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ON THE MODELLING OF DYNAMIC PROBLEMS FOR VISCOELASTIC COMPOSITES

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Abstract—The aim of this contribution is to show that the microstructure length-scale effect on the dynamic behaviour of viscoelastic composite materials plays a curcial role and cannot be neglected. To this end, the average refined model of the viscoelastic composites, involving the microstructure length parameters, has been formulated. The proposed model can be applied to the analysis of dynamic problems for viscoelastic periodic composite materials.

1. INTRODUCTION

The main analytical tool in investigations of special problems for composite materials are the averaged theories, which can be based on the asymptotic homogenization approaches as shown in [1–3] and in a series of related contributions. In the first approximation of the asymptotic approach the microstructure of the composite is scaled down and hence microstructure length-scale effect on the behaviour of the body is neglected. At the same time, the second step of the procedure leads to considerable computational difficulties which cannot be accepted in applications of the method. In this contribution it will be shown that for dynamic problems the length-scale effect cannot be neglected and hence the first order asymptotic theories for viscoelastic composites lead to the incorrect results. In order to prove this thesis, a certain new theory of viscoelastic micro-periodic heterogeneous materials, which describes the aforementioned effect, will be proposed. The method of micro-modelling applied in this research is based on the known approach to the refined macro-elastodynamics of composite structures, cf. [4]. For the sake of simplicity the considerations will be restricted to the linear theory and to the simple Voigt model fo the viscoelastic constituent materials. A more general approach is reserved for a separate contribution.

Notations

The subscripts $i, j, k, \ldots$ run over $1, 2, 3$ and are related to the Cartesian orthogonal coordinate system $Ox_1x_2x_3$ in the physical space. The points of this space are denoted by $\mathbf{x} \equiv (x_1, x_2, x_3)$ and t stands for the time coordinate. The non-tensorial superscripts $a, b, c, \ldots$ run over $1, \ldots, n$; their meaning will be explained in Section 2. Summation convention holds for both kinds of the aforementioned indices. The region in the physical space occupied by the periodic composite body in its natural state will be denoted by Ω and $V \equiv (0, l_1) \times (0, l_2) \times (0, l_3)$ stands for the representative volume element of a composite material structure. It is assumed that the microstructure length parameter, defined by $l = \max\{l_1, l_2, l_3\}$, is very small compared with the minimum characteristic length dimension L of Ω. Hence, setting $\varepsilon = l/L$ we

shall assume $\varepsilon \ll 1$. For an arbitrary V-periodic integrable function $f = f(\mathbf{x})$ we introduce the averaging operator

$$\langle f \rangle = \frac{1}{l_1 l_2 l_3} \int_V f(\mathbf{x})\, \mathrm{d}v,$$

where $\mathrm{d}v \equiv \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3$. By u_i, s_{ij}, t_i, we denote the displacements, stresses and boundary tractions, respectively. The material properties are described by elastic modulae $C_{ijkl} = C_{ijkl}(\mathbf{x})$, viscous modulae $D_{ijkl} = D_{ijkl}(\mathbf{x})$ and mass density $\rho = \rho(\mathbf{x})$, which are V-periodic functions of $\mathbf{x}$. The body forces are denoted by b_i and are assumed to be constant.

2. GOVERNING RELATIONS AND MODELLING HYPOTHESES

The starting point of the proposed method of modelling is the known governing equations of the composite body under consideration, given by the principal of virtual work

$$\int_\Omega s_{ij} \delta u_{i,j}\, \mathrm{d}v = \oint_{\partial\Omega} t_i \delta u_i\, \mathrm{d}a + \int_\Omega \rho(b_i - \ddot{u}_i)\delta u_i\, \mathrm{d}v, \tag{1}$$

which is assumed to hold for every virtual displacement δu_i and the constitutive equation of the Voigt material

$$s_{ij} = C_{ijkl} u_{(k,l)} + D_{ijkl} \dot{u}_{(k,l)}. \tag{2}$$

In micro-periodic composite materials $\rho = \rho(\mathbf{x})$, $C_{ijkl} = C_{ijkl}(\mathbf{x})$, $D_{ijkl} = D_{ijkl}(\mathbf{x})$, $\mathbf{x} \in \Omega$, are V-periodic functions, where length dimensions of V are very small as compared with those of Ω. Hence the direct description of the composite body, in which no restrictions on displacement fields are imposed, leads to the system of partial differential equations for u_i with variable highly oscillating (V-periodic) coefficients. The analytic form of these equations is not convenient for a computational analysis of special problems and that is why we look for certain simplified (averaged) mathematical models of the composites under consideration.

The method of modelling proposed below will be based on that introduced by Woźniak [4] for the linear elastic composites and generalized by Wierzbicki *et al.* [5] on the non-linear elastodynamics of composite materials. The model introduces two auxiliary concepts and is based on certain macro-modelling hypotheses. To make the paper self-consistent the foundations of the modelling approach will be recalled below.

The first auxiliary concept is that of the regular macro function. Denoting by λ_F the accuracy of computations related to the values of a function $F(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in R$, we shall refer to this function as the macro function, if for each $\mathbf{x}, \mathbf{y} \in \Omega$ and every t, such that $\mathbf{x} - \mathbf{y} \in V$, condition $|F(\mathbf{x}, t) - F(\mathbf{y}, t)| < \lambda_F$ holds. If F is continuous and has continuous derivatives then it will be called the regular macro function provided that similar conditions (with pertinent computation accuracies) also hold for all derivatives of F. Roughly speaking, increments of regular V-macro functions and their derivatives on an arbitrary cell $\mathbf{x} + V$ in the domain of their definition can be neglected when compared with values of those functions at $\mathbf{x}$.

The second auxiliary concept is that of the micro-shape function system. This is a system of n, $n \geq 1$, linear independent functions $h_a(\mathbf{x})$, $a = 1, \ldots, n$ which are V-periodic, continuous, have piecewise continuous first order derivatives and satisfy conditions: $\langle h_a \rangle = 0$, $\langle \rho h_a \rangle = 0$, $h_a(\mathbf{x}) \in O(l)$ and $h_{a,i}(\mathbf{x}) \in O(1)$, for every $\mathbf{x}$ (i.e. values of $h_{a,i}$ are independent of the microstructure length parameter l).

The avenue leading from equations (1) and (2) to the proposed macro-description of viscoelastic composite materials is based on the following hypotheses.

2.1 The kinematic hypothesis

This hypothesis states that the displacements in the composite body under consideration can be expected in the form

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) + h^a(\mathbf{x})V_i^a(\mathbf{x}, t), \qquad \mathbf{x} \in \Omega, \qquad t \geq 0, \tag{3}$$

where U_i, V_i^a are certain regular macro functions and h^a is the system of micro-shape functions, postulated in every problem under consideration.

Functions U_i and V_i^a constitute the new kinematic variables called macro-displacements and correctors, respectively. The choice of the micro-shape functions depends on the character of the material heterogeneity in the representative volume element V and on the expected accuracy of the proposed model. Generally speaking, formulae $u_i(\mathbf{x}_0, t) = U_i(\mathbf{x}_0, t) + h^a(\mathbf{x})V_i^a(\mathbf{x}_0, t)$, $\mathbf{x} \in \mathbf{x}_0 + V(\mathbf{x}_0)$ have to represent a certain discretization of an arbitrary but fixed element $\mathbf{x}_0 + V(\mathbf{x}_0)$ of the region Ω.

2.2 The virtual work hypothesis

This hypothesis assumes that the prinicple of virtual work (1) holds for

$$\delta u_i(\mathbf{x}) = \delta U_i(\mathbf{x}) + h^a(\mathbf{x})\delta V_i^a(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{4}$$

where δU_i, δV_i^a are arbitrary linear independent regular V-macro functions. Moreover, it is assumed that $\delta u_i(\mathbf{x}) = \delta U_i(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$.

This hypothesis is strictly related to the kinematic hypothesis formulated above.

2.3 The macro-approximation hypothesis

This hypothesis states that the following will be neglected:

(i) Terms $O(\lambda_F)$ in formulae involving values of F, where F is an arbitrary regular macro function.

(ii) Terms $O(\varepsilon)$ in calculations of integrals over Ω.

The above hypothesis takes into account the definition of a macro function and the condition $\varepsilon \ll 1$.

3. REFINED MACRO-MODEL

Substituting the right-hand side of equation (2) into equation (1), using equations (3) and (4) and applying the macro-approximation hypothesis, after rather lengthy calculations which will be not presented here, we arrive at the system of equations for the new kinematic variables U_i, V_i^a. This system can be written down in the form of equations of motion

$$\begin{aligned} S_{ij,j} - \langle\rho\rangle\ddot{U}_i + \langle\rho\rangle b_i &= 0, \\ l^2 J^{ab}\ddot{V}_i^b + H_i^a &= 0, \end{aligned} \tag{5}$$

where $J^{ab} \equiv \langle\rho h^a h^b\rangle l^{-2}$ are inertial modulae (the values of which are independent of the microstructure length parameter), natural boundary conditions

$$S_{ij}n_j = s_i \tag{6}$$

and constitutive equations

$$\begin{aligned} S_{ij} &= \langle C_{ijkl}\rangle U_{k,l} + \langle C_{ijlk}h^a_{,k}\rangle V_l^a + \langle D_{ijkl}\rangle \dot{U}_{k,l} + \langle D_{ijkl}h^a_{,k}\rangle \dot{V}_l^a, \\ H_i^a &= \langle C_{ijkl}h^a_{,j}h^b_{,l}\rangle V_k^b + \langle C_{ijkl}h^a_{,j}\rangle U_{k,l} + \langle D_{ijkl}h^a_{,j}h^b_{,j}h^b_{,l}\rangle \dot{V}_k^b + \langle D_{ijkl}h^a_{,j}\rangle \dot{U}_{k,l}. \end{aligned} \tag{7}$$

Fields S_{ij}, H_i^a are called the averaged stresses and microdynamic forces, respectively. After substituting the right-hand sides of equations (7) into equations (5) it can be seen that we obtain the system of differential equations for U_i, V_i^a with constant coefficients. Hence, the

resulting equations represent a certain macro-model of a periodic composite which can be applied to the analysis of special engineering problems. Moreover, this system of equations involves in the explicit form the microstructure length parameter l. The relations obtained describe the microstructure length-scale effect of the body behaviour, which is why equations (5)–(7) will represent what will be called the refined macro-model of the composite under consideration. The characteristic feature of the model is that the correctors V_i^a are governed by the ordinary differential equations, involving the second and first time derivatives of correctors. It means that the correctors are independent of the boundary conditions and can be interpreted as certain internal variables. For homogeneous bodies, by virtue of $\langle C_{ijkl}h^a_{,k}\rangle = C_{ijkl}\langle h^a_{,k}\rangle = 0$, and under homogeneous initial conditions for correctors we obtain $V_i^a(\mathbf{x}, t) = 0$ for every $\mathbf{x} \in \Omega$, $t \geq 0$. Hence the correctors describe the effect of the material inhomogeneity on the behaviour of the composite. For linear-elastic materials $D_{ijkl} = 0$ and equations (5) and (7) reduce to the form proposed by Woźniak [4] and representing the linear refined macro-elastodynamics of periodic composites.

4. LOCAL MACRO-MODEL

The local macro-model of the composite under consideration can be derived by neglecting terms $O(l^2)$ in equations (5). In this case the equations of motion reduce to the form

$$S_{ij,j} - \langle\rho\rangle\ddot{U}_i + \langle\rho\rangle b_i = 0, \tag{8}$$

natural boundary conditions have a form similar to that given by equation (6), the constitutive equations are given by

$$S_{ij} = \langle C_{ijkl}\rangle U_{k,l} + \langle C_{ijkl}h^a_{,k}\rangle V_k^a + \langle D_{ijkl}\rangle \dot{U}_{k,l} + \langle D_{ijkl}h^a_{,k}\rangle \dot{V}_l^a, \tag{9}$$

and for correctors V_i^a we obtain the system of ordinary differential equations

$$\langle D_{ijkl}h^a_{,j}h^b_{,l}\rangle \dot{V}_k^b + \langle C_{ijkl}h^a_{,j}h^b_{,l}\rangle V_k^b = -\langle D_{ijkl}h^a_{,j}\rangle \dot{U}_{k,l} - \langle C_{ijkl}h^a_{,j}\rangle U_{k,l}. \tag{10}$$

In this case correctors V_i^a are the internal variables and equation (10) can be referred to as the evolution equation. It can be observed that the above local model of viscoelastic composites is much simpler and useful to applications than that obtained by the asymptotic homogenization approach, cf. [1, 2].

Let us observe that for quasi-stationary problems (where inertia forces are neglected) the refined and local macro-models of composites coincide.

At the end of general considerations it has to be noticed that the solution U_i, V_i^a to the initial-boundary value problems for equation (5) and (7) as well as for equations (8)–(10) have a physical meaning only if U_i, V_i^a are regular V-macro functions. This condition imposes certain restriction on the boundary traction s_i in equation (6) and is also related to the boundary-layer problem for composite materials, cf. [3].

5. LAMINATED MATERIALS

Let us apply the general results of Sections 3 and 4 to the two-constituent periodic laminates. The representative layer of these composites is made of two homogeneous laminae of thicknesses l', l'', respectively. Hence $l = l' + l''$ is the microstructure length parameter. We shall assume that the x_1-axis is normal to the lamina interfaces and introduce one l-periodic micro-shape function $h = h^1(x_1)$ which in $[0, l]$ satisfies conditions $h(0) = h(l) = l/\sqrt{3}$,

$h(l') = -l/\sqrt{3}$ and is linear in $[0, l']$, $[l', l]$, where the coordinate planes $x_1 = 0$, $x_1 = l'$, $x_2 = l' + l''$ coincide with laminae interfaces. In this case equations (5) reduce to the form

$$S_{ij,j} - \langle\rho\rangle\ddot{U}_i + \langle\rho\rangle b_i = 0$$

$$l^2\langle\rho\rangle\ddot{V}_i + H_i = 0 \tag{11}$$

and equations (7) yield

$$S_{ij} = \langle C_{ijkl}\rangle U_{k,l} + \langle C_{ij1l}h_{,1}\rangle V_l + \langle D_{ijkl}\rangle \dot{U}_{k,l} + \langle D_{ij1l}h_{,1}\rangle \dot{V}_l,$$

$$H_i = \langle C_{i1k1}(h_{,1})^2\rangle V_k + \langle C_{i1kl}h_{,1}\rangle U_{k,l} + \langle D_{i1k1}(h_{,1})^2\rangle \dot{V}_k + \langle D_{i1kl}h_{,1}\rangle \dot{U}_{k,l}. \tag{12}$$

Under an extra assumption that every material constitutent is isotropic with the Lamé modulae λ, μ and the viscosity η we obtain

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$D_{ijkl} = 2\eta\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right),$$

where λ, μ, η take the values λ', μ', η' and λ'', μ'', η'' in the laminae of thicknesses l' and l'', respectively. Let c run over λ, μ, η and define

$$\langle c\rangle \equiv \frac{l'}{l}c' + \frac{l''}{l}c'',$$

$$[c] \equiv c'' - c',$$

$$\{c\} \equiv \frac{lc'}{l'} + \frac{lc''}{l''}.$$

For isotropic constituents and under the above denotations the constitutive equations (12) read

$$S_{11} = 2\langle\mu\rangle U_{1,1} + \langle\lambda\rangle(U_{1,1} + U_{2,2} + U_{3,3}) + 2\sqrt{3}\,[2\mu + \lambda]V_1 + \frac{2}{3}\langle\eta\rangle(\dot{U}_{1,1} - \dot{U}_{2,2} - \dot{U}_{3,3}) + \frac{8\sqrt{3}}{3}[\eta]\dot{V}_1,$$

$$S_{12} = S_{21} = \langle\mu\rangle(U_{1,2} + U_{2,1}) + 2\sqrt{3}\,[\mu]V_2 + \langle\eta\rangle(\dot{U}_{1,2} + \dot{U}_{2,1}) + 2\sqrt{3}\,[\eta]\dot{V}_2,$$

$$S_{13} = S_{31} = \langle\mu\rangle(U_{1,3} + U_{3,1}) + 2\sqrt{3}\,[\mu]V_3 + \langle\eta\rangle(\dot{U}_{1,3} + U_{3,1}) + 2\sqrt{3}\,[\eta]\dot{V}_3,$$

$$S_{22} = 2\langle\mu\rangle U_{2,2} + \langle\lambda\rangle(U_{1,1} + U_{2,2} + U_{3,3}) + 2\sqrt{3}\,[\lambda]V_1 + \frac{2}{3}\langle\eta\rangle(2\dot{U}_{2,2} - \dot{U}_{1,1} - \dot{U}_{3,3}) - \frac{4\sqrt{3}}{3}[\eta]\dot{V}_1,$$

$$S_{23} = S_{32} = \langle\mu\rangle(U_{2,3} + U_{3,2}) + \langle\eta\rangle(\dot{U}_{2,3} + \dot{U}_{3,2}),$$

$$S_{33} = 2\langle\mu\rangle U_{3,3} + \langle\lambda\rangle(U_{1,1} + U_{2,2} + U_{3,3}) + 2\sqrt{3}\,[\lambda]V_1 + \frac{2}{3}\langle\eta\rangle(2\dot{U}_{3,3} - \dot{U}_{1,1} - \dot{U}_{2,2}) - \frac{4\sqrt{3}}{3}[\eta]\dot{V}_1.$$

In this case the governing equations for U_i, V_i obtained from equations (11) and (12) will take the form

$$\langle 2\mu + \lambda\rangle U_{1,11} + \langle\mu\rangle(U_{1,22} + U_{1,33}) + \langle\mu + \lambda\rangle(U_{2,12} + U_{3,13}) + \langle\eta\rangle\left(\frac{4}{3}\dot{U}_{1,11} + \dot{U}_{1,22} + \dot{U}_{1,33}\right)$$

$$+ \frac{1}{3}\langle\eta\rangle(\dot{U}_{2,21} + \dot{U}_{3,13}) + \sqrt{3}\,[4\mu + 2\lambda]V_{1,1} + \sqrt{3}\,[2\mu](V_{2,2} + V_{3,3})$$

$$+ \sqrt{3}\,[2\eta]\left(\frac{4}{3}\dot{V}_{1,1} + \dot{V}_{2,2} + \dot{V}_{3,3}\right) - \langle\rho\rangle\ddot{U}_1 = 0,$$

$$\langle 2\mu+\lambda\rangle U_{2,22}+\langle\mu\rangle(U_{2,11}+U_{2,33})+\langle\mu+\lambda\rangle(U_{1,12}+U_{3,23})+\langle\eta\rangle\left(\dot{U}_{2,11}+\frac{4}{3}\dot{U}_{2,22}+\dot{U}_{2,33}\right)$$

$$+\frac{1}{3}\langle\eta\rangle(\dot{U}_{1,11}+\dot{U}_{3,23})+2\sqrt{3}\,[\mu]V_{2,1}+2\sqrt{3}\,[\lambda]V_{1,2}+2\sqrt{3}\,[\eta]\left(\dot{V}_{2,1}-\frac{2}{3}\dot{V}_{1,2}\right)-\langle\rho\rangle\ddot{U}_2=0,$$

$$\langle 2\mu+\lambda\rangle U_{3,33}+\langle\mu\rangle(U_{3,11}+U_{3,22})+\langle\mu+\lambda\rangle(U_{1,13}+U_{2,23})+\langle\mu\rangle\left(\dot{U}_{3,11}+\dot{U}_{3,22}+\frac{4}{3}\dot{U}_{3,33}\right)$$

$$+\frac{1}{3}\langle\eta\rangle(\dot{U}_{1,13}+\dot{U}_{2,23})+2\sqrt{3}\,[\mu]V_{3,1}+2\sqrt{3}\,[\lambda]V_{1,3}+2\sqrt{3}\,[\eta]\left(\dot{V}_{3,1}+\frac{2}{3}\dot{V}_{1,3}\right)-\langle\rho\rangle\ddot{U}_3=0,$$

$$\sqrt{3}\,[2\mu+\lambda]U_{1,1}+\sqrt{3}\,[\lambda](U_{2,2}+U_{3,3})+\frac{2}{3}\sqrt{3}\,[\eta](2\dot{U}_{1,1}-\dot{U}_{2,2}-\dot{U}_{3,3})$$

$$+3\{4\mu+2\lambda\}V_1 8\{\eta\}\dot{V}_1+\frac{1}{2}l^2\langle\rho\rangle\ddot{V}_1=0,$$

$$\sqrt{3}\,[\mu](U_{1,2}+U_{2,1})+\sqrt{3}\,[\eta](\dot{U}_{1,2}+\dot{U}_{2,1})+6\{\mu\}V_2+6\{\eta\}\dot{V}_2+\frac{1}{2}l^2\langle\rho\rangle\ddot{V}_2=0,$$

$$\sqrt{3}\,[\mu](U_{1,3}+U_{3,1})+\sqrt{3}\,[\eta](\dot{U}_{1,3}+\dot{U}_{3,1})+6\{\mu\}V_2+6\{\eta\}\dot{V}_3-\frac{1}{2}l^2\langle\rho\rangle\ddot{V}_3=0. \tag{13}$$

The above equations describe the laminate under consideration in the framework of the refined model. For the local model we neglect in equations (13) the terms involving the length parameter l.

6. EXAMPLE

Using the equations derived in Section 5 we shall investigate the length-scale effect on the behaviour of the composite body. To this end we shall consider the periodic laminated material occupying the region $\Omega=(0,L_1)\times(0,L_2)\times(0,L_3)$ where the microstructure length parameter l (the thickness of the repetitive layer) is assumed to be very small compared with $L\equiv\min\{L_1,L_2,L_3\}$. Let the body be subjected to the extension δ in the x_1-axis direction. Hence, the boundary conditions for U_1 are given by: $U_1=0$ for $x_1=0$, $U_1=\delta$ for $x_1=L_1$, $U_1=\delta L^{-1}x_1$ for $x_2=0$, L_2 and for $x_3=0$, L_3; at the same time $U_2=U_3=0$ on Ω. The initial conditions will be assumed in the form $U_1(\mathbf{x},0)=\delta x_1/L$, $U_1(\mathbf{x},0)=0$, $U_2(\mathbf{x},0)=U_3(\mathbf{x},0)=U_2(\mathbf{x},0)=U_3(\mathbf{x},0)=0$ and $V_i(\mathbf{x},0)=V_i(\mathbf{x},0)=0$ for every $\mathbf{x}\in\Omega$.

Let us define the non-dimensional positive parameter:

$$\xi\equiv\frac{9}{16}\langle\rho\rangle\{2\mu+\lambda\}\left(\frac{l}{\{\eta\}}\right)^2.$$

It can be shown that the solution to the aforementioned boundary value problem in the framework of the refined model is related to ξ by the following conditions:

(i) if $0<\xi<1$ then

$$V_1=\frac{\sqrt{3}}{12}\frac{[2\mu+\lambda]}{\{4\mu+2\lambda\}}\frac{\delta}{L_1}\left(-\frac{1-\sqrt{1-\xi}}{\sqrt{1-\xi}}\exp(-\alpha_1 t)+\frac{1+\sqrt{1-\xi}}{\sqrt{1-\xi}}\exp(-\alpha_2 t)-2\right),$$

$$U_1=\delta(L_1)^{-1}x_1,\qquad U_2=U_3=V_2=V_3=0$$

where

$$\alpha_1=\frac{8\{\eta\}}{l^2\langle\rho\rangle}(1+\sqrt{1-\xi}),\qquad \alpha_2=\frac{8\{\eta\}}{l^2\langle\rho\rangle}(1-\sqrt{1-\xi});$$

(ii) if $\xi = 1$ then

$$V_1 = \frac{[2\mu + \lambda]}{\{4\mu + 2\lambda\}} \frac{\delta}{L_1} (\exp(-\alpha t) + \alpha t \exp(-\alpha t) - 1),$$

$$U_1 = \delta (L_1)^{-1} x_1, \qquad U_2 = U_3 = V_2 = V_3 = 0$$

where

$$\alpha = \frac{8\{\eta\}}{l^2 \langle \rho \rangle};$$

(iii) if $\xi > 1$ then

$$V_1 = \frac{[2\mu + \lambda]}{\{4\mu + 2\lambda\}} \frac{\delta}{L_1} \left(\exp(-\alpha t) \cos \beta t + \frac{\alpha}{\beta} \exp(-\alpha t) \sin \beta t - 1 \right),$$

$$U_1 = \delta (L_1)^{-1} x_1, \qquad U_2 = U_3 = V_2 = V_3 = 0$$

where

$$\alpha = \frac{8\{\eta\}}{l^2 \langle \rho \rangle}, \qquad \beta = \frac{8\{\eta\}\sqrt{\xi - 1}}{l^2 \langle \rho \rangle}.$$

The averaged stresses calculated from the first of equations (12) are:

(i) if $0 < \xi < 1$ then

$$S_{11} = C + E_1^1 \exp(-\alpha_1 t) + E_2^1 \exp(-\alpha_2 t)$$

$$S_{22} = S_{33} = D + F_1^1 \exp(-\alpha_1 t) + F_2^1 \exp(-\alpha_2 t)$$

$$S_{ij} = 0 \quad \text{for } i \neq j \tag{14}$$

where

$$C = \left(\langle 2\mu + \lambda \rangle - \frac{[2\mu + \lambda]^2}{\{2\mu + \lambda\}} \right) \frac{\delta}{L_1},$$

$$E_1^1 = \frac{1}{\sqrt{1 - \xi}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left(-\frac{1 - \sqrt{1 - \xi}}{2} [2\mu + \lambda] + \frac{16}{3} \frac{\xi [\eta] \{\eta\}}{l^2 \langle \rho \rangle} \right) \frac{\delta}{L_1},$$

$$E_2^1 = \frac{1}{\sqrt{1 - \xi}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left(\frac{1 + \sqrt{1 - \xi}}{2} [2\mu + \lambda] - \frac{16}{3} \frac{\xi [\eta] \{\eta\}}{l^2 \langle \rho \rangle} \right) \frac{\delta}{L_1},$$

$$D = \left(\langle \lambda \rangle - \frac{[\lambda][2\mu + \lambda]}{\{2\mu + \lambda\}} \right) \frac{\delta}{L_1},$$

$$F_1^1 = \frac{1}{\sqrt{1 - \xi}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left(-\frac{1 - \sqrt{1 - \xi}}{2} [\lambda] - \frac{8}{3} \frac{\xi [\eta] \{\eta\}}{l^2 \langle \rho \rangle} \right),$$

$$F_2^1 = \frac{1}{\sqrt{1 - \xi}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left(\frac{1 - \sqrt{1 - \xi}}{2} [\lambda] + \frac{8}{3} \frac{\xi [\eta] \{\eta\}}{l^2 \langle \rho \rangle} \right) \frac{\delta}{L_1};$$

(ii) if $\xi = 1$ then

$$S_{11} = C + E_1^2 \exp(-\alpha t) + E_2^2 t \exp(-\alpha t),$$

$$S_{22} = S_{33} = D + F_1^2 \exp(-\alpha t) + F_2^2 t \exp(-\alpha t),$$

$$S_{ij} = 0 \quad \text{for } i \neq j, \tag{15}$$

where

$$E_1^2 = \frac{[2\mu + \lambda]^2}{\{2\mu + \lambda\}} \frac{\delta}{L_1},$$

$$E_2^2 = \frac{8\{\eta\}}{l^2\langle\rho\rangle} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([2\mu + \lambda] - \frac{32}{3} \frac{[\eta]\{\eta\}}{l^2\langle\rho\rangle}\right) \frac{\delta}{L_1},$$

$$F_1^2 = \frac{[2\mu + \lambda][\lambda]}{\{2\mu + \lambda\}} \frac{\delta}{L_1},$$

$$F_2^2 = \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([\lambda] + \frac{16}{3} \frac{[\eta]\{\eta\}}{l^2\langle\rho\rangle}\right) \frac{\delta}{L_1};$$

(iii) if $\xi > 1$ then

$$S_{11} = C + E_1^2 \exp(-\alpha t)\cos(\beta t) + E_2^3 \exp(-\alpha t)\sin(\beta t),$$

$$S_{22} = S_{33} = D + F_1^2 \exp(-\alpha t)\cos(\beta t) + F_2^3 \exp(-\alpha t)\sin(\beta t),$$

$$S_{ij} = 0 \quad \text{for } i \neq j \tag{16}$$

where

$$E_2^3 = \frac{1}{\sqrt{\xi - 1}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([2\mu + \lambda] - \frac{32}{3} \frac{\xi[\eta]\{\eta\}}{l^2\langle\rho\rangle}\right) \frac{\delta}{L_1},$$

$$F_2^3 = \frac{1}{\sqrt{\xi - 1}} \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([\lambda] + \frac{16}{3} \frac{\xi[\eta]\{\eta\}}{l^2\langle\rho\rangle}\right) \frac{\delta}{L_1}.$$

Since ξ depends not only on the material properties of constituents but also on the microstructure length parameter then the response of the viscoelastic body in the dynamic processes depends on the size of microstructure. Neglecting in equations (13) the terms involving the length parameter l and introducing similar boundary and initial conditions (in this case initial conditions for V_i are not taken into account), we obtain the boundary-value problem formulated in the framework of the local macro-model but describing the same physical situations. It can be verified that the solution to this problem takes the form

$$V_1 = \frac{[2\mu + \lambda]}{\{4\mu + \lambda\}} \frac{\delta}{L_1} (1 - \exp(-\gamma t)),$$

$$U_1 = \delta(L_1)^{-1} x_1, \qquad U_2 = U_3 = V_2 = V_3 = 0,$$

where

$$\gamma = \frac{3\{2\mu + \lambda\}}{4\{\eta\}},$$

for every $\mathbf{x} \in \Omega, t \geq 0$.

The pertinent averaged stresses are equal to

$$S_{11} = C + N \exp(-\gamma t),$$

$$S_{22} = S_{33} = D + R \exp(-\gamma t), \tag{17}$$

where

$$N = \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([2\mu + \lambda] - \frac{4}{3}[\eta]\gamma\right) \frac{\delta}{L_1},$$

$$R = \frac{[2\mu + \lambda]}{\{2\mu + \lambda\}} \left([\lambda] + \frac{2}{3}[\eta]\gamma\right) \frac{\delta}{L_1}.$$

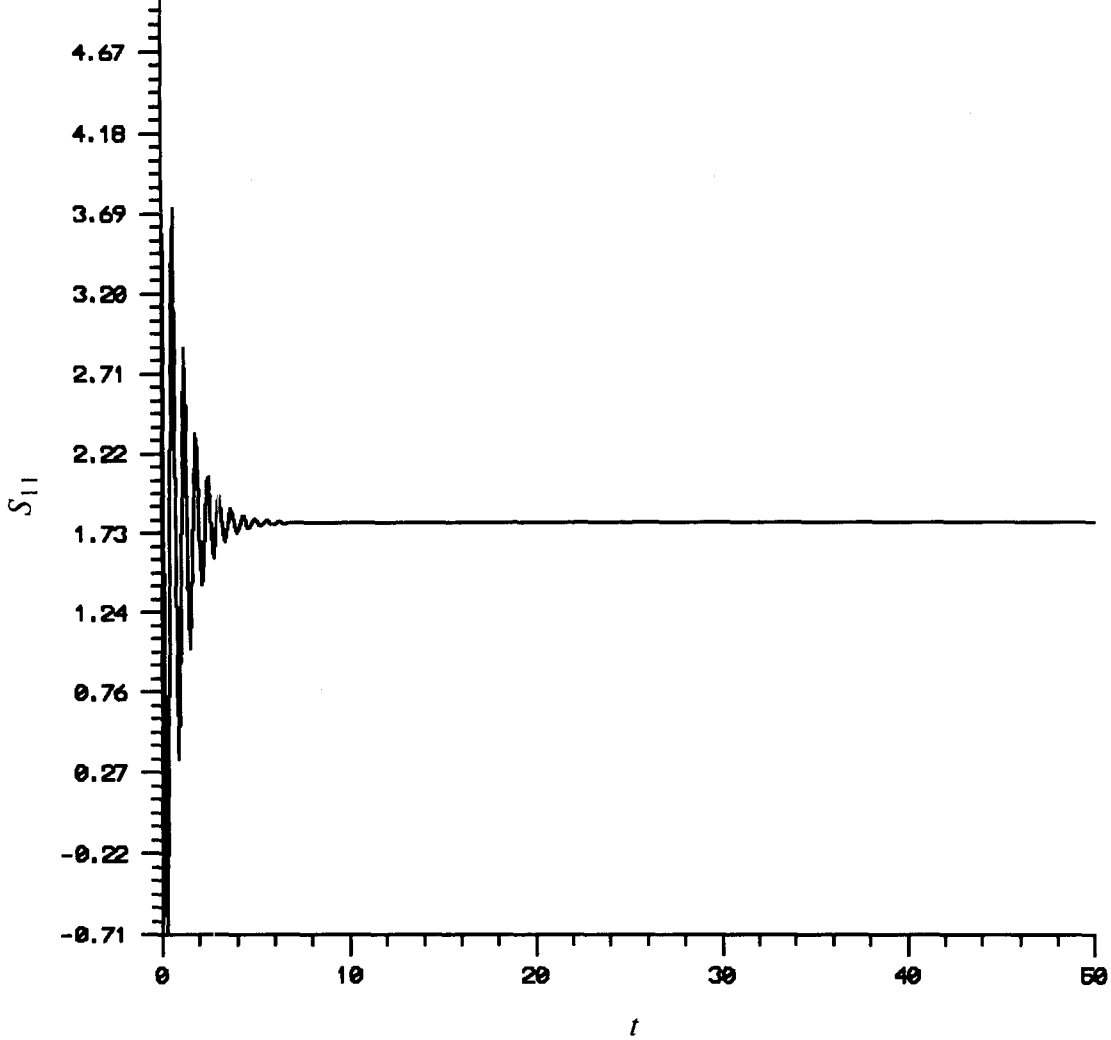

Fig. 1. Diagrams of micro-stresses S_{11} based on equations (16) and (17) for the length-scale model.

7. CONCLUSIONS

In order to show the differences between the proposed length-scale model and the homogenized model let us compare the solution (14)–(16) with that given by equations (17). It can easily be seen that the formulae for stresses S_{ij} obtained in the framework of the local

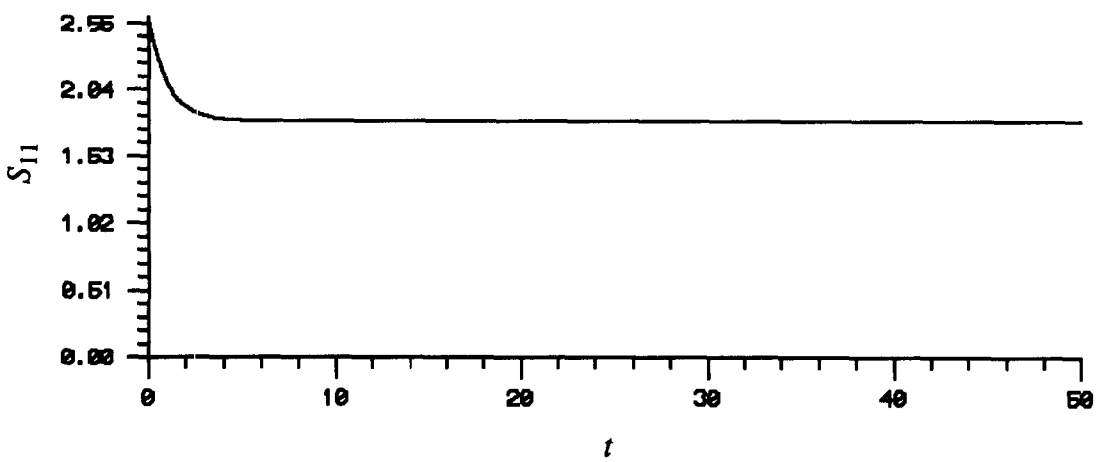

Fig. 2. Diagrams of micro-stresses S_{11} for the local model.

model cannot be treated even as the first approximation of those related to the refined model. For example, the local model is not able to describe the time oscillations of the stress field caused by the micro-heterogeneity of the composite. This fact is shown in Figs 1 and 2 where the diagrams of macro-stresses S_{11} based on equations (16) for the length-scale model and on equation (17) for the local model are shown. Thus the final conclusion is that the microstructure length-scale effect [described by the small terms $l^2\langle p\rangle V_i$ in equations (11)] on the averaged stresses in the composite body cannot be neglected. This condition holds true if we deal with the dynamic problems. However, in the quasi-stationary problems, where we neglect inertia terms $\langle\rho\rangle U_i$, $l^2\langle\rho\rangle V_i$ in equations (11), in the framework of both the models the averaged stresses are the same, being described by equations (17).

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