



**HAL**  
open science

# On a quasi-stationary model of debonding processes in layered composites

Zdzistaw Naniewicz, Czeslaw Wozniak

► **To cite this version:**

Zdzistaw Naniewicz, Czeslaw Wozniak. On a quasi-stationary model of debonding processes in layered composites. *Ingenieur-Archiv*, 1988, 58 (6), pp.403-412. hal-00944485

**HAL Id: hal-00944485**

**<https://hal.science/hal-00944485v1>**

Submitted on 18 Feb 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# On a quasi-stationary model of debonding processes in layered composites

Z. Naniewicz and Cz. Woźniak, Warszawa

**Summary:** The main aim of this contribution is to propose a mathematical model of layered composites in which the interlaminar bonding material is assumed to sustain only restricted values of the interlaminar stresses. It is shown that the boundary value problems for the quasi-stationary debonding processes lead to certain sequences of hemi-variational inequalities. The existence theorem is established.

## Über ein quasi-stationäres Modell der Schichtentrennung in Verbundwerkstoffen

**Übersicht:** In diesem Aufsatz schlagen wir ein mathematisches Modell von Verbundwerkstoffen, die nur begrenzte Belastungen übertragen, vor. Es wird bewiesen, daß die Randwertprobleme zur gewissen Semi-Variations-Ungleichungen führen.

## 1 Introduction

Throughout the paper we deal with composites made of a finite number of sheets (laminae) consisting of parallel fibres embedded in a matrix. It is assumed that every sheet is modelled as a transversely isotropic linear elastic material [1]. The sheets are bonded together by very thin layers of the bonding material. These bonding layers are modelled as surfaces, i.e., their thickness is neglected. The proposed approach is based on the assumptions that the bonding material can sustain only restricted values of the interlaminar tractions, and that it behaves elastic prior to possible delamination. Hence during the deformation discontinuities in displacement between the layers can occur. The objective of the contribution is to propose and discuss a mathematical model of debonding processes under the assumption of small displacement and velocity gradients. We show that the displacement jumps across the interlaminar surfaces can be related to the interlaminar stresses by means of some multivalued nonmonotone operators derived from the non-smooth and non-convex strain energy function via the concept of Clarke's generalized gradient [2].

At the same time the irreversibility of the debonding process is taken into account. It is also proved that the pertinent boundary value problems lead to certain hemi-variational inequalities. In the general case the governing relations are non-local in time. It is shown that after time discretization the debonding process is described by the sequence of hemi-variational inequalities and the successive problems can be treated as time independent. The existence of solutions is established by making use of the theory of generalized pseudo-monotone set valued mappings introduced by Browder and Hess [3]. The obtained results can be used as the basis for the formulation of homogenized models for debonding problems in layered composites. The proposed approach is an alternative to that given in [4].

## 2 Mathematical tools

Throughout the paper we use some concepts of the functional analysis which are summarized below; for particulars the reader is referred to [5]. We define  $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$  and  $\bar{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{0\} \cup \{+\infty\}$ . Let  $\varphi: W \rightarrow \mathbb{R}$  be a convex lowersemicontinuous function defined on a

linear topological space  $W$  and let  $W^*$  be a dual of  $W$ , i.e., the space of all linear continuous functionals defined on  $W$ . The value of  $w^* \in W^*$  on the element  $w \in W$  is denoted by  $\langle w, w^* \rangle$  and the bilinear form  $\langle \cdot, \cdot \rangle: W \times W^* \rightarrow \bar{\mathbb{R}}$  determines a dual pairing between  $W$  and  $W^*$ . The set of all  $w^* \in W^*$ , such that

$$\langle v - w, w^* \rangle \leq \varphi(v) - \varphi(w), \quad \forall v \in W,$$

and  $\varphi(w) < +\infty$ , is a subdifferential of  $\varphi$  at  $w \in W$  and is denoted by  $\partial\varphi(w)$ . If  $D$  is a closed convex set in  $W$ , then the indicator function of  $D$  given by

$$\text{ind}_D(w) = \begin{cases} 0 & \text{if } w \in D, \\ +\infty & \text{if } w \in W \setminus D \end{cases}$$

is an example of a convex lowersemicontinuous function defined on  $W$ . In the paper we deal also with Lipschitz continuous functions  $j: W \rightarrow \mathbb{R}$  which may be neither differentiable nor convex and are defined on the Hilbert space  $W$ . For such functions we use the concept of the generalized Clarke's gradient [2]. To this aid we define first the generalized Clarke's directional differential  $j^0(w; v)$  of  $j: W \rightarrow \mathbb{R}$  at  $w \in W$  and in the direction  $v \in W$  by means of

$$j^0(w; v) \equiv \limsup_{\substack{h \rightarrow 0 \\ \lambda \searrow 0}} \frac{j(w + h + \lambda v) - j(w + h)}{\lambda}.$$

Then the generalized gradient  $\bar{\partial}j(w)$  of  $j(\cdot)$  at  $w \in W$  is defined as

$$\bar{\partial}j(w) \equiv \{\bar{v} \in W: j^0(w; v) \geq (v, \bar{v}), \forall v \in W\}$$

where  $(\cdot, \cdot)$  denotes the scalar product in the Hilbert space  $W$ . As it is known [2]  $\bar{\partial}j(w)$  coincides with the subdifferential  $\partial j(w)$  of  $j(\cdot)$  at  $w \in W$ , if  $j(\cdot)$  is convex. The function  $j(\cdot)$  is said to be regular in the sense of Clarke, if

$$j^0(w; v) = j'(w; v)$$

holds for every  $w \in W$ ,  $v \in W$  with

$$j'(w; v) = \lim_{\lambda \searrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda}$$

as the one-sided directional differential of  $j(\cdot)$  at  $w \in W$  in the direction  $v \in W$ .

### 3 Basic assumptions and the general statement of the problem

Let  $\Omega$  be the regular region in  $\mathbb{R}^3$  occupied by the undeformed composite which is assumed to be made of  $S$  disjointed sheets (laminae). Hence  $\bar{\Omega} = \cup \bar{A}_K$  ( $K = 1, \dots, S$ ),  $A_K$  being the region occupied by the  $K$ -th undeformed sheet, see Fig. 1. Thus the interlaminar surfaces separating the adjacent undeformed sheets are given by  $\Pi_K = (\partial A_K \cap \partial A_{K+1}) \setminus \partial\Omega$  where  $K = 1, \dots, S - 1$ . We use the notations

$$A \equiv \bigcup_{K=1}^S A_K, \quad \Pi \equiv \bigcup_{K=1}^{S-1} \Pi_K$$

for the parts of the undeformed composite  $\Omega$  occupied by the sheets and their interfaces, respectively. To every  $\mathbf{z} \in \Pi_K$  ( $K = 1, \dots, S - 1$ ) we assign the unit normal  $\mathbf{N}(\mathbf{z})$  outward to  $A_K$  and hence inward to  $A_{K+1}$ . The unit outward normal to  $\partial\Omega$  at  $\mathbf{x} \in \partial\Omega$  is denoted by  $\mathbf{n}(\mathbf{x})$ , see Fig. 1. For an arbitrary vector field  $\mathbf{w}: \Pi \rightarrow \mathbb{R}^3$  we introduce the notation

$$w_N(\mathbf{z}) \equiv \mathbf{w}(\mathbf{z}) \cdot \mathbf{N}(\mathbf{z}), \quad \mathbf{z} \in \Pi.$$

Moreover, let  $\psi: A \rightarrow \mathbb{R}^n$  be a field such that every  $\psi|_{A_K}$  for  $K = 1, \dots, S$  (which is a field obtained from  $\psi$  by restricting its domain to  $A_K$ ) has well defined traces on  $\partial A_K$ . For an arbitrary  $\mathbf{z} \in \Pi$  we have also  $\mathbf{z} \in \Pi_K$  for some  $K = 1, \dots, S - 1$ , and hence the jump of  $\psi$  across  $\Pi$  at  $\mathbf{z} \in \Pi$

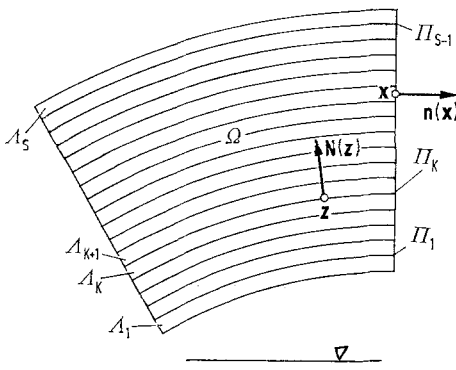


Fig. 1. Scheme of the undeformed composite

will be defined by

$$[[\psi]](\mathbf{z}) \equiv \psi^+(\mathbf{z}) - \psi^-(\mathbf{z}), \quad \mathbf{z} \in II$$

where  $\psi^+(\mathbf{z})$ ,  $\psi^-(\mathbf{z})$  are the values of traces of  $\psi|_{A_{K+1}}$ ,  $\psi|_{A_K}$ , respectively, at  $\mathbf{z} \in II_K \subset II$ .

The deformations of the composite are analysed in the time interval  $[0, \tau_f]$  with the initial time instant related to the undeformed (and unstressed) body. We assume that for every  $\tau \in [0, \tau_f]$  the body forces (per mass density)  $\mathbf{b}(\mathbf{x}, \tau)$  at  $\mathbf{x} \in A$  and the surface tractions  $\mathbf{p}(\mathbf{x}, \tau)$  at  $\mathbf{x} \in \Gamma$  are known where  $\Gamma$  is the corresponding part of the boundary  $\partial\Omega$ . We assume also that on  $\Gamma_0 \equiv \partial\Omega \setminus \Gamma$  the traces  $\mathbf{u}_0(\mathbf{x}, \tau)$ ,  $\mathbf{x} \in \Gamma_0$ , of the displacement fields  $\mathbf{u}(\cdot, \tau): A \rightarrow \mathbb{R}^3$  are known for every  $\tau \in [0, \tau_f]$  and  $\text{mes}(\Gamma_0) > 0$ . By  $\mathbf{T}(\mathbf{x}, \tau)$  we denote the Cauchy stress tensor at  $\mathbf{x} \in A$ ,  $\tau \in [0, \tau_f]$ , and by  $\mathbf{t}(\mathbf{z}, \tau)$  the interlaminar stress vectors at  $\mathbf{z} \in II$ ,  $\tau \in [0, \tau_f]$  which for  $\mathbf{z} \in II_K \subset II$  are acting on the  $K$ -th sheet  $A_K$  across the interface  $II_K$ . Every sheet is assumed to be modelled as linear-elastic homogeneous material. By  $\varrho(\mathbf{x})$ ,  $\mathbf{C}(\mathbf{x})$ , we denote the known mass density and the tensor of elastic moduli at  $\mathbf{x} \in A$ , respectively, which are constant in every  $A_K$  ( $K = 1, \dots, S$ ). We introduce the strain tensor

$$\mathbf{E}(\mathbf{u}) \equiv (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$$

with  $\nabla(\cdot)$  as the gradient operator and  $(\cdot)^T$  for transposition. Under the assumptions that in all problems the inertia terms can be neglected and that no external forces are applied to the interfaces between the sheets we have

$$\text{Div } \mathbf{T}(\mathbf{x}, \tau) + \varrho(\mathbf{x}) \mathbf{b}(\mathbf{x}, \tau) = \mathbf{0} \quad \text{at } \mathbf{x} \in A,$$

$$\mathbf{T}(\mathbf{x}, \tau) = \mathbf{C}(\mathbf{x}) \mathbf{E}(\mathbf{u}(\mathbf{x}, \tau)) \quad \text{at } \mathbf{x} \in A,$$

$$\mathbf{T}(\mathbf{x}, \tau) \mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}, \tau) \quad \text{at } \mathbf{x} \in \Gamma, \quad \tau \in [0, \tau_f] \quad (1)$$

$$\mathbf{u}(\mathbf{x}, \tau) = \mathbf{u}_0(\mathbf{x}, \tau) \quad \text{at } \mathbf{x} \in \Gamma_0,$$

$$\mathbf{T}^+(\mathbf{z}, \tau) \mathbf{N}(\mathbf{z}) = \mathbf{T}^-(\mathbf{z}, \tau) \mathbf{N}(\mathbf{z}) = \mathbf{t}(\mathbf{z}, \tau) \quad \text{at } \mathbf{z} \in II.$$

The Eqs. (1) have to be considered together with the interrelation between the interlaminar displacement jumps  $[[\mathbf{u}]](\mathbf{z}, \tau)$  and the interlaminar stresses  $\mathbf{t}(\mathbf{z}, \tau)$  where  $\mathbf{z} \in II$ ,  $\tau \in [0, \tau_f]$ . Here this interrelation will be based on the following physical assumptions:

1. The interlaminar bonding material (which in the undeformed state is modelled by the surface  $II = \cup II_K$  ( $K = 1, \dots, S-1$ )) possesses linear elastic properties before possibly debonding.

2. The local debonding conditions (i.e. the debonding conditions for the fixed  $\mathbf{z} \in II$ ) are based on the fact that the interlaminar bonding material can sustain only restricted values of interlaminar stress.

3. After debonding only unilateral contact between the adjacent sheets can be realized; for the sake of simplicity we neglect the effect of friction.

In order to specify the aforementioned assumptions and to establish the pertinent interlaminar relation we introduce the decomposition

$$\mathbf{t}(\mathbf{z}, \tau) = \mathbf{s}(\mathbf{z}, \tau) + \mathbf{r}(\mathbf{z}, \tau), \quad \mathbf{r}(\mathbf{z}, \tau) = \mathbf{N}(\mathbf{z}) r_N(\mathbf{z}, \tau) \quad \text{for } \mathbf{z} \in \Pi \quad (2)$$

where  $\mathbf{s}(\mathbf{z}, \tau)$  represents the interlaminar stress (before the local debonding of sheets) which is due to the elastic properties of the bonding material and where  $\mathbf{r}(\mathbf{z}, \tau) = \mathbf{N}(\mathbf{z}) r_N(\mathbf{z}, \tau)$  is the reaction due to the impenetrability of the adjacent sheets. The impenetrability condition has the form of the unilateral constraints  $\llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau) \geq 0$  ( $\mathbf{z} \in \Pi$ ,  $\tau \in [0, \tau_f]$ ), and  $\llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau) = 0$  implies  $r_N(\mathbf{z}, \tau) \geq 0$  as well as  $\llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau) > 0$  implies  $r_N(\mathbf{z}, \tau) = 0$ . It can be easily verified that the aforementioned conditions can be written in the simple form

$$r_N(\mathbf{z}, \tau) \in \partial \text{ind}_{\mathbb{R}^+}(\llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau)) \quad \text{for } \mathbf{z} \in \Pi, \quad \tau \in [0, \tau_f]$$

which also follows immediately from the general constraint approach [6]. Setting

$$u_N \equiv \llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau), \quad \mathbf{u}_T \equiv \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau) - \mathbf{N}(\mathbf{z}) \llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau)$$

we assume that before possible debonding the strain energy function of the bonding material for every  $\mathbf{z} \in \Pi$  is given by the function  $\pi_0: \Pi \times \mathbb{R}^3 \rightarrow \mathbb{R}$  defined as

$$\pi_0(\mathbf{z}, \llbracket \mathbf{u} \rrbracket) = \frac{1}{2} \gamma_N (u_N)^2 + \frac{1}{2} \gamma_T (\llbracket \mathbf{u}_T \rrbracket)^2 - \alpha$$

where  $\gamma_N, \gamma_T, \alpha$  are positive constants. Here  $\gamma_N, \gamma_T$  can be treated as moduli (longitudinal and shear, respectively) which characterize the properties of the bonding material. The constant  $\alpha$  characterizes the local debonding condition which is postulated in the form

$$\pi_0(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)) = 0 \quad \text{for } \mathbf{z} \in \Pi, \quad \tau \in [0, \tau_f] \quad (3)$$

provided that the inequality  $\pi_0(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \sigma)) < 0$  holds for every  $\sigma \in [0, \tau]$ . After the debonding only the unilateral contact between the layers can be realized and the strain energy of the bonding is assumed to be equal to zero. Thus in the modelling of the interlaminar relation the irreversibility of the debonding process has to be taken into account. To this aid we introduce the function  $\pi: \Pi \times \mathbb{R}^3 \rightarrow \mathbb{R}$  given by

$$\pi(\mathbf{z}, \mathbf{w}) \equiv \begin{cases} \pi_0(\mathbf{z}, \mathbf{w}) & \text{if } \pi_0(\mathbf{z}, \mathbf{w}) < 0, \\ 0 & \text{if } \pi_0(\mathbf{z}, \mathbf{w}) \geq 0 \end{cases}$$

and the functionals

$$\Theta_{0 \leq \sigma < \tau}(\mathbf{z}, \mathbf{v}(\sigma)) \equiv \begin{cases} 1 & \text{if } \pi(\mathbf{z}, \mathbf{v}(\sigma)) < 0 \quad \text{for every } \sigma \in [0, \tau], \\ 0 & \text{if } \pi(\mathbf{z}, \mathbf{v}(\sigma)) = 0 \quad \text{for some } \sigma \in [0, \tau] \end{cases}$$

defined for every  $\mathbf{z} \in \Pi$ ,  $\tau \in (0, \tau_f]$  on the set of (sufficiently regular) functions  $\mathbf{v}: [0, \tau] \rightarrow \mathbb{R}^3$ . Now setting

$$\theta(\mathbf{z}, \tau) = \Theta_{0 \leq \sigma < \tau}(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \sigma)) \quad \text{for } \mathbf{z} \in \Pi, \quad (4)$$

we observe that for an arbitrary time instant  $\tau \in [0, \tau_f]$  and in an arbitrary deformation process the strain energy of the interlaminar bonding is determined by the function  $\beta: \Pi \times \mathbb{R}^3 \times \{0, 1\} \rightarrow \mathbb{R}$ , given by

$$\beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau), \theta(\mathbf{z}, \tau)) \equiv \pi(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)) \theta(\mathbf{z}, \tau)$$

with  $\pi(\cdot)$  and  $\theta(\cdot)$  as introduced above. The functions  $\beta(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau))$  defined on  $\mathbb{R}^3$  are non-smooth and non-convex but for every  $\mathbf{z} \in \Pi$ ,  $\theta(\mathbf{z}, \tau) \in \{0, 1\}$  are regular in the sense of Clark (Sect. 2). The local debonding takes place, if  $\theta(\mathbf{z}, \tau) = 1$  and  $\llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)$  satisfies (3); in any other case the functions  $\beta(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau))$  are smooth and the values of  $\mathbf{s}(\mathbf{z}, \tau)$  can be derived from  $\beta(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau))$  as the derivatives with respect to  $\llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)$ . However, using the concept of the generalized Clarke's

gradient of  $\beta(\mathbf{z}, \cdot, \theta(\mathbf{z}, \tau))$  we also obtain the general interrelation between  $\mathbf{s}(\mathbf{z}, \tau)$  and  $\llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)$ :

$$\mathbf{s}(\mathbf{z}, \tau) \in \bar{\partial} \beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau), \theta(\mathbf{z}, \tau)); \quad (5)$$

here and in the sequel the operator  $\bar{\partial}$  is taken with respect to  $\llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau)$  for fixed  $\mathbf{z} \in \Pi$  and  $\theta(\mathbf{z}, \tau) \in \{0, 1\}$ . The parameter  $\theta(\mathbf{z}, \tau)$  is determined by (4) which describes the irreversibility of the debonding process. It has to be emphasized that for the local debonding condition (given by (3) with  $\theta(\mathbf{z}, \tau) = 1$ ) the values of  $\mathbf{s}(\mathbf{z}, \tau)$  are not uniquely determined by (5).

The formulae (1)–(5) represent the general governing relations of the debonding process for the composites under consideration. The basic unknown is  $\mathbf{u}: \mathcal{A} \times [0, \tau_f] \rightarrow \mathbb{R}^3$ . The problem of finding  $\mathbf{u}(\mathbf{x}, \tau)$  for  $\mathbf{x} \in \mathcal{A}$ ,  $\tau \in [0, \tau_f]$ , such that (1)–(5) hold, can be treated as quasi-stationary since no time derivatives enter into the governing relation. However, the general statement of the debonding problems given above is non-local in time and hence rather complicated to be the basis for more detailed discussion. That is why we propose in the next section the time-local approximation of the foregoing statement of the debonding problem.

#### 4 Time-local approximation

Now we are showing that after time discretization of the governing relations (1)–(5) we arrive at some formulation of the time-nonlocal debonding problem which, from the formal point of view, can be treated as a sequence of time-local problems. Setting  $\tau_0 \equiv 0$ ,  $\tau_M \equiv \tau_f$  where  $M$  is a sufficiently large positive integer, and introducing the finite sequence  $\tau_0, \tau_1, \dots, \tau_M$  of time instants such that  $\tau_0 < \tau_1 < \dots < \tau_M$ , we assume that the general relations (1)–(5) presented in Sect. 3 have to hold only for  $\tau \in \{\tau_0, \tau_1, \dots, \tau_M\}$ . Let us define

$$\bar{\theta}(\mathbf{z}, \tau_{m-1}) \equiv \begin{cases} 1 & \text{if } \pi(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \sigma)) < 0 \text{ for every } \sigma \in \{\tau_0, \tau_1, \dots, \tau_{m-1}\} \\ 0 & \text{if } \pi(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \sigma)) = 0 \text{ for some } \sigma \in \{\tau_0, \tau_1, \dots, \tau_{m-1}\} \end{cases}, \quad (6)$$

for every  $\mathbf{z} \in \Pi$  and for  $m = 1, \dots, M$ . Taking into account the definition of the functionals  $\Theta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \cdot))$  in (4) it is easy to see that after the time discretization the functions  $\theta(\cdot, \tau_m)$  have to be replaced by the functions  $\bar{\theta}(\cdot, \tau_{m-1})$  where  $m = 1, \dots, M$ . Thus the time discretization implies the approximation

$$\theta(\mathbf{z}, \tau_m) \sim \bar{\theta}(\mathbf{z}, \tau_{m-1}) \quad \text{with } m = 1, \dots, M \quad \text{and } \mathbf{z} \in \Pi.$$

Hence (1)–(3) and (5) take the form

$$\begin{aligned} \text{Div } \mathbf{T}(\mathbf{x}, \tau_m) + \varrho(\mathbf{x}) \mathbf{b}(\mathbf{x}, \tau_m) &= 0 \quad \text{for } \mathbf{x} \in \mathcal{A}, \\ \mathbf{T}(\mathbf{x}, \tau_m) &= \mathbf{C}(\mathbf{x}) \mathbf{E}(\mathbf{u}(\mathbf{x}, \tau_m)) \quad \text{for } \mathbf{x} \in \mathcal{A}, \\ \mathbf{T}(\mathbf{x}, \tau_m) \mathbf{n}(\mathbf{x}) &= \mathbf{p}(\mathbf{x}, \tau_m) \quad \text{for } \mathbf{x} \in \Gamma, \\ \mathbf{u}(\mathbf{x}, \tau_m) &= \mathbf{u}_0(\mathbf{x}, \tau_m) \quad \text{for } \mathbf{x} \in \Gamma_0, \\ \mathbf{T}^+(\mathbf{z}, \tau_m) \mathbf{N}(\mathbf{z}) &= \mathbf{T}^-(\mathbf{z}, \tau_m) \mathbf{N}(\mathbf{z}) = \mathbf{t}(\mathbf{z}, \tau_m) \quad \text{for } \mathbf{z} \in \Pi, \\ \mathbf{t}(\mathbf{z}, \tau_m) &= \mathbf{s}(\mathbf{z}, \tau_m) + \mathbf{r}(\mathbf{z}, \tau_m), \quad \mathbf{r}(\mathbf{z}, \tau_m) = \mathbf{N}(\mathbf{z}) r_N(\mathbf{z}, \tau_m) \quad \text{for } \mathbf{z} \in \Pi, \\ r_N(\mathbf{z}, \tau_m) &\in \text{ind}_{\bar{\mathbf{u}}_+}(\llbracket \mathbf{u} \rrbracket_N(\mathbf{z}, \tau_m)) \quad \text{for } \mathbf{z} \in \Pi, \\ \mathbf{s}(\mathbf{z}, \tau_m) &\in \bar{\partial} \beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau_m), \theta(\mathbf{z}, \tau_{m-1})) \quad \text{for } \mathbf{z} \in \Pi \end{aligned} \quad (7)$$

and (4) has to be replaced by (6). All aforementioned relations have to hold for  $m = 1, 2, \dots, M$ .

Now observe that for  $m = 1$  we have  $\bar{\theta}(\mathbf{z}, \tau_{m-1}) = \bar{\theta}(\mathbf{z}, 0) = 1$  for  $\mathbf{z} \in \Pi$ , and the basic unknown in (7) is the displacement field  $\mathbf{u}(\cdot, \tau_1)$ . Assume that  $\mathbf{u}(\cdot, \tau_1)$  has been obtained as a specific solution of (7) (in Sect. 6 we shall prove that such solution exists); then from (6) we can calculate the function  $\bar{\theta}(\cdot, \tau_1)$ . Hence for  $m = 2$  the basic unknown in (7) is the displacement field  $\mathbf{u}(\cdot, \tau_2)$  since now  $\bar{\theta}(\cdot, \tau_1)$  is the known function. Using this procedure we can calculate  $\mathbf{u}(\cdot, \tau_m)$  for

$m = 1, 2, \dots, M$  by finding the successive solutions to (7) (provided that such solutions exist) and using successively (6). All these problems can be treated, from the formal point of view, as time-local problems. Moreover, in every such problem the time instant  $\tau_m$  plays the role of the parameter.

## 5 Variational formulation

Taking into account the results of Sect. 4 we restrict ourselves to an arbitrary but fixed time instant  $\tau = \tau_m$  and we assume that  $\bar{\theta}(\mathbf{z}, \tau_{m-1})$  for  $\mathbf{z} \in II$ , has been calculated previously. Then, from the formal point of view, (7) for  $\tau = \tau_1$  can be treated as the governing relations of some time independent problem for the function  $\mathbf{u}(\cdot, \tau_m): A \rightarrow \mathbb{R}^3$ . For the sake of simplicity we omit the argument  $\tau_m$  in (7) and write  $\beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}), \bar{\theta}(\mathbf{z}))$  instead of  $\beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau_m), \bar{\theta}(\mathbf{z}, \tau_{m-1}))$ .

Let  $V$  be the space of the sufficiently regular test functions  $\mathbf{v}: A \rightarrow \mathbb{R}^3$  such that every  $\mathbf{v}|_{A_K}$  ( $K = 1, \dots, S$ ) has well defined traces on  $\partial A_K$  and  $\mathbf{v}(\mathbf{x}) = \mathbf{0}$  for  $\mathbf{x} \in \Gamma_0$ . We assume also that for every  $\mathbf{v} \in V$  the divergence theorem can be applied to  $\mathbf{v}(\mathbf{x}) \cdot \text{Div } \mathbf{T}(\mathbf{x})$ ,  $\mathbf{x} \in A$ . Taking into account the possible discontinuity of  $\mathbf{v}(\cdot)$  across  $II = \cup II_K$  ( $K = 1, \dots, S-1$ ) we obtain

$$\begin{aligned} \int_A \mathbf{v}(\mathbf{x}) \cdot \text{Div } \mathbf{T}(\mathbf{x}) \, dV &= - \int_A \text{tr} [\nabla \mathbf{v}(\mathbf{x}) \mathbf{T}(\mathbf{x})] \, dV + \int_{\Gamma} \mathbf{v}(\mathbf{x}) \cdot \mathbf{T}(\mathbf{x}) \mathbf{n}(\mathbf{x}) \, dA \\ &\quad + \sum_{K=1}^{S-1} \int_{II_K} [\mathbf{v}^-(\mathbf{z}) \cdot \mathbf{T}^-(\mathbf{z}) - \mathbf{v}^+(\mathbf{z}) \cdot \mathbf{T}^+(\mathbf{z})] \mathbf{N}(\mathbf{z}) \, dA \end{aligned}$$

where we have used the notations introduced in Sec. 3. Then it can be shown that (7) for  $\tau = \tau_m$ , in the simplified notation introduced above, is equivalent to the equations

$$\begin{aligned} \int_A \text{tr} [\mathbf{T}(\mathbf{x}) \nabla \mathbf{v}(\mathbf{x})] \, dV + \int_{II} \mathbf{t}(\mathbf{z}) \cdot \llbracket \mathbf{v} \rrbracket(\mathbf{z}) \, dA &= \int_A \varrho(\mathbf{x}) \mathbf{b}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, dV \\ + \int_{\Gamma} \mathbf{p}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, dA &\quad \text{for } \forall \mathbf{v} \in V, \end{aligned} \quad (8)$$

$$\mathbf{T}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) \mathbf{E}(\mathbf{u}(\mathbf{x})) \quad \mathbf{x} \in A,$$

$$\mathbf{t}(\mathbf{z}) = \mathbf{N}(\mathbf{z}) r_N(\mathbf{z}) + \mathbf{s}(\mathbf{z}) \quad \mathbf{z} \in II,$$

and the conditions

$$\begin{aligned} r_N(\mathbf{z}) &\in \text{div} \overline{\mathbf{R}}_+(\llbracket \mathbf{u} \rrbracket_N(\mathbf{z})), \\ &\quad \mathbf{z} \in II \\ \mathbf{s}(\mathbf{z}) &\in \bar{\partial} \beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}), \bar{\theta}(\mathbf{z})). \end{aligned} \quad (9)$$

In this section we are to show that under proper regularity conditions (8), (9) lead to some hemi-variational inequality [5]. We bear in mind that  $\varrho(\mathbf{x})$ ,  $\mathbf{C}(\mathbf{x})$  are constant in every  $A_K$  ( $K = 1, \dots, S$ ), and we assume that  $\mathbf{b} \in (L^2(\Omega))^3$ ,  $\mathbf{p} \in (L^2(\Gamma))^3$ . Using the known notations for the Sobolev spaces we define

$$V \equiv \{ \mathbf{v} \in (L^2(\Omega))^3: \mathbf{v}|_{A_K} \in (H^1(A_K))^3 \ (K = 1, \dots, S); \gamma_0 \mathbf{v} = \mathbf{0} \}$$

where  $\gamma_0 \mathbf{v}$  is the trace of  $\mathbf{v}$  on  $\Gamma_0$ , and

$$\Sigma_K \equiv \{ \mathbf{T} \in (L^2(A_K))^{3 \times 3}: \text{Div } \mathbf{T} \in (L^2(A_K))^3 \ (K = 1, \dots, S) \}$$

(for an arbitrary set  $M$  the symbol  $M^{3 \times 3}$  stand for a set of all  $3 \times 3$  symmetric matrix elements which belong to  $M$ ). In order to simplify the analysis, from now on we assume that  $\mathbf{u}_0(\mathbf{x}, \tau) = \mathbf{0}$  for  $\mathbf{x} \in \Gamma_0$ ,  $\tau \in [0, \tau_f]$  (cf. (1)). Then the displacement field  $\mathbf{u}(\cdot)$  in (8) can be treated as an element of the space  $V$  and  $\llbracket \mathbf{u} \rrbracket \in (H^{1/2}(II))^3$ . Moreover, we assume that  $\mathbf{T}|_{A_K} \in \Sigma_K$  for  $K = 1, \dots, S$ ; then  $\mathbf{t} \in (L^2(II))^3$ . Hence all integrals in the variational condition (8) have a well defined sense.

Denoting by  $\langle \cdot, \cdot \rangle$  the bilinear form defined on  $V \times V^*$  and setting

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &\equiv \int_A \operatorname{tr} [\mathbf{E}(\mathbf{v}(\mathbf{x})) \mathbf{C}(\mathbf{x}) \mathbf{E}(\mathbf{u}(\mathbf{x}))] dV, \\ \langle \mathbf{v}, f \rangle &\equiv \int_A \varrho(\mathbf{x}) \mathbf{b}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dV + \int_F \mathbf{p}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dA, \\ \langle \mathbf{v}, s^* \rangle &\equiv \int_{II} \mathbf{s}(\mathbf{z}) \cdot \llbracket \mathbf{v} \rrbracket(\mathbf{z}) dA, \\ \langle \mathbf{v}, r^* \rangle &\equiv \int_{II} r_N(\mathbf{z}) \llbracket \mathbf{v} \rrbracket_N(\mathbf{z}) dA \end{aligned} \quad \mathbf{v} \in V$$

we obtain from (8) the variational condition

$$a(\mathbf{u}, \mathbf{v}) + \langle \mathbf{v}, s^* \rangle + \langle \mathbf{v}, r^* \rangle = \langle \mathbf{v}, f \rangle \quad \text{for } \forall \mathbf{v} \in V \quad (10)$$

in which the functionals  $s^*, r^* \in V^*$  have to be interrelated with the displacement field  $\mathbf{u} \in V$  by means of the conditions (9).

In order to interrelate the functional  $r^*$  with the displacement field  $\mathbf{u}(\cdot)$  we introduce the closed convex set  $\mathbf{K}$  in  $V$  by means of

$$\mathbf{K} \equiv \{ \mathbf{v} \in V : \llbracket \mathbf{v} \rrbracket_N(\mathbf{z}) \geq 0 \text{ for almost every } \mathbf{z} \in II \}.$$

Then

$$r^* \in \partial \operatorname{ind}_{\mathbf{K}}(\mathbf{u}) \quad (11)$$

implies that the first condition from (9) holds almost everywhere on  $II$ .

Now we have to establish the interrelation between the functional  $s^*$  and the displacement  $\mathbf{u}(\cdot)$ , such that the second condition from (9) holds. To this aim we define the functional

$$J(\mathbf{v}) \equiv \int_{II} \beta(\mathbf{z}, \llbracket \mathbf{v} \rrbracket(\mathbf{z}), \bar{\theta}(\mathbf{z})) dV, \quad \mathbf{v} \in V$$

bearing in mind that  $J(\mathbf{v})$  depends also on the function  $\bar{\theta}(\cdot)$ . Taking into account the definition of  $\beta(\cdot)$  we conclude that  $J(\cdot)$  is Lipschitz continuous on  $V$  and is regular in the sense of Clarke (Sects. 2 and 3). The interrelation between  $s^*$  and  $\mathbf{u}(\cdot)$  that we are going to establish will be determined by the

*Lemma.* Let  $s^* \in \bar{\partial} J(\mathbf{u})$ . Then there exists  $\mathbf{s} \in (L^2(II))^3$  such that

- (i)  $\langle \mathbf{v}, s^* \rangle = \int_{II} \mathbf{s}(\mathbf{z}) \cdot \llbracket \mathbf{v} \rrbracket(\mathbf{z}) dA$  for  $\forall \mathbf{v} \in V$ ,
- (ii)  $\mathbf{s}(\mathbf{z}) \in \bar{\partial} \beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}), \bar{\theta}(\mathbf{z}))$  for almost every  $\mathbf{z} \in II$ .

*Proof.* Let us assume that for the fixed  $\bar{\theta}(\cdot)$  the function  $\bar{\beta}: II \times \mathbb{R}^3 \rightarrow \mathbb{R}$  given by  $\bar{\beta}(\mathbf{z}, \mathbf{w}) \equiv \beta(\mathbf{z}, \mathbf{w}, \bar{\theta}(\mathbf{z}))$  has the properties

1.  $\bar{\beta}(\cdot, \mathbf{w})$  is measurable for any  $\mathbf{w} \in \mathbb{R}^3$ ,
2.  $\bar{\beta}(\cdot, \mathbf{0}) \in L^1(II)$ ,
3.  $|\bar{\beta}(\mathbf{z}, \mathbf{w}_1) - \bar{\beta}(\mathbf{z}, \mathbf{w}_2)| \leq k(\mathbf{z}) \|\mathbf{w}_1 - \mathbf{w}_2\|$ ,  $k(\mathbf{z}) > 0$ , for every  $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$ ,  $\mathbf{z} \in II$  with  $k(\cdot) \in L^2(II)$ .

Under the conditions 1.–3. the functional  $\bar{J}: (L^2(II))^3 \rightarrow \mathbb{R}$  given by

$$\bar{J}(\mathbf{w}) \equiv \int_{II} \bar{\beta}(\mathbf{z}, \mathbf{w}(\mathbf{z})) dA, \quad \mathbf{w} \in (L^2(II))^3$$

is Lipschitzian and has the generalized gradient (Sect. 2)

$$\bar{\partial} \bar{J}(\mathbf{w}) \equiv \{ \mathbf{s} \in (L^2(II))^3 : \bar{J}^0(\mathbf{w}; \mathbf{v}) \geq \langle \mathbf{v}, \mathbf{w} \rangle \text{ for every } \mathbf{v} \in (L^2(II))^3 \}$$

where  $(\cdot, \cdot)$  is a scalar product in  $(L^2(II))^3$  and

$$\bar{J}^0(\mathbf{w}; \mathbf{v}) = \limsup_{\substack{\mathbf{h} \rightarrow \mathbf{0} \\ \lambda \searrow 0}} \frac{\bar{J}(\mathbf{w} + \mathbf{h} + \lambda \mathbf{v}) - \bar{J}(\mathbf{w} + \mathbf{h})}{\lambda}$$

is Clarke's directional differential at  $\mathbf{w}$  and in the direction  $\mathbf{v}$ . It is known [5] that, if  $\mathbf{s} \in \bar{\partial}\bar{J}(\mathbf{w})$ , then

$$\mathbf{s}(\mathbf{z}) \in \bar{\partial}\bar{\beta}(\mathbf{z}, \mathbf{w}(\mathbf{z})) \quad \text{for } \mathbf{z} \in \Pi$$

and hence

$$\mathbf{s}(\mathbf{z}) \in \bar{\partial}\beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}), \bar{\theta}(\mathbf{z}))$$

for almost every  $\mathbf{z} \in \Pi$ . On the other hand the condition  $\mathbf{s}^* \in \bar{\partial}J(\mathbf{u})$  implies the existence of  $\mathbf{s} \in (H^{1/2}(\Pi))^3$  with

$$\langle \mathbf{v}, \mathbf{s}^* \rangle = \langle \llbracket \mathbf{v} \rrbracket, \mathbf{s} \rangle_{1/2}$$

where  $\langle \cdot, \cdot \rangle_{1/2}$  is the bilinear form determined by the pertinent dual pairing. Since the space  $H^{1/2}(\Pi)$  is dense in  $L^2(\Pi)$ , it is possible to extend the functional  $\mathbf{s}^*$  defined on  $(H^{1/2}(\Pi))^3$  onto  $(L^2(\Pi))^3$  in such a way that

$$\mathbf{s} \in \bar{\partial}\bar{J}(\llbracket \mathbf{u} \rrbracket).$$

By virtue of the meaning of  $\bar{\partial}\bar{J}(\cdot)$ , the aforementioned condition leads to the conditions (i)–(ii) of the Lemma which ends the proof.

Now making use of (10), (11) and the Lemma formulated above we obtain

$$\alpha(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \langle \mathbf{v} - \mathbf{u}, f \rangle + \text{ind}_{\mathbb{K}}(\mathbf{v}) - \text{ind}_{\mathbb{K}}(\mathbf{u}) + J^0(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq 0, \quad \forall \mathbf{v} \in V \quad (12)$$

where  $\mathbf{u} \in V$ ,  $f \in V^*$ . Taking into account that the functional  $J(\cdot)$  depends on  $\bar{\theta}(\tau): \Pi \rightarrow \{0, 1\}$ ,  $J(\mathbf{v}) = J(\mathbf{v}; \bar{\theta})$ , and that  $\mathbf{u} = \mathbf{u}(\tau_m)$ ,  $f = f(\tau_m)$ ,  $\bar{\theta} = \bar{\theta}(\tau_{m-1})$  we arrive at the sequence of inequalities

$$\begin{aligned} & \alpha(\mathbf{u}(\tau_m), \mathbf{v} - \mathbf{u}(\tau_m)) - \langle \mathbf{v} - \mathbf{u}(\tau_m), f(\tau_m) \rangle \\ & + \text{ind}_{\mathbb{K}}(\mathbf{v}) - \text{ind}_{\mathbb{K}}(\mathbf{u}(\tau_m)) + J^0(\mathbf{u}(\tau_m), \mathbf{v} - \mathbf{u}(\tau_m); \bar{\theta}(\tau_{m-1})) \geq 0 \end{aligned} \quad (13)$$

for every  $\mathbf{v} \in V$ ,  $\mathbf{u}(\tau_m) \in V$ ,  $m = 1, 2, \dots, M$ ; this represents the variational form of the governing relations (7) with  $\bar{\theta}(\tau_{m-1}) \equiv \bar{\theta}(\cdot, \tau_{m-1})$  defined by (6) and  $\mathbf{u}_0(\mathbf{x}, \tau_m) = \mathbf{0}$  for every  $\mathbf{x} \in \Gamma_0$  and  $m = 0, 1, \dots, M$ . Let  $f(\tau_m)$  ( $m = 1, 2, \dots, M$ ) be known. Then the successive solutions  $\mathbf{u}(\tau_m)$  of the inequalities (13) for  $m = 1, 2, \dots, M$  (provided that they exist) can be obtained from the sequence of the time-local problems for  $\mathbf{u}(\tau_m)$ , since for every  $\tau = \tau_m$  the function  $\bar{\theta}(\tau_{m-1}) \equiv \bar{\theta}(\cdot, \tau_{m-1})$  has been previously calculated after having solved the problem for  $\tau = \tau_{m-1}$  (cf. the comments at the end of Sect. 4). Hence for every  $\tau = \tau_m$  ( $m = 1, 2, \dots, M$ ) in the problem of finding the displacement field  $\mathbf{u}(\tau_m) = \mathbf{u}(\cdot, \tau_m)$  from (13) we assume that the functional  $J(\cdot, \bar{\theta}(\tau_{m-1})) : V \rightarrow \mathbb{R}$  is known. Since  $\beta(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau_m), \theta(\mathbf{z}, \tau_{m-1})) \equiv \pi(\mathbf{z}, \llbracket \mathbf{u} \rrbracket(\mathbf{z}, \tau_m)) \bar{\theta}(\mathbf{z}, \tau_{m-1})$ , the aforementioned functional is defined by

$$J(\mathbf{v}; \bar{\theta}(\tau_{m-1})) \equiv \int_{\Pi} \pi(\mathbf{z}, \llbracket \mathbf{v} \rrbracket(\mathbf{z})) \bar{\theta}(\mathbf{z}, \tau_{m-1}) \, dA$$

for every  $\mathbf{v} \in V$  and  $m = 1, 2, \dots, M$ . The functional  $J(\cdot; \bar{\theta}(\tau_{m-1}))$  will be referred to as the superpotential of the interlaminar bonding tractions. Due to the time irreversibility of the process it depends on the argument  $\bar{\theta}(\tau_{m-1})$  given by (6). The variational formula (13) constitutes the hemivariational inequality [5] for the displacement field  $\mathbf{u}(\tau_m) \in V$ . This fact makes it possible to establish the existence of solutions to the problems under consideration which will be done in the following section of the paper. Let us remark that after replacing in (13)  $\mathbf{u}(\tau_m)$  and  $\bar{\theta}(\tau_{m-1})$  by  $\mathbf{u}(\tau)$  and  $\theta(\tau)$ , respectively, we obtain the variational form of the general governing relations (1)–(5) (for  $\mathbf{u}_0 = \mathbf{0}$ ) which has to hold for every  $\tau \in [0, \tau_f]$ .

## 6 Existence

We restrict ourselves to the problem of existence of a solution  $\mathbf{u}(\tau_m)$  to the hemivariational inequality (13) for an arbitrary but fixed  $m \in \{1, 2, \dots, M\}$  under assumption that the function  $\bar{\theta}(\tau_{m-1})$  has been previously calculated. Hence the superpotential  $J(\cdot) = J(\cdot, \bar{\theta}(\tau_{m-1}))$  is assumed

to be known and the problem is governed by the hemi-variational inequality of the form (12) where for the sake of simplicity we have neglected the time parameter  $\tau_m$  and the dependence of the superpotential  $J$  on the argument  $\bar{\theta}(\cdot, \tau_{m-1})$ . The analysis is carried out under the extra assumption that  $\text{mes}(\Gamma_0 \cap \partial A_K) > 0$  for  $K = 1, \dots, S$  and is based on some facts from the theory of generalized pseudo-monotone set valued mappings introduced in [3]. The reader who is not familiar with this theory can pass now to the theorem formulated at the end of this section.

Let us introduce the Lamé operator  $A: V \rightarrow V^*$  by means of

$$\langle \mathbf{v}, A\mathbf{u} \rangle = a(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V.$$

It is easy to see that the hemi-variational inequality (12) is equivalent to the multiequation

$$f \in A\mathbf{u} + \bar{\partial}J(\mathbf{u}) + \partial \text{ind}_{\mathbf{K}}(\mathbf{u}). \quad (14)$$

Thus in order to prove that for every  $f \in V^*$  there exist the displacement field  $\mathbf{u} \in V$  such that (14) holds, it suffices to show that

$$\text{Range}(A + \bar{\partial}J + \partial \text{ind}_{\mathbf{K}}) = V^*. \quad (15)$$

First we establish for this purpose the pseudo-monotonicity of the multioperator  $\bar{\partial}J$  and then we apply the known theorem on the range of the generalized regular pseudo-monotone mappings [3]. Using the results of [3] we show that the following three conditions hold:

- (i)  $\bar{\partial}J(\mathbf{v})$  is a non-empty, closed, convex and bounded set for every  $\mathbf{v} \in V$ ;
- (ii)  $\bar{\partial}J$  is an upper semi-continuous mapping from the strong topology on  $V$  to the weak topology on  $V^*$ ;
- (iii) if  $(\mathbf{v}_n)$  is a sequence in  $V$  weakly converging to  $\mathbf{v}$ ,  $v_n^* \in \bar{\partial}J(\mathbf{v}_n)$  and  $\limsup \langle \mathbf{v}_n - \mathbf{v}, v_n^* \rangle \leq 0$ , then for any  $\mathbf{w} \in V$  there exists  $v^*(\mathbf{w}) \in \bar{\partial}J(\mathbf{v})$  with  $\liminf \langle \mathbf{v}_n - \mathbf{w}, v_n^* \rangle \geq \langle \mathbf{v} - \mathbf{w}, v^*(\mathbf{w}) \rangle$ .

It is known that the generalized gradient of a Lipschitz function satisfies (i) and (ii) [3]. To get (iii) we use the method proposed in [3]. Let  $(\mathbf{v}_n)$  converge weakly to  $\mathbf{v}$  and let  $v_n^* \in \bar{\partial}J(\mathbf{v}_n)$ . From the estimation

$$|J(\mathbf{v}) - J(\mathbf{w})| \leq \|k\|_{L^2(\Pi)} \|\llbracket \mathbf{v} \rrbracket - \llbracket \mathbf{w} \rrbracket\|_{L^2(\Pi)^3} \leq c \|\mathbf{v} - \mathbf{w}\|_V,$$

with  $k \in L^2(\Pi)$  and  $c$  as a positive constant we obtain

$$|\langle \mathbf{v}_n - \mathbf{v}, v_n^* \rangle| \leq \|k\|_{L^2(\Pi)} \|\llbracket \mathbf{v}_n \rrbracket - \llbracket \mathbf{v} \rrbracket\|_{L^2(\Pi)^3}.$$

Since the trace operators  $\gamma_K: (H^1(A_K))^3 \rightarrow (L^2(\partial A_K))^3$  ( $K = 1, \dots, S - 1$ ) are compact, the jump operator  $\llbracket \cdot \rrbracket: V \rightarrow (L^2(\Pi))^3$  is compact as well. Thus  $\llbracket \mathbf{v}_n \rrbracket$  converges strongly to  $\llbracket \mathbf{v} \rrbracket$  in  $(L^2(\Pi))^3$  and consequently

$$\lim \langle \mathbf{v}_n - \mathbf{v}, v_n^* \rangle = 0. \quad (16)$$

Further, we show that every weak limit of  $(v_n^*)$  lies in  $\bar{\partial}J(\mathbf{v})$ . Without losing generality we may assume that  $(v_n^*)$  converges weakly to  $v^*$ . From the regularity of  $\beta(\mathbf{z}, \cdot, \bar{\theta}(\mathbf{z}))$  (cf. the definition of  $J(\mathbf{v})$  in Sect. 5) we deduce

$$J^0(\mathbf{u}; \mathbf{v}) = J'(\mathbf{u}, \mathbf{v}) = \bar{J}'(\llbracket \mathbf{u} \rrbracket, \llbracket \mathbf{v} \rrbracket) = \bar{J}^0(\llbracket \mathbf{u} \rrbracket, \llbracket \mathbf{v} \rrbracket)$$

for any  $\mathbf{u}, \mathbf{v} \in V$ . Hence

$$\bar{J}^0(\llbracket \mathbf{v}_n \rrbracket, \llbracket \mathbf{w} \rrbracket) \geq \langle \mathbf{w}, v_n^* \rangle, \quad \forall \mathbf{w} \in V.$$

Taking into account the semi-continuity of  $\bar{J}^0(\cdot, \cdot)$  [3] we obtain, by passing to the limit,

$$J^0(\mathbf{v}, \mathbf{w}) = \bar{J}^0(\llbracket \mathbf{v} \rrbracket, \llbracket \mathbf{w} \rrbracket) \geq \langle \mathbf{w}, v^* \rangle, \quad \forall \mathbf{w} \in V$$

which means that  $v^* \in \bar{\partial}J(\mathbf{v})$ . Together with (16) it follows that

$$\liminf \langle \mathbf{v}_n - \mathbf{w}, v_n^* \rangle = \liminf \langle \mathbf{v} - \mathbf{w}, v^* \rangle = \langle \mathbf{v} - \mathbf{w}, v^*(\mathbf{w}) \rangle$$

for some  $v^*(\mathbf{w}) \in \bar{\partial}J(\mathbf{v})$  which ends the proof of (iii).

The generalized gradient  $\bar{\partial}J$  being pseudo-monotone satisfies the condition

$$\langle v, v^* \rangle \geq -c \|v\|_V, \quad v^* \in \bar{\partial}J(v), \quad v \in V$$

with a constant  $c$ , since  $J(\cdot)$  is Lipschitzian on  $V$ . Thus, in the terminology of [3],  $\bar{\partial}J(\cdot)$  is a generalized regular pseudo-monotone operator. On the other hand, it is obvious that

$$0 \in \text{Domain}(A + \partial \text{ind}_{\mathbb{K}}).$$

Thus  $A + \partial \text{ind}_{\mathbb{K}} + \bar{\partial}J$  is generalized regular pseudo-monotone [3]. The coercivness of  $A$  on  $V$  follows from Korn's inequality, from the assumption that  $\text{mes}(\Gamma_0 \cap \partial A_K) > 0$  for  $K = 1, \dots, S$ , and from the strong ellipticity of the tensor of elastic moduli  $\mathbf{C}(\mathbf{x})$ ,  $\mathbf{x} \in A$ . It implies the coercivness of  $A + \partial \text{ind}_{\mathbb{K}} + \bar{\partial}J$  and by the theorem given in [3] we arrive at (15). The obtained result can be summarized in the form of the

*Theorem.* Under the regularity conditions introduced in Sect. 5 and the condition  $\text{mes}(\Gamma_0 \cap A_K) > 0$  for  $K = 1, \dots, S$ , the hemi-variational inequality (12) possesses at least one solution for an arbitrary  $f \in V^*$ .

## 7 Final remarks

The quasi-stationary model of debonding processes in layered linear-elastic composites, given by the sequence of hemi-variational inequalities (13) and by the conditions (6), constitutes the starting point for further approximations rather than for the direct analysis of special problems and numerical solutions. As a rule, the layered composites are made of a great number of sheets. Hence the great number of interfaces  $\Pi_K$  ( $K = 1, \dots, S - 1$ ), across which the displacement field may suffer discontinuity, argues against the introduced model from the point of view of engineering applications. However, the mathematical model of debonding processes proposed in this paper can serve as a foundation for the derivation of some homogenized models, e.g., those based on the microlocal modelling approach [7]. The derivation of such models will be performed in a forthcoming paper. On the other hand the governing relations obtained in this paper can be generalized by introducing in (7) the inertia forces and by interpreting the conditions (5) as the constraints imposed on  $\mathbf{s}(\mathbf{z}, \tau)$ . On this way, using the method of constraints [6], it is possible to formulate the evolutionary models of the debonding processes [4]. Let us remark that the general line of approach to the debonding problems proposed in this paper can be applied also to layered composites made of inelastic materials.

## References

1. Jones, R.: Mechanics of composite materials. New York: Mc Graw Hill 1975
2. Clarke, F. H.: Generalized gradients of Lipschitz functionals. Adv. Math. 40 (1981) 52–67
3. Browder, F. E.; Hess, P.: Nonlinear mappings of monotone types in Banach spaces. J. Funct. Anal. 11 (1972) 251–294
4. Woźniak, Cz.: Mathematical modelling of delamination effects. Arch. Mech. 40 (1988)
5. Panagiotopoulos, P. D.: Inequality problems in mechanics and applications. Basel, Boston, Stuttgart: Birkhäuser 1985
6. Woźniak, Cz.: Constraints in constitutive relations of mechanics. Mech. Teor. Stos. 22 (1984) 323–341
7. Woźniak, Cz.: A nonstandard method of modelling of thermoelastic periodic composites. Int. J. Eng. Sci. 25 (1987) 483–498

Received December 15, 1987

Dr. Z. Naniewicz  
Prof. Dr. Cz. Woźniak  
Institute of Mechanics  
University of Warsaw  
PL-00-901 Warszawa  
P. K. i N. p. 935  
Poland