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On the Non-standard Model of the Theory of Elasticity

by

Czesław WOŹNIAK

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Summary. This is a continuation of the note [1]. It yields the mass-point interpretation of the Cauchy laws of motion and stress relations for elastic and hyper-elastic materials. The approach is based on the assumption that the interactions in a non-standard mass-point system are S-local, i.e. local in the S-topology of the reference space [2].

1. Equations of motion. Interactions in the non-standard Newtonian system $\mathcal{M}$ will be called S-local if for every $P, Q \in D$, $P \neq Q$, we have $\theta_P \notin \mu(\theta_Q) \Rightarrow \mathbf{f}_{PQ}(\tilde{\mathbf{p}}(\theta_P, t), \tilde{\mathbf{p}}(\theta_Q, t)) = \mathbf{0}$ for an arbitrary S-deformation function $\tilde{\mathbf{p}}$. In what follows $\mathbf{f}_{PQ}$ will stand for $\mathbf{f}_{PQ}(\tilde{\mathbf{p}}(\theta_P, t), \tilde{\mathbf{p}}(\theta_Q, t))$, $\mathbf{f}_{PQ} \equiv \mathbf{0}$ for $P = Q$, $\mathbf{f}_P$ will stand for $\mathbf{f}_P(\tilde{\mathbf{p}}(\theta_P, t), \dot{\tilde{\mathbf{p}}}(\theta_P, t))$ and we shall deal only with S-local interactions.

Let $^*\Delta$, $^*\Delta \subset {}^*\Omega$, be an arbitrary standard region in $^*R^3$ with boundary $^*\Gamma$ and exterior $^*\Delta^e \equiv {}^*R^3 \backslash {}^*\Delta$. Let $\mathbf{v}: {}^*\bar{\Delta} \to {}^*R^3$ be an arbitrary standard differentiable function, $\mathbf{v} = (v_k)$, $k = 1, 2, 3$. From Eq. (1.1) of [1] we have (k runs over 1, 2, 3) for some S-deformation function $\tilde{\mathbf{p}}$

$$(1.1) \qquad \sum_{P \in \gamma_R(^*\Delta)} [f_P^k - m_P \ddot{\tilde{p}}(\theta_P, t)] \, v_k(\theta_P) = \sum\sum_{P, Q \in \gamma_R(^*\Delta)} f_{PQ}^k \, v_k(\theta_P) + \sum\sum_{\substack{P \in \gamma_R(^*\Delta) \\ Q \in \gamma_R(^*\Delta^e)}} f_{PQ}^k \, v_k(\theta_P).$$

We are to derive from Eq. (1.1) the well-known "standard" principle of virtual work of the continuum mechanics.

Let be given a certain internal fine partition π_Γ (cf. [2], p. 71) of $^*\Gamma$ into the surface elements S_j, $j = 1, \ldots, \breve{\omega}$, $\breve{\omega} \in {}^*N \backslash N$, and an internal set of $\breve{\omega}$ points θ_j, $\theta_j \in S_j$. Let us define for every surface element S, $S \subset {}^*\Gamma$, the set $\beta_R(S) := \{(P, Q) | (P, Q) \in {}^*\Delta \times {}^*\Delta^e \wedge \{\sigma\theta_P + (1-\sigma)\theta_Q\} \in S$ for some $\sigma \in [0, 1]\}$. Then

$$(1.2) \qquad \sum\sum_{\substack{P \in \gamma_R(^*\Delta) \\ Q \in \gamma_R(^*\Delta^e)}} f_{PQ}^k \, v_k(\theta_P) = \sum_{j=1}^{\breve{\omega}} \sum_{(P,Q) \in \beta_R(S_j)} f_{PQ}^k \, [v_k(\theta_j) + \varepsilon_k], \qquad \varepsilon_k \equiv v_k(\theta_P) - v_k(\theta_j) \in \mu(0).$$

Let us assume that for some π_Γ there exists a function $\mathbf{t}_R:\Gamma \to R^3$ such that the relations

$$(1.3) \qquad t_R^k(\boldsymbol{\theta}) = -{}^0\left(\frac{1}{\text{area } A} \sum_{(P,Q)\in\beta_R(A)} f_{PQ}^k\right), \qquad \boldsymbol{\theta} = {}^0\boldsymbol{\theta}_j \in \Gamma$$

hold for every surface element satisfying the condition $S_j \subset A \subset \mu(\boldsymbol{\theta}_j) \cap {}^*\Gamma$, $j = 1, \dots, \check{\omega}$. The function $\mathbf{t}_R:\Gamma \to R^3$ will be called an S-contact force acting across the surface Γ (from Δ^e on Δ) related to the reference configuration $\boldsymbol{\varkappa}_R$ [1]. Assuming that

$${}^0\Big(\sum_{j=1}^{\check{\omega}} \sum_{(P,Q)\in\beta_R(S_j)} f_{PQ}^k\,[v_k(\boldsymbol{\theta}_P) - v_k(\boldsymbol{\theta}_j)]\Big) = 0$$

we obtain from Eqs (1.2), (1.3) that

$$(1.4) \qquad {}^0\Big(\sum_{\substack{P\in\gamma_R({}^*\Delta)\\ Q\in\gamma_R({}^*\Delta^e)}}\!\!\!\!\!\!\sum f_{PQ}^k\, v_k(\boldsymbol{\theta}_P)\Big) = {}^0\Big(\sum_{j=1}^{\check{\omega}} t_R^k(\boldsymbol{\theta}_j)\, v_k(\boldsymbol{\theta}_j)\,\text{area } S_j\Big) = -\int_\Gamma t_R^k\, v_k\, da_R$$

under the well-known conditions for $\mathbf{t}_R$.

Now let be given a certain internal fine partition π_Δ of ${}^*\Delta$ into the parallelepipeds V_j, $j = 1, \dots, \check{\omega}$, $\check{\omega} \in {}^*N \backslash N$, and an internal set of $\check{\omega}$ points $\boldsymbol{\theta}_j$, $\boldsymbol{\theta}_j \in V_j$. Introducing certain "fictitious" points (with zero mass) situated on ∂V_j between the points of every pair $\boldsymbol{\theta}_P \in V_j$, $\boldsymbol{\theta}_Q \in {}^*\Delta \backslash V_j$, and bearing in mind that $v_k(\boldsymbol{\theta}_P) = v_k(\boldsymbol{\theta}_j) + (\theta_P^\alpha - \theta_j^\alpha)\, v_{k,\alpha}(\boldsymbol{\theta}_j) + o_k(\boldsymbol{\theta}_P - \boldsymbol{\theta}_j)$, we obtain

$$(1.5) \qquad \sum_{\substack{P\in\gamma_R({}^*\Delta)\\ Q\in\gamma_R({}^*\Delta)}}\!\!\!\!\!\!\sum f_{PQ}^k\, v_k(\boldsymbol{\theta}_P) = \sum_{j=1}^{\check{\omega}} \sum_{\substack{P\in\gamma_R(V_j)\\ Q\in\gamma_R(V_j)}}\!\!\!\!\!\!\sum f_{PQ}^k\, v_k(\boldsymbol{\theta}_P) =$$

$$= \sum_{j=1}^{\check{\omega}} \sum_{\{P,Q\}\subset\gamma_R(V_j)} f_{PQ}^k\,(\theta_P^\alpha - \theta_Q^\alpha)\, v_{k,\alpha}(\boldsymbol{\theta}_j) + \sum_{j=1}^{\check{\omega}} \sum_{\substack{P\in\gamma_R(V_j)\\ Q\in\gamma_R(V_j)}}\!\!\!\!\!\!\sum f_{PQ}^k\, v_k(\boldsymbol{\theta}_P - \boldsymbol{\theta}_j).$$

Let us assume that for some π_Δ there exist the differentiable functions $T_R^{k\alpha}(\boldsymbol{\theta}):\Delta \to R$ such that the relations

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We introduce the "fictitious" points on every ∂W, analogously as on ∂V_j.

$$(1.6)\qquad T_R^{k\alpha}(\theta) = {}^0\left(\frac{1}{\operatorname{vol} W}\sum_{\{P,Q\}\subset\gamma_R(\bar{W})} f_{PQ}^k\,(\theta_P^\alpha-\theta_Q^\alpha)\right),\qquad \theta = {}^0\theta_j\in\Delta;$$

hold for every volume element W satisfying the condition $V_j\subset W\subset\mu(\theta_j)$, where $j=1,\dots,\breve{\omega}$*). Assuming that

$$^0\Big(\sum_{j=1}^{\breve{\omega}}\sum_{\substack{P\in\gamma_R(V_j)\\ Q\in\gamma_R(V_j)}} f_{PQ}^k\, v_k(\theta_P-\theta_j)\Big)=0,$$

from Eqs (1.5), (1.6) we obtain

$$(1.7)\qquad {}^0\Big(\sum_{\substack{P\in\gamma_R(^*\Delta)\\ Q\in\gamma_R(^*\Delta)}}\sum f_{PQ}^k\, v_k(\theta_P)\Big) = {}^0\Big(\sum_{j=1}^{\breve{\omega}} T_R^{k\alpha}(\theta_j)\, v_{k,\alpha}(\theta_j)\operatorname{vol} V_j\Big) = \int_\Delta T_R^{k\alpha}\, v_{k,\alpha}\, dv_R.$$

Using an analogous procedure with respect to the LHS of Eqs (1.1) we shall postulate that for some π_Δ of $^*\Delta$ there exist the continuous functions $\mathbf{b}_R:\Delta\to R^3$, $\varrho_R:\Delta\to R^+$ such that the relations

$$(1.8)\qquad b_R^k(\theta) = {}^0\left(\frac{1}{\operatorname{vol} W}\sum_{P\in\gamma_R(W)} f_P^k\right),\qquad \varrho_R(\theta) = {}^0\left(\frac{1}{\operatorname{vol} W}\sum_{P\in\gamma_R(W)} m_P\right),$$

$$\theta\in{}^0\theta_j\in\Delta,$$

hold for every W, such that $V_j\subset W\subset\mu(\theta_j)$, where $j=1,\dots,\breve{\omega}$. We shall refer to functions defined by Eqs (1.8) as S-body force and S-density of mass, respectively. We shall also postulate that

$$(1.9)\qquad {}^0\Big(\sum_{j=1}^{\breve{\omega}}\sum_{P\in\gamma_R(V_j)}[f_P^k-m_P\,\ddot{\tilde{p}}^k(\theta_P,t)]\,[v_k(\theta_P)-v_k(\theta_j)]\Big)=0.$$

It follows that

$$(1.10)\qquad {}^0\Big(\sum_{P\in\gamma_R(^*\Delta)}[f_P^k-m_P\,\ddot{\tilde{p}}^k(\theta_P,t)]\,v_k(\theta_P)\Big)=$$

$$= {}^0\Big(\sum_{j=1}^{\breve{\omega}}[b_R^k-\varrho_R\,\ddot{p}^k(\theta_j,t)]\,v_k(\theta_j)\operatorname{vol} V_j\Big) = \int_\Delta (b_R^k-\varrho_R\,\ddot{p}^k)\,v_k\,dv_R,$$

where $\mathbf{p}(\cdot)$ is the standard part of the S-deformation function $\tilde{\mathbf{p}}(\cdot)$.

Taking into account Eqs (1.4), (1.7) and (1.10) we conclude that, under the conditions mentioned above, the relation (1.1) yields the standard well-known condition

$$(1.11)\qquad \int_\Delta (b_R^k-\varrho_R\,\ddot{p}^k)\,v_k\,dv_R+\int_\Gamma t_R^k\,v_k\,da_R=\int_\Delta T_R^{k\alpha}\,v_{k,\alpha}\,dv_R,$$

which has to hold for every $\Delta \subset \Omega$ and every differentiable function $\mathbf{v}:\Omega \to R^3$. It means that $T_R^{k\alpha}{}_{,\alpha} + b_R^k = \varrho_R \ddot{p}^k$ in Ω and $T_R^{k\alpha} n_{R\alpha} = t_R^k$ almost everywhere on $\partial\Omega$, n_R being the unit normal to $\partial\Omega$. Thus we conclude that Eq. (1.7) gives the non-standard representation of the first Piola–Kirchhoff stress tensor. Tensor $T_R^{k\alpha}$ can now be called the first Piola–Kirchhoff S-stress tensor. Bearing in mind Eq. (2.5) of [1] we also conclude that for the S-local interactions we have $d_R^k = T_R^{k\alpha}{}_{,\alpha}$ provided that Eq. (1.1) implies Eq. (1.11). If the interactions are not S-local, then the RHS of Eq. (1.1) satisfy the condition

$$^0\Big(\sum_{\substack{P\in\gamma_R(^*\Delta)\\ Q\in D}}\sum f_{PQ}^k\, v_k(\theta_P)\Big) = {}^0\Big(\sum_{j=1}^{\dot{\omega}} d_R^k(\theta_j)\, v_k(\theta_j)\,\mathrm{vol}\, V_j\Big) = \int_\Delta d_R^k\, v_k\, dv_R,$$

where

$$d_R^k(\theta) = {}^0\left(\frac{1}{\mathrm{vol}\, W}\sum_{\substack{P\in\gamma_R(W)\\ Q\in D}} f_{PQ}^k\right), \qquad \theta = {}^0\theta_j \in \Delta, \tag{1.12}$$

hold for every W, $V_j \subset W \subset \mu(\theta_j)$, $j = 1, \ldots, \check{\omega}$, where V_j is the volume element from a certain fine partition π_Δ of $^*\Delta$ and $\theta_j \in V_j$. We have tacitly assumed here that

$$^0\Big(\sum_{j=1}^{\dot{\omega}}\sum_{P\in\gamma_R(V_j)}\sum_{Q\in D} f_{PQ}^k\,[v_k(\theta_P) - v_k(\theta_j)]\Big) = 0.$$

Mind that we have obtained here the definitions (1.8), (1.12) of $\mathbf{b}_R, \varrho_R, \mathbf{d}_R$ which are slightly different from those of [1] (cf. Eqs (2.1), (2.3) of [1]).

2. Stress relations. Let $q^\beta = q^\beta(\mathbf{x}, t)$ be an inverse of the standard part $x^k = p^k(\theta, t)$ of the S-deformation function of a mass-point system $\mathcal{N} = \{D, m, \mathbf{f}, \sigma\}$. Defining $T^{\beta\alpha} \equiv T_R^{k\alpha} q^\beta{}_{,k}$, $\varrho_{PQ}^2 \equiv [\varkappa_t^k(P) - \varkappa_t^k(Q)]\,[\varkappa_t^l(P) - \varkappa_t^l(Q)]\,\delta_{kl}$ and $\alpha_{PQ}(\varrho) \equiv \sigma_{PQ}(\varrho)\,\varrho^{-1}$ (cf. [1]) by virtue of $\varkappa_t^k(P) - \varkappa_t^k(Q) = p^k{}_{,\beta}(\theta)\,(\theta_P^\beta - \theta_Q^\beta) + v^k(\theta_P - \theta_Q)$, we obtain

$$T^{\beta\alpha} = {}^0\left(\frac{1}{\mathrm{vol}\, W}\sum_{\{P,Q\}\subset\gamma_R(W)} \alpha_{PQ}(\varrho_{PQ})\,(\theta_P^\alpha - \theta_Q^\alpha)\,(\theta_P^\beta - \theta_Q^\beta)\right) \tag{2.1}$$

for every W such that $V_j \subset W \subset \mu(\theta_j)$, cf. Eq (3.6). Here and in what follows we tacitly neglect all higher-order terms. Defining the function

$$(2.2)\qquad H^{\beta\alpha}(\theta, \mathbf{C}) \equiv {}^{0}\left(\frac{1}{\operatorname{vol} W} \sum_{\{P,Q\}\subset\gamma_k(W)} \alpha_{PQ}\left(C_{\gamma\delta}(\theta^{\gamma}_P-\theta^{\gamma}_Q)(\theta^{\delta}_P-\theta^{\delta}_Q)\right)\times \right.$$
$$\left. \times(\theta^{\alpha}_P-\theta^{\alpha}_Q)(\theta^{\beta}_P-\theta^{\beta}_Q)\right),$$

we arrive at the stress relation $T^{\beta\alpha}(\theta) = H^{\beta\alpha}(\theta, \mathbf{C})$. Eq. (2.2) represents what can be called the S-stress function for the second Piola–Kirchhoff S-stress tensor (2.1).

Now assume that all interactions $\mathbf{f}_{PQ}$ have potentials $\vartheta_{PQ}(\varrho^2)$, i.e. $f^k_{PQ} = -\partial\vartheta_{PQ}(\varrho^2_{PQ})/\partial\varkappa^k_t(P)$. Putting $\tilde{\vartheta}_{PQ}(\mathbf{C}) \equiv \vartheta_{PQ}\left(C_{\gamma\delta}(\theta^{\gamma}_P-\theta^{\gamma}_Q)(\theta^{\delta}_P-\theta^{\delta}_Q)\right)$ we have

$$\frac{\partial\tilde{\vartheta}_{PQ}(\mathbf{C})}{\partial C_{\alpha\beta}} = \frac{\partial\vartheta_{PQ}}{\partial(\varrho^2_{PQ})}(\theta^{\alpha}_P-\theta^{\alpha}_Q)(\theta^{\beta}_P-\theta^{\beta}_Q).$$

At the same time

$$(2.3)\qquad T^{\beta\alpha} = {}^{0}\left(\frac{1}{\operatorname{vol} W}\sum_{\{P,Q\}\subset\gamma_k(W)} f^k_{PQ}(\theta^{\alpha}_P-\theta^{\alpha}_Q)\, q^{\beta}{}_{,k}\right) =$$
$$= -2\,{}^{0}\left(\frac{1}{\operatorname{vol} W}\sum_{\{P,Q\}\subset\gamma_k(W)} \frac{\partial\vartheta_{PQ}}{\partial(\varrho^2_{PQ})}\left[\varkappa^k_t(P)-\varkappa^k_t(Q)\right](\theta^{\alpha}_P-\theta^{\alpha}_Q)\, q^{\beta}{}_{,k}\right) =$$
$$= -2\,{}^{0}\left(\frac{1}{\operatorname{vol} W}\sum_{\{P,Q\}\subset\gamma_k(W)} \frac{\partial\tilde{\vartheta}_{PQ}(\mathbf{C})}{\partial C_{\alpha\beta}}\right).$$

Now assume that for some fine partition π_{Δ} of Δ there exists the scalar function $\varphi: R^6 \to R$, $\varphi = \varphi(\mathbf{C})$, $\mathbf{C} \equiv (C_{\alpha\beta})$, such that the relation

$$(2.4)\qquad \varphi(\theta, \mathbf{C}) \equiv -{}^{0}\left(\frac{1}{\operatorname{vol} W}\sum_{\{P,Q\}\subset\gamma_k(W)} \tilde{\vartheta}_{PQ}(\mathbf{C})\right), \qquad \theta = {}^{0}\theta_j,$$

holds for every volume element W satisfying $V_j \subset W \subset \mu(\theta_j)$, where $j = 1, \ldots, \tilde{\omega}$ (cf. Sec. **1**). Then from (2.3), (2.4) we obtain the well-known relation

$$T^{\beta\alpha} = 2\,\frac{\partial\varphi(\theta, \mathbf{C})}{\partial C_{\alpha\beta}}.$$

Thus we see that the function $\varphi(\theta, \mathbf{C})$, defined by Eq. (2.4), represents what can be called the S-strain energy function, i.e. the stress energy function derived from the potentials $\vartheta_{PQ}(\varrho)$ of the S-local interactions between the mass-points in a certain non-standard Newtonian mass-point system.

3. Conclusions. From the results obtained in the note it follows that the well-known Cauchy laws of motion as well as the stress relations of the

elasticity theory, which up to now have been assumed in the form of certain postulates, can be derived from the Newtonian mass-point mechanics. To this aim Newton's equations of motion have to be reinterpreted within a certain non-standard mathematical structure. The technique of the non-standard analysis applied to the Newtonian mass-point mechanics have led to the non-standard "discrete" interpretations of such "continuum" concepts as boundary tractions, the stress tensor, the body force, the strain–energy function, etc.

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Ч. Возьняк, **О нестандартной модели теории упругости**

Настоящая работа является продолжением сообщения [1]. В ней даётся интерпретация законов Коши движения и отношений напряжения с точки зрения систем материальной точки для упругих и гиперупругих материалов. Такой подход исходит из предпосылки, что взаимодействия в нестандартной системе материальной точки являются S-локальными, т.е. локальными в S-типологии контрольного пространства [2].