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PERSISTENCE OF SOME ADDITIVE FUNCTIONALS OF SINAI'S WALK

ALEXIS DEVULDER

ABSTRACT. We are interested in Sinai's walk $(S_n)_{n \in \mathbb{N}}$. We prove that the annealed probability that $\sum_{k=0}^n f(S_k)$ is strictly positive for all $n \in [1, N]$ is equal to $1/(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}$, for a large class of functions f , and in particular for $f(x) = x$. The persistence exponent $\frac{3-\sqrt{5}}{2}$ first appears in a non-rigorous paper of Le Doussal, Monthus and Fischer, with motivations coming from physics. The proof relies on techniques of localization for Sinai's walk and uses results of Cheliotis about the sign changes of the bottom of valleys of a two-sided Brownian motion.

1. INTRODUCTION

In this paper we consider random walks in random environments in \mathbb{Z} . Let $\omega := (\omega_i)_{i \in \mathbb{Z}}$ be a collection of independent and identically distributed random variables taking values in $(0, 1)$, with joint law η . A realization of ω is called an *environment*. Conditionally on ω , we define a Markov chain $(S_n)_{n \in \mathbb{N}}$ by $S_0 = 0$ and for $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$,

$$P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & \text{if } k = i + 1, \\ 1 - \omega_i & \text{if } k = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We say that $(S_n)_{n \in \mathbb{N}}$ is a *random walk in random environment* (RWRE). This model has many applications in physics (see e.g. Hughes [18]) and in biology (see e.g. Cocco and Monasson [10] about DNA reconstruction), and has unusual properties. Moreover, its properties are used to study several other mathematical models, see e.g. Zindy [31], Enriquez, Lucas and Simenhaus [14] and Devulder [13].

The probability P_ω is called the *quenched law*. We denote by P_ω^x the quenched law for a RWRE starting at $x \in \mathbb{Z}$ instead of 0. We also consider the *annealed law*, which is defined by

$$\mathbb{P}(\cdot) = \int P_\omega(\cdot) \eta(d\omega).$$

Notice in particular that $(S_n)_{n \in \mathbb{N}}$ is not Markovian under \mathbb{P} . We also denote by \mathbb{E} , E_ω and E_ω^x the expectations under \mathbb{P} , P_ω and P_ω^x respectively. We assume that the following ellipticity condition holds:

$$\exists \varepsilon_0 > 0, \quad \eta(\varepsilon_0 \leq \omega_0 \leq 1 - \varepsilon_0) = 1. \quad (1.1)$$

This ensures that $|\log(\frac{1-\omega_0}{\omega_0})|$ is η -a.s. bounded by $\log(\frac{1-\varepsilon_0}{\varepsilon_0})$. Solomon [27] proved that $(S_n)_{n \in \mathbb{N}}$ is recurrent for almost every environment ω if and only if

$$\int \log \left(\frac{1 - \omega_0}{\omega_0} \right) \eta(d\omega) = 0. \quad (1.2)$$

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We assume that this condition is satisfied throughout the paper. Moreover, in order to avoid the degenerate case of simple random walks, we suppose in the following that

$$\sigma := \left(\int \log^2 \left(\frac{1 - \omega_0}{\omega_0} \right) \eta(d\omega) \right)^{1/2} > 0. \quad (1.3)$$

A RWRE $(S_n)_{n \in \mathbb{N}}$ satisfying conditions (1.1), (1.2) and (1.3) is referred to as *Sinai's walk*. Sinai ([25], see also Andreatti [2] for extensions) proved that in this (recurrent) case,

$$\sigma^2 \frac{S_n}{\log^2 n} \rightarrow_{\text{law}} b_\infty$$

as $n \rightarrow +\infty$, where b_∞ is a non degenerate and non gaussian random variable and \rightarrow_{law} denotes convergence in law under \mathbb{P} . We refer to Hughes [18], Révész [21] and Zeitouni [30] for more properties of RWRE.

Sinai [26] also showed in 1992 that for a symmetric simple random walk $(R_n)_{n \in \mathbb{N}}$, we have $\mathbb{P}(\sum_{k=1}^n R_k > 0 \ \forall 1 \leq n \leq N) \asymp N^{-1/4}$ as $N \rightarrow +\infty$. In this paper, we are interested in the corresponding probability for Sinai's walk $(S_n)_{n \in \mathbb{N}}$, and more generally in the one-sided exit problem for some additive functionals of Sinai's walk under the annealed law \mathbb{P} . We say that $g(x) = o(1)$ as $x \rightarrow +\infty$ (resp $-\infty$) if $g(x) \rightarrow 0$ as $x \rightarrow +\infty$ (resp $-\infty$). Our main result is the following.

Theorem 1.1. *Let f be a function $\mathbb{Z} \rightarrow \mathbb{R}$, such that $f(0) = 0$; $\forall x > 0, f(x) \geq 1$; $\forall x < 0, f(x) \leq -1$ and $|f(x)| \leq \exp(|x|^{o(1)})$ as $x \rightarrow \pm\infty$. We consider a RWRE $(S_n)_{n \in \mathbb{N}}$ satisfying conditions (1.1), (1.2) and (1.3), and a real number $u \leq 0$. We have as $N \rightarrow +\infty$,*

$$\mathbb{P} \left(\sum_{k=0}^n f(S_k) > u \quad \forall 1 \leq n \leq N \right) = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2} + o(1)}}.$$

Let $(A_t)_{t \in D}$ be a real valued stochastic process starting from 0, where $D = \mathbb{R}_+$ or $D = \mathbb{N}$. The asymptotic study of the survival function $\mathbb{P}(\forall t \in (0, T] \cap D, A_t \geq x)$ for $x \leq 0$, when $T \rightarrow +\infty$, is called *one sided exit problem* or *persistence probability*. This problem is equivalent to the study of $\mathbb{P}(T_x > T)$, where T_x is the first passage time of the process $(-A_t)_t$ strictly above the level $y = -x \geq 0$. In many cases with physical relevance, the survival function behaves asymptotically like $1/T^{\alpha+o(1)}$ as $T \rightarrow +\infty$, with $\alpha > 0$. The exponent α is called the *persistence* or *survival* exponent. This problem, which is well known for random walks or Lévy processes, is less understood for the integrals of these processes, in particular in the discrete case. We refer to Aurzada and Simon [4] for a recent review on this subject from the mathematical point of view. Persistence properties have also received a considerable attention in physics, see e.g. Bray, Majumdar and Schehr [6] for an up-to-date survey.

In our case, the probability we obtain in Theorem 1.1 for the integrals of $(f(S_n))_n$ is a power of $\log N$ instead of N , which is quite unusual and contrasts with all the cases presented in the review paper [4]. The value of the survival exponent is $\frac{3-\sqrt{5}}{2}$; it does not depend on the function f for a wide class of functions, and it also does not depend on the law η of the environment, as long as (1.1), (1.2) and (1.3) are satisfied. It is derived from the results of Cheliotis [8] about the number of sign changes of the bottom of valleys of Brownian motion, and was first stated in a non rigorous paper of Le Doussal, Monthus and Fisher [16], with motivations coming from physics. Notice also that the strict inequality $> u$ can be replaced by a large one in Theorem 1.1.

Before giving some examples, we introduce some more notation. We denote by \mathbb{N}^* the set of positive integers, and \mathbb{Z}_-^* is the set of negative integers. We define the local time of the RWRE

$(S_n)_{n \in \mathbb{N}}$ at time $n \in \mathbb{N}$ as follows:

$$L(A, n) := \sum_{k=0}^n \mathbf{1}_{\{S_k \in A\}}, \quad L(x, n) := L(\{x\}, n)$$

for $A \subset \mathbb{Z}$ and $x \in \mathbb{Z}$. In words $L(A, n)$ is the number of visits of the random walk S to the set A in the first n steps. This quantity will be useful in the proof of Theorem 1.1, because

$$\sum_{k=0}^n g(S_k) = \sum_{x \in \mathbb{Z}} g(x) L(x, n), \quad n \in \mathbb{N}, \quad (1.4)$$

for every function g .

It can be interesting to keep in mind the first example:

Example 1. For $f(x) = \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x < 0\}}$, Theorem 1.1 gives

$$\mathbb{P} [L(\mathbb{N}^*, n) > L(\mathbb{Z}_-, n) \quad \forall 1 \leq n \leq N] = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}}.$$

The following example gives for $\alpha = 1$ the persistence of the *temporal average* or *running average* of Sinai's walk, that is $\frac{1}{n} \sum_{k=0}^n S_k$, with the terminology of Le Doussal et al. ([16] Section IV):

Example 2. Let $\alpha > 0$, $\text{sgn}(x) := \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x < 0\}}$ for $x \in \mathbb{R}$, and $f(x) = \text{sgn}(x)|x|^\alpha$ for $x \in \mathbb{Z}$. We get for $u \leq 0$,

$$\mathbb{P} \left(\frac{1}{n} \sum_{k=0}^n \text{sgn}(S_k) |S_k|^\alpha > u \quad \forall 1 \leq n \leq N \right) = \frac{1}{(\log N)^{\frac{3-\sqrt{5}}{2}+o(1)}}.$$

We recall that the corresponding probability for $\alpha = 1$ for a simple random walk is of order $N^{-1/4}$ (see Sinai [26]; see also Vysotsky [29] and Dembo, Ding and Gao [11] for recent extensions). Example 2 is also, for $\alpha > 0$ arbitrary, the analogue for Sinai's walk of the results obtained by Simon [24] for some additive functionals of stable processes with no negative jumps. We can also consider functions increasing more rapidly, such as $f(x) = \text{sgn}(x)|x|^{\log(2+|x|)^\alpha}$, $x \in \mathbb{Z}$ for $\alpha > 0$.

The rest of the paper is organized as follows. We introduce some notation and basic facts in Section 2. In Section 3 we build a set $\mathcal{B}(N)$ of *bad environments*, such that in a bad environment, $\sum_{k=0}^n f(S_k)$ is less than u for at least one integer $n \in [1, N]$ with a great quenched probability. To this aim, we approximate the potential of the environment by a two-sided Brownian motion, and we define *strong changes of sign* for the valleys of this Brownian motion. We prove that on a bad environment, the existence of such a strong change of sign forces the walk to stay a long time in \mathbb{Z}_- with a large quenched probability, leading to the upper bound of Theorem 1.1. In Section 4 we build a set $\mathcal{G}(N)$ of *good environments*. We prove, using a mathematical induction, that in such a good environment $\sum_{k=0}^n f(S_k)$ is strictly positive for $1 \leq n \leq N$ with a large quenched probability, which leads to the lower bound of Theorem 1.1. Finally, Section 5 is devoted to the proof of two technical lemmas.

Throughout the paper, $c_i, i \in \mathbb{N}$ denote positive constants.

2. PRELIMINARIES

2.1. Potential. We recall that the potential V is a function of the environment ω , which is defined on \mathbb{Z} as follows:

$$V(n) := \begin{cases} \sum_{i=1}^n \log \frac{1-\omega_i}{\omega_i} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n+1}^0 \log \frac{1-\omega_i}{\omega_i} & \text{if } n < 0. \end{cases}$$

For $p \in \mathbb{Z}$, we define the hitting time of p by $(S_n)_n$ by:

$$\tau(p) := \inf\{k \in \mathbb{N}, S_k = p\}.$$

We now give some basic estimates that will be useful throughout the paper.

Lemma 2.1. (see e.g. Zeitouni [30] formula (2.1.4) p. 196) *If $p < q < r$, then*

$$P_\omega^q[\tau(r) < \tau(p)] = \left(\sum_{k=p}^{q-1} e^{V(k)} \right) \left(\sum_{k=p}^{r-1} e^{V(k)} \right)^{-1}. \quad (2.1)$$

Lemma 2.2. (see e.g. Zeitouni [30] p. 250), *If $g < h < i$*

$$E_\omega^h[\tau(g) \wedge \tau(i)] \leq \sum_{k=h}^{i-1} \sum_{\ell=g}^k \frac{\exp[V(k) - V(\ell)]}{\omega_\ell} \leq \varepsilon_0^{-1}(i-g)^2 \exp \left[\max_{g \leq \ell \leq k \leq i-1} (V(k) - V(\ell)) \right]. \quad (2.2)$$

Moreover, the following estimate, can be found in Andreatti ([1] p. 22) and is in the spirit of Révész ([21] p278-279).

Lemma 2.3. *If $p < z < q < r$ or $p < q < z < r$,*

$$E_\omega^q[L(z, \tau(p) \wedge \tau(r))] = \frac{P_\omega^q[\tau(z) < \tau(p) \wedge \tau(r)]}{\omega_z P_\omega^{z+1}[\tau(z) > \tau(r)] + (1 - \omega_z) P_\omega^{z-1}[\tau(z) > \tau(p)]}. \quad (2.3)$$

For the sake of completeness, we recall the proof:

Proof of Lemma 2.3: By the strong Markov property

$$\begin{aligned} E_\omega^q[L(z, \tau(p) \wedge \tau(r))] &= E_\omega^q[L(z, \tau(p) \wedge \tau(r)) \mathbf{1}_{\{\tau(z) < \tau(p) \wedge \tau(r)\}}] \\ &= E_\omega^z[L(z, \tau(p) \wedge \tau(r))] P_\omega^q[\tau(z) < \tau(p) \wedge \tau(r)]. \end{aligned}$$

Since $L(z, \tau(p) \wedge \tau(r))$ is under P_ω^z a geometric random variable of parameter $\omega_z P_\omega^{z+1}[\tau(z) > \tau(r)] + (1 - \omega_z) P_\omega^{z-1}[\tau(z) > \tau(p)]$, we get (2.3). \square

2.2. x -extrema. We now recall some definitions introduced by Neveu and Pitman [20]. If w is a continuous function $\mathbb{R} \rightarrow \mathbb{R}$, $x > 0$, and $y_0 \in \mathbb{R}$, it is said that w admits an x -minimum at y_0 if there exists real numbers α and β such that $\alpha < y_0 < \beta$, $w(y_0) = \inf\{w(y), y \in [\alpha, \beta]\}$, $w(\alpha) \geq w(y_0) + x$ and $w(\beta) \geq w(y_0) + x$. It is said that w admits an x -maximum at y_0 if $-w$ admits an x -minimum at y_0 . In these two cases we say that w admits an x -extremum at y_0 .

We denote by \mathcal{W} the set of functions w from \mathbb{R} to \mathbb{R} such that the three following conditions are satisfied: **(a)** w is continuous on \mathbb{R} ; **(b)** for every $x > 0$, the set of x -extrema of w can be written $\{x_k(w, x), k \in \mathbb{Z}\}$, where $(x_k(w, x))_{k \in \mathbb{Z}}$ is strictly increasing, unbounded from above and below, with $x_0(w, x) \leq 0 < x_1(w, x)$; **(c)** for all $x > 0$ and $k \in \mathbb{Z}$, $x_{k+1}(w, x)$ is an x -maximum if and only if $x_k(w, x)$ is an x -minimum. We now consider a two-sided standard Brownian motion W . We know from Cheliotis ([8], Lemma 8) that $\eta(W \in \mathcal{W}) = 1$.

For each $x > 0$, $b_W(x)$, also denoted by $b(x)$ when no confusion is possible, is defined on $\{W \in \mathcal{W}\}$ as

$$b_W(x) := \begin{cases} x_0(W, x) & \text{if } x_0(W, x) \text{ is an } x\text{-minimum,} \\ x_1(W, x) & \text{otherwise.} \end{cases}$$

One interesting feature about b_W is that the diffusion in the random environment W , defined by Brox [7], is localized in a small neighborhood of $b_W(\log t)$ at time t with probability nearly one (see Brox [7], Tanaka [28] and Hu [17]). Such a diffusion can be viewed as a continuous time analogue of Sinai's walk (see e.g. Shi [23]), and a similar localization phenomenon arises for Sinai's walk (see Sinai [25], Golosov [15] and more recently Andreatti [2]).

For $x > 0$ and $k \in \mathbb{Z}$, the restriction of $W - W(x_k(W, x))$ to $[x_k(W, x), x_{k+1}(W, x)]$ is denoted by $T_k(x)$ and is called an x -slope. It is the translation of the trajectory of W between two consecutive x -extrema. If $x_k(W, x)$ is an x -minimum (resp. x -maximum), it is a nonnegative (resp. non-positive) function, and its maximum (resp. minimum) is attained at $x_{k+1}(W, x)$. For each x -slope $T_k(x)$, we denote by $H(T_k(x))$ its height and by $e(T_k(x))$ its excess height, that is $H(T_k(x)) := |W[x_{k+1}(W, x)] - W[x_k(W, x)]|$ and $e(T_k(x)) := H(T_k(x)) - x$. We also define $e(T_k(0)) = H(T_k(0)) = 0$, $k \in \mathbb{Z}$.

The point of view of x -extrema has been used in some recent studies of processes in random environment, see e.g. Bovier and Faggionato [5] for Sinai's walk, Cheliotis [9] for (recurrent) diffusions in a Brownian potential, and Andreatti and Devulder [3] for (transient) diffusions in a drifted Brownian potential.

3. PROOF OF THE UPPER BOUND

3.1. Strong change of sign. Let $c > 0$. Similarly as in Cheliotis ([8] Corollary 2), we denote by $(X_k)_{k \geq 1}$ the strictly increasing sequence of points for which $b_W(\cdot)$ changes its sign in $[c, +\infty)$. The proof of the following fact is deferred to Section 5:

Fact 3.1. *Almost surely,*

$$\begin{aligned} X_1 &= \inf\{x \geq c, e(T_0(x)) = 0\}, \\ X_{k+1} &= \inf\{x > X_k, e(T_0(x)) = 0\}, \quad k \in \mathbb{N}^*. \end{aligned}$$

Moreover, the sign of $b(\cdot)$ is constant on every interval $[c, X_1]$, $(X_k, X_{k+1}]$, $k \in \mathbb{N}^*$.

As a consequence, a.s. for every $x > 0$, b_W changes its sign at x if and only if $e(T_0(x)) = 0$. We can now define *strong changes of sign* as follows:

Definition 3.2. *Consider $x > 0$. For $a > 0$, we say that x is an a -strong change of sign of b_W if and only if*

$$e(T_0(x)) = 0, \quad e(T_{-1}(x)) \geq ax, \quad \text{and} \quad e(T_1(x)) \geq ax.$$

In the following lemma, we evaluate the probability that there is no a -strong change of sign x such that $b_W(x) > 0$ in $[c, X_{2k+1}]$.

Lemma 3.3. *For $a > 0$, $c \geq 1$ and $k \in \mathbb{N}^*$, we define $A(k, a, c)$ also denoted by $A_{k,a,c}$ as follows:*

$$A_{k,a,c} := \{\forall i \in \{1, \dots, 2k\}, b_W(X_i) > 0 \Rightarrow (e(T_{-1}(X_i)) < aX_i \text{ or } e(T_1(X_i)) < aX_i)\}.$$

We have,

$$\eta(A_{k,a,c}) \leq \eta(A_{1,a,c})(1 - e^{-2a})^{k-1}. \quad (3.1)$$

The proof of this lemma is deferred to Section 5.

3.2. Bad environments. Let $(\omega_i)_{i \in \mathbb{Z}}$ be a collection of independent and identically distributed random variables satisfying (1.1), (1.2) and (1.3). We now fix $\varepsilon \in (0, 1/2)$. Let $K \geq 1$. In order to transfer to our random potential V , with some approximations, some results such as the ones of Cheliotis [8], which are available for Brownian motion, but unavailable for V to the extent of our knowledge, we use the following coupling. According to the Komlós–Major–Tusnády strong approximation theorem (see Komlós et al. [19]), there exist (strictly) positive constants C_1, C_2 and C_3 , independent of $K \in \mathbb{N}^*$, such that, possibly in an enlarged probability space, there exists a two-sided standard Brownian motion $(W(t), t \in \mathbb{R})$, such that

$$\mathcal{B}_1(K) := \left\{ \sup_{-K \leq i \leq K} |V(i) - \sigma W(i)| \leq C_1 \log K \right\}$$

satisfies $\eta([\mathcal{B}_1(K)]^c) \leq C_2 K^{-C_3}$.

Throughout the end of the paper, we set $a := \frac{1}{2} \exp\left(\frac{\sqrt{5}-3}{2\varepsilon}\right)$. Moreover, for $u \in \mathbb{R}$, $\lfloor u \rfloor$ denotes the integer part of u . We define for $N > 1$ the events

$$\begin{aligned} \mathcal{B}_2(N) &:= \left\{ \begin{array}{l} \text{the number of sign changes of } b_W \\ \text{in } [(\log N)^\varepsilon, (\log N)^{1-\varepsilon}] \text{ is at least } 2\varepsilon \log_2 N + 2 \end{array} \right\}, \\ \mathcal{B}_3(N) &:= [A(\lfloor \varepsilon \log_2 N \rfloor, a, (\log N)^\varepsilon)]^c, \end{aligned}$$

where $\log_2 x := \log \log x$ for $x > 1$.

We now introduce, for every continuous process $(Z(t), t \geq 0)$,

$$\begin{aligned} \underline{Z}(t) &:= \inf\{Z(u), 0 \leq u \leq t\}, \quad t \geq 0, \\ d_Z(r) &:= \inf\{t \geq 0, Z(t) - \underline{Z}(t) \geq r\}, \quad r \geq 0. \end{aligned} \tag{3.2}$$

Then we set $W^+(t) := W(t)$ and $W^-(t) := W(-t)$ for $t \geq 0$, and consider for $N > 1$:

$$\mathcal{B}_4(N) := \{d_{\sigma W^+}(5 \log N) \leq (\log N)^4\}, \quad \mathcal{B}_5(N) := \{d_{\sigma W^-}(5 \log N) \leq (\log N)^4\}.$$

For technical reasons, we also introduce

$$\mathcal{B}_6(N) := \{\forall k \in \mathbb{Z} \cap [-\log^4 N - 1, \log^4 N), \forall t \in [k, k+1], |W(t) - W(k)| \leq \log_2 N\}.$$

This enables us to define the set $\mathcal{B}(N)$ of *bad environments* as follows:

$$\mathcal{B}(N) := \mathcal{B}_1[\lfloor (\log N)^{\frac{3-\sqrt{5}}{2C_3}+4} \rfloor] \cap \bigcap_{i=2}^6 \mathcal{B}_i(N).$$

We now estimate the probability of bad environments with the following lemma:

Lemma 3.4. *If $\varepsilon > 0$ is small enough, we have for large N ,*

$$\eta(\mathcal{B}(N)^c) \leq \frac{3}{(\log N)^{\frac{3-\sqrt{5}}{2}-\zeta(\varepsilon)}}, \tag{3.3}$$

where ζ is a function $(0, +\infty) \rightarrow \mathbb{R}$ such that $\zeta(t) \rightarrow_{t \rightarrow 0} 0$ and $\zeta(t) > 0$ for $t > 0$ small enough, which is defined in (3.4).

Proof: Denote by $k_W(e^t)$ the number of sign changes of b_W in $[1, e^t]$ for $t > 0$. Cheliotis ([8] Corollary 5) proves that the laws of $k_W(e^t)/t$, $t > 0$ satisfy a large deviation principle with speed

t and good rate function I , defined by $I(x) := x \log(2x(x + \sqrt{x^2 + 5/4})) + 3/2 - (x + \sqrt{x^2 + 5/4})$ for $x > 0$, $I(x) := +\infty$ for $x < 0$, and $I(0) := (3 - \sqrt{5})/2$. Hence by scaling, for N large enough,

$$\begin{aligned} \eta(\mathcal{B}_2(N)^c) &\leq \eta(k_W(e^{(1-2\varepsilon)\log_2 N})) \leq 3\varepsilon \log_2 N \\ &\leq \exp\{-[I(3\varepsilon/(1-2\varepsilon)) - \varepsilon](1-2\varepsilon)\log_2 N\} \\ &= (\log N)^{\zeta(\varepsilon) - \frac{3-\sqrt{5}}{2}} \end{aligned} \quad (3.4)$$

where $\zeta(t) := I(0) - [I(3t/(1-2t)) - t](1-2t) > 0$ for small $t > 0$ since $0 < I(u) < I(0)$ for small $u > 0$. Moreover, $\zeta(t) \rightarrow 0$ as $t \rightarrow 0$, $t > 0$, since I is right-continuous at 0. Lemma 3.3 gives for large N , since $1 - e^{-t} \leq t$ for $t \in \mathbb{R}$,

$$\eta(\mathcal{B}_3(N)^c) = \eta(A(\varepsilon \log_2 N, a, (\log N)^\varepsilon)) \leq (\log N)^{\zeta(\varepsilon) - \frac{3-\sqrt{5}}{2}}, \quad (3.5)$$

if $\varepsilon > 0$ is small enough so that $\zeta(\varepsilon) > 0$. Notice that for $r \geq 0$ and $T > 0$,

$$\eta(d_{W^+}(r) > T) \leq \eta(W^+(T) - \underline{W}^+(T) \leq r) = \eta(|W(T)| \leq r) \leq 2r/\sqrt{T},$$

since $W^+(T) - \underline{W}^+(T) =_{law} |W(T)|$ (see Lévy's theorem e.g. in Revuz and Yor [22] th VI.2.3). This gives

$$\eta[\mathcal{B}_4(N)^c] = \eta[\mathcal{B}_5(N)^c] \leq 10/(\sigma \log N). \quad (3.6)$$

Moreover for large N , we get since $\sup_{0 \leq t \leq 1} W(t) =_{law} |W(1)|$,

$$\eta(\mathcal{B}_6(N)^c) \leq 3(\log^4 N) \eta(\sup_{0 \leq t \leq 1} |W(t)| > \log_2 N) \leq 12(\log^4 N) \exp(-(\log_2 N)^2/2) \leq (\log N)^{-2}. \quad (3.7)$$

Combining this with (3.4), (3.5), (3.6) and $\eta(\mathcal{B}_1(K)^c) \leq \frac{C_2}{K^{C_3}}$ proves the lemma. \square

3.3. Random walk in a bad environment. In the following lemma, we show that in a bad environment, the quenched probability that $\sum_{k=0}^n f(S_k)$ is greater than $u \leq 0$ for all n between 1 and N is small:

Lemma 3.5. *Let f be as in Theorem 1.1, and $u \leq 0$. For large N ,*

$$\forall \omega \in \mathcal{B}(N), \quad P_\omega \left(\forall n \in [1, N], \sum_{k=0}^n f(S_k) > u \right) \leq 4(\log N)^{-2}. \quad (3.8)$$

Proof of Lemma 3.5: We assume that $\omega \in \mathcal{B}(N)$, and we prove that in such a bad environment, there exists a time $t \in [1, N]$ such that $\sum_{k=1}^t f(S_k) \leq u$, with a large enough quenched probability.

First, define $C_4 := \sigma + \frac{3-\sqrt{5}}{2C_3}C_1 + 4C_1$. Since $\omega \in \mathcal{B}_6(N) \cap \mathcal{B}_1[(\log N)^{\frac{3-\sqrt{5}}{2C_3}+4}]$, we have

$$\forall u \in [-\log^4 N, \log^4 N], \quad |V(\lfloor u \rfloor) - \sigma W(u)| \leq C_4 \log_2 N. \quad (3.9)$$

Notice that since $\omega \in \mathcal{B}_3(N)$, there exists $k_N \in \{1, \dots, 2\lceil \varepsilon \log_2 N \rceil\}$ such that $\mathbf{h}_N := X_{k_N}$ is an a -strong change of sign for W and $b_W(\mathbf{h}_N) > 0$, where the $(X_k)_k$ are the ones in Fact 3.1 with $c = (\log N)^\varepsilon$. Moreover, since $\omega \in \mathcal{B}_2(N) \cap \mathcal{B}_3(N)$,

$$(\log N)^\varepsilon \leq \mathbf{h}_N = X_{k_N} < X_{k_N+1} < X_{k_N+2} \leq X_{2\lceil \varepsilon \log_2 N \rceil+2} \leq (\log N)^{1-\varepsilon}.$$

To simplify the notation, we set $x_i := x_i(W, \mathbf{h}_N)$ and $y_i := \lfloor x_i \rfloor$ for $i \in \{-2, \dots, 2\}$. We also define

$$\begin{aligned} v_{-2} &:= \max\{k \in \mathbb{Z}, k \leq y_{-1}, V(k) \geq V(y_0)\}, \\ v_2 &:= \min\{k \in \mathbb{Z}, k > y_1, V(k) \geq \sigma W(x_0) + (7 + C_4) \log_2 N\}. \end{aligned}$$

Since $b_W(\mathbf{h}_N) > 0$, x_1 is an \mathbf{h}_N -minimum for W , and consequently x_0 and x_2 are \mathbf{h}_N -maxima for W and x_{-1} is an \mathbf{h}_N -minimum for W . Moreover, $e(T_0(\mathbf{h}_N)) = 0$, $e(T_1(\mathbf{h}_N)) \geq a\mathbf{h}_N$ and $e(T_{-1}(\mathbf{h}_N)) \geq a\mathbf{h}_N$ since \mathbf{h}_N is an a -strong change of sign. Due to these properties, we get

$$W(x_0) = \sup\{W(t), t \in [x_{-1}, x_1]\} \geq 0, \quad (3.10)$$

$$W(x_1) = \inf\{W(t), t \in [x_0, x_2]\} \leq 0, \quad (3.11)$$

$$W(x_0) - W(x_1) = \mathbf{h}_N, \quad (3.12)$$

$$W(x_2) - W(x_1) \geq (1+a)\mathbf{h}_N, \quad (3.13)$$

$$W(x_0) - W(x_{-1}) \geq (1+a)\mathbf{h}_N, \quad (3.14)$$

$$W(x_{-1}) = \inf\{W(t), t \in [x_{-2}, x_0]\} < W(x_1). \quad (3.15)$$

The following lemma will allow us to apply (3.9) to some x_i , y_i and v_i .

Lemma 3.6. *For N large enough,*

$$\begin{aligned} \forall \omega \in \mathcal{B}(N), \quad & -(\log N)^4 \leq v_{-2} \leq x_{-1} < x_0 \leq 0 < x_1 < v_2 < x_2 \leq (\log N)^4, \\ \forall \omega \in \mathcal{B}(N), \quad & v_{-2} + 3 \leq y_{-1} \leq y_0 - 3 \leq -3. \end{aligned} \quad (3.16)$$

Proof: First, it is clear by definition that $x_{-2} < x_{-1} < x_0 \leq 0 < x_1 < x_2$.

Moreover, x_1 is an X_{k_N} -minimum, whereas $x_1(W, X_{k_N+1})$ is an (X_{k_N+1}) -maximum because $X_{k_N} = \mathbf{h}_N$ and X_{k_N+1} are consecutive changes of sign for b_W . So $x_1 \neq x_1(W, X_{k_N+1})$. Since $x_1(W, X_{k_N+1})$ is also an X_{k_N} -maximum, and x_2 is the smallest positive X_{k_N} -maximum, we get $x_2 \leq x_1(W, X_{k_N+1})$. Now,

$$W^+[x_1(W, X_{k_N+1})] - \underline{W}^+[x_1(W, X_{k_N+1})] \leq W[x_1(W, X_{k_N+1})] - W[x_0(W, X_{k_N+1})] = X_{k_N+1}.$$

Since $X_{k_N+1} \leq (\log N)^{1-\varepsilon} \leq (5/\sigma) \log N$ for N such that $5 \log N > \sigma (\log N)^{1-\varepsilon}$, and $\omega \in \mathcal{B}_4(N)$, this yields to

$$0 < x_2 \leq x_1(W, X_{k_N+1}) \leq d_{\sigma W^+}(5 \log N) \leq (\log N)^4.$$

Since $v_2 > y_1 = \lfloor x_1 \rfloor$, we have $x_1 < v_2$. Moreover, we can now apply (3.9) to x_2 together with (3.12) and (3.13), which gives $V(y_2) \geq \sigma W(x_2) - C_4 \log_2 N \geq \sigma W(x_0) + \sigma a \mathbf{h}_N - C_4 \log_2 N$, which is greater than $\sigma W(x_0) + (7 + C_4) \log_2 N + 2 \log[(1 - \varepsilon_0)/\varepsilon_0]$ uniformly on $\mathcal{B}(N)$ for N large enough. This gives $v_2 < y_2 \leq x_2$.

Moreover, $v_{-2} \leq y_{-1} \leq x_{-1}$. Now, similarly as before, $x_0(W, X_{k_N+2}) < x_0(W, X_{k_N+1}) < x_0(W, X_{k_N})$, and since all of them are X_{k_N} -extrema, this yields to $x_0(W, X_{k_N+2}) \leq x_{-2}$. Now, we have $W^-(-x_0(W, X_{k_N+2})) - \underline{W}^-(-x_0(W, X_{k_N+2})) \leq e(T_0(X_{k_N+2})) = X_{k_N+2} \leq (\log N)^{1-\varepsilon}$, which gives as previously $x_{-2} \geq x_0(W, X_{k_N+2}) \geq -d_{\sigma W^-}(5 \log N) \geq -(\log N)^4$.

We already know that $x_0(W, X_{k_N+2}) \leq x_{-1} < 0 < x_2 \leq x_1(W, X_{k_N+1}) < x_1(W, X_{k_N+2})$, which leads to $W[x_0(W, X_{k_N+2})] \geq W(x_2) \geq W(x_0) + a\mathbf{h}_N$ since $x_0(W, X_{k_N+2})$ is an (X_{k_N+2}) -maximum. Applying (3.9) to $x_0(W, X_{k_N+2}) \geq -\log^4 N$ and to $x_0 \geq x_0(W, X_{k_N+2}) \geq -\log^4 N$, this gives $V(\lfloor x_0(W, X_{k_N+2}) \rfloor) \geq \sigma W(x_0(W, X_{k_N+2})) - C_4 \log_2 N \geq \sigma W(x_0) + C_4 \log_2 N \geq V(y_0)$ for N such that $\sigma a \mathbf{h}_N \geq 2C_4 \log_2 N$, which yields to $v_{-2} \geq \lfloor x_0(W, X_{k_N+2}) \rfloor \geq -(\log N)^4$.

Finally, notice that by (3.9) and (3.14),

$$V(y_0) - V(y_{-1}) \geq \sigma W(x_0) - \sigma W(x_{-1}) - 2C_4 \log_2 N \geq \sigma(1+a)\mathbf{h}_N - 2C_4 \log_2 N,$$

which is, for large N uniformly on $\mathcal{B}(\mathcal{N})$, strictly larger than $-3 \log \varepsilon_0 \geq 3 \sup_{k \in \mathbb{Z}} |V(k) - V(k-1)|$ since $\mathbf{h}_N \geq (\log N)^\varepsilon$. This and $x_{-1} \leq x_0 \leq 0$ give the second inequality in (3.16). The first one is obtained similarly. \square

Let

$$E_1 := \{\tau(y_{-1}) < \tau(v_2)\}, \quad E_2 := \{L((0, v_2], \tau(y_{-1}) \wedge \tau(v_2)) \leq (\log N)^{18+2C_4} e^{\sigma \mathbf{h}_N}\}.$$

We prove the following lemma:

Lemma 3.7. *For large N ,*

$$\forall \omega \in \mathcal{B}(N), \quad P_\omega(E_1^c) \leq (\log N)^{-2}, \quad P_\omega(E_2^c) \leq (\log N)^{-2}.$$

Proof: First, $-(\log N)^4 \leq y_{-1} \leq -3$ uniformly on $\mathcal{B}(N)$ for large N , and equations (1.1), (2.1), (3.9) and (3.10) yield to

$$P_\omega(E_1^c) \leq |y_{-1}| \max_{y_{-1} \leq k \leq -1} e^{V(k)-V(v_2-1)} \leq \varepsilon_0^{-1} (\log N)^{4+C_4} \exp[\sigma W(x_0) - V(v_2)] \leq (\log N)^{-2},$$

for every $\omega \in \mathcal{B}(N)$ for large N , which proves the first part of the lemma.

Thanks to $x_0 \leq 0 < v_2 < x_2$ and to (3.11), we have $W(z) \geq W(x_1)$ for all $z \in (0, v_2)$. Moreover, $V(k) \leq \sigma W(x_0) + (7 + C_4) \log_2 N$ for every $k \in [y_{-1}, v_2 - 1]$ by the definition of v_2 , (3.10), and (3.9). This, Lemma 3.6, (2.1), (3.12) and (3.9) again give for $z \in (0, v_2)$,

$$P_\omega^{z-1}[\tau(z) > \tau(y_{-1})] = \frac{e^{V(z-1)}}{\sum_{k=y_{-1}}^{z-1} e^{V(k)}} \geq \frac{\varepsilon_0 e^{\sigma W(x_1) - C_4 \log_2 N}}{2(\log N)^4 e^{\sigma W(x_0) + (C_4+7) \log_2 N}} \geq \frac{\varepsilon_0 e^{-\sigma \mathbf{h}_N}}{2(\log N)^{11+2C_4}}.$$

Applying (2.3) and observing that $v_2 \leq (\log N)^4$, $P_\omega[\tau(z) < \tau(y_{-1})] \leq 1$ and $y_{-1} \leq -1$, we obtain for every $\omega \in \mathcal{B}(N)$ for large N ,

$$\begin{aligned} E_\omega[L((0, v_2], \tau(y_{-1}) \wedge \tau(v_2))] &\leq \sum_{z=1}^{v_2-1} \frac{P_\omega[\tau(z) < \tau(y_{-1})]}{\omega_z P_\omega^{z+1}[\tau(z) > \tau(v_2)] + (1 - \omega_z) P_\omega^{z-1}[\tau(z) > \tau(y_{-1})]} + 1 \\ &\leq 2\varepsilon_0^{-2} (\log N)^{15+2C_4} e^{\sigma \mathbf{h}_N} + 1. \end{aligned}$$

Using Markov's inequality, we get $P_\omega(E_2^c) \leq (\log N)^{-2}$ for large N . \square

Now, let $T := \inf\{k > \tau(y_{-1}), S_k \in \{v_{-2}, y_0 - 1\}\}$ be the first exit time from the interval $(v_{-2}, y_0 - 1)$ by the random walk S after $\tau(y_{-1})$. We introduce $n_1 := \lfloor \frac{\varepsilon_0^2 \exp(\sigma(1+a)\mathbf{h}_N)}{2(\log N)^{2+2C_4}} \rfloor$ and the events

$$E_3 := \{T \geq \tau(y_{-1}) + n_1\}, \quad E_4 := \{\tau(y_{-1}) + n_1 < N\}.$$

Lemma 3.8. *For N large enough,*

$$\forall \omega \in \mathcal{B}(N), \quad P_\omega(E_3^c) \leq (\log N)^{-2}, \quad P_\omega(E_4 \cap E_1) \leq (\log N)^{-2}.$$

Proof: Recall that $v_{-2} < y_{-1} < y_0 - 1$ on $\mathcal{B}(N)$ for N large enough by (3.16), and that $\tau(y_{-1}) < \infty$ \mathbb{P} -a.s. since $(S_n)_n$ is recurrent. We first consider $L(y_{-1}, T)$ and notice that it is under P_ω a geometrical random variable of parameter

$$\begin{aligned} p_1 &:= \omega_{y_{-1}} P_\omega^{y_{-1}+1}[\tau(y_{-1}) > \tau(y_0 - 1)] + (1 - \omega_{y_{-1}}) P_\omega^{y_{-1}-1}[\tau(y_{-1}) > \tau(v_{-2})] \\ &= \omega_{y_{-1}} e^{V(y_{-1})} \left(\sum_{k=y_{-1}}^{y_0-2} e^{V(k)} \right)^{-1} + (1 - \omega_{y_{-1}}) e^{V(y_{-1}-1)} \left(\sum_{k=v_{-2}}^{y_{-1}-1} e^{V(k)} \right)^{-1} \\ &\leq \varepsilon_0^{-2} \exp[V(y_{-1}) - V(y_0)] \\ &\leq \varepsilon_0^{-2} e^{-\sigma(1+a)\mathbf{h}_N} (\log N)^{2C_4} =: p_2, \end{aligned}$$

thanks to (2.1) and the definition of v_{-2} , and where the last inequality comes from (3.9) and (3.14). This ensures that for large N , uniformly on $\mathcal{B}(N)$ since $\mathbf{h}_N \geq (\log N)^\varepsilon$,

$$\log P_\omega[L(y_{-1}, T) \geq n_1] = (n_1 - 1) \log(1 - p_1) \geq -2n_1 p_1 \geq -2n_1 p_2 \geq -(\log N)^{-2}.$$

Since $1 - e^{-t} \leq t$ for $t \in \mathbb{R}$, this yields $P_\omega[L(y_{-1}, T) < n_1] \leq (\log N)^{-2}$. Finally, we have $T \geq \tau(y_{-1}) + L(y_{-1}, T)$, which gives $P_\omega(E_3^c) \leq P_\omega[L(y_{-1}, T) < n_1] \leq (\log N)^{-2}$.

We now turn to E_4 . Notice that uniformly on $\mathcal{B}(N)$ for large N , thanks to Lemma 3.6, (3.9), (3.10), (3.11), (3.15) and the definition of v_2 , we have

$$\forall k \in [y_{-1}, v_2 - 1], \quad \sigma W(x_{-1}) - C_4 \log_2 N \leq V(k) \leq \sigma W(x_0) + (7 + C_4) \log_2 N. \quad (3.17)$$

Since $H(T_0(X_{k_N})) = X_{k_N} < X_{k_{N+1}}$, x_0 and x_1 are not $(X_{k_{N+1}})$ -extrema. Hence, $[x_{-1}, x_2] \subset [x_0(W, X_{k_{N+1}}), x_1(W, X_{k_{N+1}})]$, and then $W(x_2) - W(x_{-1}) \leq X_{k_{N+1}}$. Moreover, $\log_2 N = o(\mathbf{h}_N)$ uniformly on $\mathcal{B}(N)$ and $W(x_0) \leq W(x_2) - a\mathbf{h}_N$ by (3.12) and (3.13), so (3.17) gives for large N ,

$$\max\{V(k) - V(\ell), y_{-1} \leq \ell \leq k \leq v_2 - 1\} \leq \sigma(W(x_2) - W(x_{-1})) \leq \sigma X_{k_{N+1}} \leq \sigma(\log N)^{1-\varepsilon}.$$

This together with (2.2) and $|v_2 - y_{-1}| \leq 2(\log N)^4$ yield $E_\omega(\tau(y_{-1})\mathbf{1}_{E_1}) < \sqrt{N}$ uniformly on $\mathcal{B}(N)$ for large N . Since $E_\omega(n_1\mathbf{1}_{E_1}) < \sqrt{N}$ because $\mathbf{h}_N \leq (\log N)^{1-\varepsilon}$ on $\mathcal{B}(N)$, this yields $P_\omega(E_4^c \cap E_1) \leq (\log N)^{-2}$ for every $\omega \in \mathcal{B}(N)$ for large N by Markov's inequality. \square

We now consider f satisfying the hypotheses of Theorem 1.1. For every $\omega \in \mathcal{B}(N)$, we have on $E_5 := \cap_{i=1}^4 E_i$, since $f(t) \leq 0$ for every $t \leq 0$,

$$\begin{aligned} \sum_{k=0}^{\tau(y_{-1})-1} f(S_k) &\leq \left[\max_{k \in (0, v_2]} f(k) \right] L((0, v_2], \tau(y_{-1}) \wedge \tau(v_2)) \\ &\leq \left[\max_{k \in (0, v_2]} f(k) \right] (\log N)^{18+2C_4} e^{\sigma \mathbf{h}_N}. \end{aligned} \quad (3.18)$$

For every $\Delta \subset \mathbb{Z}$ and $0 \leq s \leq t$, we define $L(\Delta, s \rightsquigarrow t) := \sum_{k=s}^t \mathbf{1}_{\{S_k \in \Delta\}}$, which is the number of visits of $(S_n)_{n \in \mathbb{N}}$ to the set Δ between times s and t .

For every $\omega \in \mathcal{B}(N)$ and each integer $k \in [\tau(y_{-1}), \tau(y_{-1}) + n_1]$, we have $\tau(y_{-1}) \leq k \leq T$ on E_3 , so $S_k \leq y_0 - 1 \leq -1$, thus $f(S_k) \leq -1$. As a consequence for large N ,

$$\sum_{k=\tau(y_{-1})}^{\tau(y_{-1})+n_1} f(S_k) \leq -n_1 - 1 \leq -\varepsilon_0^2 \frac{\exp[\sigma \mathbf{h}_N + \sigma a(\log N)^\varepsilon]}{2(\log N)^{2+2C_4}}, \quad (3.19)$$

since $\mathbf{h}_N \geq (\log N)^\varepsilon$. Combining (3.18), (3.19), and $\max_{k \in (0, v_2]} f(k) \leq \max_{k \in (0, (\log N)^4]} f(k) \leq \exp((\log N)^{\varepsilon/2})$ for large N , we get $\sum_{k=0}^{\tau(y_{-1})+n_1} f(S_k) \leq u$ on E_5 for every $\omega \in \mathcal{B}(N)$ for large N . Moreover, $1 \leq \tau(y_{-1}) + n_1 \leq N$ on E_5 , hence for large N , for every $\omega \in \mathcal{B}(N)$, we have $E_5 \subset \{\exists n \in [1, N], \sum_{k=0}^n f(S_k) \leq u\}$. Consequently, the left hand side of (3.8) is less than $P_\omega(E_5^c) \leq 4(\log N)^{-2}$ for every $\omega \in \mathcal{B}(N)$ for large N by Lemmas 3.7 and 3.8, which proves Lemma 3.5. \square

Finally, integrating (3.8) on the set of bad environments $\mathcal{B}(N)$ gives by Lemma 3.4:

$$\begin{aligned} \mathbb{P} \left(\forall n \in [1, N], \sum_{k=0}^n f(S_k) > u \right) &\leq \int_{\mathcal{B}(N)} P_\omega \left(\forall n \in [1, N], \sum_{k=0}^n f(S_k) > u \right) \eta(d\omega) + \eta(\mathcal{B}(N)^c) \\ &\leq \frac{4}{(\log N)^{\frac{3-\sqrt{5}}{2}-\zeta(\varepsilon)}} \end{aligned}$$

for large N . Now, let $\varepsilon \rightarrow 0$. This gives the upper bound in Theorem 1.1. \square

4. PROOF OF THE LOWER BOUND

4.1. Definition of the set $\mathcal{G}(N)$ of good environments. We consider a collection $(\omega_i)_{i \in \mathbb{Z}}$ of independent and identically distributed random variables, satisfying (1.1), (1.2) and (1.3).

We notice that due to (1.2) and (1.3), there exist $\gamma > 0$ and $\delta > 0$ such that $\eta(-2\delta \leq \log \frac{1-\omega_0}{\omega_0} \leq -\delta) =: e^{-\gamma} > 0$. We fix $\varepsilon > 0$ such that $\varepsilon\delta/4 < 4$. Let $N \in \mathbb{N}$ such that $N \geq 2$. In the spirit of Devulder [12], we first define

$$\mathcal{G}_1(N) := \left\{ \forall k \in \{1, \dots, \lfloor \varepsilon \log_2 N \rfloor\}, \quad -2\delta \leq \log \frac{1-\omega_k}{\omega_k} \leq -\delta \right\},$$

and we introduce

$$\begin{aligned} h(N) &:= (\log N)^{\varepsilon\delta/32}, \\ \theta_0 &:= \inf\{k \geq \lfloor \varepsilon \log_2 N \rfloor, \quad V(k) \leq -5h(N)\}, \\ \mathcal{G}_2(N) &:= \{\forall k \in \{\lfloor \varepsilon \log_2 N \rfloor, \dots, \theta_0\}, \quad V(k) \leq -\frac{\delta\varepsilon}{2} \log_2 N\}, \\ \mathcal{G}_3(N) &:= \{\theta_0 \leq \lfloor \varepsilon \log_2 N \rfloor + (\log N)^{\varepsilon\delta/4}\}. \end{aligned} \tag{4.1}$$

We set $\widehat{V}(i) := V(i + \theta_0) - V(\theta_0)$ for $i \geq 0$, and $\widehat{V}(i) := V(i)$ for $i < 0$. By the strong Markov property, \widehat{V} has the same law as V and is independent of $(V(i), 0 \leq i \leq \theta_0)$. Let $K \geq 1$. As before, according to the Komlós–Major–Tusnády strong approximation theorem (see Komlós et al. [19]), possibly in an enlarged probability space, there exists a standard two-sided Brownian motion $(W(t), t \in \mathbb{R})$ such that the set

$$\mathcal{G}_4(K) := \left\{ \sup_{-K \leq i \leq K} |\widehat{V}(i) - \sigma W(i)| \leq C_1 \log K \right\}$$

satisfies $\eta(\mathcal{G}_4(K)^c) \leq C_2/K^{C_3}$. Moreover, we can choose $(W(t), t \in \mathbb{R})$ so that it is independent of $(V(i), 0 \leq i \leq \theta_0)$ since \widehat{V} is independent of $(V(i), 0 \leq i \leq \theta_0)$. In the following, we take $K = (\log N)^{\frac{3-\sqrt{5}}{2C_3}+4}$. We introduce

$$\begin{aligned} \mathcal{G}_5(N) &:= \{\max\{d_{\sigma W^+}(5 \log N), d_{\sigma W^-}(5 \log N), d_{-\sigma W^-}(5 \log N)\} \leq (\log N)^4\}, \\ \mathcal{G}_7(N) &:= \{\forall x \in [1/\sigma, 5(\log N)/\sigma], \quad b_W(x) > 0\}, \end{aligned}$$

and define $\mathcal{G}_6(N)$ by the same formula as $\mathcal{B}_6(N)$.

We can now define the set $\mathcal{G}(N)$ of *good environments* as follows (see Figure 1):

$$\mathcal{G}(N) := \mathcal{G}_4 \left[(\log N)^{\frac{3-\sqrt{5}}{2C_3}+4} \right] \cap \bigcap_{i=1, \dots, 7, i \neq 4} \mathcal{G}_i(N).$$

When no confusion is possible we write \mathcal{G} instead of $\mathcal{G}(N)$ and \mathcal{G}_i instead of $\mathcal{G}_i(N)$, $i \neq 4$.

4.2. Probability of the set $\mathcal{G}(N)$ of good environments.

Lemma 4.1. *We have for large N ,*

$$\eta(\mathcal{G}(N)) \geq \frac{c_1 \varepsilon \log_2 N}{(\log N)^{\frac{3-\sqrt{5}}{2} + \varepsilon(\gamma + \delta/32)}}. \tag{4.2}$$

Proof: First, observe that

$$\eta(\mathcal{G}_1) \geq (e^{-\gamma})^{\varepsilon \log_2 N} = (\log N)^{-\varepsilon\gamma}.$$

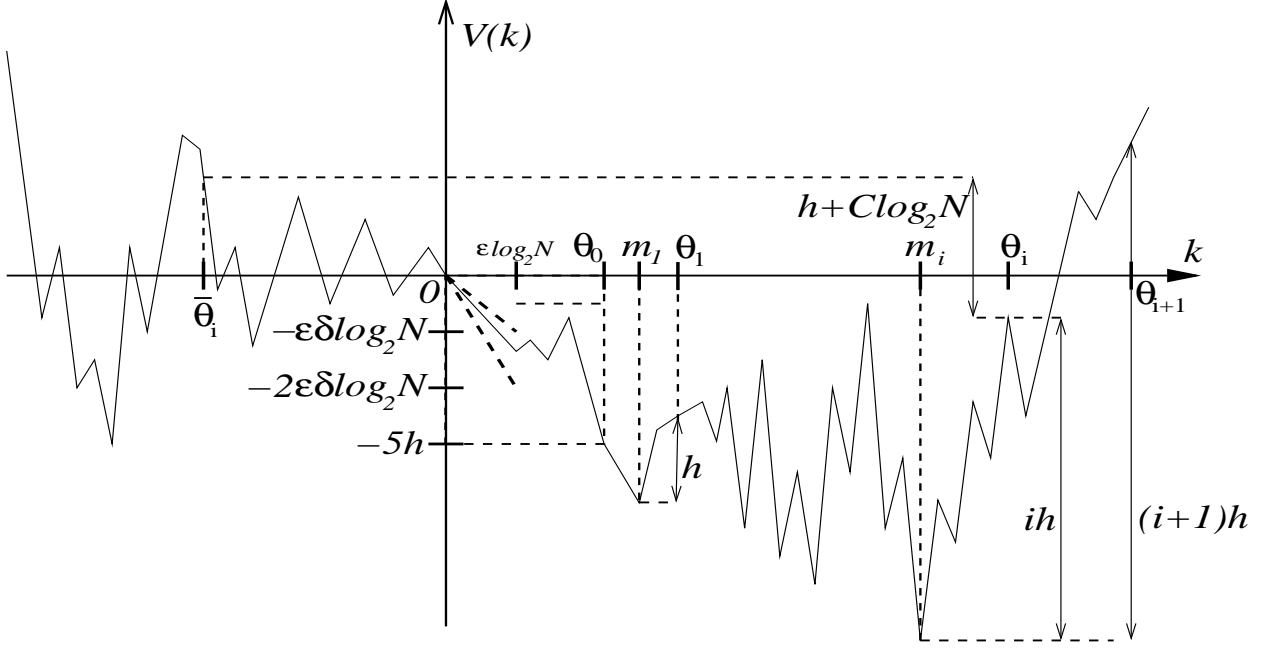


FIGURE 1. Schema of the potential V for a “good” environment $\omega \in \mathcal{G}(N)$ in the case $m_i = m_{i+1}$, where h denotes $h(N)$.

We now prove that

$$\eta(\mathcal{G}_2 \cap \mathcal{G}_3 \mid \mathcal{G}_1) \geq \frac{\delta \varepsilon \log_2 N}{40h(N)} \quad (4.3)$$

for large N . To this aim, we define $A := \log((1 - \varepsilon_0)/\varepsilon_0)$, so $|V(k+1) - V(k)| \leq A$ a.s. for every $k \in \mathbb{Z}$ thanks to (1.1). For $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $a < 0 < b$, let $T_{a,b} := \inf\{k \geq 0, V(k) \notin (a, b)\} < \infty$ a.s. We recall that thanks to the optimal stopping theorem, $\eta[V(T_{a,b}) < 0] \geq b/(b - a + A)$ (see e.g. Zindy [31] Lemma 2.1 and apply it to $-V$). In particular, we get on \mathcal{G}_1 uniformly for N large enough,

$$\eta[\mathcal{G}_2 \mid V(\lfloor \varepsilon \log_2 N \rfloor)] \geq \delta \varepsilon \log_2 N / (20h(N)),$$

which yields to $\eta(\mathcal{G}_2 \mid \mathcal{G}_1) \geq \delta \varepsilon \log_2 N / (20h(N))$. Moreover, we have on \mathcal{G}_1 by the Markov property

$$\begin{aligned} \eta(\mathcal{G}_2 \cap \mathcal{G}_3^c \mid V(\lfloor \varepsilon \log_2 N \rfloor)) &\leq \eta(V(\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor) \in [-5h(N), 2\delta \varepsilon \log_2 N]) \\ &\leq \eta\left(\frac{|V(\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor)|}{\sigma \sqrt{\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor}} \leq \frac{5h(N)}{\sigma \sqrt{\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor}}\right) \end{aligned}$$

for N large enough. By Berry-Esseen, we get with $Y =_{law} \mathcal{N}(0, 1)$,

$$\begin{aligned} \eta(\mathcal{G}_2 \cap \mathcal{G}_3^c \mid V(\lfloor \varepsilon \log_2 N \rfloor)) &\leq \eta\left(|Y| \leq \frac{5h(N)}{\sigma \sqrt{\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor}}\right) + \frac{c_2}{\sqrt{\lfloor (\log N)^{\delta \varepsilon / 4} \rfloor}} \\ &\leq \frac{11h(N)}{\sigma \sqrt{2\pi} (\log N)^{\delta \varepsilon / 8}} + \frac{2c_2}{(\log N)^{\delta \varepsilon / 8}} = o(\eta(\mathcal{G}_2 \mid \mathcal{G}_1)) \end{aligned}$$

as $N \rightarrow +\infty$. Consequently $\eta(\mathcal{G}_2 \cap \mathcal{G}_3^c \mid \mathcal{G}_1) = o(\eta(\mathcal{G}_2 \mid \mathcal{G}_1))$, which gives (4.3) for large N .

Since W and \widehat{V} are independent of $(V(i), 0 \leq i \leq \theta_0)$, we get

$$\eta(\mathcal{G}(N)) = \eta(\mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3) \eta(\mathcal{G}_8) \geq \frac{c_3 \varepsilon \log_2 N}{(\log N)^{\varepsilon \gamma} h(N)} \eta(\mathcal{G}_8) \quad (4.4)$$

where $\mathcal{G}_8(N) := \mathcal{G}_4[(\log N)^{\frac{3-\sqrt{5}}{2C_3}+4}] \cap \bigcap_{i=5,\dots,7} \mathcal{G}_i(N)$. We now need the following result:

Theorem 4.2. (*Cheliotis [8], Corollary 1*)

$$\eta(\{t \mapsto b_W(t)\} \text{ keeps the same sign in } [1, x]) / x^{(-3+\sqrt{5})/2} \xrightarrow{x \rightarrow +\infty} 1/2 + 7\sqrt{5}/30 =: c_4.$$

Hence $\eta(\mathcal{G}_7) \sim_{N \rightarrow +\infty} c_4 / [2(5 \log N)^{(3-\sqrt{5})/2}]$ by scaling. Moreover, $\eta[\mathcal{G}_5^c] \leq 30 / (\sigma \log N)$ by (3.6), $\eta[(\mathcal{G}_4(K))^c] \leq C_2 / K^{C_3}$, and $\eta[\mathcal{G}_6(N)^c] \leq (\log N)^{-2}$ by (3.7), so

$$\eta(\mathcal{G}_8^c) \leq 1 - c_5 / (\log N)^{\frac{3-\sqrt{5}}{2}}$$

for N large enough for some $c_5 > 0$, since $\frac{3-\sqrt{5}}{2} < 1$. Hence, $\eta(\mathcal{G}_8) \geq c_5 / (\log N)^{\frac{3-\sqrt{5}}{2}}$ for large N . This, combined with (4.4), gives (4.2). \square

4.3. Random walk in a good environment. In this subsection, we prove the following lemma, and then the lower bound of Theorem 1.1. Notice that we just have to consider the case $u = 0$. In all the rest of this section, the function f satisfies the hypotheses of Theorem 1.1.

Lemma 4.3. *There exists a constant $c_6 > 0$ such that for N large enough,*

$$\forall \omega \in \mathcal{G}(N), \quad P_\omega \left(\sum_{k=0}^n f(S_k) > 0 \quad \forall 1 \leq n \leq N \right) \geq c_6. \quad (4.5)$$

Before proving this lemma, we introduce some more notation. We consider $N \in \mathbb{N}^*$ and a good environment $\omega \in \mathcal{G}(N)$. We introduce for $i \in \mathbb{N}^*$,

$$\begin{aligned} t_i &:= \inf\{t > 0, \sigma W(t) - \sigma \underline{W}(t) \geq ih(N)\} = d_{\sigma W^+}(ih(N)), \\ \theta_i &:= \lfloor t_i \rfloor + \theta_0, \\ m_i &:= \inf \left\{ k \in \mathbb{N}, \quad V(k) = \inf_{0 \leq \ell \leq \theta_i} V(\ell) \right\}, \end{aligned}$$

where θ_0 is defined in (4.1). In particular, $\sigma W(t_i) = \sigma \underline{W}(t_i) + ih(N)$ by continuity of W . Moreover, $\omega \in \mathcal{G}_7$, so $x_0(\sigma W, ih(N)) = x_0(W, ih(N)/\sigma)$ is an $ih(N)$ -maximum for σW and $x_1(\sigma W, ih(N))$ an $ih(N)$ -minimum for σW for every integer $i \geq 1$ such that $1 \leq ih(N) \leq 5 \log N$. Consequently for such i , $t_i \geq x_1(\sigma W, ih(N))$, otherwise there would be an $ih(N)$ -maximum for σW in $(0, x_1(\sigma W, ih(N)))$, which is not possible. Moreover, $\sigma W[x_2(\sigma W, ih(N))] - \sigma W[x_1(\sigma W, ih(N))] \geq ih(N)$, which gives $t_i \leq x_2(\sigma W, ih(N))$. Hence,

$$x_0(\sigma W, ih(N)) \leq 0 < x_1(\sigma W, ih(N)) < t_i \leq x_2(\sigma W, ih(N)), \quad (4.6)$$

then

$$\sup\{W(t), x_0(\sigma W, ih(N)) \leq t \leq t_i\} = W[x_0(\sigma W, ih(N))], \quad (4.7)$$

$$\inf\{W(t), x_0(\sigma W, ih(N)) \leq t \leq t_i\} = W[x_1(\sigma W, ih(N))] \quad (4.8)$$

and $\sigma W[x_0(\sigma W, ih(N))] \geq \sigma W[x_1(\sigma W, ih(N))] + ih(N) = \sigma W(t_i)$. We set similarly as in (3.2),

$$\underline{V}(n) := \inf\{V(k), 0 \leq k \leq n\}, \quad n \in \mathbb{N}.$$

We recall that $C_4 = \sigma + \frac{3-\sqrt{5}}{2C_3}C_1 + 4C_1$ and notice that similarly as in (3.9),

$$\forall u \in [-(\log N)^4, (\log N)^4], \quad |\widehat{V}(\lfloor u \rfloor) - \sigma W(u)| \leq C_4 \log_2 N. \quad (4.9)$$

We also introduce $i_{\max}(N) := \max\{i \in \mathbb{N}, ih(N) \leq 4 \log N\}$. Since $\varepsilon\delta/4 < 4$ and $\mathcal{G}(N) \subset \mathcal{G}_3(N) \cap \mathcal{G}_5(N)$, we get uniformly on $\mathcal{G}(N)$ for large N ,

$$\forall 1 \leq i \leq i_{\max}(N), \quad 0 \leq m_i \leq \theta_i \leq \lfloor d_{\sigma W^+}(5 \log N) \rfloor + \theta_0 \leq 2(\log N)^4; \quad 0 \leq t_i \leq (\log N)^4. \quad (4.10)$$

We now define for $1 \leq i \leq i_{max}(N)$, with $\nu := 8 + 2C_4$,

$$F_i(N) := \left\{ \sum_{k=0}^n f(S_k) > 0 \quad \forall 1 \leq n \leq \tau(\theta_i) \right\} \cap \left\{ \sum_{k=0}^{\tau(\theta_i)} f(S_k) \geq \frac{\exp(ih(N))}{2(\log N)^\nu} \right\}.$$

Our aim in the following is to prove, by induction, a lower bound for $P_\omega(F_i(N))$ for $1 \leq i \leq i_{max}(N)$. We start with $i = 1$.

Lemma 4.4. *There exists a constant $c_7 > 0$ such that for N large enough,*

$$\forall \omega \in \mathcal{G}(N), \quad P_\omega(F_1(N)) \geq c_7 - 4(\log N)^{-6}. \quad (4.11)$$

Proof: Recall that $\varepsilon_0 \leq e^{V(-1)} \leq \varepsilon_0^{-1}$. Moreover, we have for $\omega \in \mathcal{G}(N)$, $V(k) \leq -\delta k$ for $0 \leq k \leq \lfloor \varepsilon \log_2 N \rfloor$, whereas $V(k) \leq -(\delta\varepsilon/2) \log_2 N$ for $\lfloor \varepsilon \log_2 N \rfloor < k \leq \theta_0$, and for $\theta_0 < k \leq \theta_1$,

$$V(k) = V(\theta_0) + \widehat{V}(k - \theta_0) \leq -5h(N) + \sigma W(k - \theta_0) + C_4 \log_2 N \leq -4h(N) + C_4 \log_2 N,$$

thanks to (4.9) since $t_1 \leq (\log N)^4$ by (4.10) for N large enough so that $i_{max}(N) \geq 1$. Let $c_7 := \varepsilon_0(\varepsilon_0^{-1} + 2(1 - e^{-\delta})^{-1})^{-1}$. We have $P_\omega[\tau(\theta_1) < \tau(-1)] = e^{V(-1)} \left(\sum_{k=-1}^{\theta_1-1} e^{V(k)} \right)^{-1}$, which is, due to the previous remarks, greater than or equal to

$$\varepsilon_0 \left[\varepsilon_0^{-1} + \sum_{k=0}^{\lfloor \varepsilon \log_2 N \rfloor} e^{-\delta k} + (\theta_0 - \lfloor \varepsilon \log_2 N \rfloor)(\log N)^{-(\delta\varepsilon/2)} + (\theta_1 - \theta_0)e^{-4h(N)}(\log N)^{C_4} \right]^{-1} \geq c_7, \quad (4.12)$$

for every $\omega \in \mathcal{G}(N)$ for large N since $\theta_0 - \lfloor \varepsilon \log_2 N \rfloor \leq (\log N)^{\varepsilon\delta/4}$ on $\mathcal{G}_3(N)$ and due to (4.10).

Moreover on $\mathcal{G}_1(N)$, $\theta_1 \geq m_1 \geq \lfloor \varepsilon \log_2 N \rfloor$, which is greater than 1 for large N , so $f(m_1) \geq 1$. Observe that on $\{\tau(\theta_1) < \tau(-1)\}$, due to (1.4) and since $f(m_1) \geq 1$ and $f \geq 0$ on \mathbb{N} ,

$$\sum_{k=0}^{\tau(\theta_1)} f(S_k) \geq L(m_1, \tau(\theta_1) \wedge \tau(-1)), \quad \sum_{k=0}^n f(S_k) \geq f(1) > 0, \quad 0 < n \leq \tau(\theta_1). \quad (4.13)$$

In order to give a lower bound of $L(m_1, \tau(\theta_1) \wedge \tau(-1))$, notice that thanks to (4.9) and $t_1 \leq (\log N)^4$, $\widehat{V}(m_1 - \theta_0) \leq \sigma W(t_1) + C_4 \log_2 N \leq \widehat{V}(\lfloor t_1 \rfloor) - h(N) + 2C_4 \log_2 N$. Consequently, uniformly on $\mathcal{G}(N)$ for large N , we have $m_1 + 1 < \theta_1$ and

$$P_\omega^{m_1+1}[\tau(\theta_1) < \tau(m_1)] = e^{V(m_1)} \left(\sum_{k=m_1}^{\theta_1-1} e^{V(k)} \right)^{-1} \leq e^{V(m_1)-V(\theta_1-1)} \leq \varepsilon_0^{-1} e^{-h(N)} (\log N)^{2C_4},$$

$$P_\omega^{m_1-1}[\tau(-1) < \tau(m_1)] = e^{V(m_1-1)} \left(\sum_{k=-1}^{m_1-1} e^{V(k)} \right)^{-1} \leq e^{V(m_1-1)-V(0)} \leq \varepsilon_0^{-1} e^{-h(N)} (\log N)^{2C_4}$$

since $V(m_1) \leq V(\theta_0) \leq -5h(N) \leq -h(N) + 2C_4 \log_2 N$. We know that $L(m_1, \tau(-1) \wedge \tau(\theta_1))$ is under $P_\omega^{m_1}$ a geometric random variable of parameter $P_\omega^{m_1}[\tau(-1) \wedge \tau(\theta_1) < \tau^*(m_1)]$, where $\tau^*(m_1) := \inf\{k \in \mathbb{N}^*, S_k = m_1\}$ is the first return time to m_1 . Hence,

$$P_\omega^{m_1}[L(m_1, \tau(-1) \wedge \tau(\theta_1)) > k] \geq (P_\omega^{m_1}[\tau(-1) \wedge \tau(\theta_1) > \tau^*(m_1)])^k \geq \left(1 - \frac{\varepsilon_0^{-1} (\log N)^{2C_4}}{e^{h(N)}} \right)^k.$$

Taking $k = k_1 := \lfloor \frac{\exp(h(N))}{2(\log N)^\nu} \rfloor$, we obtain uniformly on $\mathcal{G}(N)$ for large N ,

$$\log P_\omega^{m_1}[L(m_1, \tau(-1) \wedge \tau(\theta_1)) > k_1] \geq -2k_1 \varepsilon_0^{-1} e^{-h(N)} (\log N)^{2C_4} \geq -(\log N)^{-6}.$$

Hence,

$$P_\omega^{m_1}[L(m_1, \tau(-1) \wedge \tau(\theta_1)) \leq k_1] \leq 1 - \exp(-(\log N)^{-6}) \leq (\log N)^{-6}. \quad (4.14)$$

Since $f(k) \geq 1$ for $k \geq 1$ and $f(0) = 0$, we have thanks to (4.13),

$$\begin{aligned} P_\omega[\tau(\theta_1) < \tau(-1)] &= P_\omega \left[\sum_{k=0}^n f(S_k) > 0 \forall 1 \leq n \leq \tau(\theta_1), \tau(\theta_1) < \tau(-1) \right] \\ &\leq P_\omega[F_1(N)] + P_\omega[\tau(\theta_1) < \tau(-1), L(m_1, \tau(\theta_1) \wedge \tau(-1)) \leq k_1]. \end{aligned}$$

We get in particular for large N by the strong Markov property, (4.12) and (4.14),

$$\begin{aligned} \forall \omega \in \mathcal{G}(N), \quad P_\omega[F_1(N)] &\geq P_\omega[\tau(\theta_1) < \tau(-1)] - P_\omega^{m_1}[L(m_1, \tau(\theta_1) \wedge \tau(-1)) \leq k_1] \\ &\geq c_7 - (\log N)^{-6}. \end{aligned}$$

This gives (4.11) for N large enough. \square

We now set $C_5 := 11 + 2C_4$. By Lemma 4.4, there exists $N_\varepsilon \in \mathbb{N}$ such that for every $N \geq N_\varepsilon$, inequality (4.17) holds for $i = 1$, (4.10) holds for every $\omega \in \mathcal{G}(N)$, $[\varepsilon \log_2 N] \geq 1$, and the following conditions are satisfied:

$$\forall N \geq N_\varepsilon, \quad \log N \geq h(N) \geq (C_5 + 17 + 8C_4) \log_2 N \geq 4 + 3\varepsilon_0^{-2}, \quad (4.15)$$

$$\forall N \geq N_\varepsilon, \quad \min_{[-(\log N)^4, 0]} f \geq -\exp((\log^4 N)^{\varepsilon\delta/2^7}) = -e^{h(N)}. \quad (4.16)$$

We prove by induction on i the following lemma:

Lemma 4.5. *For all $N \geq N_\varepsilon$ and for every $1 \leq i \leq i_{max}(N)$,*

$$\forall \omega \in \mathcal{G}(N), \quad P_\omega[F_i(N)] \geq c_7 - 4i(\log N)^{-6}. \quad (4.17)$$

Proof: We fix $N \geq N_\varepsilon$. We already know that (4.17) is true for $i = 1$. Now, assume (4.17) is true for an integer i such that $1 \leq i \leq i_{max}(N) - 1$, and let us prove it is true for $i + 1$. We fix $\omega \in \mathcal{G}(N)$.

We notice that $\theta_i < \theta_{i+1}$. Indeed, if $\underline{W}(t_i) = \underline{W}(t_{i+1})$, we have $\sigma W(t_{i+1}) = \sigma W(t_i) + h(N)$, which gives, since $N \geq N_\varepsilon$, $\widehat{V}(\lfloor t_{i+1} \rfloor) \geq \widehat{V}(\lfloor t_i \rfloor) + h(N) - 2C_4 \log_2 N > \widehat{V}(\lfloor t_i \rfloor)$ by (4.9) and (4.10), so $\theta_{i+1} \neq \theta_i$. If $\underline{W}(t_i) \neq \underline{W}(t_{i+1})$, there exists $u \in [t_i, t_{i+1}]$ such that $|\sigma W(u) - \sigma W(t_i)| > ih(N)$, and $\theta_i = \theta_{i+1}$ would imply $|u - t_i| \leq |t_{i+1} - t_i| \leq 1$ and then contradict $\omega \in \mathcal{G}_6(N)$ for $N \geq N_\varepsilon$.

First step : Define

$$\begin{aligned} \bar{\theta}_i &:= \max\{k \in \mathbb{Z}, k < \theta_i, V(k) \geq V(\theta_i) + h(N) + C_5 \log_2 N\}, \\ E_{6,i} &:= \{\inf\{k \geq \tau(\theta_i), S_k = \theta_{i+1}\} < \inf\{k \geq \tau(\theta_i), S_k = \bar{\theta}_i\}\} = \{\tau(\theta_{i+1}) < \tau(\theta_i, \bar{\theta}_i)\}, \end{aligned}$$

where

$$\forall (a, b) \in \mathbb{Z}^2, \quad \tau(a, b) := \inf\{k \geq \tau(a), S_k = b\}.$$

We prove that $P_\omega(E_{6,i}^c) \leq (\log N)^{-6}$. First, notice that since $\underline{W}(t) \leq \underline{W}(t_i)$ for $t_i \leq t \leq t_{i+1}$, applying twice (4.9) gives

$$\max_{[\theta_i, \theta_{i+1}]} V \leq V(\theta_i) + h(N) + 2C_4 \log_2 N. \quad (4.18)$$

Hence, applying the Markov property at time $\tau(\theta_i)$, we get since $\theta_{i+1} \leq 2(\log N)^4$ by (4.10),

$$P_\omega(E_{6,i}^c) = \frac{\sum_{k=\theta_i}^{\theta_{i+1}-1} e^{V(k)}}{\sum_{k=\bar{\theta}_i}^{\theta_{i+1}-1} e^{V(k)}} \leq \frac{2(\log N)^{4+2C_4} e^{V(\theta_i)+h(N)}}{e^{V(\theta_i)+h(N)} (\log N)^{C_5}} \leq (\log N)^{-6}. \quad (4.19)$$

Second step : We consider

$$E_{7,i} := \{L((\bar{\theta}_i, 0), \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1})) < \exp((i-3)h(N))\}$$

and we show that $P_\omega(E_{7,i}^c) \leq (\log N)^{-6}$. Since this is obvious when $\bar{\theta}_i \geq -1$, we just have to consider the case $\bar{\theta}_i < -1$.

First, notice that since $x_0(\sigma W, ih(N))$ is a $ih(N)$ -maximum for σW , we have by (4.9) since $x_0(\sigma W, ih(N)) \geq -d_{-\sigma W} - (5 \log N) \geq -(\log N)^4$,

$$\begin{aligned} V(\lfloor x_0(\sigma W, ih(N)) \rfloor) &\geq \sigma W[x_0(\sigma W, ih(N))] - C_4 \log_2 N \\ &\geq \sigma W[x_1(\sigma W, ih(N))] + ih(N) - C_4 \log_2 N \\ &\geq \sigma W[x_1(\sigma W, ih(N))] + ih(N) - C_4 \log_2 N + 5h(N) + V(\theta_0). \end{aligned} \quad (4.20)$$

Moreover, $\sigma W(t_i) = \sigma \underline{W}(t_i) + ih(N)$, and $\underline{W}(t_i) = W[x_1(\sigma W, ih(N))]$ due to (4.6). This together with (4.9) and $t_i \leq (\log N)^4$ (see (4.10)) gives

$$V(\theta_i) - V(\theta_0) = \widehat{V}(\lfloor t_i \rfloor) \leq \sigma W(t_i) + C_4 \log_2 N = \sigma W[x_1(\sigma W, ih(N))] + ih(N) + C_4 \log_2 N. \quad (4.21)$$

Hence, (4.20) and $N \geq N_\varepsilon$ lead to

$$V(\lfloor x_0(\sigma W, ih(N)) \rfloor) \geq V(\theta_i) + 5h(N) - 2C_4 \log_2 N > V(\bar{\theta}_i).$$

Consequently, $\lfloor x_0(\sigma W, ih(N)) \rfloor < \bar{\theta}_i < \theta_i < \theta_{i+1}$ by definition of $\bar{\theta}_i$. If $\bar{\theta}_i < -1$, let $z \in (\bar{\theta}_i, 0) \cap \mathbb{Z}$. We get by Lemma 2.3,

$$\begin{aligned} E_\omega^{\theta_i}[L(z, \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1}))] &= \frac{P_\omega^{\theta_i}[\tau(z) < \tau(\theta_{i+1})]}{\omega_z e^{V(z)} \left(\sum_{k=z}^{\theta_{i+1}-1} e^{V(k)} \right)^{-1} + (1 - \omega_z) P_\omega^{z-1}[\tau(z) > \tau(\bar{\theta}_i)]} \\ &\leq \varepsilon_0^{-1} e^{-V(z)} \sum_{k=z}^{\theta_{i+1}-1} e^{V(k)} \\ &\leq 3\varepsilon_0^{-1} (\log N)^4 \exp\left(-V(z) + \max_{[z, \theta_{i+1}]} V\right), \end{aligned} \quad (4.22)$$

since $\theta_{i+1} \leq 2(\log N)^4$ by (4.10) and $z > \bar{\theta}_i \geq \lfloor x_0(\sigma W, ih(N)) \rfloor \geq -(\log N)^4$. We notice that by (4.21),

$$V(\theta_i) = V(\theta_0) + \widehat{V}(\theta_i - \theta_0) \leq -5h(N) + \sigma W[x_1(\sigma W, ih(N))] + ih(N) + C_4 \log_2 N. \quad (4.23)$$

Since $-\log^4 N \leq x_0(\sigma W, ih(N)) \leq \bar{\theta}_i < z < 0 < x_1(\sigma W, ih(N)) \leq t_i \leq \log^4 N$ by (4.6) and (4.10), equations (4.8), (4.9) and (4.23) give

$$V(z) \geq \sigma W(x_1(\sigma W, ih(N))) - C_4 \log_2 N \geq (5-i)h(N) + V(\theta_i) - 2C_4 \log_2 N. \quad (4.24)$$

Moreover by definition of $\bar{\theta}_i$ and (1.1),

$$\max_{[\bar{\theta}_i, \theta_i]} V \leq V(\theta_i) + h(N) + C_5 \log_2 N - \log \varepsilon_0.$$

Combining this with (4.18), (4.22) and (4.24) gives since $N \geq N_\varepsilon$,

$$\begin{aligned} E_\omega^{\theta_i}[L(z, \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1}))] &\leq 3\varepsilon_0^{-1} (\log N)^4 \exp(-V(z) + V(\theta_i) + h(N) + C_5 \log_2 N - \log \varepsilon_0) \\ &\leq 3\varepsilon_0^{-2} (\log N)^{C_5 + 2C_4 + 4} e^{(i-4)h(N)} \\ &\leq (\log N)^{-10} e^{(i-3)h(N)}. \end{aligned}$$

Summing this over z , we get $E_\omega^{\theta_i}[L((\bar{\theta}_i, 0), \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1}))] \leq (\log N)^{-6} e^{(i-3)h(N)}$ since $\bar{\theta}_i \geq -(\log N)^4$. We get $P_\omega(E_{7,i}^c) = P_\omega^{\theta_i}(E_{7,i}^c) \leq (\log N)^{-6}$ by Markov's inequality and property.

Third step : we define

$$E_{8,i} := \left\{ L(m_{i+1}, \tau(\theta_i)) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1}) > \frac{\exp((i+1)h(N))}{(\log N)^\nu} \right\}.$$

We prove that

$$P_\omega(E_{8,i}^c) \leq 2(\log N)^{-6}. \quad (4.25)$$

To this aim, we first show that

$$P_\omega^{\theta_i}[\tau(m_{i+1}) > \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})] \leq (\log N)^{-6}. \quad (4.26)$$

This is true if $\theta_i \leq m_{i+1} \leq \theta_{i+1}$ by (4.19). Else, $m_i = m_{i+1} < \theta_i$ and then $\sigma \underline{W}(t_{i+1}) \geq \sigma \underline{W}(t_i) - 2C_4 \log_2 N$ by (4.9) and (4.10), which leads to $V(\theta_{i+1}) \geq V(\theta_i) + h(N) - 4C_4 \log_2 N$. Hence we get successively, again by (4.9) and (4.10), for every $m_i \leq k \leq \theta_i$,

$$\begin{aligned} \sigma W(m_i - \theta_0) &\leq \widehat{V}(m_i - \theta_0) + C_4 \log_2 N = \inf_{[0, [t_i]]} \widehat{V} + C_4 \log_2 N \leq \sigma \underline{W}(t_i) + 2C_4 \log_2 N, \\ \widehat{V}(k - \theta_0) &\leq \sigma [W(k - \theta_0) - W(m_i - \theta_0)] + \sigma W(m_i - \theta_0) + C_4 \log_2 N \\ &\leq ih(N) + \sigma \underline{W}(t_i) + 3C_4 \log_2 N. \end{aligned}$$

As a consequence,

$$\max_{[m_i, \theta_i]} V \leq V(\theta_0) + \sigma W(t_i) + 3C_4 \log_2 N \leq V(\theta_i) + 4C_4 \log_2 N \leq V(\theta_{i+1}) - h(N) + 8C_4 \log_2 N. \quad (4.27)$$

Now we have in this case $m_i = m_{i+1} < \theta_i$, since $N \geq N_\varepsilon$ and by (2.1),

$$\begin{aligned} P_\omega^{\theta_i}[\tau(m_{i+1}) > \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})] &= P_\omega^{\theta_i}[\tau(m_{i+1}) > \tau(\theta_{i+1})] \\ &\leq 2(\log N)^4 \varepsilon_0^{-1} \exp \left[\max_{[m_i, \theta_i]} V - V(\theta_{i+1}) \right] \end{aligned}$$

which together with (4.27) gives (4.26) since $N \geq N_\varepsilon$.

Moreover, we prove that $P_\omega^{m_{i+1}}(E_{9,i}^c) \leq (\log N)^{-6}$, where

$$E_{9,i} := \left\{ L(m_{i+1}, \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})) > \frac{\exp((i+1)h(N))}{(\log N)^\nu} \right\}.$$

We know that $\bar{\theta}_i < m_i \leq m_{i+1} < \theta_{i+1}$ thanks to the first two inequalities of (4.27), which are true in every case. So, $L(m_{i+1}, \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1}))$ is under $P_\omega^{m_{i+1}}$ a geometrical r.v. with parameter

$$\begin{aligned} q_1 &:= \omega_{m_{i+1}} P_\omega^{m_{i+1}+1}[\tau(m_{i+1}) > \tau(\theta_{i+1})] + (1 - \omega_{m_{i+1}}) P_\omega^{m_{i+1}-1}[\tau(m_{i+1}) > \tau(\bar{\theta}_i)] \\ &\leq \omega_{m_{i+1}} \varepsilon_0^{-1} e^{V(m_{i+1})-V(\theta_{i+1})} + (1 - \omega_{m_{i+1}}) \varepsilon_0^{-1} e^{V(m_{i+1})-V(\bar{\theta}_i)}. \end{aligned} \quad (4.28)$$

Moreover, we obtain successively the following inequalities:

$$V(m_{i+1}) \leq V(\theta_0) + \sigma \underline{W}(t_{i+1}) + C_4 \log_2 N \leq V(\theta_{i+1}) - (i+1)h(N) + 2C_4 \log_2 N, \quad (4.29)$$

$$V(\theta_i) \geq V(\theta_0) + \sigma \underline{W}(t_{i+1}) + ih(N) - C_4 \log_2 N \geq V(m_{i+1}) + ih(N) - 2C_4 \log_2 N, \quad (4.30)$$

$$V(\bar{\theta}_i) \geq V(\theta_i) + h(N) + C_5 \log_2 N \geq V(m_{i+1}) + (i+1)h(N) + 11 \log_2 N, \quad (4.31)$$

where we used $V(m_{i+1}) \leq V(\theta_0 + [z_{i+1}])$ with z_{i+1} such that $W(z_{i+1}) = \underline{W}(t_{i+1})$ in (4.29), and $\underline{W}(t_i) \geq \underline{W}(t_{i+1})$ in (4.30). It follows from (4.28), (4.29) and (4.31) that

$$q_1 \leq \varepsilon_0^{-1} \exp(-(i+1)h(N) + 2C_4 \log_2 N) =: q_2.$$

Now, define $n_2 := \lfloor \frac{\exp((i+1)h(N))}{(\log N)^\nu} \rfloor$. We have for $N \geq N_\varepsilon$,

$$\log P_\omega^{m_{i+1}}(E_{9,i}) = n_2 \log(1 - q_1) \geq n_2 \log(1 - q_2) \geq -2n_2 q_2 \geq -(\log N)^{-6}.$$

Indeed, $q_2 \in (0, 1/2)$ hence $\log(1 - q_2) \geq -2q_2$. Since $1 - e^{-t} \leq t$ for $t \in \mathbb{R}$, this yields $P_\omega^{m_{i+1}}(E_{9,i}^c) \leq (\log N)^{-6}$. Hence by the strong Markov property,

$$\begin{aligned} P_\omega(E_{8,i}^c) &= P_\omega^{\theta_i}[E_{8,i}^c, \tau(m_{i+1}) > \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})] + P_\omega^{\theta_i}[E_{8,i}^c, \tau(m_{i+1}) \leq \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})] \\ &\leq P_\omega^{\theta_i}[\tau(m_{i+1}) > \tau(\bar{\theta}_i) \wedge \tau(\theta_{i+1})] + P_\omega^{m_{i+1}}(E_{9,i}^c) \\ &\leq 2(\log N)^{-6}, \end{aligned}$$

where the last inequality comes from (4.26). This gives (4.25). Moreover, notice that in the particular case $i = i_{max}(N) - 1$, we get on $E_{8,i}$ since $N \geq N_\varepsilon$,

$$\tau(\theta_{i_{max}(N)}) \geq L(m_{i_{max}(N)}, \tau(\theta_{i_{max}(N)-1}) \rightsquigarrow \tau(\theta_{i_{max}(N)-1}, \bar{\theta}_{i_{max}(N)-1}) \wedge \tau(\theta_{i_{max}(N)})) \geq N. \quad (4.32)$$

Forth step: conclusion. First, let $\tau(\theta_i) < n \leq \tau(\theta_{i+1})$. We have

$$\sum_{k=0}^n f(S_k) = \sum_{k=0}^{\tau(\theta_i)-1} f(S_k) + \left(\sum_{z \leq \bar{\theta}_i} + \sum_{\bar{\theta}_i < z < 0} + \sum_{z \geq 0} \right) f(z) L(z, \tau(\theta_i) \rightsquigarrow n). \quad (4.33)$$

The second sum of the right hand side is 0 on $E_{6,i}$, and the last sum is at least $f(\theta_i)$ because $f \geq 0$ on \mathbb{N} . Since $f < 0$ on \mathbb{Z}_- and $\bar{\theta}_i \geq -(\log N)^4$, we get on $F_i(N) \cap E_{6,i} \cap E_{7,i}$,

$$\begin{aligned} \sum_{k=0}^n f(S_k) &\geq \sum_{k=0}^{\tau(\theta_i)} f(S_k) + \left(\min_{[-(\log N)^4, 0]} f \right) L((\bar{\theta}_i, 0), \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1})) \\ &\geq \frac{\exp(ih(N))}{2(\log N)^\nu} - \exp(h(N)) \exp((i-3)h(N)) > 0 \end{aligned} \quad (4.34)$$

because for $N \geq N_\varepsilon$, $0 > \min_{[-(\log N)^4, 0]} f \geq -e^{h(N)}$ by (4.16) and $e^{h(N)} \geq \min\{(\log N)^\nu, 4\}$.

We now consider $\sum_{k=0}^{\tau(\theta_{i+1})} f(S_k)$ which is, on $E_{6,i}$, equal to

$$\sum_{k=0}^{\tau(\theta_i)-1} f(S_k) + \left(\sum_{z \leq \bar{\theta}_i} + \sum_{\bar{\theta}_i < z < 0} + \sum_{z \in \mathbb{N} - \{m_{i+1}\}} + \sum_{z \in \{m_{i+1}\}} \right) f(z) L(z, \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1})). \quad (4.35)$$

The potential V is decreasing on $[0, \lfloor \varepsilon \log_2 N \rfloor]$ since $\omega \in \mathcal{G}_1(N)$, hence $m_{i+1} \geq \lfloor \varepsilon \log_2 N \rfloor \geq 1$ and then $f(m_{i+1}) \geq 1$. Consequently, the last sum in the right hand side of (4.35) is at least $L(m_{i+1}, \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1}))$. Moreover, the first term is positive on $F_i(N)$, the second one is 0 on $E_{6,i}$, and the forth one is nonnegative since $f \geq 0$ on \mathbb{N} . So, we have on $F_i(N) \cap E_{6,i} \cap E_{7,i} \cap E_{8,i}$ for $N \geq N_\varepsilon$, since $\bar{\theta}_i \geq -(\log N)^4$,

$$\begin{aligned} \sum_{k=0}^{\tau(\theta_{i+1})} f(S_k) &\geq L(m_{i+1}, \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1})) \\ &\quad + \left(\min_{[-(\log N)^4, 0]} f \right) L((\bar{\theta}_i, 0), \tau(\theta_i) \rightsquigarrow \tau(\theta_i, \bar{\theta}_i) \wedge \tau(\theta_{i+1})) \\ &\geq \frac{\exp((i+1)h(N))}{(\log N)^\nu} - \exp((i-2)h(N)) \\ &\geq \frac{\exp((i+1)h(N))}{2(\log N)^\nu}. \end{aligned}$$

This and (4.34) yield $F_i(N) \cap E_{6,i} \cap E_{7,i} \cap E_{8,i} \subset F_{i+1}(N)$. Consequently, our induction hypothesis $P_\omega[F_i(N)] \geq c_7 - 4i(\log N)^{-6}$ and inequalities (4.19), (4.25) and $P_\omega(E_{7,i}^c) \leq (\log N)^{-6}$ give for

every $\omega \in \mathcal{G}(N)$,

$$P_\omega[F_{i+1}(N)] \geq P_\omega[F_i(N)] - P_\omega(E_{6,i}^c) - P_\omega(E_{7,i}^c) - P_\omega(E_{8,i}^c) \geq c_7 - 4(i+1)(\log N)^{-6}. \quad (4.36)$$

This ends the induction for all $N \geq N_\varepsilon$. Hence (4.17) is true for every $N \geq N_\varepsilon$ and $1 \leq i \leq i_{max}(N)$. \square

Proof of Lemma 4.3: Notice that thanks to (4.32), and the inclusion before (4.36) for $i = i_{max}(N) - 1$, we also show similarly as in (4.36) that $P_\omega[F_{i_{max}(N)}(N) \cap \{\tau(\theta_{i_{max}(N)}) \geq N\}] \geq c_7 - \frac{4i_{max}(N)}{(\log N)^6} \geq c_7 - \frac{16}{(\log N)^5}$ for all $N \geq N_\varepsilon$ and $\omega \in \mathcal{G}(N)$. Consequently, for N large enough,

$$\begin{aligned} \forall \omega \in \mathcal{G}(N), \quad & P_\omega \left(\sum_{k=0}^n f(S_k) > 0 \forall 1 \leq n \leq N \right) \\ & \geq P_\omega[F_{i_{max}(N)}(N) \cap \{\tau(\theta_{i_{max}(N)}) \geq N\}] \geq \frac{c_7}{2} =: c_6, \end{aligned}$$

which proves Lemma 4.3. \square

Now, integrating (4.5) on $\mathcal{G}(N)$ and applying Lemma 4.1 gives

$$\mathbb{P} \left(\sum_{k=0}^n f(S_k) > 0 \forall 1 \leq n \leq N \right) \geq c_6 \eta(\mathcal{G}(N)) \geq \frac{c_6 c_1 \varepsilon \log_2 N}{(\log N)^{\frac{3-\sqrt{5}}{2} + \varepsilon(\gamma + \delta/32)}}$$

for N large enough. Now, let $\varepsilon \rightarrow 0$; this proves the lower bound of Theorem 1.1 for $u = 0$ and then for every $u \leq 0$.

5. PROOFS OF FACT 3.1 AND LEMMA 3.3

5.1. Proof of Lemma Fact 3.1. We first study the left continuity of some functions.

Lemma 5.1. *On $\{W \in \mathcal{W}\}$, for all $k \in \mathbb{Z}$, the functions $x_k(W, \cdot)$, $e(T_k(\cdot))$ and $H(T_k(\cdot))$ are left-continuous on $(0, +\infty)$. More precisely, for all realization of W in \mathcal{W} , for every $n \in \mathbb{N}^*$ and $x > 0$, there exists $K_{x,n} \in (0, x)$ such that all the functions $x_k(W, \cdot)$, $k \in \{-n, \dots, n\}$, are constant on $[K_{x,n}, x]$.*

Proof: We assume throughout the proof that $W \in \mathcal{W}$. Let $x > 0$. We first notice that $\lim_{k \rightarrow \pm\infty} |x_k(W, x/2)| = +\infty$, so there is a finite number of $(x/2)$ -extrema on every compact set, and in particular on $[x_0(W, x), x_1(W, x)]$. Now, we can denote the $(x/2)$ -extrema in this interval by $x_0(W, x) = x_{K_0}(W, x/2) < \dots < x_{K_1}(W, x/2) = x_1(W, x)$ for some integers $K_0 < K_1$.

Let $i \in \{K_0 + 1, \dots, K_1 - 1\}$. We define $H_i := \sup\{y > 0, x_i(W, x/2) \text{ is an } y\text{-extremum}\}$. Assume for example that $x_i(W, x/2)$ is an $(x/2)$ -minimum and that $x_0(W, x)$ is an x -minimum. There exists an increasing sequence $(y_n)_n$, converging to H_i as $n \rightarrow +\infty$, and such that for every $n \in \mathbb{N}$, $x_i(W, x/2)$ is an y_n -extremum, and so an y_n -minimum. So, W being continuous, there exist $\alpha_n < x_i(W, x/2) < \beta_n$ such that

$$W[x_i(W, x/2)] = \inf_{[\alpha_n, \beta_n]} W, \quad W(\alpha_n) = W[x_i(W, x/2)] + y_n = W(\beta_n).$$

Since $x_0(W, x) < x_i(W, x/2) < x_1(W, x)$, $x_i(W, x/2)$ is not an x -extremum, so $x \geq H_i \geq y_n$. If $\alpha_n < x_0(W, x)$, then $W[x_i(W, x/2)] \leq W[x_0(W, x)]$ so $x_i(W, x/2)$ would be an x -minimum, which is not the case, so $\alpha_n \in [x_0(W, x), x_1(W, x)]$. If $W(\beta_n) \leq W[x_1(W, x)]$ and $\beta_n > x_1(W, x)$, we can replace β_n by another $\beta_n \leq x_1(W, x)$. If $W(\beta_n) > W[x_1(W, x)]$ and $\beta_n > x_1(W, x)$, we would have $W(\alpha_n) = W(\beta_n) > W[x_1(W, x)]$, which is the supremum of W in $[x_0(W, x), x_1(W, x)]$, and this is not possible. Hence (α_n, β_n) belongs to the compact $[x_0(W, x), x_1(W, x)]^2$, thus there

exists a strictly increasing sequence n_p and $(\alpha, \beta) \in \mathbb{R}^2$ such that $(\alpha_{n_p}, \beta_{n_p})_{p \rightarrow +\infty} \rightarrow (\alpha, \beta)$. By continuity of W , $W[x_i(W, x/2)] = \inf_{[\alpha, \beta]} W$, and $W(\alpha) = W[x_i(W, x/2)] + H_i = W(\beta)$. Hence $x_i(W, x/2)$ is an H_i -minimum. Since $x_i(W, x/2)$ is not an x -extremum, this gives $H_i < x$. The other cases are treated similarly.

Now, let $H'_x := \max_{K_0 < i < K_1} H_i$; we have $x/2 \leq H'_x < x$. For $y \in (H'_x, x)$, the only possible y -extrema in $(x_0(W, x), x_1(W, x))$ are the $(x/2)$ -extrema, that is the $x_i(W, x/2)$, $K_0 < i < K_1$, but they are not y -extrema since $y > H_i$. So, there is no y -extrema in $(x_0(W, x), x_1(W, x))$, and then $x_0(W, y) = x_0(W, x)$ and $x_1(W, y) = x_1(W, x)$. Hence for every $x > 0$, there exists $H'_x < x$ such that the functions $x_0(W, \cdot)$ and $x_1(W, \cdot)$ are constant on $[H'_x, x]$, and consequently, they are left-continuous. More generally, we prove similarly that for all $n \in \mathbb{N}^*$, there exists $K_{x,n} \in (0, x)$ such that all the functions $x_k(W, \cdot)$, $k \in \{-n, \dots, n\}$ are constant on $[K_{x,n}, x]$. Hence all the functions $x_k(W, \cdot)$, $k \in \mathbb{Z}$ are left-continuous, and as a consequence, $e(T_k(\cdot))$ and $H(T_k(\cdot))$ are also left-continuous. \square

Proof of Fact 3.1: Assume that we are on $\{W \in \mathcal{W}\}$, and let $x > 0$. We saw in the proof of Lemma 5.1 that there exists an interval $[y, x]$ such that $x_0(W, \cdot)$ and $x_1(W, \cdot)$ are constant on this interval, and so is $b(\cdot)$, therefore $b(\cdot)$ does not change its sign on $[y, x]$.

Define $H_{p,q} := |\sum_{k=p}^{q-1} (-1)^k H(T_k(c))|$ for $p < q$ and $\mathcal{H} := \{\forall p < q < r < s, H_{p,q} \neq H_{r,s}\} \cap \{W \in \mathcal{W}\}$. Since the r.v. $H(T_k(c))$, $k \in \mathbb{Z}$ are independent (see [20] Proposition of Section 1) and have a density (see [8] (8) p. 1768 and (11) p. 1770), it follows that the r.v. $H_{p,q} - H_{r,s}$, $p < q < r < s$ also have densities, thus $\eta(\mathcal{H}) = 1$. Moreover, for every trajectory $W \in \mathcal{W}$, every $x \geq c$ and $m < n$, there exist $p < q < r < s$ such that $H(T_m(x)) = H_{p,q}$ and $H(T_n(x)) = H_{r,s}$. Consequently, on \mathcal{H} , for every $x \geq c$, all the $H(T_i(x))$, $i \in \mathbb{Z}$ are different.

Now, assume we are on \mathcal{H} . Let $x \geq c$. The $e(T_i(x))$, $i \in \{-3, \dots, 3\}$ are all different, so for $\varepsilon > 0$ small enough, at most one of them is less than ε . As was shown in the proof of Lemma 2 of Cheliotis ([8] p. 1772), for such $\varepsilon > 0$, $b(x)$ and $b(x + \varepsilon)$ have a different sign iff $e[T_0(x)] < \varepsilon$. So there exists a non empty interval $[x, x + \varepsilon]$ without any change of sign of b for some $\varepsilon > 0$ iff $e(T_0(x)) > 0$.

Hence on \mathcal{H} there is a change of sign of b at x iff $e(T_0(x)) = 0$, which proves Fact 3.1. \square

5.2. Proof of Lemma 3.3. We consider a two-sided Brownian motion W defined on a probability space $(\Omega, \mathcal{A}, \eta)$. We know that $\eta(\mathcal{H} \cap \{W \in \mathcal{W}\}) = 1$. This enables us to replace, in the rest of the paper, Ω by $\Omega \cap \mathcal{H} \cap \{W \in \mathcal{W}\}$.

We denote by \mathcal{F}_x the completion of the σ -field $\sigma(W(s)\mathbf{1}_{\{x_0(W,x) \leq s \leq x_1(W,x)\}}, s \in \mathbb{R})$ for $x > 0$, and by \mathcal{F}_0 and \mathcal{F}_∞ the completions of $\sigma(\emptyset)$ and $\sigma(W(s), s \in \mathbb{R})$ respectively. For $0 < y \leq x$, $[x_0(W, y), x_1(W, y)] \subset [x_0(W, x), x_1(W, x)]$ and $x_0(W, y)$ and $x_1(W, y)$ are \mathcal{F}_x -measurable (see 5.3 Appendix), so $\mathcal{F}_y \subset \mathcal{F}_x$. Hence $(\mathcal{F}_x)_{x \geq 0}$ is a filtration. Notice that W is not adapted to $(\mathcal{F}_x)_{x \geq 0}$. Moreover, for $k \in \mathbb{Z}$, $x \mapsto e[T_k(x)]$ is a.s. left-continuous by Lemma 5.1, but it is not right-continuous, and $(\mathcal{F}_x)_{x \geq 0}$ is not the natural filtration of one of these processes. We now give an elementary proof of Lemma 3.3. We start with the following lemma.

Lemma 5.2. *For every $k \geq 1$, X_k is a $(\mathcal{F}_x)_{x \geq 0}$ -stopping time.*

Proof: Instead of trying to prove whether the filtration $(\mathcal{F}_x)_x$ is right-continuous, we give an elementary proof. Notice that $e[T_0(y)] = (\sup_{\mathbb{R}} - \inf_{\mathbb{R}})(W\mathbf{1}_{[x_0(W,y), x_1(W,y)]}) - y$ is \mathcal{F}_y -measurable for every $y > 0$, that means, the processes $(e[T_0(y)])_y$ and then $(H[T_0(y)])_y$ are adapted to the filtration $(\mathcal{F}_y)_y$. Moreover, the function $e[T_0(\cdot)]$ has a jump at $y \in [c, x]$ if and only if $x_0(W, y)$ or $x_1(W, y)$ is a y -extremum but is not a z -extremum for $z > y$, and in this case the number of z -extrema in $[x_0(W, x), x_1(W, x)]$ decreases by at least 1 between $z = y$ and every

$z > y$. So, the number of discontinuities of $e[T_0(\cdot)]$ in $[c, x]$ is less than the number of c -extrema in $[x_0(W, x), x_1(W, x)]$, which is finite on $\{W \in \mathcal{W}\}$.

Hence, the process $e(T_0(\cdot))$ is left-continuous with a finite number of discontinuities in $[c, x]$, is nonnegative, and it is strictly decreasing between two consecutive discontinuities and then has right limits. Moreover on $\{W \in \mathcal{W}\}$, $H(T_0(\cdot))$ is nondecreasing and so only has positive jumps, and then $e(T_0(\cdot))$ also has only positive jumps. As a consequence, $e(T_0(\cdot))$, which is left-continuous with right limits, is lower semi-continuous on $(0, +\infty)$.

Recalling that $\{X_1 \leq x\} = \{\exists y \in [c, x], e[T_0(y)] = 0\}$ by Fact 3.1 since $\Omega \subset \mathcal{H}$, we claim that for $x \geq c$,

$$\{X_1 \leq x\} = \bigcap_{p \in \mathbb{N}^*} \{\exists y \in [c, x], e[T_0(y)] < 1/p\} \quad (5.1)$$

$$= \bigcap_{p \in \mathbb{N}^*} \bigcup_{y \in ([c, x] \cap \mathbb{Q}) \cup \{c\}} \{e[T_0(y)] < 1/p\}. \quad (5.2)$$

Indeed for the first line, inclusion \subset is clear. For the inclusion \supset , on the event in RHS of (5.1), where RHS stands for right hand side, there is a sequence $y_n \in [c, x]$, $n \in \mathbb{N}^*$ such that $e[T_0(y_n)] < 1/n$ for $n \in \mathbb{N}^*$. Since $[c, x]$ is compact, there exists a subsequence $(y_{p_n})_n$, which converges to an $y \in [c, x]$. Hence, $0 \leq e[T_0(y)] \leq \liminf_{n \rightarrow +\infty} e[T_0(y_{p_n})] = 0$ by lower semi-continuity, which proves the inclusion. For line (5.2), inclusion (RHS of (5.1)) \supset (RHS of (5.2)) is clear, whereas inclusion \subset follows from the left-continuity of $e(T_0(\cdot))$.

Hence $\{X_1 \leq x\} \in \mathcal{F}_x$ for every $x \geq c$, and $\{X_1 \leq x\} = \emptyset \in \mathcal{F}_x$ for $0 \leq x < c$, so X_1 is a $(\mathcal{F}_x)_{x \geq 0}$ -stopping time. Let $k \geq 1$. Since $\lim_{u \rightarrow X_k, u > X_k} e[T_0(u)] > 0$ because $e[T_0(X_k)] = 0$ and so there is a positive jump at x for $e[T_0(\cdot)]$, we show similarly that for $x \geq c$,

$$\begin{aligned} \{X_{k+1} \leq x\} &= \{X_k < x\} \cap \bigcap_{p \in \mathbb{N}^*} \{\exists y \in (X_k, x], e(T_0(y)) < 1/p\} \\ &= \{X_k < x\} \cap \bigcap_{p \in \mathbb{N}^*} \bigcup_{y \in ((c, x] \cap \mathbb{Q})} [\{y > X_k\} \cap \{e(T_0(y)) < 1/p\}]. \end{aligned}$$

Hence it follows by induction that X_k is a $(\mathcal{F}_x)_{x \geq 0}$ -stopping time for every $k \geq 1$. \square

We can then consider the σ -fields \mathcal{F}_{X_k} for $k \geq 1$.

We now fix $k \geq 1$. First, we notice that $A_{k+1, a, c} = A_{k+1, a, c}^+ \cup A_{k+1, a, c}^-$, where $A_{k+1, a, c}^+ := A_{k+1, a, c} \cap \{b(c) > 0\}$ and $A_{k+1, a, c}^- := A_{k+1, a, c} \cap \{b(c) \leq 0\}$. We start with $A_{k+1, a, c}^+$, and notice that

$$A_{k+1, a, c}^+ = A_{k, a, c} \cap \{b(X_1) > 0\} \cap [\{e(T_{-1}(X_{2k+1})) < aX_{2k+1}\} \cup \{e(T_1(X_{2k+1})) < aX_{2k+1}\}]. \quad (5.3)$$

Let $n_0 \in \mathbb{N}^*$. We define a sequence $(R_n)_{n \geq n_0}$ by induction as follows:

$$\begin{aligned} R_{n_0} &:= 2^{-n_0} (\lfloor 2^{n_0} X_{2k} \rfloor + 1) \mathbf{1}_{\{X_{2k+1} > 2^{-n_0} (\lfloor 2^{n_0} X_{2k} \rfloor + 1)\}}, \\ R_n &:= 2^{-n} \lfloor 2^n H[T_0(R_{n-1})] \rfloor \mathbf{1}_{\{X_{2k+1} > 2^{-n_0} (\lfloor 2^{n_0} X_{2k} \rfloor + 1)\}}, \quad n > n_0. \end{aligned}$$

In particular, we have $c \leq X_{2k} < R_{n_0} < X_{2k+1}$ on $B_{k+1, a, c}^{+, n_0} := \{X_{2k+1} > 2^{-n_0} (\lfloor 2^{n_0} X_{2k} \rfloor + 1)\} = \{R_{n_0} \neq 0\}$. Moreover $R_n \in (2^{-n} \mathbb{N})$ for all $n \geq n_0$. We have $R_n \leq H[T_0(R_{n-1})] \leq R_n + 2^{-n}$ on $B_{k+1, a, c}^{+, n_0}$ and $R_n = 0$ on $(B_{k+1, a, c}^{+, n_0})^c$ for $n \geq n_0$. We now prove the two following lemmas:

Lemma 5.3. *The sequence $(R_n)_{n \geq n_0}$ is nondecreasing. It converges a.s. to a r.v. R_∞ , and*

$$R_\infty = X_{2k+1} \mathbf{1}_{B_{k+1, a, c}^{+, n_0}}.$$

Proof: Since $H[T_0(x)] \geq x$ for every $x \geq 0$ and $(2^n R_{n-1}) \in \mathbb{N}$ for $n > n_0$, we get on $B_{k+1, a, c}^{+, n_0}$,

$$R_{n-1} = 2^{-n} \lfloor 2^n R_{n-1} \rfloor \leq 2^{-n} \lfloor 2^n H[T_0(R_{n-1})] \rfloor = R_n, \quad n > n_0.$$

So, $(R_n)_{n \geq n_0}$ is a nondecreasing sequence on $B_{k+1,a,c}^{+,n_0}$, and also on $(B_{k+1,a,c}^{+,n_0})^c$ on which $R_n = 0$ for every $n \geq n_0$. Hence, it converges a.s. to $R_\infty := \lim_{n \rightarrow +\infty} R_n$.

Let $n \geq n_0 + 1$. If $R_{n-1} < x < R_n$, we have

$$e[T_0(x)] = H[T_0(x)] - x \geq H[T_0(R_{n-1})] - x \geq R_n - x > 0. \quad (5.4)$$

Assume that $R_{n_0} \neq 0$ and that there exists $n \geq n_0$ such that $e[T_0(R_n)] = 0$, and let n_1 denote the smallest such n . Then, $H[T_0(R_{n_1})] = R_{n_1} + e[T_0(R_{n_1})] = R_{n_1}$, so

$$R_{n_1+1} = 2^{-(n_1+1)} [2^{n_1+1} H[T_0(R_{n_1})]] = 2^{-(n_1+1)} [2^{n_1+1} R_{n_1}] = R_{n_1}$$

since $R_{n_1} \in 2^{-n_1}\mathbb{N}$. We prove similarly by induction that $R_n = R_{n_1}$ for every $n \geq n_1$, so $R_\infty = R_{n_1}$ and then $e[T_0(R_\infty)] = 0$. Moreover, by (5.4), $e(T_0(\cdot)) > 0$ on $[R_{n_0}, R_\infty)$. Furthermore we know that on $B_{k+1,a,c}^{+,n_0}$, $X_{2k} < R_{n_0} < X_{2k+1}$, so $e(T_0(\cdot)) > 0$ on $(X_{2k}, R_{n_0}]$ by Fact 3.1 and then on (X_{2k}, R_∞) . Hence $R_\infty = \inf\{x > X_{2k}, e[T_0(x)] = 0\} = X_{2k+1}$ in this case.

Else, assume that $R_{n_0} \neq 0$ and $e[T_0(R_n)] \neq 0$ for every $n \geq n_0$. Then $(R_n)_{n \geq n_0}$ is a nondecreasing sequence such that $e[T_0(\cdot)] > 0$ on each interval (R_{n-1}, R_n) , $n > n_0$ by (5.4), and then $e[T_0(\cdot)] > 0$ on $[R_{n_0}, R_\infty)$. As in the previous case, we get $e[T_0(\cdot)] > 0$ on (X_{2k}, R_∞) . Since $e[T_0(X_{2k+1})] = 0$ and $X_{2k} < X_{2k+1}$, this yields to $R_\infty \leq X_{2k+1} < \infty$.

Moreover in this case, $0 \leq e[T_0(R_{n-1})] = H[T_0(R_{n-1})] - R_{n-1} \leq R_n + 2^{-n} - R_{n-1} \rightarrow_{n \rightarrow +\infty} 0$. Since $e[T_0(\cdot)]$ is a left-continuous function on \mathcal{W} and $(R_n)_n$ is nondecreasing and converging to $R_\infty < \infty$, this gives $e[T_0(R_\infty)] = \lim_{n \rightarrow +\infty} e[T_0(R_{n-1})] = 0$. As in the previous case, we conclude that $R_\infty = X_{2k+1}$. Since $R_n = 0 \forall n \geq n_0$ if $R_{n_0} = 0$, this proves the lemma. \square

Lemma 5.4. For all $n \geq n_0$,

$$\forall m \in \mathbb{N}^*, \quad \{R_n = m2^{-n}\} \in \mathcal{F}_{m2^{-n}}. \quad (5.5)$$

Proof: We prove this lemma by induction. We start with R_{n_0} , and observe that for $m \in \mathbb{N}^*$,

$$\begin{aligned} \{R_{n_0} = m2^{-n_0}\} &= \{X_{2k+1} > 2^{-n_0}([2^{n_0}X_{2k}] + 1)\} \cap \{[2^{n_0}X_{2k}] = m - 1\} \\ &= \{X_{2k+1} > m2^{-n_0}\} \cap \{(m-1)2^{-n_0} \leq X_{2k} < m2^{-n_0}\}, \end{aligned}$$

which belongs to $\mathcal{F}_{m2^{-n_0}}$ since X_{2k} and X_{2k+1} are $(\mathcal{F}_x)_{x \geq 0}$ -stopping times. This gives (5.5) for $n = n_0$. Now, assume that (5.5) is true for some $n \geq n_0$. Then for $m \in \mathbb{N}^*$,

$$\begin{aligned} \{R_{n+1} = m2^{-(n+1)}\} &= \{[2^{n+1}H[T_0(R_n)]] = m\} \cap B_{k+1,a,c}^{+,n_0} \\ &= \cup_{p \in \mathbb{N}^*} \{R_n = p2^{-n}, [2^{n+1}H[T_0(R_n)]] = m\} \\ &= \cup_{p \in \mathbb{N}^*, p2^{-n} \leq m2^{-(n+1)}} \{R_n = p2^{-n}\} \cap \{[2^{n+1}H[T_0(p2^{-n})]] = m\}. \end{aligned}$$

The second equality comes from $\{R_n \neq 0\} = \{R_{n+1} \neq 0\} = B_{k+1,a,c}^{+,n_0}$. The third one is a consequence of $x \leq H[T_0(x)]$. If $0 < p2^{-n} \leq m2^{-(n+1)}$, our induction hypothesis gives $\{R_n = p2^{-n}\} \in \mathcal{F}_{p2^{-n}} \subset \mathcal{F}_{m2^{-(n+1)}}$, and $\{[2^{n+1}H[T_0(p2^{-n})]] = m\} \in \mathcal{F}_{p2^{-n}} \subset \mathcal{F}_{m2^{-(n+1)}}$ since $(H[T_0(y)], y \geq 0)$ is adapted to $(\mathcal{F}_y)_{y \geq 0}$. Consequently, $\{R_{n+1} = m2^{-(n+1)}\} \in \mathcal{F}_{m2^{-(n+1)}}$ for every $m \in \mathbb{N}^*$, which ends the induction. \square

In view of (5.3), we define for $n \geq n_0$,

$$C_{k+1,a,c}^{+,n} := A_{k,a,c} \cap \{b(X_1) > 0\} \cap \{e[T_{-1}(R_n)] < aR_n\} \cup \{e[T_1(R_n)] < aR_n\}.$$

Assume that we are on $B_{k+1,a,c}^{+,n_0} \cap A_{k+1,a,c}^+$. There exists $i \in \{-1, 1\}$ such that $e(T_i(X_{2k+1})) < aX_{2k+1}$, that is $H[T_i(X_{2k+1})] < (a+1)X_{2k+1}$. On the one hand, $R_n \rightarrow_{n \rightarrow +\infty} X_{2k+1}$, $R_n \leq X_{2k+1}$, then by Lemma 5.1, for n large enough, $R_n \in [K_{X_{2k+1},2}, X_{2k+1}]$, then $x_j(W, R_n) = x_j(W, X_{2k+1})$ for $-1 \leq j \leq 2$ and so $H[T_i(R_n)] = H[T_i(X_{2k+1})]$. On the other hand, $(a+1)(X_{2k+1} - R_n)$ tends

to 0 as $n \rightarrow \infty$ by Lemma 5.3 and then is strictly less than $(a+1)X_{2k+1} - H[T_i(X_{2k+1})] > 0$ for n large enough. So for large n ,

$$H[T_i(R_n)] - (a+1)R_n = (a+1)(X_{2k+1} - R_n) - [(a+1)X_{2k+1} - H[T_i(X_{2k+1})]] < 0.$$

Then for large n , $\mathbb{1}_{C_{k+1,a,c}^{+,n} \cap B_{k+1,a,c}^{+,n_0}} = 1$.

Hence, in every case, $\mathbb{1}_{B_{k+1,a,c}^{+,n_0} \cap A_{k+1,a,c}^+} \leq \liminf_{n \rightarrow +\infty} \mathbb{1}_{B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n}}$. Then by Fatou's lemma,

$$\eta(B_{k+1,a,c}^{+,n_0} \cap A_{k+1,a,c}^+) \leq \int_{\Omega} (\liminf_{n \rightarrow +\infty} \mathbb{1}_{B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n}}) d\eta \leq \liminf_{n \rightarrow +\infty} \eta(B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n}). \quad (5.6)$$

Let $n \geq n_0$. We now have to estimate

$$\eta(B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n}) = \sum_{m \in \mathbb{N}, m \geq c2^n} \eta(B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n} \cap \{R_n = m2^{-n}\}). \quad (5.7)$$

For $m \geq c2^n$, we have, since $m > 0$,

$$\begin{aligned} & \eta(B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n} \cap \{R_n = m2^{-n}\}) \\ &= \eta(C_{k+1,a,c}^{+,n} \cap \{R_n = m2^{-n}\}) \\ &= \eta([A_{k,a,c} \cap \{b(X_1) > 0\}] \cap \cup_{i=\pm 1} \{e[T_i(R_n)] < aR_n\} \cap \{R_n = m2^{-n}\}) \\ &= \eta(A_{k,a,c} \cap \{b(c) > 0\} \cap \{X_{2k} < m2^{-n}\} \cap \{R_n = m2^{-n}\} \cap \cup_{i=\pm 1} \{e[T_i(m2^{-n})] < am2^{-n}\}), \end{aligned} \quad (5.8)$$

where the last equality comes from $X_{2k} < R_{n_0} \leq R_n$ on $\{R_n > 0\} = B_{k+1,a,c}^{+,n_0}$.

For $\ell \geq 1$, we have on $\{X_\ell < x\}$, $[x_{-1}(W, X_\ell), x_2(W, X_\ell)] \subset [x_0(W, x), x_1(W, x)]$ since $x_0(W, X_\ell)$ and $x_1(W, X_\ell)$ are not x -extrema on \mathcal{H} . Hence, the random variables $e[T_i(X_\ell)]$, $i \in \{-1, 1\}$ are measurable with respect to $\mathcal{F}_{X_\ell+} = \{A \in \mathcal{F}_\infty, \forall x \geq 0, A \cap \{X_\ell < x\} \in \mathcal{F}_x\}$ (see 5.3 Appendix). As a consequence, $A_{k,a,c} \in \mathcal{F}_{X_{2k+}}$ for every $k \geq 1$. which gives in particular $[A_{k,a,c} \cap \{X_{2k} < m2^{-n}\}] \in \mathcal{F}_{m2^{-n}}$ for every $m \in \mathbb{N}$.

Moreover, let $m \in \mathbb{N}$ such that $c \leq m2^{-n}$. We have $\{b(c) > 0\} \in \mathcal{F}_c \subset \mathcal{F}_{m2^{-n}}$. Since $\{R_n = m2^{-n}\} \in \mathcal{F}_{m2^{-n}}$, we get $[A_{k,a,c} \cap \{X_{2k} < m2^{-n}\} \cap \{b(c) > 0\} \cap \{R_n = m2^{-n}\}] \in \mathcal{F}_{m2^{-n}}$. But $e[T_1(m2^{-n})]$, $e[T_{-1}(m2^{-n})]$ and $\mathcal{F}_{m2^{-n}}$ are independent by Neveu et al. ([20], prop. 1), so

$$\begin{aligned} \text{RHS of (5.8)} &= \eta[A_{k,a,c} \cap \{b(c) > 0\} \cap \{X_{2k} < m2^{-n}\} \cap \{R_n = m2^{-n}\}] \\ &\quad \times \eta(\cup_{i=\pm 1} \{e[T_i(m2^{-n})] < am2^{-n}\}) \\ &= (1 - e^{-2a}) \eta[A_{k,a,c} \cap \{b(c) > 0\} \cap \{R_n = m2^{-n}\}] \end{aligned} \quad (5.9)$$

since $e[T_i(m2^{-n})]/(m2^{-n})$, $i \neq 0$, are independent exponential r.v. with mean 1 (also by Neveu et al. [20], prop. 1) and $X_{2k} < R_n$ on $\{R_n \neq 0\}$. So, (5.7), (5.8) and (5.9), give

$$\begin{aligned} \eta(B_{k+1,a,c}^{+,n_0} \cap C_{k+1,a,c}^{+,n}) &= (1 - e^{-2a}) \sum_{m \in \mathbb{N}, m \geq c2^n} \eta[A_{k,a,c} \cap \{b(c) > 0\} \cap \{R_n = m2^{-n}\}] \\ &\leq (1 - e^{-2a}) \eta[A_{k,a,c}^+]. \end{aligned}$$

Consequently, (5.6) leads to

$$\eta(A_{k+1,a,c}^+) \leq \eta(A_{k+1,a,c}^+ \cap B_{k+1,a,c}^{+,n_0}) + \eta[(B_{k+1,a,c}^{+,n_0})^c] \leq (1 - e^{-2a}) \eta[A_{k,a,c}^+] + \eta[(B_{k+1,a,c}^{+,n_0})^c].$$

But $c \leq X_{2k}$ and $X_{2k+1}/X_{2k} > 1$ a.s., so

$$\eta[(B_{k+1,a,c}^{+,n_0})^c] \leq \eta[X_{2k+1} \leq X_{2k} + 2^{-n_0}] \leq \eta[X_{2k+1}/X_{2k} \leq 1 + 2^{-n_0}/c] \rightarrow_{n_0 \rightarrow +\infty} 0$$

As a consequence,

$$\eta(A_{k+1,a,c}^+) \leq (1 - e^{-2a}) \eta(A_{k,a,c}^+).$$

Since we get similarly $\eta(A_{k+1,a,c}^-) \leq (1 - e^{-2a})\eta(A_{k,a,c}^-)$, this yields to $\eta(A_{k+1,a,c}) \leq (1 - e^{-2a})\eta(A_{k,a,c})$. We obtain (3.1) by induction on k , which proves Lemma 3.3. \square

5.3. Appendix : measurability. We fix $x > 0$. We define

$$Z(s) = W(s)\mathbb{1}_{\{x_0(W,x) \leq s \leq x_1(W,x)\}} \quad (5.10)$$

so $\mathcal{F}_x = \sigma(Z(s), s \in \mathbb{R})$. Let $y < x$, and

$$z_0 = z_0(y) := \inf\{t \in \mathbb{R}, Z(t) \neq 0\} = x_0(W, x), \quad z_\infty := \sup\{t \in \mathbb{R}, Z(t) \neq 0\} = x_1(W, x).$$

We define recursively for $k \in \mathbb{N}$, (with $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$)

$$\begin{aligned} u_{2k+1}(y) &:= \inf\{t > z_{2k}(y), Z(t) - \inf\{Z(u), z_{2k}(y) \leq u \leq t\} \geq y\} \mathbb{1}_{\{Z(z_0) > Z(z_\infty)\}} \\ &\quad + \inf\{t > z_{2k}(y), \sup\{Z(u), z_{2k}(y) \leq u \leq t\} - Z(t) \geq y\} \mathbb{1}_{\{Z(z_0) < Z(z_\infty)\}}, \\ z_{2k+1}(y) &:= [\inf\{t > z_{2k}(y), Z(t) = \inf\{Z(u), z_{2k}(y) \leq u \leq u_{2k+1}(y)\} \wedge z_\infty\} \mathbb{1}_{\{Z(z_0) > Z(z_\infty)\}} \\ &\quad + [\inf\{t > z_{2k}(y), Z(t) = \sup\{Z(u), z_{2k}(y) \leq u \leq u_{2k+1}(y)\} \wedge z_\infty\} \mathbb{1}_{\{Z(z_0) < Z(z_\infty)\}}, \\ u_{2k+2}(y) &:= \inf\{t > z_{2k+1}(y), \sup\{Z(u), z_{2k+1}(y) \leq u \leq t\} - Z(t) \geq y\} \mathbb{1}_{\{Z(z_0) > Z(z_\infty)\}} \\ &\quad + \inf\{t > z_{2k+1}(y), Z(t) - \inf\{Z(u), z_{2k+1}(y) \leq u \leq t\} \geq y\} \mathbb{1}_{\{Z(z_0) < Z(z_\infty)\}}, \\ z_{2k+2}(y) &:= [\inf\{t > z_{2k+1}(y), Z(t) = \sup\{Z(u), z_{2k+1}(y) \leq u \leq u_{2k+2}(y)\} \wedge z_\infty\} \mathbb{1}_{\{Z(z_0) > Z(z_\infty)\}} \\ &\quad + [\inf\{t > z_{2k+1}(y), Z(t) = \inf\{Z(u), z_{2k+1}(y) \leq u \leq u_{2k+2}(y)\} \wedge z_\infty\} \mathbb{1}_{\{Z(z_0) < Z(z_\infty)\}}. \end{aligned}$$

Consequently, all these r.v. $z_i(y)$, $i \geq 0$ are \mathcal{F}_x -measurable and so are the r.v. $Z(z_k(y))$, $k \in \mathbb{N}$. Moreover it follows from the definition of y and x -extrema that the y -extrema in $[x_0(W, x), x_1(W, x)]$ are exactly the $z_k(y)$ (with repetitions at z_∞). In particular, $x_0(W, y) = \sum_{k \in \mathbb{N}} z_k(y) \mathbb{1}_{\{z_k(y) \leq 0 < z_{k+1}(y)\}}$ and $x_1(W, y) = \sum_{k \in \mathbb{N}} z_{k+1}(y) \mathbb{1}_{\{z_k(y) \leq 0 < z_{k+1}(y)\}}$ are \mathcal{F}_x -measurable.

We now aim to prove that the random variables $e(T_i(X_k))$, $i \in \{-1, 1\}$ are measurable with respect to \mathcal{F}_{X_k+} , where $\mathcal{F}_{X_k+} = \{A \in \mathcal{F}_\infty, \forall x \geq 0, A \cap \{X_k < x\} \in \mathcal{F}_x\}$. let $K(y) := \sum_{\ell \in \mathbb{N}} \ell \mathbb{1}_{\{z_\ell(y) \leq 0 < z_{\ell+1}(y)\}}$, so $x_i(W, y) = z_{K(y)+i}(y)$ for every $i \in \mathbb{N}$ such that $x_i(W, y) \in [x_0(W, x), x_1(W, x)]$, and $K(y)$ is \mathcal{F}_x -measurable. For $i \in \mathbb{N}$ (with $z_i(y) := z_0(y)$ for $i < 0$),

$$h_i(y) := |Z(z_{K(y)+i}(y)) - Z(z_{K(y)+i+1}(y))| = \sum_{k \in \mathbb{N}} \mathbb{1}_{K(y)=k} |Z(z_{k+i}(y)) - Z(z_{k+i+1}(y))| \quad (5.11)$$

is also \mathcal{F}_x -measurable (for every $0 < y < x$). And $h_i(y) = H(T_i(y))$ if the support of slope $T_i(y)$ is included in $[x_0(W, x), x_1(W, x)]$, since in this case, $Z(z_{K(y)+i}(y)) = Z(x_i(W, y)) = W(x_i(W, y))$ and $Z(z_{K(y)+i+1}(y)) = Z(x_{i+1}(W, y)) = W(x_{i+1}(W, y))$.

We start with $H(T_1(X_k))$. Let $a \in \mathbb{R}$; we have to prove that $\{H(T_1(X_k)) \leq a\} \in (\mathcal{F}_{X_k+})$, which means that $\{H(T_1(X_k)) \leq a\} \cap \{X_k < x\} \in \mathcal{F}_x$ for every $x \geq 0$. This is obvious for $0 \leq x < c$. We now fix $x \geq c$ and define for $p > 1/c$ (h_1 is defined in (5.11))

$$D_p(x) := \sum_{i=1}^{\infty} h_1(x - i/p) \mathbb{1}_{\{0 < x - i/p\}} \mathbb{1}_{\{x - i/p \leq X_k\}} \mathbb{1}_{\{X_k < x - (i-1)/p\}},$$

which is \mathcal{F}_x -measurable. Moreover, on $\{X_k < x\}$, there exists a unique (random) $j = j(p) \geq 1$ such that $x - j/p \leq X_k < x - (j-1)/p \leq x$, and then $x - j/p > 0$ since $X_k \geq c > 1/p$. We have

$$[x_{-1}(W, x - j/p), x_2(W, x - j/p)] \subset [x_{-1}(W, X_k), x_2(W, X_k)] \subset [x_0(W, x), x_1(W, x)].$$

The last inclusion comes from the fact that X_k is a change of sign of $b(\cdot)$, and $x > X_k$, so $e(T_0(X_k)) = 0$ and $x_0(W, X_k)$ and $x_1(W, X_k)$ are not x -extrema

Let $y_p := (x - j(p)/p) \mathbb{1}_{\{X_k < x\}}$. So on $\{X_k < x\}$, $D_p(x) = h_1(y_p) = H(T_1(y_p))$ (see the comments after (5.11) since the support of slope $T_1(x - j/p)$ is included in $[x_0(W, x), x_1(W, x)]$). Since

$y_p \in (X_k - 1/p, X_k]$, $y_p \xrightarrow{p \rightarrow +\infty} X_k$ on $\{X_k < x\}$, and since $H(T_1(\cdot))$ is left-continuous on $(0, +\infty)$ on \mathcal{W} , $H(T_1(X_k)) = \lim_{p \rightarrow +\infty} H(T_1(y_p)) = \lim_{p \rightarrow +\infty} D_p(x)$ on $\{X_k < x\}$. Hence,

$$\{H(T_1(X_k)) \leq a\} \cap \{X_k < x\} = \left\{ \lim_{p \rightarrow +\infty} D_p(x) \leq a \right\} \cap \{X_k < x\}.$$

Since $\lim_{p \rightarrow +\infty} D_p(x)$ is the limit of a sequence of \mathcal{F}_x -measurable r.v., it is also \mathcal{F}_x -measurable, and then $\{\lim_{p \rightarrow +\infty} D_p(x) \leq a\} \in \mathcal{F}_x$. Since $\{X_k < x\} \in \mathcal{F}_x$, we get $\{H(T_1(X_k)) \leq a\} \cap \{X_k < x\} \in \mathcal{F}_x$, and this is true for every $x \geq 0$. So $\{H(T_1(X_k)) \leq a\} \in \mathcal{F}_{X_k+}$ for every $a \in \mathbb{R}$.

Hence $H(T_1(X_k))$ and then $e(T_1(X_k))$ are (\mathcal{F}_{X_k+}) -measurable. We prove similarly that $H(T_{-1}(X_k))$ and then $e(T_{-1}(X_k))$ are (\mathcal{F}_{X_k+}) -measurable.

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