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► **To cite this version:**

Grzegorz Mieczarek, Czeslaw Wozniak. On the dynamic modelling of fibrous composites. Journal of Technical Physics, 1995, 36 (1), pp.103-111. hal-00944499

HAL Id: hal-00944499

<https://hal.science/hal-00944499>

Submitted on 18 Feb 2014

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ON THE DYNAMIC MODELLING OF FIBROUS COMPOSITES (*)

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The aim of this contribution is to propose a new simplified approach to the wave propagation problem in periodic fibre-reinforced composites. The analysis is based on the concepts of the refined macrodynamics, [6, 7]. The main advantage of the proposed approach is that it yields simple analytical formulae for spectral lines and hence avoids troublesome calculations connected with numerical solutions to the boundary-value problems on the unit cell. The investigations are restricted to the linear-elastic composites with perfect bonding between constituents.

1. Introduction

The problem of finding and investigating the dispersion relation for fibre-reinforced periodic composites was treated in a series of papers by ACHENBACH and SUN, 1972, [1], HJALMARS, 1981, [2], ABOUDI, 1981, [3], TOLF, 1982, [4], ŹÓRAWSKI, 1982, [5], BRAGA, 1992, [11] and others. For a Floquet type of wave, propagating in a plane normal to the fibres, Fig. 1, the displacement components were assumed in the form $u_i(x_1, x_2, t) = v_i(x_1, x_2) \cos(\omega t - k_1 x_1 - k_2 x_2)$, $i = 1, 2$, where $v_i(\cdot)$ are periodic functions of periodicity determined by the composite structure, k_i are the wave numbers and ω is the wave frequency. By means of the periodicity of functions $v_i(\cdot)$, the analysis can be restricted to the representative rectangular cell of the composite. The governing equations of the linear elasticity theory, together with periodic boundary conditions as well as the stress continuity conditions across the interfaces between the fibres and the matrix, yield the system of equations which describe the dynamic behaviour of the entire periodic body. After discretization we arrive at the system of homogeneous linear algebraic equations. For a nontrivial solution, the determinant of the matrix of coefficients in these equations must vanish, yielding the dispersion condition interrelating parameters ω and k_i . It is easy to see that this general procedure involves many troublesome numerical calculations which have to be performed independently for every particular composite structure under consideration.

In this paper a new simplified approach to the formation of the dispersion relation is proposed which makes it possible to obtain the formulae for spectral lines in an explicit analytical form. It is based on equations of the refined macrodynamics for periodic linear elastic composites, WOŹNIAK *et al.*, 1993, [6,7]. The general considerations are specified for slab-reinforced composites, cf. Fig. 1. For the general discussion of spectral lines and phase velocities for the slab-reinforced composites, the reader is referred to [4]. Here we restrict ourselves to the derivation of analytical formulae for transversal macro-waves propagating normally to the fibres in direction of the x_1 -axis.

(*) The research was supported by KBN, Warsaw (Poland), under grant No 3 3310 92 03.

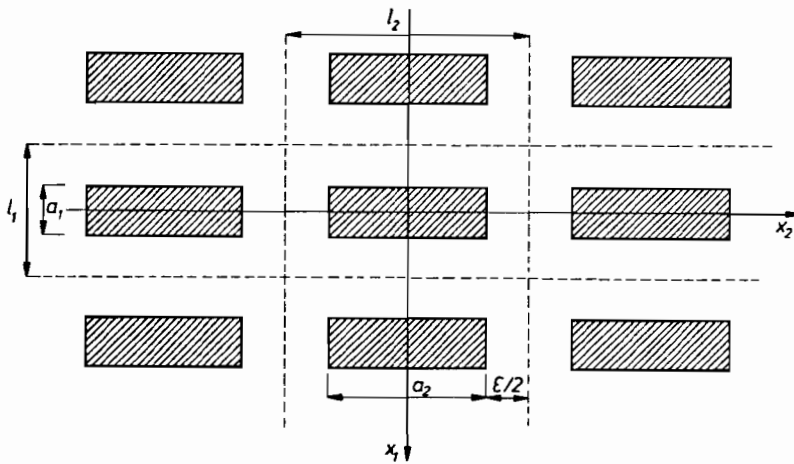


FIG. 1. Scheme of the slab-reinforced composite.

2. Analysis

In order to specify the general equations of refined macrodynamics (cf. Eqs. (5.1) in WOŹNIAK, 1993, [6]) we have to restrict the kinematics of the composite body to a class of motions which can be expected and will be studied in the problem under consideration. To this end define by $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$ the representative array of the composite on $0x_1x_2$ plane (cf. Fig. 1) and assume that a transversal wave is propagating across the unbounded medium in the direction of x_1 -axis.

In the first approximation, proposed in this contribution, the displacements $u_i(x_1, x_2, t)$, $i = 1, 2$, will be assumed in the form

$$(2.1) \quad \begin{aligned} u_1(x_1, x_2, t) &= h_2(x_2, x_1)Q_1^2(x_1, t), \\ u_2(x_1, x_2, t) &= U_2(x_1, t) + h_1(x_1, x_2)Q_2^1(x_1, t), \end{aligned}$$

where U_2 and Q_2^1 , Q_1^2 are macro-field variables (called macrodisplacements and correctors, respectively), i.e., together with all their derivatives, they have negligibly small increments within every interval $(x_1, x_1 + l)$, $x_1 \in R$. The continuous functions h_i , $i = 1, 2$ (called micro-shape functions) are periodic, with periods l_i in x_i -axis directions, respectively, and have to be assumed in every problem under consideration; they describe, from the qualitative viewpoint, the expected micro-disturbances of displacements caused by the inhomogeneity of the medium. Moreover, values of h_i are assumed to be of order $O(l)$, l being the microstructure length parameter (in the sequel $l \equiv l_1$), and values of their derivatives must be independent of l . It is also assumed that the averages of micro-shape functions and their first derivatives over the representative array are equal to zero. Generally speaking, micro-shape functions play a role similar to that of the shape functions in the finite element discretization procedures. Investigating a slab-reinforced composite shown in Fig. 1, we can assume ($i = 1, 2$, no summation):

$$h_i = (l_i - a_i) \frac{x_i + l_i/2}{l_i - a_i} \quad \text{if } x_i \in \left(-\frac{l_i}{2}, -\frac{a_i}{2}\right),$$

$$h_i = (l_i - a_i) \frac{x_i}{a_i} \quad \text{if } x_i \in \left(-\frac{a_i}{2}, \frac{a_i}{2}\right),$$

$$h_i = (l_i - a_i) \frac{x_i - l_i/2}{a_i - l_i} \quad \text{if } x_i \in \left(\frac{a_i}{2}, \frac{l_i}{2}\right).$$

Let $\mu = \mu(x_1, x_2)$, $\rho = \rho(x_1, x_2)$ stand for the shear modulus and the mass-density functions, respectively, defined by $\mu = \mu_M$, $\rho = \rho_M$ for the matrix, and $\mu = \mu_R$, $\rho = \rho_R$ for the reinforcement.

For an arbitrary integrable function $f(x_1, x_2)$ with periods l_1, l_2 , in directions of the coordinate axes, we introduce the averaging operator

$$\langle f \rangle \equiv \frac{1}{l_1 l_2} \int_{-l_1/2}^{l_1/2} \int_{-l_2/2}^{l_2/2} f(x_1, x_2) dx_2 dx_1$$

and setting $l \equiv l_1$, we define the averaged moduli

$$(2.2) \quad \begin{aligned} \tilde{\rho} &\equiv \langle \rho \rangle, & \rho_{11} &\equiv \langle \rho h_1 h_1 \rangle / l^2, & \rho_{22} &\equiv \langle \rho h_2 h_2 \rangle / l^2, \\ \tilde{\mu} &\equiv \langle \mu \rangle, & \mu_{11} &\equiv \langle \rho h_{1,1} h_{1,1} \rangle, & \mu_{22} &\equiv \langle \mu h_{2,2} h_{2,2} \rangle, \\ \mu_{12} &\equiv \langle \mu h_{1,1} h_{2,2} \rangle, & \mu_1 &\equiv \langle \mu h_{1,1} \rangle, & \mu_2 &\equiv \langle \mu h_{2,2} \rangle. \end{aligned}$$

From the conditions imposed on the averaged strain energy function (cf. WOŹNIAK *et al.*, 1993, [6, 7]) it follows that the matrix

$$M = \begin{bmatrix} \tilde{\mu} & \mu_1 & \mu_2 \\ \mu_1 & \mu_{11} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_{22} \end{bmatrix}$$

has to be positive definite; this condition will be used in the subsequent investigations. It can also be seen that all the averaged moduli defined by (2.2) (including ρ_{11} and ρ_{22}) are independent of the microstructure length parameters l_1, l_2 . Bearing in mind Eqs. (2.1), (2.2), from the governing relations of refined macrodynamics (cf. Eqs. (5.1) in [6]) we obtain the following system of equations for macro-displacements U_2 and correctors Q_1^2, Q_2^1 :

$$(2.3) \quad \begin{aligned} \tilde{\rho} \ddot{U}_2 - \tilde{\mu} U_{2,11} - \mu_1 Q_{2,1}^1 - \mu_2 Q_{1,1}^2 &= 0, \\ l^2 \rho_{11} \ddot{Q}_2^1 + \mu_{11} Q_2^1 + \mu_{12} Q_1^2 + \mu_1 U_{2,1} &= 0, \\ l^2 \rho_{22} \ddot{Q}_1^2 + \mu_{22} Q_1^2 + \mu_{12} Q_2^1 + \mu_2 U_{2,1} &= 0. \end{aligned}$$

Let us assume

$$(2.4) \quad U_2 = A \cos(\omega t - k x_1), \quad Q_2^1 = B_1 \sin(\omega t - k x_1), \quad Q_1^2 = B_2 \sin(\omega t - k x_1),$$

where A, B_1, B_2 are arbitrary constants. Substituting (2.4) into (2.3) we arrive at the system of three homogeneous linear algebraic equations for A, B_1, B_2 . The nontrivial solutions exist only if the determinant of this system is equal to zero. This condition yields the dispersion relation between the wave number k and the frequency ω .

With the additional notations

$$\begin{aligned}\tilde{\alpha} &\equiv \frac{\tilde{\mu}}{\tilde{\rho}}, & \alpha &\equiv \frac{\mu_{11}}{\rho_{11}} + \frac{\mu_{22}}{\rho_{22}}, & \beta &\equiv \frac{\mu_{11}\mu_{22} - (\mu_{12})^2}{\rho_{11}\rho_{22}}, \\ \gamma_1 &\equiv \frac{1}{\tilde{\rho}} \left(\frac{(\mu_1)^2}{\rho_{11}} + \frac{(\mu_2)^2}{\rho_{22}} \right), & \gamma_2 &\equiv \frac{\mu_{11}(\mu_2)^2 + \mu_{22}(\mu_1)^2 - 2\mu_{12}\mu_1\mu_2}{\tilde{\rho}\rho_{11}\rho_{22}},\end{aligned}$$

the obtained dispersion relation can be written as

$$l^4\omega^6 - (\tilde{\alpha}k^2l^4 + \alpha l^2)\omega^4 + [(\alpha\tilde{\alpha} - \gamma_1)k^2l^2 + \beta]\omega^2 - (\tilde{\alpha}\beta - \gamma_2)k^2 = 0.$$

Introducing the dimensionless wave number $q \equiv kl$ and setting $\Omega \equiv \omega l$, the above relation reads

$$(2.5) \quad \Omega^6 - (\tilde{\alpha}q^2 + \alpha)\Omega^4 + [(\alpha\tilde{\alpha} - \gamma_1)q^2 + \beta]\Omega^2 - (\tilde{\alpha}\beta - \gamma_2)q^2 = 0.$$

The effect of a material inhomogeneity on the dynamic behaviour of the composite is described by moduli γ_1, γ_2 which are linear functions of $(\mu_1)^2$, $(\mu_2)^2$ and $\mu_1\mu_2$. For a homogeneous medium $\mu_M = \mu_R = \mu = \text{const}$, $\mu_i = \langle \mu h_{i,i} \rangle = \mu \langle h_{i,i} \rangle = 0$ and also $\gamma_1 = \gamma_2 = 0$. Since the matrix M is positive definite, then $\mu_{11}\mu_{22} - (\mu_{12})^2 > 0$. Using the above condition it can also be shown that $\mu_{11}(\mu_2)^2 + \mu_{22}(\mu_1)^2 - 2\mu_{12}\mu_1\mu_2 \geq 0$. It follows that $\beta > 0$ and $\gamma_2 \geq 0$, where $\gamma_2 = 0$ only if $\mu_R = \mu_M = \text{const}$. It has to be emphasized that solutions U_2, Q_1^2, Q_2^2 of Eqs. (2.3), which were assumed in the form (2.4), have to be macrofields. Such situation takes place if the dimensionless wave number $q = kl$ is small compared to 1. The conclusion is that q (and hence also l) can be treated as small parameters. Moreover, the second and third of the governing equations (2.3) involve terms $O(l^2)$ which, in the framework of the refined macrodynamics, can not be neglected. Hence we shall assume that in solving the Eq. (2.5), terms of order $O(q^4)$ can be neglected as compared with the terms independent of q , but terms $O(q^2)$ have to be retained. This statement will be referred to as *the macro-wave approximation*. Using this approximation we shall look for a solution of Eq. (2.5), related to the acoustic branch of the dispersion relation, in the form

$$(2.6) \quad (\Omega_1)^2 = c_1q^2 + d_1q^4 + O(q^6),$$

where c_1, d_1 are constants independent of the small parameter q . Substituting the right-hand side of Eq. (2.6) into the dispersion relation (2.5) and equating terms with the same power of q , we calculate constants c_1, d_1 . This result leads to the explicit form of the formula for the spectral line. Introducing the effective shear modulus

$$(2.7) \quad \mu^{\text{eff}} \equiv \tilde{\mu} - \frac{\gamma_2\tilde{\rho}}{\beta} = \tilde{\mu} - \frac{\mu_{11}(\mu_2)^2 + \mu_{22}(\mu_1)^2 - 2\mu_{12}\mu_1\mu_2}{\mu_{11}\mu_{22} - (\mu_{12})^2},$$

we obtain

$$(2.8) \quad (\Omega_1)^2 = \frac{\mu^{\text{eff}}}{\tilde{\rho}}q^2 \left(1 - \frac{\alpha\gamma_2 - \beta\gamma_1}{\beta^2}q^2 \right) + O(q^6).$$

By simple calculations it can be shown that

$$(2.9) \quad \tilde{\gamma} \equiv \frac{\alpha\gamma_2 - \beta\gamma_1}{\beta^2} = \left(\frac{\mu_{11}\mu_2 - \mu_{12}\mu_1}{\mu_{11}\mu_{22} - \mu_{12}^2} \right)^2 \frac{\rho_{22}}{\tilde{\rho}} + \left(\frac{\mu_{22}\mu_1 - \mu_{12}\mu_2}{\mu_{11}\mu_{22} - \mu_{12}^2} \right)^2 \frac{\rho_{11}}{\tilde{\rho}} > 0,$$

and the equality $\tilde{\gamma} = 0$ holds only if the shear moduli μ_R, μ_M for the matrix and the reinforcement coincide: $\mu_R = \mu_M$. If $\mu_R \neq \mu_M$, then the constant $\tilde{\gamma}$ describes the effect of dispersion on the dynamic behaviour of the composite.

From (2.7) it follows that $\mu^{\text{eff}} \det \mu_{ij} \det M$, where $\det \mu_{ij} = \mu_{11}\mu_{22} - (\mu_{12})^2$. Since $\det \mu_{ij} > 0$, $\det M > 0$, then $\mu^{\text{eff}} > 0$. On the other hand, $\frac{\gamma_2 \tilde{\rho}}{\beta} \geq 0$ and from (2.7) we obtain

$$0 < \mu^{\text{eff}} \leq \tilde{\mu},$$

where $\mu^{\text{eff}} = \tilde{\mu}$ only if $\mu_R = \mu_M$. In the framework of the macro-wave approximation, terms $O(q^6)$ in Eq. (2.8) can be neglected as compared with terms $O(q^2)$, and the explicit formula interrelating ω and k has the form

$$(2.10) \quad (\omega_1)^2 = \frac{\mu^{\text{eff}}}{\tilde{\rho}} k^2 [1 - \tilde{\gamma}(kl)^2]$$

and holds for k satisfying the condition $1 + (kl)^4 \simeq 1$. Equation (2.10) represents the acoustic branch of the spectral line.

REMARK. For a long-wave approximation, defined by $1 + O(q^2) \simeq 1$, we have to neglect the term $\tilde{\gamma}(kl)^2$ in Eq. (2.10), obtaining the low frequency limit given by the well-known formula $(\omega_1)^2 = \frac{\mu^{\text{eff}}}{\tilde{\rho}} k^2$.

In order to find the optical branches (the second and third modes of the dispersion relation), we assume

$$(2.11) \quad (\Omega_{2,3})^2 = c_{2,3} + d_{2,3}q^2 + O(q^4),$$

where $c_{2,3}, d_{2,3}$ are constants independent of q . Substituting the right-hand side of Eq. (2.11) into the dispersion relation (2.5) and equating terms with the same power of q , with notation

$$\delta^2 \equiv \alpha^2 - 4\beta = \left(\frac{\mu_{11}}{\rho_{11}} - \frac{\mu_{22}}{\rho_{22}} \right)^2 + 4 \frac{(\mu_{12})^2}{\rho_{11}\rho_{22}}$$

and some transformations, we obtain (subscripts 2,3 are related to signs $-$, $+$, respectively):

$$(2.12) \quad (\Omega_{2,3})^2 = \frac{\alpha \mp \delta}{2} + \frac{2(\alpha \mp \delta)\gamma_1 - 4\gamma_2}{3(\alpha \mp \delta)^2 - 4(\alpha \mp \delta)\alpha + 4\beta} q^2 + O(q^4).$$

The macro-wave approximation yields the formula

$$(2.13) \quad (\omega_{2,3})^2 = \frac{\alpha \mp \delta}{2} \frac{1}{l^2} + \frac{2(\alpha \mp \delta)\gamma_1 - 4\gamma_2}{3(\alpha \mp \delta)^2 - 4(\alpha \mp \delta)\alpha + 4\beta} k^2$$

which holds true for values of k satisfying $1 + (kl)^4 \simeq 1$. The second term on the right-hand side of Eq. (2.13) disappears for a homogeneous medium since $\gamma_1 = \gamma_2 = 0$. However, in many cases formula (2.13) does not constitute a good approximation of the optical branches of the dispersion relation, since higher frequencies may be not described properly in the framework of the "first approximation" theory based on Eq. (2.1); the detailed analysis of this problem is reserved for a separate study (cf. also the comments at the end of Sec. 3.1).

3. Special Cases

3.1. The homogeneous medium. In order to test the obtained formulae, let us assume that the shear moduli of the matrix and reinforcement coincide: $\mu_M = \mu_R = \mu$. In this case $\gamma_1 = \gamma_2 = 0$ and the dispersion relation (2.5) can be written in the form:

$$(\Omega^2 - \tilde{\alpha}q^2)(\Omega^4 - \alpha\Omega^2 + \beta) = 0.$$

For a homogeneous medium (where also $\tilde{\rho} = \rho$) we arrive at the well-known formula $\omega^2 = \omega_1^2 = \frac{\mu}{\rho}k^2$. This result can also be obtained from Eq.(2.10) since $\mu^{\text{eff}} = \mu$ and $\tilde{\gamma} = 0$ if $\mu_M = \mu_R$.

The remaining solutions of the above equation lead to $(\omega_{2,3})^2 = \frac{\alpha \mp \delta}{2I^2}$, what can be also derived from (2.13) by substituting $\gamma_1 = \gamma_2 = 0$. Introducing the micro-shape functions in the form $h_i = l_i \sin \frac{2\pi x_i}{l_i}$, $i = 1, 2$, we obtain $(\omega_{2,3})^2 = \frac{\mu}{\rho} \left(\frac{2\pi}{l_i} \right)^2$. It is the known exact result for waves propagating in a homogeneous medium with wavenumbers $2\pi/l_i$. Hence we conclude that the aforementioned form of micro-shape functions can be also applied to the calculations of moduli $\alpha, \beta, \delta, \gamma_1, \gamma_2$ on the right-hand side of Eqs.(2.13).

3.2. The laminate-type medium. Let us define $\varepsilon \equiv l_2 - a_2$, cf. Fig. 1, and assume that ε is sufficiently small as compared to l_2 .

In this case the slab-reinforced composite will be referred to as the laminate-type medium. Taking into account the micro-shape functions h_i introduced in Sec. 2, it can be shown that for this medium conditions $\mu_{22} \in O(\varepsilon)$, $\rho_{22} \in O(\varepsilon^2)$ hold.

Multiplying Eq. (2.5) by ρ_{22} , neglecting terms $O(\varepsilon^2)$, for a negligibly small ε we obtain from (2.5) the following relation:

$$(3.1) \quad -\Omega^4 + (\tilde{\alpha}q^2 + \tilde{\beta}) \Omega^2 - (\tilde{\alpha}\tilde{\beta} - \gamma) q^2 = 0,$$

where we have introduced the notations:

$$\tilde{\beta} \equiv \frac{\mu_{11}}{\rho_{11}}, \quad \gamma \equiv \frac{(\mu_1)^2}{\tilde{\rho}\rho_{11}}.$$

It can be shown that Eq. (3.1) can be derived independently for a two-constituent laminated medium. Moreover, it has the form similar to that given by SUN *et al.*, 1968, [8], on the basis of the effective stiffness theory.

Looking for the solution Ω of Eq. (3.1) (related to the acoustic branch of the dispersion relation) in the form

$$(\Omega)^2 = cq^2 + dq^4 + O(q^6),$$

where c, d are constants independent of the parameter q , and using the procedure described in the previous section, after simple transformations we arrive at the formula

$$(\Omega)^2 = \frac{\tilde{\alpha}\tilde{\beta} - \gamma}{\tilde{\beta}} q^2 \left(1 - \frac{\gamma}{\tilde{\beta}^2} q^2 \right) + O(q^6).$$

Within the framework of the macro-wave approximation, terms $O(q^6)$ in the above formula can be neglected as small compared to $O(q^2)$. Introducing the effective shear modulus μ^{eff} for the laminate-type medium, defined by

$$\mu^{\text{eff}} = \tilde{\mu} - \frac{(\mu_1)^2}{\mu_{11}},$$

the resulting dispersion relation between ω and k will take the form

$$(3.2) \quad (\omega)^2 = \frac{\mu^{\text{eff}}}{\tilde{\rho}} k^2 \left[1 - \frac{(\mu_1)^2 \rho_{11}}{(\mu_{11})^2 \tilde{\rho}} (kl)^2 \right],$$

and can be applied for small kl , so that condition $1 + (kl)^4 \simeq 1$ holds.

Looking for the solution of Eq.(3.1) in the form $(\Omega)^2 = c_0 + d_0 q^2 + O(q^4)$, where c_0, d_0 are constants independent of q , and using the same procedure, in the framework of the macro-wave approximation we obtain

$$(3.3) \quad (\omega)^2 = \frac{\mu_{11}}{\rho_{11}} \frac{1}{l^2} - \frac{(\mu_1)^2}{\mu_{11} \tilde{\rho}} (k)^2.$$

Equation (3.3) describes the optical branch of the dispersion relation (cf. comments following Eq.(2.13)).

For a shear wave propagating in the direction normal to the interfaces of the lamina, with notations $c_T \equiv \sqrt{\frac{\mu_M}{\rho_M}}$, $\kappa \equiv \frac{\mu_R}{\mu_M}$, $\varphi \equiv \frac{\rho_R}{\rho_M}$, the exact form of the dispersion relation is, [9]:

$$(3.4) \quad 4\sqrt{\kappa\varphi} \cos q + (\sqrt{\kappa\varphi} - 1)^2 \cos \left(1 - \frac{a_1}{l} \left(1 + \sqrt{\frac{\varphi}{\kappa}} \right) \right) \frac{\Omega}{c_T} - (\sqrt{\kappa\varphi} + 1)^2 \cos \left(1 - \frac{a_1}{l} \left(1 - \sqrt{\frac{\varphi}{\kappa}} \right) \right) \frac{\Omega}{c_T} = 0.$$

Within the framework of the macro-wave approximation, the form of the acoustic branches of spectral lines determined by Eq. (3.2) has been compared with that derived from the exact relation (3.4). The calculations have been carried out for $\kappa = 2, 10$, $\varphi = 1.5$, $\frac{a_1}{l} = 0.5$; the results are shown in Fig. 2.

It is easy to see that, in this special case, the spectral lines obtained by means of the present approach are close to those resulting from the exact solution, not only in the framework of the macro-wave approximation (where we can assume $q < 0.5$) but nearly in the whole first Brillouin zone ($q < \pi$).

3.3. Quadratic cross-section. Finally let us assume that $a_1 = a_2$, and restrict our calculations to the composites reinforced by fibres of quadratic cross-sections. Let us also assume that the representative array is quadratic, setting $l_1 = l_2$. In this case $\rho_{11} = \rho_{22}$, $\mu_{11} = \mu_{22}$ and $\mu_1 = \mu_2$.

The first mode of the spectral line can be derived from Eq. (2.10); after simple calculations

$$(3.5) \quad (\omega_1)^2 = \frac{\mu^{\text{eff}}}{\tilde{\rho}} k^2 \left[1 - \frac{2\rho_{11}}{\tilde{\rho}} \frac{(\mu_1)^2}{(\mu_{11} + \mu_{12})^2} (kl)^2 \right],$$

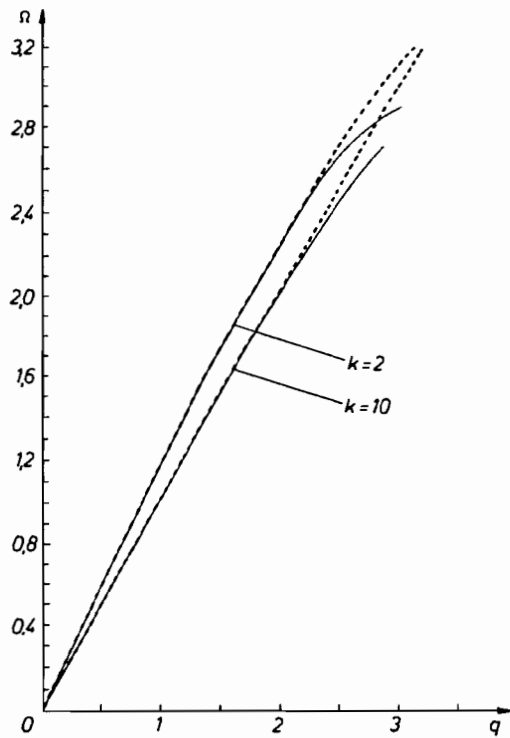


FIG. 2. Spectral lines obtained from the exact solution (continuous line) and from the present approach (dashed line) for a laminate-type structure.

where now the effective shear modulus is given by

$$\mu^{\text{eff}} = \tilde{\mu} - \frac{2(\mu_1)^2}{\mu_{11} + \mu_{12}}.$$

Similarly, from Eq. (2.13) we obtain the formulae for the second and third modes

$$(3.6) \quad \begin{aligned} (\omega_2)^2 &= \frac{\mu_{11} - \mu_{12}}{\rho_{11}} \frac{1}{l^2}, \\ (\omega_3)^2 &= \frac{\mu_{11} + \mu_{12}}{\rho_{11}} \frac{1}{l^2} + \frac{(\mu_1)^2}{\tilde{\rho}(\mu_{11} + \mu_{12})} k^2. \end{aligned}$$

Equations (3.5), (3.6) hold for small values of kl satisfying the condition $1 + (kl)^4 \cong 1$. It can be shown that, from a qualitative viewpoint, the results obtained are similar to those derived by the numerical approach in [4], where also the discussion of a dynamic behaviour for the composites under consideration can be found.

4. Concluding remarks

The obtained results lead to the general conclusion that the proposed approach, based on equations of the refined macrodynamics, can be effectively applied to the formulation

and analysis of the dispersion relation for fibre-reinforced materials. The main advantage of the approach is a simple form of the resulting formulae (2.10), (2.13), (3.2), (3.3) and (3.4), (3.5) describing the spectral lines. Generally speaking, by using the procedures proposed in this paper we avoid many troublesome calculations connected with the discretization of the representative array of a periodic material structure and numerical solutions to the boundary-value problems in this array.

It has to be remembered that the obtained results are restricted to the macro-wave approximation in which higher powers of the dimensionless wave number $q = kl$ have to be neglected. As a drawback of the method, we can mention the choice of micro-shape functions which, in the first approximation, is often based on the intuition of the researcher.

A more detailed analysis, also related to the propagation of short waves, requires larger number of shape functions and will be carried out in a separate paper. The procedure described in this contribution was also applied to materials reinforced by fibres of circular cross-section (cf. KONIECZNY *et al.*, 1994, [10]) and can be used in the analysis of the dynamic behaviour of particular types of composites.

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Received November 17, 1994.