*

Denote by K_i the nilpotent filtration on $\tilde{K} = K_1$ and by Q_i the one on $Q(K)$, $i \geq 1$. The result is proved by induction.

Lemma 3.9 *Let M be in $\mathcal{N}il_s$, then $R_s(M) \in \mathcal{U}_k \setminus \mathcal{U}_{k-1}$ if and only if $\bar{T}^k(M)^s$ is non trivial and $\bar{T}^\ell(M)^s$ is trivial for all $\ell > k$.*

Note that $\bar{T}^i(M)$ is $(s-1)$-connected for all i. The lemma follows from 2.4 and the definitions.
Let us come back to the proof of the proposition and start the induction. Consider the exact sequence (note that the hypothesis that $\tilde{K} \in \mathcal{N}il_1$ is used)

$$K_1 \otimes K_1 \xrightarrow{\mu} K_1 \to Q_1$$

apply $\bar{T}^2$. In degree 1 one gets

$$\{0\} \to \bar{T}^2(K_1)^1 \to \bar{T}^2(Q_1)^1 = \{0\}$$

this case follows. More generally let M_i be the inverse image of K_i in $K_1 \otimes K_1$ *via* the product, $M_1 = M_2 = K_1 \otimes K_1$. Filter the module $K_1 \otimes K_1$ by the sub-modules $\left\{L_n = \sum_{\ell \geq 1} K_\ell \otimes K_{n-\ell}\right\}_{n\geq 2}$ and assume that the result is true at step $i-1$.
The exact sequence

$$M_i \to K_i \to Q_i$$

shows it is enough to prove triviality of $\bar{T}^{i+1}(M_i)^i$. The exact sequence:

$$\{0\} \to L_i \to M_i \to K_1 \otimes K_1/L_i$$

reduces to the case of L_i and of $(K_1 \otimes K_1)/L_i$. Both of these modules are by induction hypothesis in $\mathcal{U}_i$, being filtered with sub-quotients $\Sigma^h(R_{h-k}(K) \otimes R_k(K))$, $k \leq i$. The result follows.

Definition 3.10 *Let X be a space, define $w_X = w_{H^*X}$ and $q_X = w_{QH^*X}$.*

4 Lannes' theorem and Kuhn's reduction, proof of 1.5 and 1.4

Let X be p-complete, 1-connected, assume that TH^*X is finite dimensional in each degree and 1-connected. The following theorem of Lannes is the main input in the proof of 1.4 and of 1.5. The evaluation map:

$$B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, X) \to X$$

induces a map in cohomology

$$H^*X \to H^*B\mathbb{Z}/p \otimes H^*map(B\mathbb{Z}/p, X)$$

and by adjunction a map

$$TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X)$$

Theorem 4.1 *[L92] Under the hypothesis mentioned above the natural map* $TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X)$ *is an isomorphism of unstable algebras.*

Kuhn considers the homotopy cofiber $\Delta(X)$, of the natural map $X \to \mathrm{map}(B\mathbb{Z}/p, X)$. Theorem 4.1 yields

Proposition 4.2

$$H^*(\Delta(X)) \cong \bar{T}H^*X$$

and by definition

$$w_{\Delta(X)} = w_X - 1$$

If $H^*X \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, then $H^*\Delta(X) \in \mathcal{U}_{n-1} \setminus \mathcal{U}_{n-2}$.
Given an augmented unstable algebra K, as T is exact and commutes with tensor product the indecomposable functor Q commutes with T ([S94] Lemma 6.4.2)

$$T(Q(K)) \cong Q(TK)$$

This is not true with $\bar{T}$. However if Z be an H-space, one has a homotopy equivalence:

$$\mathrm{map}(B\mathbb{Z}/p, Z) \cong Z \times \mathrm{map}_*(B\mathbb{Z}/p, Z)$$

and it follows clearly that [CCS07]

Proposition 4.3 $QH^*\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z) = \bar{T}^n QH^*Z$.

In order to prove 1.5 we will proceed as follows. Assume given a space X

- such that $\tilde{H}^*X \in \mathcal{N}il_1$,
- and such that q_X is not 0 and less or equal to Id,

we will show, considering mapping spaces, that this is impossible.

First part of the proof
Applying 3.8, we can suppose $w_X \leq Id$. Then, replacing X by $\Delta^k(X)$, and thus w_X by $w_X - k$ (take $k+1 = min(Id - w_X)$) allows to suppose that

- $R_i(H^*X) \in \mathcal{U}_0$, $i < s$,
- $R_s(H^*X) \in \mathcal{U}_1 \setminus \mathcal{U}_0$,
- $w_X(s+i-1) \leq i$.

This is Kuhn's reduction. Collapsing down a low dimensional skeleton (and after that taking p-completion) allows to assume that $R_i(H^*X) = \{0\}$, $0 < i < s$. Next let Z be $\Omega(\Sigma X)\hat{}_p$, then $H^*Z \cong \mathbb{T}(\tilde{H}^*X)$. Observe that Lannes' theorem applies to Z.
With the hypothesis $R_i(H^*X) \in \mathcal{U}_0$, $i < s$ and $w_X(s+i-1) \leq i$ one gets easily (using 3.6),

Corollary 4.4 $w_Z(j) = 0$ *if* $0 < j < s$, $w_Z(s+i-1) \leq i$, $i \geq 1$.

The corollary below is a key step of the proof.

Corollary 4.5 $\bar{T}^n H^*Z$ *is* $(n+s-2)$*-connected.*

It follows that:

Corollary 4.6 *The space* $map_*(B/Z/p^{\wedge n}, Z)$ *is* $(n+s-2)$*-connected.*

This uses [BK87], in particular chapters 6 and 9 to show there are no phantom maps.
Because the largest sub-module N_s of H^*Z in $\mathcal{N}il_{s+1}$ is an ideal, one gets a non trivial algebraic map of unstable algebras:

$$\varphi_s^* : H^*Z \to H^*Z/N_{s+1} \cong \mathbb{F}_2 \oplus \Sigma^s R_s H^*Z \to \mathbb{F}_2 \oplus \Sigma^s F(1) \subset \mathbb{F}_2 \oplus \Sigma^s \tilde{H}^*B\mathbb{Z}/p\ .$$

It cannot factor through $H^*\Sigma^{s-1}K(\mathbb{Z}/p, 2)$, because there are no non trivial map from an s-suspension (and thus from an unstable module in $\mathcal{N}il_s$) to an $(s-1)$-suspension of a reduced module, and because:

Proposition 4.7 *$H^*K(\mathbb{Z}/p,2)$ is reduced.*

This result is known to experts, and in fact true for any $H^*K(\mathbb{Z}/p,n)$. As we do not know a reference we give at the end a proof (for $n=2$).

Second part of the proof
The contradiction comes from the fact that using obstruction theory one can construct a factorization.
The existence of a map realising φ_s^* is a consequence, using Lannes' theorem, of the Hurewicz theorem because $\mathrm{map}_*(B\mathbb{Z}/p, Z)$ is $(s-1)$-connected.
$K(\mathbb{Z}/p,2)$ is built up, starting with $\Sigma B\mathbb{Z}/p$, as follows (Milnor's construction). There is a filtration $* = C_0 \subset C_1 = \Sigma B\mathbb{Z}/p \subset C_2 \subset \ldots \subset \cup_n C_n = K(\mathbb{Z}/p,2)$, a diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & B^{*n+1} & \longrightarrow & B^{*n+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \cdots \end{array}$$

and cofibrations, up to homotopy

$$\Sigma^{n-1}B^{\wedge n} \to C_{n-1} \to C_n$$

$$\Sigma^{n-2+s}B^{\wedge n} \to \Sigma^{s-1}C_{n-1} \to \Sigma^{s-1}C_n$$

The obstructions to extend $\varphi_s\colon \Sigma^s B\mathbb{Z}/p \to Z$ to $\Sigma^{s-1}K(\mathbb{Z}/p,2)$ are in the groups

$$[\Sigma^{n+s-2}(B\mathbb{Z}/p)^{\wedge n}, Z] = \pi_{n+s-2}\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$$

. Indeed on has to solve the extension problem

$$\begin{array}{ccccc} \Sigma^{n-2+s}B^{\wedge n} & \longrightarrow & \Sigma^{s-1}C_{n-1} & \longrightarrow & \Sigma^{s-1}C_n \\ & & \downarrow & \swarrow & \\ & & Z & & \end{array}$$

These groups are trivial. It follows one can do the extension, this is a contradiction.
To complete the proof of 1.4 it is now enough to use again $w_{\Delta(X)} = w_X - 1$.

Proof of 4.7
The cohomology $H^*K(\mathbb{Z}/p,2)$ is isomorphic to (with obvious notations)

$$\mathbb{F}_p[x, \beta P^{I_h}\beta(x)] \otimes E(P^{I_\ell}\beta(x)) \cong \mathbb{F}_p[x, x_h; h \geq 1] \otimes E(y_\ell; \ell \geq 0)$$

with $|x| = 2$, $I_h = (p^{h-1}, p^{h-2}, \ldots, p, 1)$, $h \geq 1$, and $\beta(x) = y_0$. One gets $\beta(y_h) = x_h$, $h \geq 1$. It is enough to show that for any non zero element $z \in \tilde{H}^*K(\mathbb{Z}/p,2)$ one can find a operation θ so that $\theta(z) \in \mathbb{F}_p[x, x_h]$ and $\theta(z) \neq 0$. Any element z writes as a sum $\sum_{0 \leq i \leq t}\sum_j P_{i,j}(x, x_h) \otimes L_{i,j}(y_\ell)$. If the element z has degree $2n$, and in its above decomposition there are some non trivial terms where the exterior part is of degree 0, a straightforward application of the Cartan formula shows that P^n does the job. Indeed it is zero on any element an exterior generator. Otherwise it is enough to create such elements. One applies some operations which decreases (non trivially) the length of exterior factors that occur. Consider in the exterior factors $L_{i,j}$ of minimal length, and in there the occurrence of y_ℓ for the smallest possible value of ℓ. Let Q_i be the usual Milnor derivation at an odd prime. The bracket $[\beta, Q_i]$ is also a derivation. It acts trivially on the x_h, sends x onto y_{i+1}, and y_ℓ onto $x_{i-\ell+1}^{p^\ell}$. It is easily shown using a standard lexicographic order argument that it does the job for i large enough.

5 The Eilengerg- Moore spectral sequence, triviality of $\mathrm{Ext}^1_{\mathcal{U}}(-,-)$.

In the sequel the spaces X are still p-complete, 1-connected, such that $\tilde{H}^*X \in \mathcal{N}il_1$ and of finite dimension in each degree. One will also assume that for any i the reduced unstable modules R_iH^*X are finitely generated. The object of this last section is to prove the following:

Theorem 5.1 *Let X be a space such that $\tilde{H}^*X \in \mathcal{N}il_1$. Assume moreover that $deg(f(R_1H^*X)) = d > 0$ then $deg(R_2H^*X) \geq 2d$.*

The value of the result is to give some control on R_2H^*X, starting from some information on R_1H^*X. The hypothesis $d > 0$ is not significant, the result being obviously true if $d = 0$. This theorem is proved only for $p = 2$, the difficulties that occur in [S94], [S10] occur again, even if they look more manageable.
One would like also to extend the result and get one of the following type. One assumes that $R_1H^*X, \dots, R_{k-1}H^*X \in \mathcal{U}_0$, $R_kH^*X \in \mathcal{U}_d$, and conclude that R_kH^*X, for some $k > d$. The possibility of such an extension is likely but does not look direct.
The proof does not depend on Lannes' mapping space theorem, but on minimal information on the Eilenberg-Moore spectral sequence. It still depends on the algebraic properties of T_V. It should be also said that this proof should be compared, and was motivated by:

Proposition 5.2 *Let M be a connected reduced unstable module such that $deg(f(M))$ is finite non zero. The unstable module $\mathbb{T}(M)$ does not carry the structure of an unstable algebra.*

The proof of this proposition is left to the reader.
The proof of the theorem is divided in three steps. One assumes there exists a space for which the theorem does not hold. In the two first parts one shows the triviality of certain cup-products. Using this information, in the last part one shows that an extension in the Eilenberg-Moore spectral sequence is trivial, this leads immediately to a contradiction.

First part of the proof: triviality of cup-products I.
Assume that $deg(f(R_1H^*X)) = d > 0$ and suppose that $deg(f(R_2H^*X)) \leq 2d-1$. Denote by H_k the largest sub-module of H^*X which belongs to $\mathcal{N}il_k$, $H_k/H_{k+1} \cong \Sigma^k R_kH^*X$.
This hypothesis, and 2.7, implies that the non-trivial classes in R_1H^*X live in degrees i with $\alpha(i) \leq d$, and that there exist non-trivial classes in degrees i such that $\alpha(i) = d$. Denote by R the sub-module generated in R_1H^*X by these latter classes. For a class $x \in R_1H^*X$ denote by $\sigma(x)$ the suspension in ΣR_1H^*X. By the very construction the sub-module R has the property to be the smallest sub-module in R_1H^*X such that $R_1H^*X/R \in \mathcal{U}_{d-1}$. For a while it will be easier to establish results in the category of functors, here is the property that defines $f(R)$:

Lemma 5.3 *The functor $f(R)$ is the smallest sub-functor of $f(R_1H^*X)$ such that the quotient $f(R_1H^*X/R))$ is of degree strictly less than d. In particular any non trivial quotient of R is of degree d and not of degree less or equal than $d-1$.*

Corollary 5.4 *Any non-trivial quotient of the functor $f(R \otimes R) \cong f(R) \otimes f(R)$ is of degree $2d$ and not of degree less or equal than $2d-1$.*

This property follows from the lemma and various references about the Taylor series of a tensor product of functors, see [K94-1] see also [Dj07] 1.13 (applying Kuhn duality) for example.
Thus if $deg(f(R_2H^*X)) \leq 2d-1$ any morphism $f(R) \otimes f(R) \to f(R_2H^*X)$ is trivial.
Choose a *finite* set of classes g_i, $i \in I$, in H^*X which projects onto generators γ_i of ΣR. These classes generate a sub-module $U \subset H^*X$. As the tensor product of nilpotents modules belongs to $\mathcal{N}il_2$ the cup-product defines a morphism $U \otimes U \to H_2$. In fact:

Proposition 5.5 *The cup-product takes values in H_3.*

This is a direct consequence of what precedes. If it is not true this would provide a non-trivial morphism $f(R) \otimes f(R) \to f(R_2H^*X)$, and a non trivial quotient of $f(R \otimes R) \cong f(R) \otimes f(R)$ of degree less or equal to $2d-1$. Indeed, the composite $U \otimes U \to \Sigma R \otimes \Sigma R \to \Sigma^2 R_2H^*X$ would be non-trivial, which implies the result.

Second part of the proof: triviality of cup-products II.
First recall, for an unstable module M, the operation $Sq_1 \colon M \to M$, $x \mapsto Sq^{|x|-1}x$ and some of its properties: the derivation formula $Sq_1(x \otimes y) = Sq_0(x) \otimes Sq_1(y) + Sq_1(x) \otimes Sq_0(y)$, and the Adem relations $Sq_1Sq_0 = 0$.
For a given integer k the classes $Sq_1^k(g_i)$, $i \in I$, generate a sub-module U_k which projects onto $\Sigma\Phi^k R$. This because if $p \colon U \to \Sigma R_1H^*X$ is the canonical map, and $p(g_i) = \sigma\gamma_i$ then $p(Sq_1^k(g_i)) = \sigma(Sq_0^k\gamma_i)$.

Corollary 5.6 *All cup-products of classes in U_k, k large enough, are trivial.*

It is better now to return in the category $\mathcal{U}$ and to use 2.8 to prove the corollary. The cup product of the image of two classes in U_k is of degree $2 + 2^k(i+j)$, $i, j > 0$, for some i and j. The cup-products belongs to a quotient S of the second symmetric power $S^2(U)$ which is finitely generated, and thus has a it finite nilpotent filtration, denote it S_ℓ, 3.4. Thus, for some h if $\ell > h$ then S_ℓ is trivial. Moreover each quotient $S_\ell/S_{\ell+1} \cong \Sigma^\ell R_\ell S$ is finitely generated, it belongs to $\mathcal{U}_{n(\ell)}$ for some $n(\ell)$. Let t be the supremum of these $n(\ell)$ for $\ell \leq h$.

In conclusion the degree a non trivial cup-product should satisfy the equation:

$$2 + 2^k(i + j) = \ell + u$$

with $3 \le t\ell \le h$ and $\alpha(u) \le t$. It rewrites

$$2^k(i + j) + 2 - \ell = u$$

Clearly such an equation cannot hold if k is large enough because of the length of the 2-adic expansions on both sides.

Third part of the proof, triviality of $Ext^1(-,-)$
Consider the term $F_\infty^{-2,*} \subset H^*\Omega X$ of the Eilenberg-Moore filtration. The columns -1 and -2 of the spectral sequence yield an extension:

$$\{0\} \to \Sigma E_\infty^{-1,*} \to \Sigma^2 F_\infty^{-2,*} \to E_\infty^{-2,*} \to \{0\}$$

As $\Sigma^2 R_1 H^*X$ is a quotient of $QH^*X \cong E_2^{-1,*}$, there is an epimorphism $E_\infty^{-1,*} \to R_1 H^*X$ one gets by composition an extension:

$$\{0\} \to \Sigma^2 R_1 H^*X \to \Sigma^2 E_0 \to E_\infty^{-2,*} \to \{0\}$$

The term $E_\infty^{-2,*}$ is a quotient of $Tor_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2)$, this gives rise to another extension after suspending twice:

$$\{0\} \to \Sigma^2 R_1 H^*X \to \Sigma^2 E_1 \to Tor_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2) \to \{0\}$$

Next $Tor_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2)$ receives, because of 5.6, $U_k \otimes U_k$. Thus there is a map $\Sigma\Phi^k R \otimes \Sigma\Phi^k R \to Tor_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2)$ and one gets an extension:

$$\{0\} \to \Sigma^2 R_1 H^*X \to E_2 \to \Sigma^2(\Phi^k R \otimes \Phi^k R) \to \{0\}$$

Proposition 5.7 *For k large enough*

$$Ext^1_{\mathcal{U}}(\Sigma^2(\Phi^k R \otimes \Phi^k R), \Sigma^2 R_1 H^*X)$$

is trivial.

In a first step one uses the derivative Ω_1 of Ω to reduce to the computation of $Ext^1_{\mathcal{U}}(\Phi^k R \otimes \Phi^k R, R_1 H^*X)$. One uses in particular the following: Ω_1 applied to a second suspension is a third suspension, and Ω_1 applied to a suspension is a suspension ([Mil84], [S94]). Then, recall from [CS13]

Lemma 5.8 *Let R be a finitely generated reduced unstable module, M be a finitely generated unstable module. For k large enough one has*

$$Ext^1_{\mathcal{U}}(\Phi^k R, M) \cong Ext^1_{\mathcal{U}}(\Phi^k R, M/N) \cong Ext^1_{\mathcal{F}}(f(R), f(M))$$

where N is the largest nilpotent submodule of M.

Lemma 5.9 *Let S be a functor of degree n such that all simple quotients are of degree n, such that $S(\{0\}) = \{0\}$, and F a functor of degree less than $2n-1$. Then, $Ext^1_{\mathcal{F}}(S \otimes S, F)$ is trivial.*

For this lemma use [FFSS99], the category $bi - \mathcal{F}$ and the hypothesis that $f(R)$ has no quotient of degree strictly less than d.

Proof of the theorem
Choose a non zero class $g \in R$ such that $\alpha(|g|) = d$. This defines a class γ of the same degree in $H^*\Omega X$. Be careful that the notations g_i and γ_i used above in part II were for classes originated in H^*X, thus there is degree shift because of the suspension. The class γ is not nilpotent, as Sq_0 acts non nilpotently on g, $\gamma^{1+2^h} \neq 0$. If h is large enough $\alpha(|\gamma^{1+2^h}|) = 2d$, in such a degree $E_\infty^{-1,*}/\mathcal{N}il$ is trivial. So the cup product γ^{1+2^h} is detected in the Eilenberg-Moore spectral sequence by $\sigma(g) \otimes \sigma(Sq_1^{2^k} g) + \sigma(Sq_1^{2^k} g) \otimes \sigma(g)$. All of these classes are in a sub-module of the cohomology $H^*\Omega X$, which projects onto $\Phi^K R \otimes \Phi^k R \oplus R_1 H^*X$ because the extension above splits. The class γ^{1+2^h} projects to 0 on the summand $R_1 H^*X$. There should not exist an operation such that $\theta(\gamma^{1+2^h}) = \gamma^{2^{h+1}}$. This is a contradiction because such an operation exists and the class γ projects to a non-zero class on this summand as well as all the classes $Sq_0^k(\gamma)$.
The key point (as in [S94]) is the triviality of the -1-column in a certain degree. It is worth to observe that when computing cup-products in $H^*\Omega\Sigma X$ the above phenomenon does not occur, because the cup-product is given by the following formula:

- $x \cup y = x \otimes y + y \otimes x + xy$,

in this formula $x, y, xy \in H^*X$, which are considered by abuse as classes in $H^*\Omega\Sigma X$ using duality and the identification as Hopf algebras of $H^*\Omega\Sigma X$ (X connected) with the tensor algebra $\mathbb{T}(\tilde{H}_*X)$, the elements of $\tilde{H}_*X$ being primitive following Bott-Samelson.

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