Refined macro-dynamics of periodic structures

Czeslaw Wozniak

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CZ. WOŹNIAK (WARSZAWA)

IN THIS CONTRIBUTION a certain non-asymptotic approach to the macro-dynamic modelling of micro-periodic composites is proposed. The obtained equations of macro-dynamics describe phenomena related to the micro-heterogeneity of materials, e.g. dispersion and scale effects. An example of applications to micro-vibration and wave propagation problems illustrates the usefulness of the proposed approach.

1. Introduction

AS IT IS KNOWN, by micromechanics of composite materials we mean a study of composite material behaviour wherein the *effects of the constituent materials are detected only as averaged apparent properties of the composite*, [1]. For periodic material structures which are made of a large number of repetitive micro-heterogeneous cells, the basis for averaging is a certain representative volume element (r.v.e.) of the structure. The equations of macro-mechanics can be obtained by the known asymptotic approaches to the macro-modelling of micro-heterogeneous materials, cf. [2–8], where inertial properties of a composite are uniquely determined by the averaged mass density. Hence, the asymptotic equations of macro-dynamics can be applied solely to problems in which the time-dependent excitations of the structure produce the wave-lengths much larger than the maximum length dimension of r.v.e. In order to eliminate this drawback, a certain non-asymptotic method of macro-modelling for dynamics of periodic structures will be proposed. The result of this approach will be referred to as a *refined macro-dynamics* of periodic material structures. The equations of refined macro-dynamics can also describe vibration and wave propagation problems with the wave-lengths of an order of the cell length dimensions.

The analysis will be restricted to the small displacement gradient theory and to the linear-elastic materials. The equations of refined macro-dynamics investigated in this paper will be obtained by simple calculations of averages and do not require any solution to boundary value problem on r.v.e. Hence, the proposed approach can be effectively applied to engineering problems.

Notations

Throughout the paper subscripts i, j, k, l run over the sequence 1, 2, 3 and are related to the Cartesian orthogonal coordinate system in the reference space. Indices a, b run over $1, \ldots, n$. The summation convention holds for i, j, k, l as well as for a, b unless otherwise stated. Points in the reference space $\mathbf{R}^3$ are denoted by $\mathbf{x}$, points in the region $V \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ by $\mathbf{y}$. For any differentiable

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function $f(\mathbf{x})$ we denote $f_{,i} \equiv \partial f/\partial x_i$. Symbol $\cong$ denotes an approximation admissible from a computational viewpoint. The time coordinate is denoted by τ, $\tau \in [\tau_0, \tau_f]$ and $\dot{u}(\cdot, \tau) \equiv \partial u/\partial \tau$. For an arbitrary integrable function $\phi(\cdot)$ defined almost everywhere in the region Ω in $\mathbf{R}^3$ we introduce the well-known averaging operator

$$\langle \phi \rangle(\mathbf{x}) \equiv \frac{1}{\mathrm{vol}(V)} \int_V \phi(\mathbf{x}+\mathbf{y}) dv(\mathbf{y}), \tag{1.1}$$

for every $\mathbf{x} \in \Omega_0$, where $\Omega_0 \equiv \{\mathbf{z} \in \mathbf{R}^3 : V(\mathbf{z}) \subset \Omega\}$ and $dv(\mathbf{y}) = dy_1 dy_2 dy_3$.

2. Preliminary concepts

Let the heterogeneous periodic material structure under consideration in its undeformed configuration occupy region Ω in the reference space, and $V \subset \Omega$ can be taken as a certain r.v.e. of the periodic structure under consideration. We assume that $l \equiv \max\{l_1, l_2, l_3\}$ is sufficiently small as compared to the minimum characteristic length dimension of Ω. Hence, we shall deal with a certain micro-periodic composite structure. Every element $V(\mathbf{x}) = \mathbf{x} + V$, such that $V(\mathbf{x}) \subset \Omega$, will be referred to as a cell of this structure. For the sake of simplicity we assume that $\overline{\Omega} = \cup \overline{V}(\mathbf{z})$, $\mathbf{z} \in \Lambda$, where Λ is the lattice of points in Ω which are centers of mutually disjoined cells $V(\mathbf{z})$.

In order to formulate basic hypotheses leading to macromodels of the micro-periodic body, we shall introduce two auxiliary concepts. The first of them is that of a V-macro function. A continuous function $F(\cdot)$ defined on Ω which, for every $\mathbf{x}, \mathbf{z} \in \Omega$ and $\mathbf{z} - \mathbf{x} \in V$, satisfies condition $F(\mathbf{x}) \cong F(\mathbf{z})$ is called a *continuous V-macro function*. Similarly, a continuous function $F(\cdot)$ having continuous derivatives up to k-th order, is called V-macro function if $F(\cdot)$, together with its derivatives up to k-th order, are continuous V-macro functions. Moreover, for every continuous V-macro function defined on Ω and for every integrable function $f(\cdot)$ defined on Ω we assume

$$\langle fF \rangle(\mathbf{x}) \cong \langle f \rangle(\mathbf{x}) F(\mathbf{x}) \tag{2.1}$$

for every $\mathbf{x} \in \Omega_0$. If $\langle f \rangle(\cdot)$ is a continuous V-macro function defined on Ω_0 and $F(\cdot)$ is V-macro function defined on Ω such that $F(\mathbf{x}) \cong \langle f \rangle(\mathbf{x}), \mathbf{x} \in \Omega_0$, then

$$\int_\Omega fG\, dv \cong \sum_{z \in \Lambda} \langle f \rangle(\mathbf{z}) G(\mathbf{z}) \mathrm{vol}(V) \cong \int_\Omega FG\, dv \tag{2.2}$$

holds for an arbitrary continuous V-macro function G defined on Ω. Generally speaking, V-macro functions describe the macroscopic behaviour of the body.

The second auxiliary concept we are to introduce is that of a micro-shape function. We assume that the phenomena related to heterogeneous micro-periodic material structure, from the qualitative point of view, will be described by means of independent continuous functions $h_a(\cdot)$, $a = 1, \ldots, n$, defined on $\mathbf{R}^3$ (which have physical sense only for $\mathbf{x} \in \Omega$), satisfying the conditions:

(i) $h_a(\mathbf{x}) = h_a(\mathbf{x} + l_i \mathbf{e}_i)$, $i = 1, 2, 3$ (no summation with respect to i) for every $\mathbf{x} \in \mathbf{R}^3$, where $\mathbf{e}_i$ is a versor of x_i-axis,

(ii) $\langle h_a \rangle(\mathbf{x}) = 0$, $\langle h_{a,i} \rangle(\mathbf{x}) = 0$, for every $\mathbf{x} \in \mathbf{R}^3$,

(iii) $\langle \varrho h_a \rangle(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Omega_0$, where $\varrho(\cdot)$ is a mass density function defined almost everywhere on Ω.

Condition (i) is equivalent to the statement that $h_a(\cdot)$ are V-periodic. Functions $h_a(\cdot)$ will be called *micro-oscillatory shape functions*. The choice of these shape functions has to be postulated a priori in every special problem and depends on the character of micro-oscillations which we are going to analyze. As a simple example of functions $h_a(\cdot)$ we can take trigonometric functions $\sin(2\pi a x_i/l_i)$, $i = 1, 2, 3$, $a = 1, \ldots, n$ (no summation over i).

3. Basic assumptions

The proposed method of macro-modelling for micro-periodic composite body under consideration will be based on two fundamental assumptions. The first of them takes into account the physically reasonable hypotheses that the motion of micro-periodic structure can be obtained by a superimposition of micro-oscillations on a certain macro-motion. This macro-motion is represented by a V-macro field $U_i(\cdot, \tau)$. The micro-oscillations will be described by means of the micro-oscillatory shape functions $h_a(\cdot)$ and by certain arbitrary fields $Q_i^a(\cdot, \tau)$. Denoting by $u_i(\cdot, \tau)$ a displacement field at a time instant τ from the undeformed configuration of the body, the first of the basic assumptions will be referred to as:

Micro-macro localization hypothesis

Every motion of a micro-periodic body under consideration can be represented in the form

$$u_i(\mathbf{x}, \tau) = U_i(\mathbf{x}, \tau) + h_a(\mathbf{x}) Q_i^a(\mathbf{x}, \tau), \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f], \tag{3.1}$$

where $U_i(\cdot, \tau)$, $Q_i^a(\cdot, \tau)$ are arbitrary continuous V-macro fields together with their first and second order space and time derivatives, and $h_a(\cdot)$ are postulated *a priori* linear independent micro-shape functions.

Fields $U_i(\cdot, \tau)$ and $Q_i^a(\cdot, \tau)$ will be called *macro-displacements* and *correctors*, respectively.

The second basic assumption takes into account the fact that in displacement gradients $u_{i,j}(\mathbf{x}, \tau)$ which can be obtained from Eq. (3.1) the terms involving $h_a(\mathbf{x})$ are small as compared to terms involving $h_{a,i}(\mathbf{x})$. Hence, denoting by $\varepsilon_{ij}(\mathbf{x}, \tau)$ the linearized strain tensor, we formulate the following assumption:

Micro-strain assumption

For the micro-periodic body under consideration the components $\varepsilon_{ij}(\mathbf{x}, \tau)$ of the linearized strain tensor will be assumed in the form

$$\varepsilon_{ij}(\mathbf{x}, \tau) = U_{(i,j)}(\mathbf{x}, \tau) + h_{a,(j}(\mathbf{x}) Q_{i)}^a(\mathbf{x}, \tau), \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f]. \tag{3.2}$$

Denoting by $\sigma_{ij}(\mathbf{x}, \tau)$ components of the stress tensor and by $A_{ijkl}(\mathbf{x})$ components of the tensor of elastic moduli, the stress-strain relations for a micro-periodic body are:

$$\sigma_{ij}(\mathbf{x}, \tau) = A_{ijkl}(\mathbf{x})[U_{(k,l)}(\mathbf{x}, \tau) + h_{a,(k}(\mathbf{x}) Q_{l)}^a(\mathbf{x}, \tau)], \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f]. \tag{3.3}$$

It has to be remembered that the tensor of elastic moduli $A_{ijkl}(\cdot)$ as well as the mass density $\varrho(\cdot)$ are V-periodic fields having the periods l_i along x_i-axes, respectively.

4. Analysis

The macro-modelling approach of the composite structure under consideration will be based on the assumptions formulated in Sec. 3, on the properties of V-macro functions (2.1), (2.2) and on the well-known principle of virtual work:

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv = \oint_{\partial\Omega} p_i\delta u_i da + \int_{\Omega} (\varrho b_i - \varrho\ddot{u}_i)\delta u_i dv\,, \tag{4.1}$$

where p_i, b_i are surface tractions and body forces, respectively. Condition (4.1) has to hold for every admissible virtual strain field $\delta\varepsilon_{ij}$ and every virtual displacement field δu_i. In the sequel we assume that body forces are constant and that $\delta u_i = 0$ on the boundary $\partial\Omega$. By means of Eqs. (3.1) and (3.2) we obtain $\delta u_i = \delta U_i + h_a\delta Q_i^a$, $\delta\varepsilon_{ij} = \delta U_{(i,j)} + h_{a,(j}\,\delta Q_{i)}^a$, where δU_i, δQ_i^a are arbitrary sufficiently regular V-macro functions defined on Ω, such that $\delta U_i = 0$, $\delta Q_i^a = 0$ on the boundary $\partial\Omega$. Using Eqs. (2.1) and (2.2), the left-hand side of Eq. (4.1) can be transformed as follows:

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv \cong \sum_{\mathbf{z}\in\Lambda}[\langle\sigma_{ij}\rangle(\mathbf{z},\tau)\delta U_{(i,j)}(\mathbf{z}) + \langle\sigma_{ij}h_{a,(j}\rangle(\mathbf{z},\tau)\delta Q_{i)}^a(\mathbf{z})]\,\mathrm{vol}(V)\,.$$

From Eq. (3.3) and (2.1) it follows that $\langle\sigma_{ij}\rangle(\cdot,\tau)$ and $\langle\sigma_{ij}h_{a,j}\rangle(\cdot,\tau)$ are V-macro fields defined on Ω_0. Hence, there exist V-macro fields $S_{ij}(\cdot,\tau)$, $H_{ai}(\cdot,\tau)$ defined on Ω such that

$$S_{ij}(\mathbf{x},\tau) \cong \langle\sigma_{ij}\rangle(\mathbf{x},\tau), \quad H_{ai}(\mathbf{x},\tau) \cong \langle\sigma_{ij}h_{a,j}\rangle(\mathbf{x},\tau), \quad \mathbf{x}\in\Omega_0\,. \tag{4.2}$$

By means of Eq. (2.2) we conclude that

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv \cong \int_{\Omega} (S_{ij}\delta U_{(i,j)} + H_{ai}\delta Q_i^a)dv\,. \tag{4.3}$$

For the right-hand side of Eq. (4.1), using Eqs. (2.1), (2.2) and bearing in mind that $\delta u_i = 0$ on $\partial\Omega$, we obtain

$$\int_{\partial\Omega} p_i\delta u_i da + \int_{\Omega} (\varrho b_i - \varrho\ddot{u}_i)\delta u_i dv \cong \int_{\Omega} [(\langle\varrho\rangle b_i - \langle\varrho\rangle\ddot{U}_i)\delta U_i - \langle\varrho h_a h_b\rangle\ddot{Q}_i^a\delta Q_i^b]dv\,. \tag{4.4}$$

V-macro fields $S_{ij}(\cdot,\tau)$ and $H_{ai}(\cdot,\tau)$ will be called *macro-stresses* and *structural forces*, respectively, and they represent an averaged state of stress in a micro-periodic body under consideration. On the basis of formulae (4.1), (4.3), (4.4) we shall postulate the following:

Macro-approximation of virtual work principle

Macro-stress field $S_{ij}(\cdot,\tau)$ and structural force fields $H_{ai}(\cdot,\tau)$ are interrelated with macro-displacement field $U_i(\cdot,\tau)$ and corrector fields $Q_i^a(\cdot,\tau)$ by means of condition

$$\int_{\Omega} (S_{ij}\delta U_{(i,j)} + H_{ai}\delta Q_i^a)dv = \int_{\Omega} [(\langle\varrho\rangle b_i - \langle\varrho\rangle\ddot{U}_i)\delta U_i - \langle\varrho h_a h_b\rangle\ddot{Q}_i^a\delta Q_i^b]dv\,, \tag{4.5}$$

which has to hold for arbitrary continuous (with their first derivatives) V-macro fields $\delta U_i(\cdot)$, $\delta Q_i^a(\cdot)$ defined on Ω, such that $\delta U_i = 0$, $\delta Q_i^a = 0$ on the boundary $\partial\Omega$.

Under the assumption that fields $\delta U_i(\cdot)$, $\delta Q_i^a(\cdot)$ are independent, from Eq. (4.5), we obtain

$$
\begin{aligned}
&\int_{\Omega} (S_{ij,j} + \langle \varrho \rangle b_i - \langle \varrho \rangle \ddot{U}_i) \delta U_i dv = 0, \\
&\int_{\Omega} (H_{ai} + \langle \varrho h_a h_b \rangle \ddot{Q}_i^b) \delta Q_i^a dv = 0,
\end{aligned} \tag{4.6}
$$

for every (sufficiently regular) V-macro fields δU_i, δQ_i^a, which are equal to zero on the boundary $\partial\Omega$. It can be shown that the restriction of virtual fields δU_i, δQ_i^a to V-macro fields in Eqs. (4.6) is irrelevant. Hence, using the du Bois–Reymond lemma, conditions (4.6) are equivalent to

$$
\begin{aligned}
&S_{ij,j}(\mathbf{x}, \tau) + \langle \varrho \rangle b_i - \langle \varrho \rangle \ddot{U}_i(\mathbf{x}, \tau) = 0, \\
&H_{ai}(\mathbf{x}, \tau) + \langle \varrho h_a h_b \rangle \ddot{Q}_i^b(\mathbf{x}, \tau) = 0, \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f].
\end{aligned} \tag{4.7}
$$

It can be easily observed that from Eqs. (4.2), (3.3) and using Eq. (2.1) we can assume fields $S_{ij}(\cdot, \tau)$, $H_{ai}(\cdot, \tau)$ in the form

$$
\begin{aligned}
S_{ij}(\mathbf{x}, \tau) &= \langle A_{ijkl} \rangle U_{(k,l)}(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,k} \rangle Q_l^a(\mathbf{x}, \tau), \\
H_{ai}(\mathbf{x}, \tau) &= \langle A_{ijkl} h_{a,j} \rangle U_{(k,l)}(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q_k^b(\mathbf{x}, \tau),
\end{aligned} \tag{4.8}
$$

where $\mathbf{x} \in \Omega$ and $\tau \in [\tau_0, \tau_f]$. Equations (4.7), (4.8) involve exclusively V-macro fields and constitute the final result of the foregoing analysis. The proposed method of macro-modelling in its part related to the modelling of the inertial properties of the body does not involve any asymptotic approximation. At the same time the material macro-properties of the body are obtained by using micro-strain assumption (3.2) where terms of order $O(l)$ were neglected, i.e. we have applied here an asymptotic approximation. That is why the proposed method can be referred to as a semi-asymptotic method of macro-modelling.

5. Conclusions

The macro-modelling approach introduced in Secs. 3 and 4 describes a certain generalized elastic continuum governed by the field equations (4.7) and constitutive relations (4.8). It is easy to see that the form of Eqs. (4.7) is independent of the material properties of the body under consideration, hence, Eqs. (4.7) will be called the equations of motion. Similarly, Eqs. (4.8) will be referred to as macro-constitutive equations. Combining Eqs. (4.7) and (4.8) we obtain the system of equations for macro-displacement and corrector fields

$$
\begin{aligned}
&\langle A_{ijkl} \rangle U_{k,lj}(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,l} \rangle Q_{k,j}^a(\mathbf{x}, \tau) + \langle \varrho \rangle b_i = \langle \varrho \rangle \ddot{U}_i(\mathbf{x}, \tau), \\
&\langle \varrho h_a h_b \rangle \ddot{Q}_i^b(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q_k^b(\mathbf{x}, \tau) = -\langle A_{ijkl} h_{a,j} \rangle U_{k,l}(\mathbf{x}, \tau), \\
&\qquad\qquad \mathbf{x} \in \Omega, \tau \in [\tau_0, \tau_f].
\end{aligned} \tag{5.1}
$$

These equations describe on the macro-level a micro-periodic material structure, the inertial properties of which are specified not only by the averaged mass density $\langle \varrho \rangle$ but also by *micro-inertial moduli* $\langle \varrho h_a h_b \rangle$. It has to be emphasized that the micro-inertial moduli $\langle \varrho h_a h_b \rangle$ involve the length dimension of the r.v.e. Hence, Eqs. (5.1) can be a basis for investigations of scale effects in micro-periodic composites. Moreover, the obtained equations allow to analyze micro-vibrations and propagation of micro-waves in composite

materials, what takes into account the dispersion effects due to the micro-heterogeneity of the medium. The aforementioned problems can not be investigated by means of the asymptotic homogenized continuum, cf. [2–8]; that is why we shall consider Eqs. (4.7), (4.8) and (5.1) as equations of the refined macro-dynamics of micro-periodic elastic composites.

The form of the derived equations depends on the choice and number of micro-shape functions $h_a(\cdot)$, $a = 1, \ldots, n$. Let us observe that on the micro-level (inside every cell $V(\mathbf{x})$, $\mathbf{x} \in \Omega_0$) the model is described by a proper choice of micro-shape functions; in such a manner we can obtain more or less exact descriptions of a problem. Let us also observe that the corrector fields Q_i^a are governed by the second of Eqs. (5.1) which are ordinary differential equations involving only time derivatives of correctors.

In every special problem governing equations (5.1) have to be considered together with the boundary conditions for macro-displacements U_i and with initial conditions for macro displacement U_i and correctors Q_i^a. The form of these conditions depends on the physical character of the problem and takes into account Eqs. (3.1) or (3.3); it has to be emphasized that the solutions to the boundary value problem for Eqs. (5.1) have a physical sense only if the obtained solutions U_i, Q_i^a, together with their first and second order derivatives, are continuous V-macro fields.

The refined macro-dynamics constitutes a certain generalization of the macro-modelling approach proposed in [5] and developed in [6–8], where the micro-inertial moduli were not taken into account. Disregarding in Eqs. (5.1) the terms involving $\langle \varrho h_a h_b \rangle$ as small of order $O(l^2)$, we obtain the asymptotic model of micro-periodic composites described in [5–8] and given by

$$\langle A_{ijkl} \rangle U_{k,lj}(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,l} \rangle Q^a_{k,j}(\mathbf{x}, \tau) + \langle \varrho \rangle b_i = \langle \varrho \rangle \ddot{U}_i(\mathbf{x}, \tau),$$

$$\langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q^b_k(\mathbf{x}, \tau) = -\langle A_{ijkl} h_{a,j} \rangle U_{k,l}(\mathbf{x}, \tau), \mathbf{x} \in \Omega, \tau \in [\tau_0, \tau_f].$$

In this asymptotic case correctors Q_i^a are governed by a system of $3n$ linear algebraic equations and can be eliminated from the foregoing system of equations. Such situation does not hold for Eqs. (5.1) of refined macro-dynamics, where correctors are governed by a system of ordinary differential equations, involving second-order time derivatives of correctors.

It has to be emphasized that for the homogeneous material structures we obtain $\langle A_{ijkl} h_{a,l} \rangle = A_{ijkl} \langle h_{a,l} \rangle = 0$ and hence, Eqs. (5.1) yields two independent systems of equations

$$A_{ijkl} U_{k,lj}(\mathbf{x}, \tau) + \varrho b_i = \varrho \ddot{U}_i(\mathbf{x}, \tau),$$

$$\varrho \langle h_a h_b \rangle \ddot{Q}^b_i(\mathbf{x}, \tau) + A_{ijkl} \langle h_{a,j} h_{b,l} \rangle Q^b_k(\mathbf{x}, \tau) = 0.$$

Under initial conditions $Q_i^a(\mathbf{x}, \tau_0) = 0$, $\dot{Q}_i^a(\mathbf{x}, \tau_0) = 0$, $\mathbf{x} \in \Omega$, we obtain $Q_i^a(\mathbf{x}, \tau) = 0$ for $\tau \in [\tau_0, \tau_f]$, $\mathbf{x} \in \Omega$. Hence, we conclude that the oscillatory terms in Eqs. (3.1) are due to the micro-heterogeneity of a material composite structure.

An illustrative example of applications of the refined dynamics will be given in Sec. 6.

6. Applications

The proposed model of the refined macro-dynamics will now be applied to the problem of a straight micro-periodic linear-elastic bar treated as a uniaxial structure. The representative element is now given by the interval $(-l/2, l/2)$ of the x-axis, where $x \equiv x_1$. We

assume that the Young modulus $E(x)$ and the mass density $\varrho(x)$ are piecewise constant periodic functions in $(-a/2, a/2)$ are equal to E_1, ϱ_1, and in $(-l/2, l/2)/(-a/2, a/2)$ are equal to E_2, ϱ_2, respectively. For the sake of simplicity we introduce only one micro-shape function $h(x) \equiv h_1(x)$, $x \in \mathbb{R}$, which is continuous, linear in $(-l/2, -a/2)$, $(-a/2, a/2)$, $(a/2, l/2)$, and in $[-l/2, l/2]$ takes the values $h(-l/2) = h(0) = h(l/2) = 0, h(-a/2) = 1, h(a/2) = -1$, and has the period l. By $U(\cdot, \tau)$, $Q(\cdot, \tau)$ we denote macro-displacement and corrector fields, respectively, defined on a certain interval $(-L, L)$ of the x-axis, where $L \gg l$. The derivatives of $F(\cdot)$ with respect to $x \equiv x_1$ will be denoted by $F'(\cdot)$, and the averaging operator $\langle\cdot\rangle$ will now be related to a segment $(-l/2, l/2)$ of the x-axis. From Eqs. (5.1), neglecting the body forces, we obtain

$$
\begin{aligned}
\langle E\rangle U''(x,\tau) + \langle Eh'\rangle Q'(x,\tau) &= \langle\varrho\rangle\ddot{U}(x,\tau),\\
\langle\varrho hh\rangle\ddot{Q}(x,\tau) + \langle Eh'h'\rangle Q(x,\tau) &= -\langle Eh'\rangle U'(x,\tau).
\end{aligned}
\tag{6.1}
$$

Setting $\phi_1 \equiv a/l$, $\phi_2 \equiv (l-a)/l$ we obtain

$$
\begin{gathered}
\langle\varrho\rangle = \phi_1\varrho_1 + \phi_2\varrho_2, \quad \langle E\rangle = \phi_1 E_1 + \phi_2 E_2,\\
\langle Eh'\rangle = 2(E_1 - E_2), \quad \langle Eh'h'\rangle = 4(E_1/\phi_1 + E_2/\phi_2),\\
\langle\varrho hh\rangle = l^2/3 \times \langle\varrho\rangle = l^2/3 \cdot (\phi_1\varrho_1 + \phi_2\varrho_2).
\end{gathered}
$$

The aim of the following section is to analyze the problems of free vibrations and wave propagation related to the micro-periodic bar under consideration. The analysis will be based on Eqs. (6.1).

6.1. Vibrations

Let $x \in (-L/2, L/2)$ where $L \gg l$ and $\tau \in (-\infty, \infty)$. It can be observed that Eqs. (6.1) have a solution of the form

$$
U(x,\tau) = 0, \quad Q(x,\tau) = A\cos\mu\tau + B\sin\mu\tau,
$$

where A, B are arbitrary constants and $\mu^2 = \langle Eh'h'\rangle/\langle\varrho hh\rangle$. The positive constant μ will be referred to as the free micro-vibration frequency, and will play an important role in the subsequent analysis.

First we shall look for the solution of Eqs. (6.1) in the form $U(x,\tau) = U_0(x)\exp(i\omega\tau)$, $Q(x,\tau) = Q_0(x)\exp(i\omega\tau)$. Hence

$$
\begin{aligned}
\langle E\rangle U_0''(x) + \langle\varrho\rangle\omega^2 U_0(x) + \langle Eh'\rangle Q_0'(x) &= 0,\\
(\langle\varrho hh\rangle\omega^2 - \langle Eh'h'\rangle)Q_0(x) - \langle Eh'\rangle U_0'(x) &= 0;
\end{aligned}
\tag{6.2}
$$

after simple manipulations, defining $E^{\text{eff}} \equiv \langle E\rangle - \dfrac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle} = (\phi_1/E_1 + \phi_2/E_2)^{-1}$ (cf. [1]) and introducing the micro-vibration frequency μ we obtain

$$
\left[\frac{E^{\text{eff}}}{\langle E\rangle} - \left(\frac{\omega}{\mu}\right)^2\right]U_0''(x) + \frac{\langle\varrho\rangle}{\langle E\rangle}\omega^2\left[1 - \left(\frac{\omega}{\mu}\right)^2\right]U_0(x) = 0;
\tag{6.3}
$$

a similar equation we obtain also for $Q_0(x)$.

REMARK. The effective modulus E^{eff} introduced above makes it possible in the asymptotic approximation ($l \to 0$), to represent the first equation of Eq. (4.1) in the form $E^{\text{eff}}U''(x,\tau) = \langle\varrho\rangle\ddot{U}(x,\tau)$. Let us also observe that $E^{\text{eff}}/\langle E\rangle < 1$.

From Eq. (6.3) it follows that (A, B are arbitrary constants):

(i) if $(\omega/\mu)^2 < E^{\text{eff}}/\langle E\rangle$ or $(\omega/\mu)^2 > 1$ then there exist sinusoidal vibrations $U_0(x) = A\cos kx$, $Q_0(x) = B\sin kx$;

(ii) if $E^{\text{eff}}/\langle E\rangle < (\omega/\mu)^2 < 1$ then there exist exponential vibrations $U_0(x) = A\cosh kx$, $Q_0(x) = B\sinh kx$;

(iii) if $(\omega/\mu)^2 = 1$ or $(\omega/\mu)^2 = E^{\text{eff}}/\langle E\rangle$ then we arrive at the degenerate or trivial case, respectively.

This classification holds if $\langle Eh'\rangle \neq 0$; if $\langle Eh'\rangle = 0$ then only sinusoidal vibrations are possible.

In the case (i) of sinusoidal waves, by substituting $U_0(x) = A\cos kx$, $Q_0(x) = B\sin kx$ into Eqs. (4.2) we obtain nontrivial solutions ($A \neq 0$, $B \neq 0$) only if

$$\begin{vmatrix} \omega^2\langle\varrho\rangle - k^2\langle E\rangle & k\langle Eh'\rangle \\ k\langle Eh'\rangle & \omega^2\langle\varrho hh\rangle - \langle Eh'h'\rangle \end{vmatrix} = 0\,.$$

Introducing the micro-vibration frequency μ defined above we obtain finally

$$\omega^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}k^2 + \left(\omega^2 - \frac{\langle E\rangle}{\langle\varrho\rangle}k^2\right)\left(\frac{\omega}{\mu}\right)^2. \tag{6.4}$$

The second term on the RHS of Eq. (6.4) describes the dispersion effect due to the micro-heterogeneous structure of the bar, i.e. the nonlinear interrelation between ω and k. For a homogeneous bar $E^{\text{eff}} = \langle E\rangle$ and Eq. (6.4) yields $\omega^2\langle\varrho\rangle = \langle E\rangle k^2$, $\langle E\rangle = E$. If $l \to 0$ then $\mu \to \infty$ and the dispersion effect disappears. It has to be emphasized that $U_0(\cdot)$, $Q_0(\cdot)$ have to be V-macro fields (cf. Sec. 2), where now V reduces to the line segment $(-l/2, l/2)$. Hence, for sinusoidal waves the obtained results have a physical sense only if $lk \ll 1$. Treating lk as a small parameter we can derive from Eq. (6.4) the formula

$$\omega^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}k^2\left[1 - \frac{1}{3}(lk)^2\frac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle^2}\right] + o(k^2l^2)\,,$$

which represents ω^2 in the explicit form.

In the case (ii) of exponential waves, after substituting $U_0(x) = A\cosh kx$, $Q_0(x) = B\sinh kx$ into Eqs. (6.2), we obtain nontrivial solutions only if

$$\begin{vmatrix} \omega^2\langle\varrho\rangle + k^2\langle E\rangle & k\langle Eh'\rangle \\ -k\langle Eh'\rangle & \omega^2\langle\varrho hh\rangle - \langle Eh'h'\rangle \end{vmatrix} = 0\,.$$

Hence, after introducing μ, we obtain

$$\omega^2 = \frac{\langle E\rangle}{\langle\varrho\rangle}k^2\left[\left(\frac{\omega}{\mu}\right)^2 - \frac{E^{\text{eff}}}{\langle E\rangle}\right] + \omega^2\left(\frac{\omega}{\mu}\right)^2. \tag{6.5}$$

It has to be emphasized that Eq. (6.5) has a physical sense only for micro-heterogeneous bars, because in the case of homogeneity $E^{\text{eff}}/\langle E\rangle = 1$ and there are no exponential vibrations.

Summing up we conclude that the micro-periodic heterogeneity of bars leads to the dispersion effects and to the exponential vibrations which can not be treated on the basis of the asymptotic homogenized models of periodic structures.

At the end of this example the wave propagation problem in a infinite micro-heterogeneous bar will be discussed. To this end we look for a solution of Eqs. (6.1) in the form $U(x,\tau) = F(x - c\tau)$, $Q(x,\tau) = G(x - c\tau)$, where c is the wave propagation velocity. Setting $\zeta \equiv x - c\tau$ we shall use the notation $F' = dF/d\zeta$, $G' \equiv dG/d\zeta$. Then from Eqs. (6.1) after manipulations, we obtain

$$(\langle E\rangle - \langle\rho\rangle c^2)\frac{c^2}{\mu^2}F'' + (E^{\text{eff}} - \langle\rho\rangle c^2)F = 0\,, \tag{6.6}$$

and a similar equation for G. Eq. (6.6) implies the following special cases of wave propagation in a micro-periodic bar:

(i) sinusoidal waves if $c^2 < (E^{\text{eff}}/\langle\varrho\rangle)$ or $c^2 > (\langle E\rangle/\langle\varrho\rangle)$;

(ii) exponential waves if $E^{\text{eff}}/\langle\varrho > c^2 < \langle E\rangle/\langle\varrho\rangle$;

(iii) degenerate case if $c^2 = E^{\text{eff}}/\langle\varrho\rangle$ or $c^2 = \langle E\rangle/\langle\varrho\rangle$.

Conditions (i)–(iii) hold if $\langle Eh'\rangle \neq 0$; for $\langle Eh'\rangle = 0$, $c^2 = \langle E\rangle/\langle\varrho\rangle$. In the case (i), substituting $U(x,\tau) = A\sin k(x - c\tau)$, $Q(x,\tau) = B\cos k(x - c\tau)$, $k = 2\pi/L$, where L is the wavelength and A, B are arbitrary constants, into Eqs. (6.1) we obtain nontrivial solutions only if

$$c^2\langle\varrho\rangle = E^{\text{eff}} + (\langle\varrho\rangle c^2 - \langle E\rangle)\frac{c^2k^2}{\mu^2}\,. \tag{6.7}$$

The second term on the RHS (6.7) describes the effect of dispersion. The obtained result has a physical sense only if $lk = 2\pi l/L \ll 1$ because functions $U(\cdot,\tau)$, $Q(\cdot,\tau)$ have to be V-macro functions, cf. Sec. 2. Treating kl as a small parameter and bearing in mind that $\mu^2 = \langle Eh'h'\rangle/\langle\varrho hh\rangle = 3\langle Eh'h'\rangle/l^2\langle\varrho\rangle$ from Eq. (6.7) we obtain

$$c^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}\left[1 - \frac{1}{3}(kl)^2\frac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle^2}\right] + o(k^2l^2)\,.$$

It can be seen that for $kl \to 0$ the dispersion effect disappears.

Summing up we see that the micro-heterogeneity of a bar implies the existence of exponential waves and dispersion. These effects can not be investigated on a basis of asymptotic homogenization equations (for $l \to 0$), i.e. for the known homogenized models of a bar.

7. Final remarks

The example given in Sec. 6 illustrates the fact that the refined macro-dynamics can be successfully applied to dynamic problems for micro-periodic structures. Among the advantages of the proposed macro-modelling approach we can mention the relatively simple form of the resulting equations (5.1). Moreover, the governing equations of refined macro-dynamics (4.7) and (4.8) can be obtained without any solution to a boundary value problem on the r.v.e. The main drawback lies in an imprecise choice of the micro-shape functions $h_a(\cdot)$ based often on the intuition of the researcher. Different applications of the proposed method of modelling as well as the possible generalization of this approach are now under consideration and will be presented separately.

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