On the force density method for slack cable nets

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A B S T R A C T

The design of cable nets and light tenso-structures requires a non conventional mechanical analysis, due either to the various sources of non linearity (large displacements, unilateral behaviour of the cables, non conservative loads) and to the fact that the initial configuration of the cable net is not known, depending on the prestress applied and, in general, on the dead load acting on it. As a consequence, the first problem that the engineer has to face is to determine the initial state of the structure under its own weight compatible with a set of fixed supports (the so called zero state). This problem is known as form finding.

The paper examines the force density method for form finding, and it is presented a generalization that uses the exact expressions of the equilibrium derived from the equation of the catenary. The method allows to obtain an exact configuration that may be used as a starting point for subsequent incremental non linear analyses.

In the paper it is shown that the use of the exact equilibrium conditions leads to a form finding method that is very similar to the FDM, but yields significant differences in the initial form when the weight of the cables is not negligible. A dimensionless parameter is introduced as degree of freedom of the form.

1. Introduction

Cable nets are employed for large roofs either as bearing structures (e.g. cable trusses) or as support for the fabric that constitutes the tenso-structures. In many cases the design process of light roofs starts with the project of a large cable net, then the inner cables are substituted by equivalent membranes. The subject bears therefore a significant engineering relevance.

From a mechanical point of view, the design of cable nets and tenso-structures requires non linear analyses. Accounting for large displacements and the unilateral behaviour of the cables. Furthermore, the initial configuration of the cable is not univocal, but depends on the prestress and on dead load. As a consequence, the first problem that the engineer has to face is to determine the initial state of the structure under its own weight compatible with a set of fixed supports (the so called zero state). This problem is known as form finding, and it holds either for cable-nets and tenso-structures. Different kinds of approaches for form finding exist in literature, the most used methods being the dynamical relaxation, the minimal surface method and the force density method (FDM).

In the dynamic relaxation method, starting from an arbitrary non equilibrated configuration the initial form is sought by means of an iterative pseudo-dynamical process, with each iteration based on an update of the geometry, see Day (1965). The minimal surfaces are equilibrated surfaces having a uniform isotropic positive membrane stress distribution. This method was proposed by Bletzinger and Ramm (1999), who proved the existence of minimal surfaces under assigned boundary conditions. The obvious advantage of having a uniform stress state in the membrane is however counterbalanced by the fact that the forms obtained usually present very flat surfaces with extremely high curvatures in the proximity of the supports. These kinds of forms are subjected to dynamic instabilities due to aero-elastic effects and to other engineering problems. Wüchner and Bletzinger (2005) have extended the method to non isotropic stress states, and, subsequently, to the case of heavy structures, using Finite Element approximations for the membrane (Bletzinger et al., 2005). It is interesting to observe that in Wüchner and Bletzinger (2005) the authors proved the equivalence of the FDM with the minimal surface method, and proposed an iterative strategy for obtaining a minimal surface using a sequence of force density steps.

The FDM searches for an initial equilibrated solution using the coordinates of the nodes as unknowns. In its original version to each cable is assigned a ratio, called force density, between the normal force acting in an equivalent truss element and the length of the element itself, see Schek (1974), Linkwitz (1999) and Grundig and Bahndorf (1988). Usually, in order to obtain reasonable forms, the force densities are assigned constant everywhere except for the boundary cables. Once an equilibrated configuration has been obtained, it is necessary to perform a fully non linear analysis for subsequent loads that may act on the roofs, including self

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weight, wind and snow loads. In most of the available computer codes, in the case of complex cable nets, the single element is modelled by a non linear truss, especially in the case that the thrust is high.

In the paper is presented an enhancement of the FDM, that is particularly useful when slack cables or very heavy elements are present. In this case, indeed, the initial configuration determined with the equivalent truss element can be very far from the effective catenary configuration. This goal is reached using the exact equilibrium equations of the heavy cable. It is shown that the use of the exact equilibrium conditions leads to a form finding method that is very similar to the standard FDM, although it requires the solution of a non linear system of equations.

The proposed method uses as degrees of freedom for the form a dimensionless parameter η (see Eq. (40)), analogous to the force density, but that includes also the weight of the cable. The initial form is sought within the class of the configurations having the prescribed value for the parameter η that, as will be shown in the paper, can be related to the form of the cable. The proposed procedure is different from using the standard FDM for truss structures, followed by a non linear analysis that accounts for the self weight. In the latter case, as a matter of fact, during the non linear analysis the degrees of freedom of the form (i.e., the force densities), are not kept constant. In our case, on the contrary, the degrees of freedom for the form retain their prescribed values in the final exact equilibrated configuration. In the paper are used the exact expressions of the vertical nodal forces and of the length of the cable, see Peyrot and Goulois (1979) and Jayaraman and Knudson (1981). It is shown that the proposed method yields significant differences in the initial form when the weight of the cables is not negligible compared to other methods.

In the paper we first present the basic equations of the heavy cable and we obtain the exact expressions for the length of the cable and its end forces (Sections 2 through 4). Then we present the standard FDM and its improvements for obtaining an exactly equilibrated configuration (Section 5). Section 6 illustrates the use of the method with some examples, comparing the results with those obtained using standard FDM.

2. Equilibrium equations for cable elements

2.1. Variational principle of a cable element

Let $\boldsymbol{p} = \boldsymbol{p}(s)$ be the parametric configuration of the cable at a generic instant, with s the arc-length. The tangent space $T_{\boldsymbol{p}}\mathcal{B}_{\sqcup}$ at point $\boldsymbol{p}$ is generated by the unitary triad constituted by the tangent vector $\hat{\boldsymbol{t}} = \partial_s \boldsymbol{p}$, the unit $\hat{\boldsymbol{n}} = \frac{\partial_s \hat{\boldsymbol{t}}}{\|\partial_s \hat{\boldsymbol{t}}\|}$ and the unit bi-normal vector $\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$. We denote with $\boldsymbol{\tau}$ the resultant of the component along $\hat{\boldsymbol{t}}$ of the stress vectors associated to $\hat{\boldsymbol{t}}$, defined by $\boldsymbol{\tau} = \tau \hat{\boldsymbol{t}}$. Indicating with L the current length of the cable for any virtual displacement $\boldsymbol{v}$ the principle of virtual work is given by

$$\int_0^L \boldsymbol{\tau} \cdot \partial_s(\boldsymbol{v})\, ds = \int_0^L \boldsymbol{q} \cdot \boldsymbol{v}\, ds + \mathbf{F}_0 \cdot \boldsymbol{v}_0 + \mathbf{F}_L \cdot \boldsymbol{v}_L \tag{1}$$

integrating the first term of Eq. (1) we have

$$[\boldsymbol{\tau} \cdot \boldsymbol{v}]_0^L - \int_0^L \partial_s(\boldsymbol{\tau}) \cdot \boldsymbol{v}\, ds = \int_0^L \boldsymbol{q} \cdot \boldsymbol{v}\, ds + \mathbf{F}_0 \cdot \boldsymbol{v}_0 + \mathbf{F}_L \cdot \boldsymbol{v}_L. \tag{2}$$

The field equations in $[0,L]$ is

$$-\partial_s(\boldsymbol{\tau}) = \boldsymbol{q} \tag{3}$$

and the boundary conditions are

$$\begin{aligned} &-\boldsymbol{\tau}(0) = \mathbf{F}_0 \quad \text{or} \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \\ &\boldsymbol{\tau}(L) = \mathbf{F}_L \quad \text{or} \quad \boldsymbol{v}(L) = \boldsymbol{v}_L. \end{aligned} \tag{4}$$

From the last condition the boundary forces must be tangent to the configuration of the cable.

2.2. Intrinsic representation of the equilibrium equations

Projecting the equilibrium Eq. (3) in the intrinsic tangent space we have

$$-\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{t}} = \boldsymbol{q} \cdot \hat{\boldsymbol{t}} = q_{\hat{t}}, \quad -\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = \boldsymbol{q} \cdot \hat{\boldsymbol{n}} = q_{\hat{n}}, \quad -\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = \boldsymbol{q} \cdot \hat{\boldsymbol{n}} = q_{\hat{b}}. \tag{5}$$

Using Frenet's formula and considering that $\boldsymbol{\tau} = \tau \hat{\boldsymbol{t}}$ the component of $grad_{\|}(\boldsymbol{\tau})$ are

$$\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{t}} = \partial_s \tau, \quad \partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = \tau \chi, \quad \partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = 0, \tag{6}$$

where $\chi = \|\partial_s \hat{\boldsymbol{t}}\|$ is the curvature of the funicular curve.

Finally the intrinsic representation of the equilibrium equations (3) is

$$\begin{aligned} &-\partial_s \tau(s) = q_{\hat{t}}(s), \\ &-\tau(s)\chi(s) = q_{\hat{n}}(s), \\ &q_{\hat{b}}(s) = 0, \end{aligned} \tag{7}$$

with the boundary conditions

$$\begin{aligned} &-\boldsymbol{\tau}(0) = \mathbf{F}_0 \quad \text{or} \quad \boldsymbol{v}(0) = \boldsymbol{v}_0, \\ &\boldsymbol{\tau}(L) = \mathbf{F}_L \quad \text{or} \quad \boldsymbol{v}(L) = \boldsymbol{v}_L. \end{aligned} \tag{8}$$

2.3. Cartesian representation of the equilibrium equations

Projecting the equilibrium Eq. (3) on the Euclidean spatial frame we obtain, (noting that $\partial_s \boldsymbol{\tau} \cdot \boldsymbol{e}_i = \partial_s(\boldsymbol{\tau} \cdot \boldsymbol{e}_i)\ \forall i = 1,2,3.$)

$$-\partial_s(\tau \hat{\boldsymbol{t}} \cdot \boldsymbol{e}_x) = \boldsymbol{q} \cdot \boldsymbol{e}_x, \quad -\partial_s(\tau \hat{\boldsymbol{t}} \cdot \boldsymbol{e}_y) = \boldsymbol{q} \cdot \boldsymbol{e}_y, \quad -\partial_s(\tau \hat{\boldsymbol{t}} \cdot \boldsymbol{e}_z) = \boldsymbol{q} \cdot \boldsymbol{e}_z \tag{9}$$

and remembering the definition of the tangent vector $\hat{\boldsymbol{t}} = \frac{\partial x}{\partial s}\boldsymbol{e}_x + \frac{\partial y}{\partial s}\boldsymbol{e}_y + \frac{\partial z}{\partial s}\boldsymbol{e}_z$ we obtain

$$\begin{aligned} &-\tfrac{\partial}{\partial s}\left(\tau(s)\tfrac{\partial x}{\partial s}(s)\right) = q_x(s), \\ &-\tfrac{\partial}{\partial s}\left(\tau(s)\tfrac{\partial y}{\partial s}(s)\right) = q_y(s), \\ &-\tfrac{\partial}{\partial s}\left(\tau(s)\tfrac{\partial z}{\partial s}(s)\right) = q_z(s). \end{aligned} \tag{10}$$

The projections of the internal traction stress resultant $\boldsymbol{\tau}$ along the Cartesian directions are usually called thrust and shears

$$\begin{aligned} &\mathcal{H}(s) = \boldsymbol{\tau}(s) \cdot \boldsymbol{e}_x = \tau(s)\tfrac{\partial x}{\partial s}(s), \\ &\mathcal{K}(s) = \boldsymbol{\tau}(s) \cdot \boldsymbol{e}_y = \tau(s)\tfrac{\partial y}{\partial s}(s), \\ &\mathcal{V}(s) = \boldsymbol{\tau}(s) \cdot \boldsymbol{e}_z = \tau(s)\tfrac{\partial z}{\partial s}(s). \end{aligned} \tag{11}$$

Using the definitions (11) the Cartesian equilibrium equations (10) assume the compact form

$$-\partial_s \mathcal{H}(s) = q_x(s), \quad -\partial_s \mathcal{K}(s) = q_y(s), \quad -\partial_s \mathcal{V}(s) = q_z(s). \tag{12}$$

By a first integration along s we have

$$\begin{aligned} &\mathcal{H}(s) = \mathcal{H}_0 - \int_0^s q_x(s) ds, \\ &\mathcal{K}(s) = \mathcal{K}_0 - \int_0^s q_y(s) ds, \\ &\mathcal{V}(s) = \mathcal{V}_0 - \int_0^s q_z(s) ds, \end{aligned} \tag{13}$$

where we have indicated $\mathcal{H}_0 = \mathcal{H}(0),\ \mathcal{V}_0 = \mathcal{V}(0),\ \mathcal{K}_0 = \mathcal{K}(0)$. A new integration along s yields the parametric representation of the funicular configuration

$$x(s) = \int_0^s \frac{\mathcal{H}_0 - \int_0^s q_x(s)ds}{\tau(s)}\, ds + x(0),$$
$$y(s) = \int_0^s \frac{\mathcal{K}_0 - \int_0^s q_y(s)ds}{\tau(s)}\, ds + y(0),$$
$$z(s) = \int_0^s \frac{\mathcal{V}_0 - \int_0^s q_z(s)ds}{\tau(s)}\, ds + z(0), \tag{14}$$

where the tangent component of the resultant stress traction is defined by

$$\tau(s) = \sqrt{\left(\mathcal{H}_0 - \int_0^s q_x ds\right)^2 + \left(\mathcal{K}_0 - \int_0^s q_y ds\right)^2 + \left(\mathcal{V}_0 - \int_0^s q_z ds\right)^2}. \tag{15}$$

3. Formulation of the elastic catenary element

In this section, the simplification of the equilibrium equation to the case of an elastic catenary obeying Hooke's law is shown, suspended at its ends and subjected only to its self weight. A discussion on a wide variety of elastic catenaries can be found in Ahmadi-Kashani and Bell (1981), Tibert (1998), Jayaraman and Knudson (1981), Peyrot and Goulois (1979) and Irvine (1982).

3.1. Assumptions

The basic hypotheses of the present formulation are:

1. Small strains only are considered (but large displacements).
2. Linear-elastic constitutive behaviour only is considered ($\tau = EA_0\,\varepsilon$).
3. Conservation of mass of the cable element during the deformation process is assumed, i.e. the value of the weight per unit-length varies in agreement with the mass conservation (the associated catenary model Ahmadi-Kashani and Bell (1981) is considered).
4. Bending stiffness is neglected.
5. Only the distributed vertical load (along the z direction) due to self weight is considered, so that the geometry of the configuration of the cable is plane. These hypotheses define the elastic catenary element.

3.2. Equations of the elastic cable element

A total Lagrangian approach is used. As reference configuration we adopt the inextensible catenary configuration of the cable and we denote with $s_0 \in [0, L_0]$ the arc-length coordinate, referred to the length L_0 of the non-deformed cable.

Since we consider that the only external action is the self weight q_s, along the z-direction, we have from Eq. (13)

$$\mathcal{H}(s) = \mathcal{H}_0, \quad \mathcal{K}(s) = \mathcal{K}_0, \quad -\partial_s \mathcal{V}(s) = q_z(s). \tag{16}$$

Eq. (12) reduce to

$$\tau(s)\frac{\partial x}{\partial s}(s) = \mathcal{H}_0, \quad \tau(s)\frac{\partial y}{\partial s}(s) = \mathcal{K}_0, \quad -\frac{\partial}{\partial s}\left(\tau(s)\frac{\partial z}{\partial s}\right) = q_z(s), \tag{17}$$

with

$$\tau(s_0) = \sqrt{\mathcal{H}_0^2 + \mathcal{K}_0^2 + \left(\mathcal{V}_0 - \int_0^{s_0} q_z \frac{ds}{ds_0} ds_0\right)^2}$$
$$= \sqrt{\mathcal{H}_0^2 + \mathcal{K}_0^2 + \left(\mathcal{V}_0 - \frac{W}{L_0}s_0\right)^2} = \Lambda\sqrt{1 + \left(\frac{V_0 - \frac{W}{L_0}s_0}{\Lambda}\right)^2}, \tag{18}$$

where $\Lambda = \sqrt{\mathcal{H}_0^2 + \mathcal{K}_0^2}$ and W is the total weight of the cable, that by virtue of the mass conservation can be represented $W = \int_0^L q_z ds = \int_0^{L_0} q_z \frac{ds}{ds_0} ds_0 = \int_0^{L_0} q_{z,0} ds_0$. Integrating the previous equations on the Lagrangian configuration we have

$$x(s_0) = \int_0^{s_0} \frac{\mathcal{H}_0}{\tau(s_0)}\frac{ds}{ds_0} ds_0 + x_0,$$
$$y(s_0) = \int_0^{s_0} \frac{\mathcal{K}_0}{\tau(s_0)}\frac{ds}{ds_0} ds_0 + y_0, \tag{19}$$
$$z(s_0) = \int_0^{s_0} \frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\tau(s_0)}\frac{ds}{ds_0} ds_0 + z_0.$$

Considering that $\varepsilon = (\frac{ds}{ds_0} - 1)$ and assuming a linear constitutive relation $\tau = EA_0\left(\frac{ds}{ds_0} - 1\right)$ we have $\frac{ds}{ds_0} = \frac{\tau}{EA_0} + 1$, then equations (19) become

$$x(s_0) = \mathcal{H}_0 \int_0^{s_0} \left(\frac{1}{EA_0} + \frac{1}{\tau(s_0)}\right) ds_0 + x_0,$$
$$y(s_0) = \mathcal{K}_0 \int_0^{s_0} \left(\frac{1}{EA_0} + \frac{1}{\tau(s_0)}\right) ds_0 + y_0, \tag{20}$$
$$z(s_0) = \int_0^{s_0} \frac{V_0 - \frac{W}{L_0}s_0}{\tau(s_0)}\left(\frac{\tau(s_0)}{EA_0} + 1\right) ds_0 + z_0$$

and integrating we have

$$x(s_0) - x_0 = \frac{\mathcal{H}_0 s_0}{EA_0} + \frac{\mathcal{H}_0 L_0}{W}\left(\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0}{\Lambda}\right] - \mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\Lambda}\right]\right), \tag{21}$$

$$y(s_0) - y_0 = \frac{\mathcal{K}_0 s_0}{EA_0} + \frac{\mathcal{K}_0 L_0}{W}\left(\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0}{\Lambda}\right] - \mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\Lambda}\right]\right) \tag{22}$$

and

$$z(s_0) - z_0 = \frac{s_0}{EA_0}\left(\mathcal{V}_0 - \frac{1}{2}\frac{W}{L_0}s_0\right) + \frac{\Lambda L_0}{W}\left[\sqrt{\left(\frac{\mathcal{V}_0}{\Lambda}\right)^2 + 1} - \sqrt{\left(\frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\Lambda}\right)^2 + 1}\right]. \tag{23}$$

The components of the vector joining the ends of the elastic catenary element are described by Eqs. (21)–(23), which are summarized as

$$(\boldsymbol{P} - \boldsymbol{P}_0)\cdot\boldsymbol{e}_x = l_x = x(L_0) - x_0 = f_x(\mathcal{H}_0, \mathcal{K}_0, \mathcal{V}_0, L_0),$$
$$(\boldsymbol{P} - \boldsymbol{P}_0)\cdot\boldsymbol{e}_y = l_y = y(L_0) - y_0 = f_y(\mathcal{H}_0, \mathcal{K}_0, \mathcal{V}_0, L_0), \tag{24}$$
$$(\boldsymbol{P} - \boldsymbol{P}_0)\cdot\boldsymbol{e}_z = h = z(L_0) - z_0 = g(\mathcal{H}_0, \mathcal{K}_0, \mathcal{V}_0, L_0).$$

Fig. 1 shows the nodal forces in the plane of the catenary.

The total length of the deformed catenary is given by the sum of the undeformed length L_0 and the total elongation ΔL

$$L(\Lambda, \mathcal{V}_0, L_0) = L_0 + \Delta L(\Lambda, \mathcal{V}_0, L_0) = L_0 + \int_0^{L_0}\left(\frac{\tau}{EA_0}\right) ds_0. \tag{25}$$

Using the relation (18) and integrating we obtain the expression for the global elongation of the cable that can be formulated in either one of the following equivalent expressions:

$$\Delta L(\Lambda, \mathcal{V}_0, L_0) = \frac{1}{2EAq_z}\left[\mathcal{V}_0\sqrt{\Lambda^2 + \mathcal{V}_0^2} - (V_0 - q_z L_0)\sqrt{\Lambda^2 + (\mathcal{V}_0 - q_z L_0)^2}\right.$$
$$\left. + \Lambda^2 \mathrm{Log}\left(\frac{(\mathcal{V}_0 - q_z L_0) + \sqrt{\Lambda^2 + (\mathcal{V}_0 - q_z L_0)^2}}{\mathcal{V}_0 + \sqrt{\Lambda^2 + \mathcal{V}_0^2}}\right)\right], \tag{26}$$

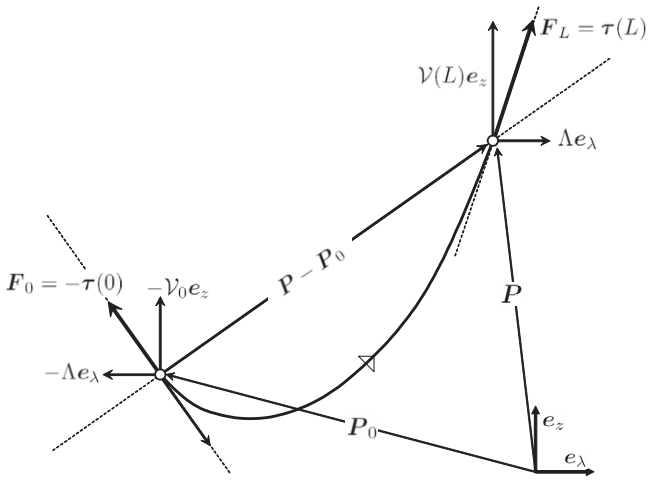

Fig. 1. Representation of the nodal forces in the catenary plane z-λ.

$$\Delta L(\Lambda,\mathcal{V}_0,L_0)=\frac{1}{2EAq_z}\left[\mathcal{V}_0\sqrt{\mathcal{V}_0^2+\Lambda^2}-(\mathcal{V}_0-q_zL_0)\sqrt{(\mathcal{V}_0-q_zL_0)^2+\Lambda^2}\right.$$
$$\left.+\Lambda^2\mathrm{ArcSinh}\left(\frac{\mathcal{V}_0}{\Lambda}\right)-\Lambda^2\mathrm{ArcSinh}\left(\frac{\mathcal{V}_0-q_zL_0}{\Lambda}\right)\right]. \quad (27)$$

In the case of non extensible cable in the equations (19) the ratio $\frac{ds}{ds_0}=1$ so that the parametric equations of the undeformable catenary element are obtained for Eqs. (21)–(23) in the limit $EA_0\to\infty$

$$x(s_0)-x_0=\frac{\mathcal{H}_0L_0}{W}\left(\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0}{\Lambda}\right]-\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0-\frac{W}{L_0}s_0}{\Lambda}\right]\right), \quad (28)$$

$$y(s_0)-y_0=\frac{\mathcal{K}_0L_0}{W}\left(\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0}{\Lambda}\right]-\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0-\frac{W}{L_0}s_0}{\Lambda}\right]\right) \quad (29)$$

and

$$z(s_0)-z_0=\frac{\Lambda L_0}{W}\left[\sqrt{\left(\frac{\mathcal{V}_0}{\Lambda}\right)^2+1}-\sqrt{\left(\frac{\mathcal{V}_0-\frac{W}{L_0}s_0}{\Lambda}\right)^2+1}\right]. \quad (30)$$

4. Vertical forces at the ends of the cable

In this section, explicit formulas for the vertical forces transmitted by the cable to the end nodes are derived. In addition to the exact expressions, approximated ones will also be proposed. These results will be used in the formulations proposed in Section 5.

4.1. Exact catenary element

Squaring and adding the first two of the catenary equilibrium relations (17), we have

$$\frac{d\lambda}{ds}=\frac{\Lambda}{\tau}=\frac{\sqrt{H_0^2+K_0^2}}{\tau},\quad \frac{d\mathcal{V}}{ds}=-q_z, \quad (31)$$

where $ds=\sqrt{d\lambda^2+dz^2}$, with $d\lambda=\sqrt{dx^2+dy^2}$. Manipulating we have

$$\frac{ds}{d\lambda}=\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2},\quad \Lambda=\frac{\tau}{\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}}. \quad (32)$$

Similarly $\frac{d\mathcal{V}}{d\lambda}=-q_z\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}$ and remembering the definition of $\mathcal{V}=\tau\frac{dz}{d\lambda}\frac{d\lambda}{ds}$, we have

$$\frac{d}{d\lambda}\left(\frac{dz}{d\lambda}\frac{\tau}{\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}}\right)=-q_z\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}. \quad (33)$$

Using Eq. (32), considering that Λ is constant, we have an alternative cartesian representation of the equilibrium equation along the z direction

$$\Lambda\frac{d^2z}{d\lambda^2}=-q_z\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}, \quad (34)$$

where $\lambda\in[0,l]$, with $l=\sqrt{l_x^2+l_y^2}$. Letting $\frac{dz}{d\lambda}=f(\lambda)$ the previous equation assumes the form $\frac{df}{d\lambda}=-\frac{q_z}{\Lambda}\sqrt{1+f(\lambda)^2}$ that has the solution

$$\frac{dz}{d\lambda}=-\mathrm{Sinh}\left[\frac{q_z\lambda}{\Lambda}-c_1\right] \quad (35)$$

and observing that $\frac{dz}{d\lambda}=\frac{\mathcal{V}}{\Lambda}$

$$\mathcal{V}(\lambda)=-\Lambda\mathrm{Sinh}\left[\frac{q_z\lambda}{\Lambda}-c_1\right] \quad (36)$$

so it is found that

$$\mathcal{V}_0=\Lambda\,\mathrm{Sinh}[c_1]. \quad (37)$$

Integrating Eq. (35) we obtain the Cartesian representation of the catenary

$$z(\lambda)=-\frac{\Lambda}{q_z}\mathrm{Cosh}\left[\frac{q_z\lambda}{\Lambda}-c_1\right]+c_2, \quad (38)$$

with boundary conditions, for $\lambda=0$ and for $\lambda=l$

$$\begin{aligned} z_0&=-\frac{\Lambda}{q_z}\mathrm{Cosh}[c_1]+c_2 \quad \text{or} \quad \mathcal{V}(0)=\Lambda\frac{dz}{d\lambda}\Big|_{\lambda=0}=\mathcal{V}_0,\\ z(l)&=-\frac{\Lambda}{q_z}\mathrm{Cosh}\left[\frac{q_zl}{\Lambda}-c_1\right]+c_2 \quad \text{or} \quad \mathcal{V}(l)=\Lambda\frac{dz}{d\lambda}\Big|_{\lambda=l}=\mathcal{V}_{\mathcal{L}}. \end{aligned} \quad (39)$$

In the case of fixed supports, subtracting the first from the second equation and introducing the dimensionless parameter

$$\eta=\frac{q_zl}{2\Lambda}, \quad (40)$$

we have

$$\begin{aligned} h=z(l)-z_0&=-\frac{\Lambda}{q_z}\left[\mathrm{Cosh}\left[\frac{q_zl}{\Lambda}-c_1\right]-\mathrm{Cosh}[c_1]\right]\\ &=l\left(\frac{2\Lambda}{q_zl}\right)\mathrm{Sinh}\left[c_1-\frac{q_zl}{2\Lambda}\right]\mathrm{Sinh}\left[\frac{q_zl}{2\Lambda}\right]=\frac{l}{\eta}\mathrm{Sinh}[c_1-\eta]\,\mathrm{Sinh}[\eta] \end{aligned} \quad (41)$$

From the last relation and the first of Eq. (39) the constants c_1 and c_2 are obtained

$$c_1=\mathrm{Sinh}^{-1}\left[\frac{\eta}{\mathrm{Sinh}[\eta]}\frac{h}{l}\right]+\eta \quad (42)$$

and

$$c_2=z_0+\frac{l}{2\eta}\left[\mathrm{Cosh}[\eta]\sqrt{1+\left(\frac{\eta}{\mathrm{Sinh}[\eta]}\frac{h}{l}\right)^2}+\frac{\eta h}{l}\right]. \quad (43)$$

Note that, since $\eta>0$, for any value of η, h and l, c_1 is a positive constant. Therefore the equation of the catenary is

$$z(\lambda)=z_0+\frac{1}{q_z}\left(2\Lambda\mathrm{Sinh}\left[\frac{\eta\lambda}{l}\right]\mathrm{Sinh}\left[\eta\left(1-\frac{\lambda}{l}\right)+\mathrm{ArcSinh}\left[\frac{\eta h}{l}\mathrm{Csch}[\eta]\right]\right]\right). \quad (44)$$

The length of the catenary, (for the deformable and the undeformable case), is given by the relation $L=\int_0^l\sqrt{1+\left(\frac{dz}{d\lambda}\right)^2}d\lambda$, where l is the horizontal span of the catenary. From the equality $\frac{dz}{d\lambda}=\frac{\mathcal{V}(\lambda)}{\Lambda}=-\mathrm{Sinh}\left[\frac{q_z\lambda}{\Lambda}-c_1\right]$ and using the expression (42), after some manipulation we have

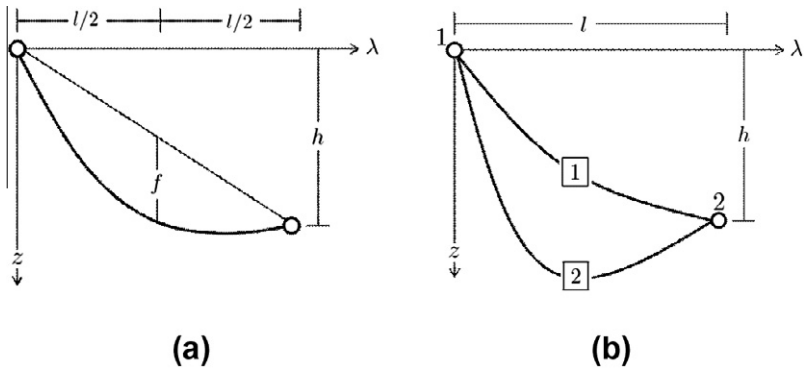

Fig. 2. Definition of the sag ratio for the catenary (a), different configurations of the catenary (b).

$$L^2 = (L_0 + \Delta L)^2 = \frac{l^2}{\eta^2}\mathrm{Sinh}^2[\eta] + h^2. \tag{45}$$

In the case of the undeformable cable $\Delta L = 0$ then we have

$$L_0^2 = \frac{l^2}{\eta^2}\mathrm{Sinh}^2[\eta] + h^2. \tag{46}$$

The vertical forces at the extremities of the cable are obtained from Eq. (37), that, inserting the expression (46) for the length, can be written as

$$\mathcal{V}_0 = \frac{q_z L_0}{2} + \frac{q_z h}{2}\frac{\mathrm{Cosh}[\eta]}{\mathrm{Sinh}[\eta]} \tag{47}$$

and the shear at the second extremity is given by

$$\mathcal{V}(L_0) = -\frac{q_z L_0}{2} + \frac{q_z h}{2}\frac{\mathrm{Cosh}[\eta]}{\mathrm{Sinh}[\eta]}. \tag{48}$$

Similar relations can be found in Tibert (1998), Jayaraman and Knudson (1981), Peyrot and Goulois (1979), Ahmadi-Kashani and Bell (1981) and Ahmadi-Kashani (1988).

The dimensionless parameter η is related to the sag of the cable, that can be defined as the ratio f/l between the sag related to the chord and the horizontal span of the cable (see Fig. 2(a)). Since $f = z(\frac{l}{2}) - \frac{h}{2}$ from Eq. (44) it is readily found

$$\frac{f}{l} = \frac{1}{\eta}\left(\sinh\frac{\eta}{2}\sinh\left[\frac{\eta}{2} + \mathrm{ArcSinh}\left[\frac{\eta h}{l}\mathrm{Csch}\eta\right]\right]\right) - \frac{1}{2}\frac{h}{l}. \tag{49}$$

In Fig. 3 is plotted the sag ratio against η, for some values of $\frac{h}{l}$.

For fixed h and l the cable can assume either of the two configurations shown in Fig. 2(b) that are characterized by having the tangents at the extremities of the same sign, or of opposite signs. The former case occurs when the parameter η is such that

$$\frac{\cosh^2\eta}{\sinh^2\eta} - \left(\frac{l}{h}\right)^2\frac{\sinh^2\eta}{\eta^2} - 1 > 0. \tag{50}$$

In this case the maximum axial force in the cable occurs at the extremity, and is equal to

$$\tau_{max} = \tau(0) = \sqrt{\Lambda^2 + \mathcal{V}_0^2} = \Lambda\sqrt{1 + \eta^2\left(\frac{L_0}{l} + \frac{h}{l}\coth\eta\right)^2}. \tag{51}$$

In the latter case, the maximum axial force is equal to Λ and it occurs at the point of abscissa

$$\frac{s_0}{l} = \frac{1}{2}\left(\frac{L_0}{l} + \frac{h}{l}\coth\eta\right). \tag{52}$$

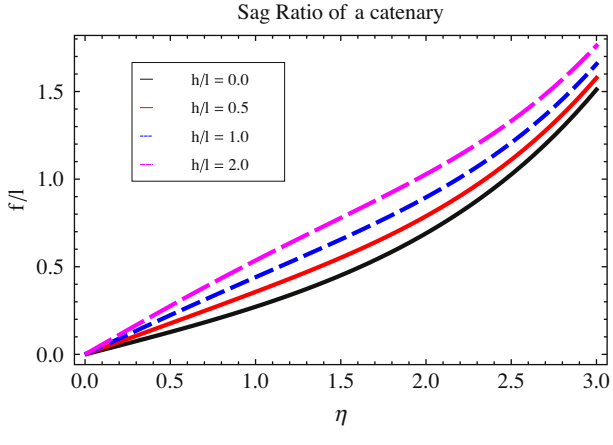

Fig. 3. Sag ratio for a catenary as function of the parameter η.

4.2. Approximated parabolic element

From expressions (44), (47) and (48) approximated forms of the relevant parameters of the cable can be obtained. The solution of the catenary equation depends on the parameter $\eta = \frac{q_z l}{2\Lambda}$, the ratio between the weight of the cable and the horizontal thrust. Then in the limit as $\eta \to 0$ we can expand expressions (44) and (36) in Taylor series at the first order in η

$$\mathrm{lin}[z(\lambda)] = z_0 + d_\eta(z(\lambda)) = z_0 + h\frac{\lambda}{l} + \eta\sqrt{l^2 + h^2}\left(1 - \frac{\lambda}{l}\right)\frac{\lambda}{l}, \tag{53}$$

$$\mathrm{lin}[\mathcal{V}(\lambda)] = \mathcal{V}(\lambda)|_{\eta=0} + d_\eta\mathcal{V}(\lambda) = \frac{h}{l}\Lambda + \frac{\sqrt{l^2 + h^2}}{l}\Lambda\left(1 - 2\frac{\lambda}{l}\right)\eta. \tag{54}$$

The shears at the ends of the cable are then given by

$$\mathrm{lin}[\mathcal{V}_0] = \frac{h}{l}\Lambda + \frac{\sqrt{l^2 + h^2}}{l}\Lambda\eta = \frac{h}{l}\Lambda + \frac{q_z\sqrt{l^2 + h^2}}{2}, \tag{55}$$

$$\mathrm{lin}[\mathcal{V}_{L_0}] = \frac{h}{l}\Lambda - \frac{\sqrt{l^2 + h^2}}{l}\Lambda\eta = \frac{h}{l}\Lambda - \frac{q_z\sqrt{l^2 + h^2}}{2}. \tag{56}$$

Observation. The results (53)–(56) can be obtained linearizing the catenary Eq. (34) for small sagging of the cable, in which case we have:

$$-\frac{d^2z}{d\lambda^2} = \frac{q_z}{\Lambda}\sqrt{\frac{l^2 + h^2}{l^2}} = \frac{q_z}{\Lambda}\frac{k}{l} = 2\eta\frac{k}{l^2} \tag{57}$$

the solution of which can be expressed in the parametric form

$$z(\lambda) = -\eta k\left(\frac{\lambda}{l}\right)^2 + (h + \eta k)\frac{\lambda}{l} + z_0. \tag{58}$$

Similarly for the length of the cable we have

$$\mathrm{lin}[L] = \sqrt{l^2 + h^2}. \tag{59}$$

In this work we have also used a second order approximated parabolic model developed by Deng et al. (2005) in which for the length of the parabola, in place of Eq. (45) the current length of the parabola itself is used, i.e.

$$L = \int_0^l \sqrt{1 + \left(\frac{dz}{d\lambda}\right)^2}\, d\lambda \tag{60}$$

in which $z(\lambda)$ is given by the expression (58). Performing the integral the length assumes the form

$$L = \frac{l}{4k\eta}\left[\sqrt{\frac{l^2 + (h - k\eta)^2}{l^2}}(k\eta - h) + \sqrt{\frac{l^2 + (h + k\eta)^2}{l^2}}(k\eta + h) + l\,\mathrm{Arcsinh}\left(\frac{h + k\eta}{l}\right) - l\,\mathrm{Arcsinh}\left(\frac{h - k\eta}{l}\right)\right] \tag{61}$$

In this approximation the shear components at the ends are given by the same expressions as (55) and (56)

$$\mathcal{V}_0 = \frac{q_z L_0}{2} + \Lambda\frac{h}{l}, \quad \mathcal{V}(L_0) = -\frac{q_z L_0}{2} + \Lambda\frac{h}{l}. \tag{62}$$

In the case of a deformable parabolic element the length becomes

$$L = L_0 + \Delta L, \quad \Delta L = \frac{1}{EA}\int_0^{L_0} \sqrt{\Lambda^2 + (\mathcal{V}_0 - q_z s_0)^2}\, ds_0, \tag{63}$$

while in the case of undeformable parabola $\Delta L = 0$ then $L = L_0$, i.e. the current length is the undeformable length.

4.3. The straight cable element

If $q_z \to 0$, i.e. light cable net, the equilibrium equation becomes

$$\frac{d^2 z}{d\lambda^2} \approx 0, \quad z(\lambda) = \frac{h}{l}\lambda + z_0, \quad \mathcal{V}_0 = \mathcal{V}(l) = \Lambda\frac{h}{l}, \tag{64}$$

that is, a straight truss is recovered.

5. The force density method

The force density method was developed by Schek (1974) who successively developed the constrained force density method. He considered weightless cables, so that they could be approximated with truss elements (approximation (64)), ad demonstrated that the form of the net could be obtained directly solving the linear equilibrium equations in the unknowns positions of the nodes, using as degrees of freedom of the form the ratios T_i/k_i (T_i being the axial force in the truss), called force density of the element. He proved that the procedure yields a set of minimal length if the axial forces T_i are taken equal in all the branches.

Later Haber and Abel (1982) pointed out that the force density corresponds to the initial geometric stiffness of the truss, clarifying the interpretation of the axial force T_i as prestress. Bletzinger and Ramm (1999) and Wüchner and Bletzinger (2005) generalized the idea of Schek to the case of membranes, using as parameter for the form finding the second Piola–Kirchhoff stresses, that are iteratively adjusted to leading the prescribed Chachy stresses. They proved that a uniform isotropic Cauchy stress state leads to membranes of minimal surface. The method was then extended to non isotropic stress states for improving the shape of the membrane. Bletzinger et al. (2005) also studied the effects of self weight adding an elastic stress to the prestress. These procedures were particularized to the case of cables, using the straight element approximation. In the latter case, the self weight of the cables are imposed as external loads on the form previously obtained.

In this work we propose a generalization of the form finding procedure to the case of heavy cables, that is, to the case of considerably slack cables, using the exact solution for heavy cables (catenary). The solution sought in this way is an exact one, so it can be used as starting point of an incremental analysis. Since the equilibrium equations become non linear in the node coordinates, the solution is sought by means of iterative techniques. At the end of the paper we will discuss how the present method can also be used for obtaining nets with uniform thrusts.

In this section, starting from the equilibrium equations of the net, first the standard FDM, will be recalled, then two non linear implementations similar to the one proposed by Haber and Abel (1982) will be outlined, and finally the new proposal will be presented.

Let i be the generic free node of the net, identified by the (unknown) position vector $\boldsymbol{P}_i$. Let r be the number of cable elements

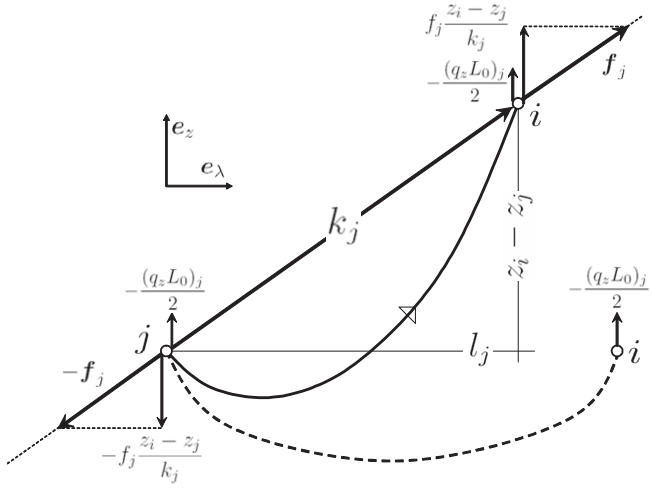

Fig. 4. Representation of the effective traction force $\boldsymbol{f}_j$ in the plane z-λ in the case of the truss, parabolic and catenary element.

Table 1
Fixed node.

	x [m]	y [m]	z [m]
$\boldsymbol{P}_1$	0	0	0
$\boldsymbol{P}_2$	1	0	0
$\boldsymbol{P}_4$	0	1	0
$\boldsymbol{P}_6$	1	1	1

Table 2
Coordinates of the free nodes, case $Q_\Lambda = 1$ [daN/m].

[daN/m]		Node	x [m]	y [m]	z [m]
$q_z = 0$	FDM	3	0.5	0.25	0.125
		5	0.5	0.75	0.375
$q_z = 1$	nl-FDM	3	0.5	0.25	−0.381649
		5	0.5	0.75	−0.202515
	P-FDM	3	0.5	0.25	−0.402939
		5	0.5	0.75	−0.223149
	C-FDM	3	0.5	0.25	−0.348097
		5	0.5	0.75	−0.161213
$q_z = 1.5$	nl-FDM	3	0.5	0.25	−0.983278
		5	0.5	0.75	−0.868457
	P-FDM	3	0.5	0.25	−1.11243
		5	0.5	0.75	−0.996926
	C-FDM	3	0.5	0.25	−0.693050
		5	0.5	0.75	−0.542846

attached to the ith node and, as done previously, indicate by $k_j = \|\boldsymbol{P}_i - \boldsymbol{P}_j\|$ the length of the segment joining the element ends.

The forces acting at the ith extremity of the cable have the components $\mathcal{H}, \mathcal{K}, \mathcal{V}$. Recalling that $\Lambda = \|\mathcal{H}_0 \boldsymbol{e}_x + \mathcal{K}_0 \boldsymbol{e}_y\|$, and using the expressions for the shear found previously (Eqs. (64), (55), (47) for the straight cable approximation, parabolic approximation, exact catenary respectively), the cartesian projection of the equilibrium equations of the ith node are

$$\begin{aligned}
&\sum_{j=1}^{r} \Lambda_j \frac{x_i - x_j}{l_j} = p_{x,i},\\
&\sum_{j=1}^{r} \Lambda_j \frac{y_i - y_j}{l_j} = p_{y,i},\\
&\sum_{j=1}^{r} \underbrace{-\delta_c \frac{(q_z L_0)_j}{2} + f_j \frac{z_i - z_j}{k_j}}_{\mathcal{V}(L_0)_j} = p_{z,i},
\end{aligned} \tag{65}$$

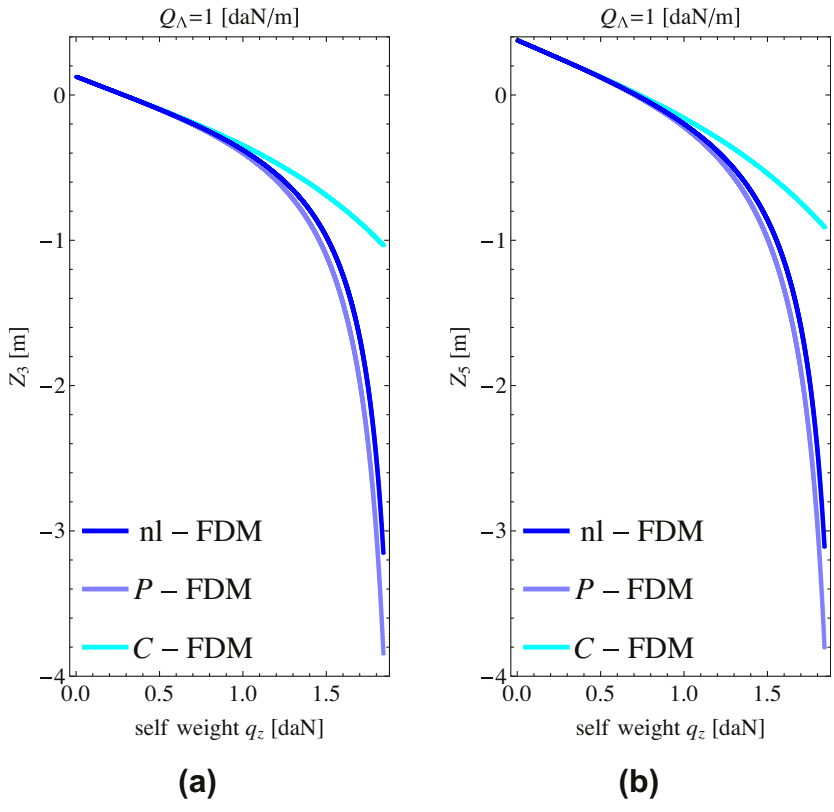

Fig. 5. Z-coordinate respectively of node-3 (a) and of node-5 (b) for increasing self weight of the cables.

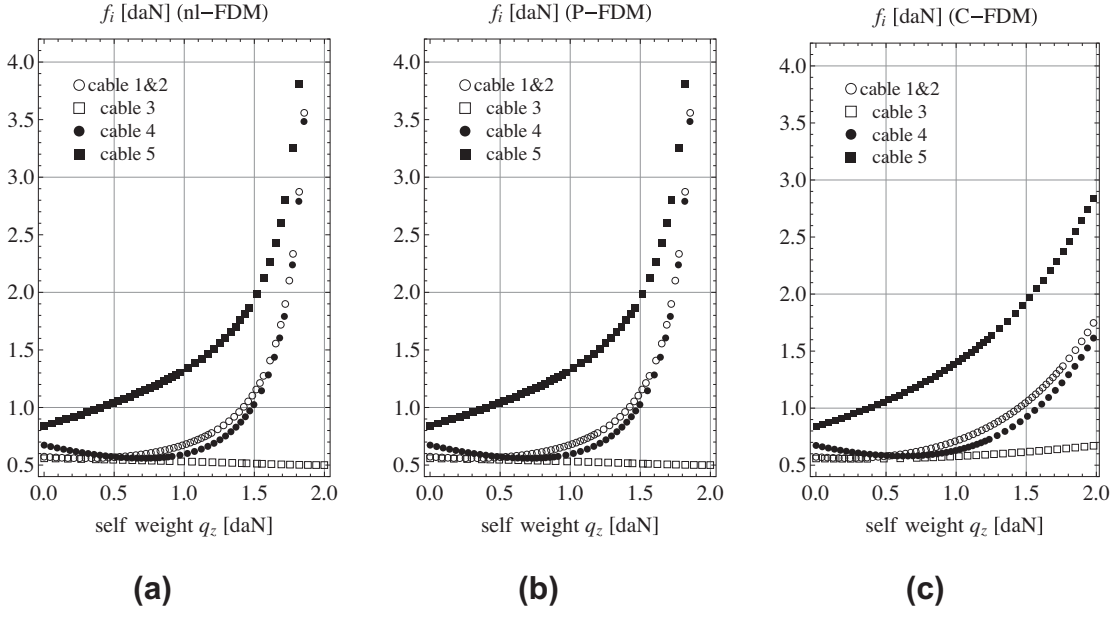

Fig. 6. Values of f_i for each cable, in the case of nl-FDM (a), P-FDM (b), ad C-FDM (c). $Q_\Lambda = 1$ [daN/m], $\eta = q_z/(2Q_\Lambda)$, see Fig. 7.

where the force f_j is shown in Fig. 4 and is given for the truss, parabolic and catenary elements, respectively, by

$$f_{j_t} = \Lambda_j \frac{k_j}{l_j}, \quad f_{j_p} = \Lambda_j \frac{k_j}{l_j}, \quad f_{j_c} = \Lambda_j \frac{k_j}{l_j} \eta_j \frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]} \tag{66}$$

and in the truss approximation the self weight is omitted ($\delta_c = 0$ for the truss, $\delta_c = 1$ otherwise).

In Eq. (65) there appear two force density quantities, the ratio $Q_{\Lambda_j} = \frac{\Lambda_j}{l_j}$ and the ratio $Q_{\mathcal{V}_j} = \frac{f_j}{k_j}$, that by means of definitions (66) is given for the truss, parabolic and catenary element, respectively, by

$$Q_{\mathcal{V}_j,t} = \frac{\Lambda_j}{l_j}, \quad Q_{\mathcal{V}_j,p} = \frac{\Lambda_j}{l_j}, \quad Q_{\mathcal{V}_j,c} = \frac{\Lambda_j}{l_j} \eta_j \frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]}. \tag{67}$$

In this way we get three versions of the FDM. The truss approximation is the standard FDM in which the length $L_0 = k = \sqrt{l^2 + h^2}$, and the self weight is neglected. In the case of small but finite self weight, we obtain the (P-FDM) parabolic form of the force density method in which the length L_0 can assume any of the forms (60) or (61). In these two force density methods appear only one kind of force density. Finally in the case of the catenary force density method (C-FDM), we have a new kind of force density, that contains the dimensionless parameter η.

The standard linear FDM. In the linear FDM we assign the force densities $Q_{\Lambda_j} = \frac{\Lambda_j}{l_j}$ and let $q_{z_j} = 0$ everywhere; in this manner the Eqs. (65) reduce to a set of linear equations

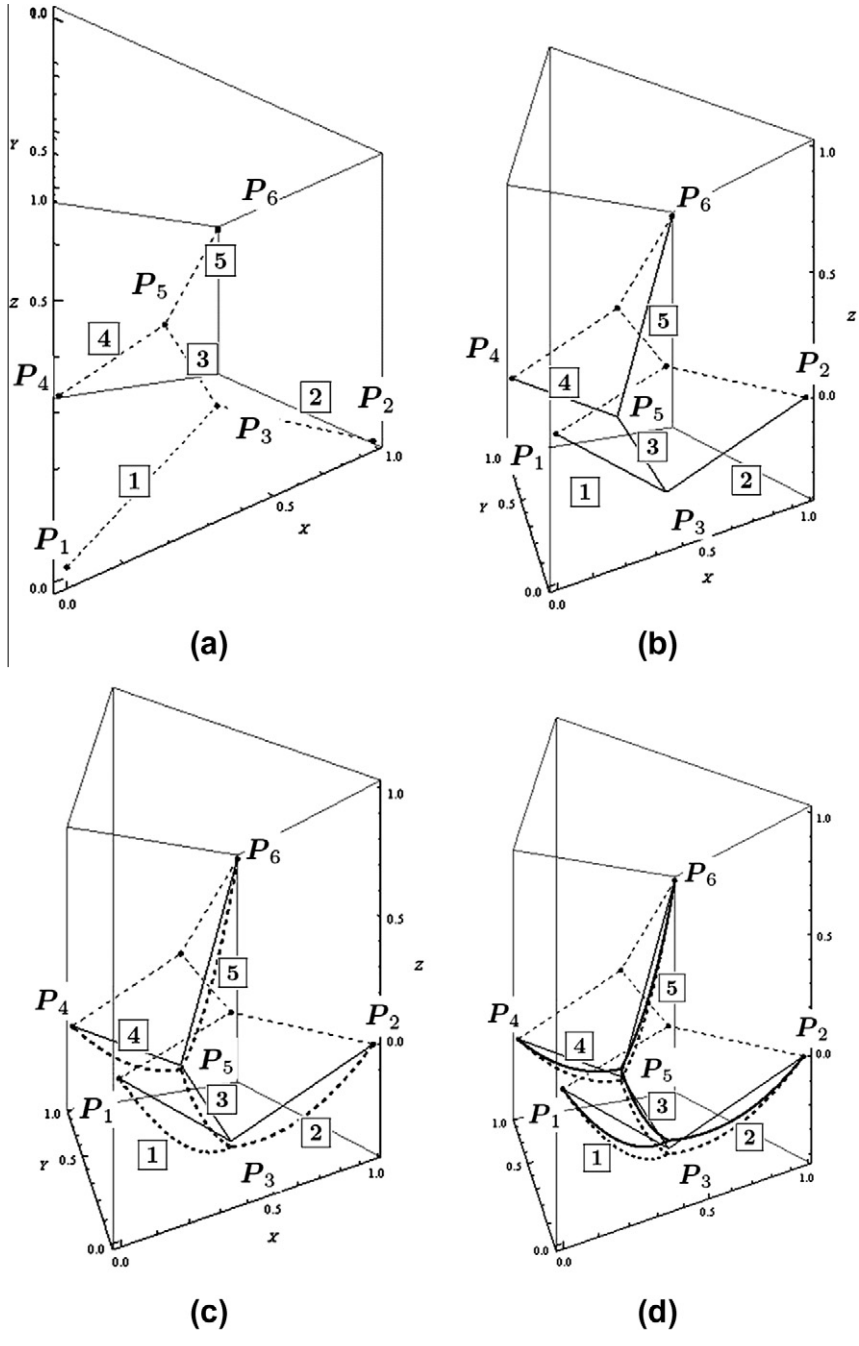

Fig. 7. Graphics representation of the configuration of a 5-cables net obtained for a self weight $q_{z_j} = 1\,[\mathrm{daN/m}],\ j = 1, 2, n$; in (a) is plotted the solution of the linear force density method, (FDM), in (b) is plotted the solution of the non linear force density method, (nlFDM), in (c) is plotted the solution of the parabolic force density method, (P-FDM) and in (d) is plotted the solution of the catenary force density method, (C-FDM).

Table 3
Forces and length of the cable for $q_z = 1$ and $q_z = 1.5$ [daN/m], case $Q_v = 1$[daN/m].

		El.	f [daN]	$\mathcal{H}$ [daN]	$\mathcal{K}$ [daN]	$\mathcal{V}_0$ [daN]	$\mathcal{V}(L_0)$ [daN]	Λ [daN]	L_0 [m]
$q_z = 0$ [daN/m]	FDM	1	0.572822	0.5	0.25	0.125	–	0.559017	0.572822
		2	0.572822	−0.5	0.25	0.125	–	0.559017	0.572822
		3	0.559017	0.0	−0.50	−0.250	–	0.500000	0.559017
		4	0.673146	0.5	−0.25	0.375	–	0.559017	0.673146
		5	0.838525	−0.5	−0.25	−0.625	–	0.559017	0.838525
$q_z = 1$ [daN/m]	nl-FDM	1	0.676872	0.5	0.25	−0.720085	–	0.559017	0.676872
		2	0.676872	−0.5	0.25	−0.720085	–	0.559017	0.676872
		3	0.531121	0.0	−0.50	−0.444695	–	0.500000	0.531121
		4	0.594569	0.5	−0.25	−0.499799	–	0.559017	0.594569
		5	1.326100	−0.5	−0.25	−1.86556	–	0.559017	1.326100
	P-FDM	1	0.689101	0.5	0.25	−0.757741	−0.036493	0.559017	0.709604
		2	0.689101	−0.5	0.25	−0.757741	−0.036493	0.559017	0.709604
		3	0.531342	0.0	−0.50	−0.455852	0.103040	0.500000	0.552125
		4	0.601910	0.5	−0.25	−0.535596	0.097506	0.559017	0.624894
		5	1.344840	−0.5	−0.25	−1.901150	−0.535086	0.559017	1.356000
	C-FDM	1	0.712522	0.5	0.25	−0.715969	−0.037296	0.559017	0.678673
		2	0.712522	−0.5	0.25	−0.715969	−0.037296	0.559017	0.678673
		3	0.577542	0.0	−0.5	−0.479001	0.074592	0.500000	0.553594
		4	0.629492	0.5	−0.25	−0.476676	0.127819	0.559017	0.604496
		5	1.394410	−0.5	−0.25	−1.905990	−0.606821	0.559017	1.299170
$q_z = 1.5$ [daN/m]	nl-FDM	1	1.131080	0.5	0.25	−1.831590	–	0.559017	1.131080
		2	1.131080	−0.5	0.25	−1.831590	–	0.559017	1.131080
		3	0.513014	0.0	−0.5	−0.499581	–	0.500000	0.513014
		4	1.032820	0.5	−0.25	−1.643070	–	0.559017	1.032820
		5	1.950290	−0.5	−0.25	−3.33118	–	0.559017	1.950290
	P-FDM	1	1.244987	0.5	0.25	−2.070830	−0.125399	0.559017	1.277870
		2	1.244987	−0.5	0.25	−2.070830	−0.125399	0.559017	1.277870
		3	0.513167	0.0	−0.5	−0.539051	0.318595	0.500000	0.564733
		4	1.142962	0.5	−0.25	−1.880560	−0.083488	0.559017	1.178180
		5	2.073696	−0.5	−0.25	−3.568090	−0.405560	0.559017	2.094890
	C-FDM	1	1.051410	0.5	0.25	−1.512270	−0.124474	0.559017	0.921960
		2	1.051410	−0.5	0.25	−1.512270	−0.124474	0.559017	0.921960
		3	0.616478	0.0	−0.5	−0.603676	0.248948	0.500000	0.568410
		4	0.920121	0.5	−0.25	−1.255070	−0.026944	0.559017	0.818750
		5	1.937730	−0.5	−0.25	−3.066693	−0.576732	0.559017	1.660130

$$\sum_{j=1}^{r} Q_{\Lambda_j}(x_i - x_j) = p_{x,i},$$
$$\sum_{j=1}^{r} Q_{\Lambda_j}(y_i - y_j) = p_{y,i}, \tag{68}$$
$$\sum_{j=1}^{r} Q_{\Lambda_j}(z_i - z_j) = p_{z,i}.$$

Note that in this case the cable reduces to a truss element, so that

$$Q_{\Lambda_j} = \frac{\Lambda_j}{l_j} = \frac{f_j}{k_j} = \frac{T_j}{k_j}. \tag{69}$$

Solving the Eqs. (68) we obtain for each free node j an initial position $\{x_0, y_0, z_0\}_j^{LFDM}$, from which is possible to define the linear length L_0^{LFDM} of each cables. An usual strategy adopted is to choose Q_{Λ_j} constant everywhere except in the boundary cables, where it is chosen one order of magnitude larger.

The nonlinear standard FDM. The previous solution can be used to initialize the non linear force density method (nlFDM) defined by the equations

$$\sum_{j=1}^{r} \frac{\Lambda_j}{l_j}(x_i - x_j) = p_{x,i},$$
$$\sum_{j=1}^{r} \frac{\Lambda_j}{l_j}(y_i - y_j) = p_{y,i}, \tag{70}$$
$$\sum_{j=1}^{r} -\frac{q_{z_j} L_{0_j}}{2} + \frac{\Lambda_j}{l_j}(z_i - z_j) = p_{z,i},$$

where the conditions on the lengths are defined only by the relative positions of the free nodes by means of the relations

$$L_{0_j}^2 = l_j^2 + h_j^2 \tag{71}$$

with auxiliary conditions on the force densities

$$Q_{\Lambda_j} = \frac{\Lambda_j}{l_j}. \tag{72}$$

We have $3j + 2n$ equations in $3j + 2n$ variables, the $3j$ equilibrium equations, with the n conditions on the length and n conditions on the force densities Q_{Λ_j}, in the $3j$ independent variables $\{x_j, y_j, z_j\}$, the n independent variables $\{L_{0j}\}$ and the n variables $\{\Lambda_j\}$. We adopt a Newton–Raphson strategy to solve these equations, in which the initial solution is represented by the LFDM-solution. The solution of the nlFDM is represented by the set of values $\{x_j, y_j, z_j\}^{nlFDM}$, $\{L_{0j}\}^{nlFDM}$ and $\{\Lambda_j\}^{nlFDM}$.

The (non linear) parabolic FDM. The parabolic force density method is defined by the equilibrium equations (70) in which the length of the element coincides with the length of the parabola (61); then we have the equilibrium equations

Table 4
Fixed node.

	x [m]	y [m]	z [m]
$\boldsymbol{P}_1$	0	0	0
$\boldsymbol{P}_2$	0.5	0	0
$\boldsymbol{P}_4$	0	1	0
$\boldsymbol{P}_6$	1	1	1

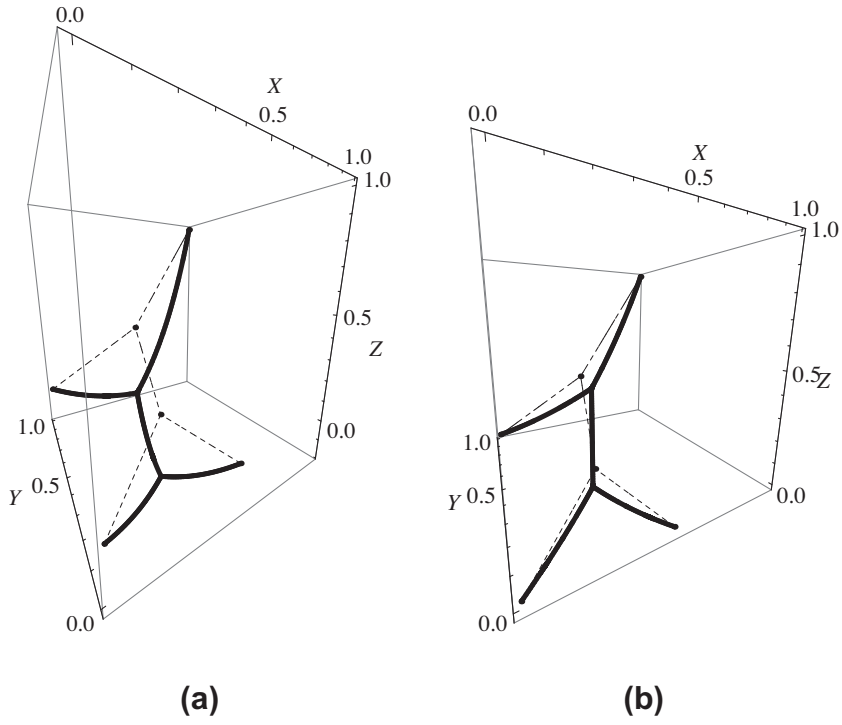

Fig. 8. C-FDM vs l-FDM. (a): form obtained with the C-FDM; (b) form obtained with the l-FDM plus incremental analysis.

Table 5
C-FDM and incremental analysis.

	C-FDM			Incr.	Anal.
	Λ [daN]	η	$Q_{\mathcal{V}}$ [daN/m]	Λ [daN]	Q_{Λ} [daN/m]
Cable 1	0.400195	0.25	1.020750	0.957653	2.271180
Cable 2	0.312500	0.25	1.020750	0.511282	1.516510
Cable 3	0.515388	0.50	1.081980	1.149480	2.183010
Cable 4	0.503891	0.25	1.020750	1.023470	1.823050
Cable 5	0.615554	0.25	1.081980	1.509320	2.795330

$$\begin{aligned}
&\sum_{j=1}^{r}\frac{\Lambda_j}{l_j}(x_i-x_j)=p_{x,i},\\
&\sum_{j=1}^{r}\frac{\Lambda_j}{l_j}(y_i-y_j)=p_{y,i},\\
&\sum_{j=1}^{r}-\frac{q_{z_j}L_{0_j}}{2}+\frac{\Lambda_j}{l_j}(z_i-z_j)=p_{z,i},
\end{aligned}\tag{73}$$

the auxiliary equations on the length (here written in an alternative form to (61))

$$\begin{aligned}
L_{0_j}=&\frac{\sqrt{4l^2\Lambda^2+(lq_zL_0-2h\Lambda)^2}}{8q_zL_0}\left(\frac{q_zL_0}{\Lambda}-2\frac{h}{l}\right)\\
&+\frac{\sqrt{4l^2\Lambda^2+(lq_zL_0+2h\Lambda)^2}}{8q_zL_0}\left(\frac{L_0q_z}{\Lambda}+2\frac{h}{l}\right)\\
&-\frac{l\Lambda}{2q_zL_0}\left[\mathrm{Sinh}^{-1}\left(\frac{h}{l}-\frac{q_zL_0}{2\Lambda}\right)-\mathrm{Sinh}^{-1}\left(\frac{h}{l}+\frac{q_zL_0}{2\Lambda}\right)\right]
\end{aligned}\tag{74}$$

and the expressions of the force densities

$$Q_{\Lambda_j}=\frac{\Lambda_j}{l_j}.\tag{75}$$

We have $3j+2n$ equations in $3j+2n$ variables, the $3j$ equilibrium equations, with the n conditions on the force densities Q_{Λ_j} and n conditions on the length $\{L_{0_j}\}$, in the $3j$ independent variables $\{x_j,y_j,z_j\}$, the n independent variables $\{\Lambda_j\}$ and n variables $\{L_{0_j}\}$. We adopt a Newton–Raphson strategy to solve these equations. The solution of the P-FDM is represented by the set of values $\{x_j,y_j,x_j\}^{P-FDM}$ with $\{\Lambda_j\}^{P-FDM}$ and the length $\{L_{0_j}\}^{P-FDM}$.

Table 6
Coordinates of the free nodes, case $q_{z_j}=1$ [daN/m].

[daN/m]	η	Node	x [m]	y [m]	z [m]
$q_z=1$	0.125	3	0.5	0.25	0.014154
		5	0.5	0.75	0.250837
	0.25	3	0.5	0.25	−0.097443
		5	0.5	0.75	0.123501
	0.5	3	0.5	0.25	−0.348097
		5	0.5	0.75	−0.161213
	1	3	0.5	0.25	−1.242040
		5	0.5	0.75	−1.130390

The (non linear) catenary FDM. The equilibrium equations for the catenary elements are:

$$\begin{aligned}
&\sum_{j=1}^{r}\Lambda_j\frac{x_i-x_j}{l_j}=p_{x,i},\\
&\sum_{j=1}^{r}\Lambda_j\frac{y_i-y_j}{l_j}=p_{y,i},\\
&\sum_{j=1}^{r}\left(-\frac{q_{z_j}L_{0_j}}{2}+Q_{\mathcal{V}_j}(z_i-z_j)\right)=p_{z,i},
\end{aligned}\tag{76}$$

where the length is given by the condition

$$L_0^2=\frac{l_j^2}{\eta_j^2}\mathrm{Sinh}^2[\eta_j]+h_j^2\tag{77}$$

and the force densities are

$$Q_{\mathcal{V}_j}=\frac{\Lambda_j}{l_j}\eta_j\frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]}=\frac{q_{z_j}}{2}\frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]}.\tag{78}$$

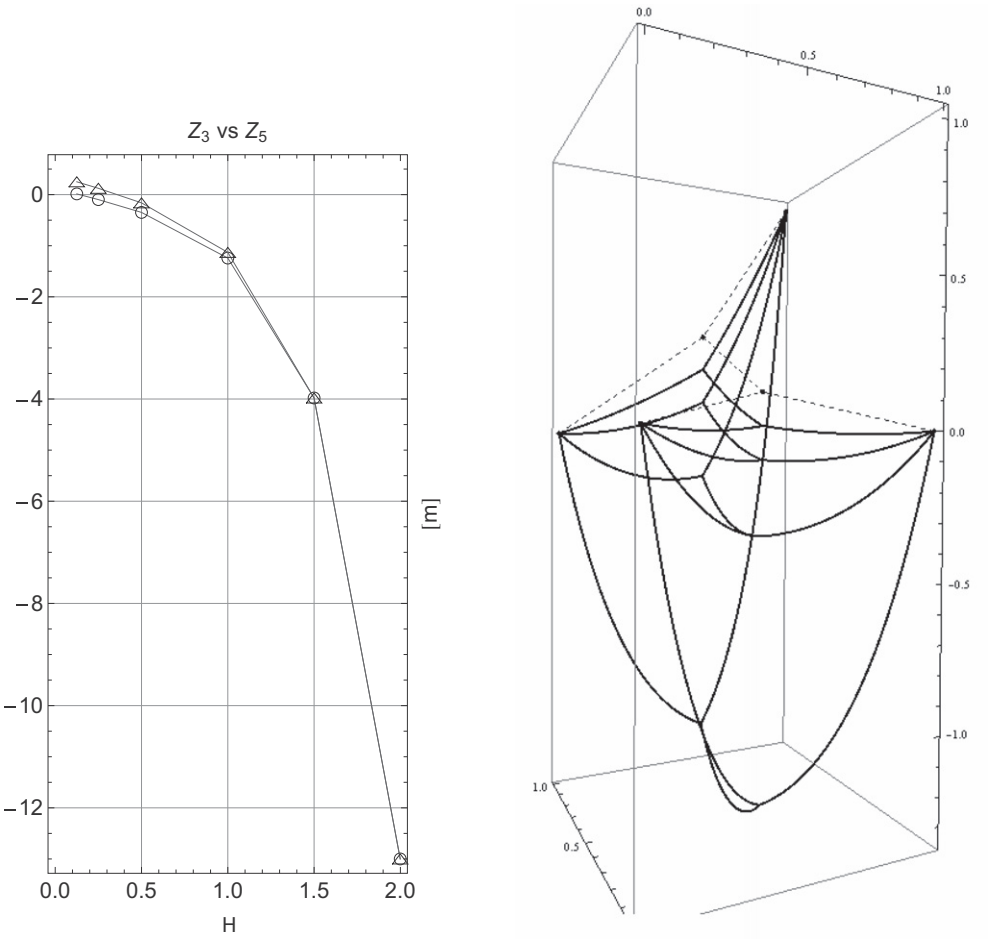

Fig. 9. Dependency of the form from the parameter η (C-FDM for $q_{z_j} = 1$). In (a) are plotted the coordinates z_3 (box-markers) and z_5 (triangle-markers) with respect to η; in (b) are plotted the different configurations of the net for the values of η considered in Table 6.

Table 7
Forces and length of the cable for $q_{z_j} = 1$ [daN/m].

		El.	$\|\boldsymbol{f}\|$ [daN]	$\mathcal{H}$ [daN]	$\mathcal{K}$ [daN]	$\mathcal{V}_0$ [daN]	$\mathcal{V}(L_0)$ [daN]	Λ [daN]	L_0 [m]
$q_z = 1$ [daN/m]	$\eta = 0.125$	1	2.248420	2.0	1.0	−0.223415	0.337237	2.236070	0.560653
		2	2.248420	−2.0	1.0	−0.223415	0.337237	2.236070	0.560653
		3	2.224270	0.0	−2.0	−1.228840	−0.674475	2.0	0.554368
		4	2.463610	2.0	−1.0	0.701548	1.315590	2.236070	0.614044
		5	3.758430	−2.0	−1.0	−3.480050	−2.544430	2.236070	0.935615
	$\eta = 0.25$	1	1.158440	1.0	0.5	−0.485530	0.087672	1.118030	0.573202
		2	1.158440	−1.0	0.5	−0.485530	0.087672	1.118030	0.573202
		3	1.115960	0.0	−1.0	−0.726768	−0.175344	1.0	0.551424
		4	1.168750	1.0	−0.5	−0.036974	0.541228	1.118030	0.578202
		5	2.122320	−1.0	−0.5	−2.310740	−1.268000	1.118030	1.042740
	$\eta = 0.5$	1	0.712522	0.5	0.25	−0.715969	−0.037296	0.559017	0.678673
		2	0.712522	−0.5	0.25	−0.715969	−0.037296	0.559017	0.678673
		3	0.577542	0.0	−0.50	−0.479001	0.0745926	0.5	0.553594
		4	0.629492	0.5	−0.25	−0.476676	0.127819	0.559017	0.604496
		5	1.394410	−0.5	−0.25	−1.905990	−0.606821	0.559017	1.299170
	$\eta = 1$	1	0.894205	0.25	0.125	−1.51796	−0.11288	0.279508	1.405080
		2	0.894205	−0.25	0.125	−1.51796	−0.11288	0.279508	1.405080
		3	0.336343	0.0	−0.25	−0.372354	0.225759	0.25	0.598113
		4	0.827913	0.25	−0.125	−1.39584	−0.088407	0.279508	1.307430
		5	1.44599	−0.25	−0.125	−2.51333	−0.283947	0.279508	2.229390

Eqs. (76) can be cast in the dimensionless form:

$$\begin{aligned}
&\sum_{j=1}^{r}\gamma_j\frac{x_i-x_j}{\eta_i}=2\frac{p_{x,i}}{q_{ref}},\\
&\sum_{j=1}^{r}\gamma_j\frac{y_i-y_j}{\eta_j}=2\frac{p_{y,i}}{q_{ref}},\\
&\sum_{j=1}^{r}\gamma_j\left(-L_{0_j}+\frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]}(z_i-z_j)\right)=2\frac{p_{z,i}}{q_{ref}},
\end{aligned}\tag{79}$$

where $\gamma_j=\frac{q_{z,j}}{q_{ref}}$ is the ratio between the unit weight of each cable and a reference unit weight (for instance, the unit weight of the lightest cable adopted) and $\eta_j=\frac{q_{z_j}}{2Q_{\Lambda_j}}=\frac{q_{z_j}l_j}{2\Lambda_j}$.

We assign the dimensionless parameters η_j, that can be chosen on the basis of the desired slackness of the cables as pointed out at the end of Section 4.1. Then, using either Eqs. (76) or (79), we have $3j+n$ equations in $3j+n$ variables, the $3j$ equilibrium equations, with the n conditions on the length $\{L_{0_j}\}$, in the $3j$ independent variables $\{x_j,y_j,z_j\}$ and the n independent $\{L_{0_j}\}$.

We adopt a Newton–Raphson strategy for solving these equations. The initial guess is given by the solution of the linearized expressions of problem (79) obtained disregarding the weight of the cables, i.e. disregarding the term L_{0_j} in the third of Eq. (79).

$$\begin{aligned}
&\sum_{j=1}^{r}\gamma_j\frac{x_i-x_j}{\eta_i}=2\frac{p_{x,i}}{q_{ref}},\quad \sum_{j=1}^{r}\gamma_j\frac{y_i-y_j}{\eta_j}=2\frac{p_{y,i}}{q_{ref}}\\
&\sum_{j=1}^{r}\gamma_j\frac{\mathrm{Cosh}[\eta_j]}{\mathrm{Sinh}[\eta_j]}(z_i-z_j)=2\frac{p_{z,i}}{q_{ref}},
\end{aligned}\tag{80}$$

The C-FDM solution yields an exact distribution of the nodal forces accounting for the geometric non linearity that can be directly used in the analysis of the net subjected to variable loads.

6. Numerical examples

In this section we present some simple examples in order to illustrate the form finding method proposed for slack cable nets

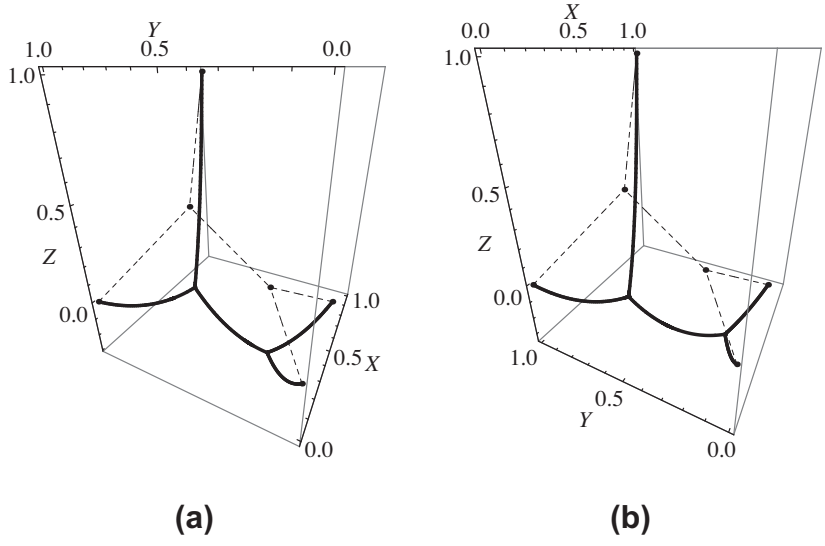

Fig. 10. Initial and converged shape of the net for the example of Section 6.2.

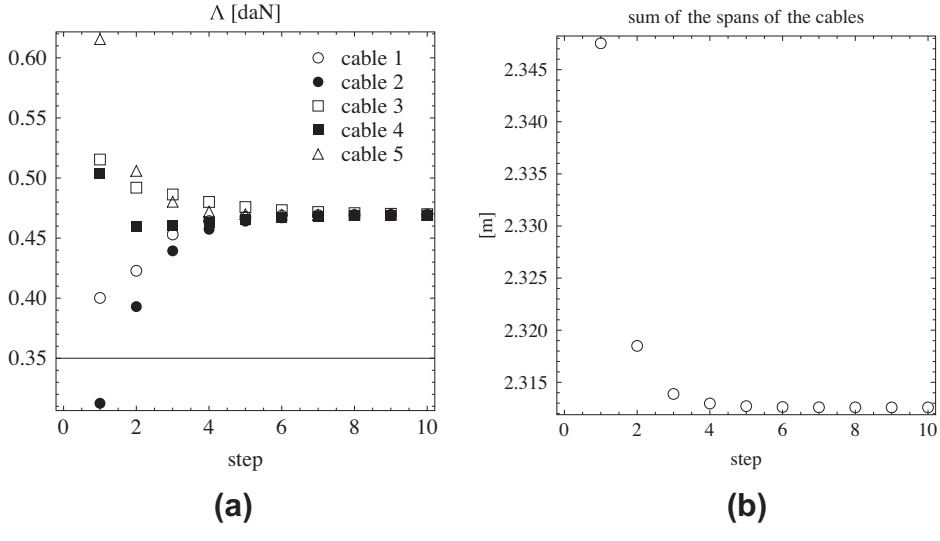

Fig. 11. URS strategy: (a) Convergence of the thrusts to the common value. (b) Minimization of the sum of the horizontal spans of the cables.

and to compare it with the methods based on the truss approximation.

6.1. A simple 3D 5-cable net

We examine a simple 3D net composed by five undeformable cables ($EA \to \infty$) as shown in Fig. 7(a), considering the weight of each cable varying in the range from zero to the value $q_z = 2$ [daN/m]. The free nodes are denoted by $\boldsymbol{P}_3$ and $\boldsymbol{P}_5$, while the other nodes are fixed, their coordinates are reported in Table 1.

The problem has been solved using the force density methods exposed in Section 5, using $Q_x = 1$ [daN/m], and the C-FDM, setting for η the value $\eta = q_z/(2Q_x)$.

The solution for the coordinates of the free nodes in the case $q_z = 1$ and $q_z = 1.5$ [daN/m] are listed in Table 2 for the three case of the truss, parabola and catenary FDM, and the forms found for the case $q_z = 1$ [daN/m] are plotted in Fig. 7(a) and (b), Fig. 7(c) and (d) respectively.

In Fig. 5(a) and (b) the vertical coordinates of the free nodes as a function of the self weight of the cables are plotted. We observe that for nl-FDM and P-FDM there exists an asymptotic point in the solution associated to the value of the self weight $q_z = 2$ [daN/m]. This asymptotic trend appears also in the plot of the effective axial forces f_i of the cables, see Fig. 6. This trend means that, in this case, the class of solutions having a fixed values of the force densities $Q_y = 1$ [daN/m] is unable to generate equilibrated solution for self weight $q_z = 2$.

The form finding method based on the choice of the parameter η, instead, yields reasonable forms for all values of the weight examined. Indeed, in this case, while the force density that appears in the horizontal equilibrium equations Q_x remains constant, the vertical force density Q_y adjusts according to the weight. As can be seen from Figs. 5 and 6, for high values of the weight the proposed method leads to a less slack net with respect to the methods based on the truss approximation, and also the forces in the elements are smaller.

Table 3 reports for every cable the length and the relevant static quantities for the initial case $q_z = 0$, for the case $q_z = 1$ and $q_z = 1.5$ [daN/m]. In the first column are listed the values of the quantity f_j. The results show that with the C-FDM the coordinates of the nodes and the static quantities differ from the other cases the more the greater the weight of the cables. This is also true for the parabolic solution, that in Deng et al. (2005) has been suggested as a valid alternative to the linear form finding method for slack structures.

6.2. A net with cables of different weight

The next example concern a 5 cables net having two free nodes and initial positions of the fixed nodes slightly different than in the previous case, as listed in Table 4.

The weight of the cables has been set to $q_z = 0.5$ [daN/m] for cables 1, 2, 4 and to $q_z = 1$ [daN/m] for cables 3 and 5. We have found an initial form with the C-FDM fixing $\eta = q_z/(2Q_x)$, with $Q_x = 1$ [daN/m] for all cables. Then we have compared it with the form obtained using a different procedure. Namely, first it has been found an initial form with the linear FDM, that is, using the truss approximation. Then it has been performed a non linear incremental analysis for imposing the self weight of the cables, using catenary elements with fixed lengths. They have been determined as the lengths of the catenary elements having the prescribed weight and the coordinates of the nodes obtained with the initial form finding.

The two procedures clearly yield different results (Fig. 8(a) and (b)); the C-FDM, maintaining constant the parameter η, keeps constant the geometric stiffness and respects the required sags of the cables. In the second procedure, during he incremental steps the force density increases, and the effect can be significant for very heavy cables.

In Table 5 are compared the thrusts found in the cables with both procedures. The non linear incremental procedure leads to much higher thrusts than the C-FDM. Also the final value of the force densities Q_x increase with respect to the initial value, while in the C-FDM they remain constant. In the table also the values of the parameter Q_y are reported, that represent the geometric stiffness of the catenary.

In this case either the vertical and the horizontal coordinates of the free nodes are different using the different procedures examined.

6.3. Dependency of the form from the parameter η

We consider the 5-cables net of Fig. 7, for which each cable has the same self weight, then $\gamma_j = 1,\ j = 1, 2, \ldots, n$, and solve the form finding problem for the cases $\eta_j = 0.125,\ \eta_j = 0.25,\ \eta_j = 0.5$ and $\eta_j = 1$.

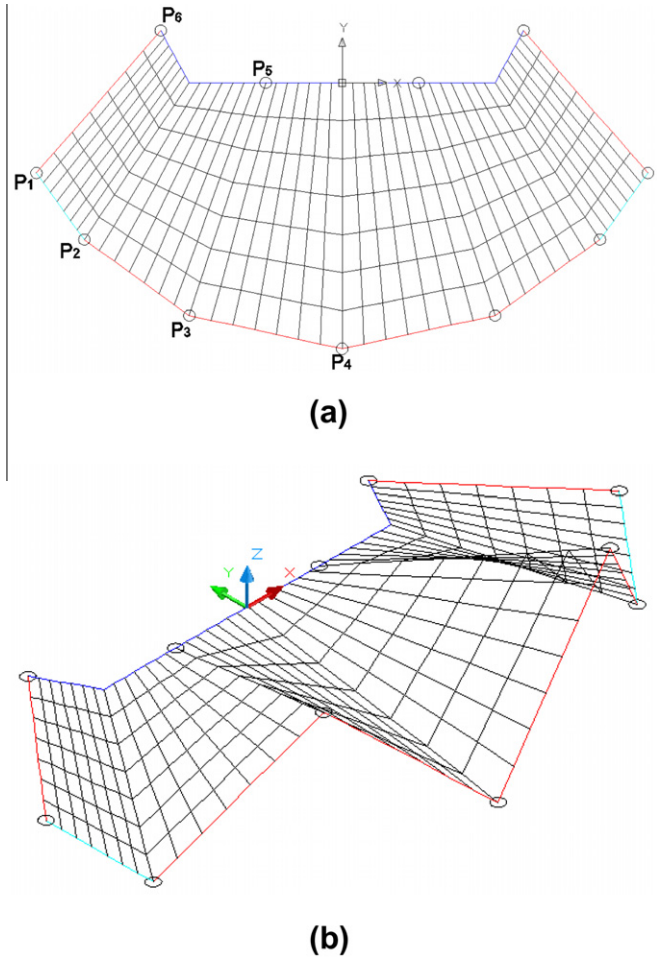

Fig. 12. Initial form. The plan view of the initial net configuration with the fixed points is shown in (a); an axonometric view of the initial net configuration is shown in (b).

Table 8
Fixed nodes.

	x [m]	y [m]	z [m]
$\boldsymbol{P}_1$	−32	−9.5	0
$\boldsymbol{P}_2$	−27	−16.5	−5
$\boldsymbol{P}_3$	−16	−24.5	10
$\boldsymbol{P}_4$	0	−28	5
$\boldsymbol{P}_5$	−8	0	0
$\boldsymbol{P}_6$	−19	5.5	0

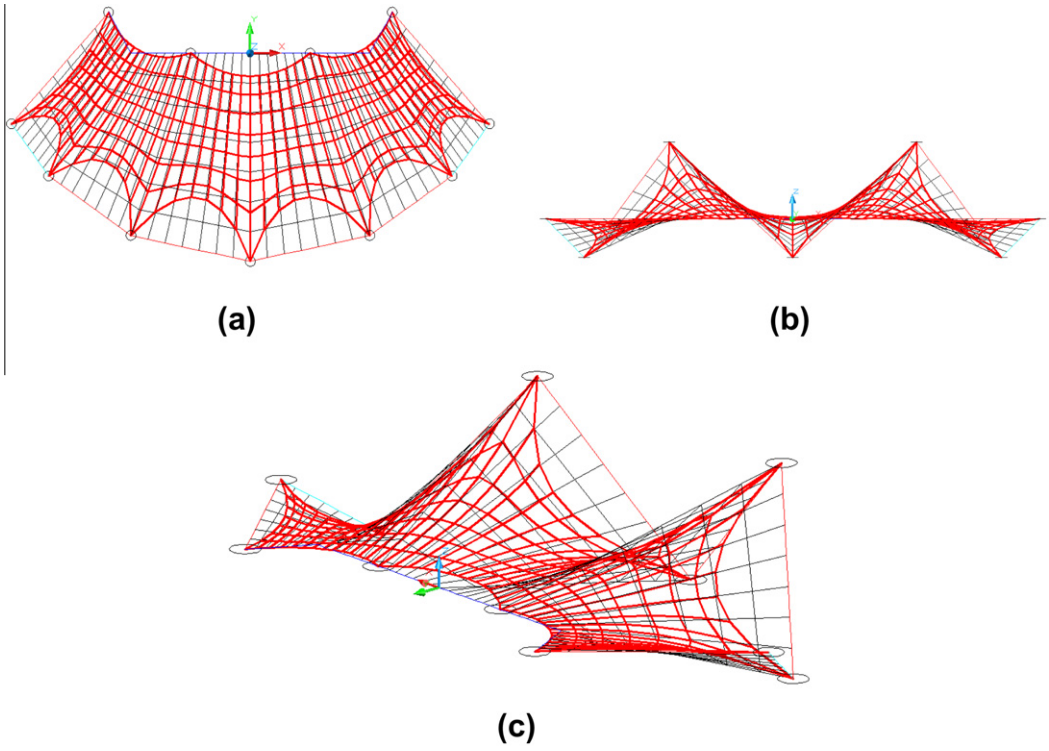

Fig. 13. Final form. In this figure we show various view for the final equilibrium form obtained by C-FDM; the plan view in (a), the front view in (b) an axonometric view (c).

Convergence, starting from the solution of system (80) is very fast. The results for the chosen values of the ratios η_j are presented in Table 6 and graphically plotted in Fig. 9. The relevant static quantities are reported in Table 7 with the same symbols of Table 3.

From the results reported in Tables 6 and 7 it is clear that assigning the values of the self weight q_{z_j} and of the η_j is equivalent to assign the values of the trust Λ_j for each cable.

6.4. Form finding for assigned thrusts

In this section it is shown how it is possible to implement an iterative strategy for obtaining a net with thrusts everywhere equal using the procedure based on catenary elements. The strategy is the same as the one proposed by Bletzinger and Ramm (1999), that is, an initial value of η is selected for the cables, and a first form is obtained. Then it has been evaluated the average of the thrusts, $\Lambda_{ave} = \frac{\sum_{j,1,n} \Lambda_j}{n}$, and the parameters η have been updated as $\eta_j^{k+1} = \frac{q_{z_j} l_j}{2\Lambda_{ave}}$, and the procedure has been iterated till convergence.

The method is applied to the same net used in the previous section. In Fig. 10 the initial and the converged shaped of the net are reported. Fig. 11(a) shows how the thrusts Λ_j converge for the various cables of the net, and in Fig. 11(b) the sum of the horizontal projections of the cables l_j is reported, clearly showing that the method yields a net for which the latter sum is minimal. This geometric property, that generalizes an analogous properties of nets with equal axial forces, can be easily proved examining the equilibrium Eq. (79).

6.5. Form finding of a complex net with the C-FDM

In this case we consider a large span membrane roof having a complex form. The membrane is modelled by a catenary cable net. The initial non equilibrated starting geometry is shown in Fig. 12 where the fixed points are indicated by a circle. The coordinates of the fixed point are listed in the Table 8.

We consider for the internal cables a mean value of $q_{z_j} = 0.2$ [daN/m] while for the boundary cables we consider $q_{z_j} = 0.3$ [daN/m]. We have set for each internal cable $\eta_{int} = 0.3$.

With reference to the Fig. 12(a), for the back boundary cables we adopt $\eta_{1,2} = 0.015$, $\eta_{2,3} = \eta_{3,4} = 0.03$, for the front central cable $\eta_{5,0} = 0.002$ while for the up lateral front cable $\eta_{6,5} = 0.01$, while for the lateral boundary cable $\eta_{1,6} = 0.02$.

The final catenary form of the net is compared in the Fig. 13 with the initial starting form.

7. Conclusions

The paper has shown an improvement of the force density method for form finding of an heavy cable net. The method employs the catenary element, so that equilibrium is exactly satisfied, and it can be easily extended to deformable cables.

The proposed method leads to an initial form that preserves the value of a dimensionless parameter, that takes the place of the force density, and that is related to the sag and to the geometric stiffness of the catenary. The example proposed in Section 6 have shown the difference between the present method and the form finding procedure that uses the truss FDM followed by a non linear analysis able to account for the weight of the cables.

In the paper has also been proposed an iterative procedure for obtaining a net with uniform thrusts. Similar procedures are also possible for imposing other constraints to the equilibrium form of the net, or for assigning constraints on the axial forces acting on the cables, that can be employed for optimizing the total weight of the net.

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