# Nonlinear macro-elastodynamics of micro-periodic composites

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by

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*Presented by Z. WESOŁOWSKI on July 5, 1993*

**Summary.** In this paper a new non-asymptotic method of macro-modelling for dynamics of micro-periodic heterogeneous bodies is proposed. The obtained equations can be applied to the problems in which scale and dispersion effects, due to the micro-heterogeneous structure of a body, have to be taken into account.

**1. Introduction.** Asymptotic methods of macro-modelling for micro-periodic materials and structures are based on the assumption that the macro-behavior of a body is independent of length dimensions of a periodicity cell, [1–6]. Hence, asymptotic equations are not able to describe scale and dispersion effects due to the micro-heterogeneity of a composite and can be applied solely to the problems in which the time-dependent excitations of the structure produce the wavelength much larger than the maximal characteristic length dimension of a periodicity cell (long-wave approximation). In this contribution the non-asymptotic method of macro-modelling is proposed in the framework of which problems of vibration and wave propagation with the wavelength of order of a cell dimension can be described (short-wave approximation).

D e n o t a t i o n s. Indices $i$, $j$ run over the sequence 1, 2, 3 and are related to the orthogonal cartesian coordinates $x^i$ in the reference space. Indices $\alpha, \beta$ also run over 1, 2, 3 but are related to the material coordinates $X^\alpha$. The partial differentiation with respect to the material coordinates is denoted by a comma. Subscripts and superscripts $a, b, \ldots$ run over $1, 2, 3, \ldots, n$. Summation convention holds for all aforementioned indices. An orthonormal vector basis in the reference space will be denoted by

Key words: nonlinear macro-elastodynamics, micro-periodic composities

$(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. For averaging operator we use the notation

$$\langle f \rangle(\boldsymbol{Z}) \equiv \frac{1}{|\boldsymbol{V}_R|} \int_{\boldsymbol{V}_R} f(\boldsymbol{Z}+\boldsymbol{Y}) dv_R(\boldsymbol{Y}), \quad dv_R(\boldsymbol{Y}) = dY_1 dY_2 dY_3,$$

where $V_R \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ and $f(\cdot)$ is an arbitrary integrable function defined on $V(\boldsymbol{Z}) \equiv \boldsymbol{Z} + V_R$, for some $\boldsymbol{Z} \in \mathbb{R}^3$.

**2. Preliminary concepts.** Let $\Omega_R$ be a region occupied by a body (or by its part) in the known reference configuration. For sake of simplicity we assume that $\overline{\Omega}_R = \bigcup \overline{V(Z)}$, $\boldsymbol{Z} \in \Lambda$, where $\Lambda$ is a lattice of points in $\Omega_R$ with a basis $(l_1\boldsymbol{e}_1, l_2\boldsymbol{e}_2, l_3\boldsymbol{e}_3)$. Material and inertial properties of the body will be determined by the strain energy density $\varepsilon_R(\cdot, \boldsymbol{F})$ and the mass density $\rho_R(\cdot)$, respectively, where $\boldsymbol{F}$ is an arbitrary nonsingular $3 \times 3$ matrix. The body is assumed to be $V_R$-periodic, i.e. $\varepsilon_R(\boldsymbol{X}, \boldsymbol{F}) = \varepsilon_R(\boldsymbol{X} + l_i\boldsymbol{e}_i, \boldsymbol{F})$, $\rho_R(\boldsymbol{X}) = \rho_R(\boldsymbol{X} + l_i\boldsymbol{e}_i)$ for $i = 1, 2, 3$, (no summation with respect to "$i$"!). It means that every $V(\boldsymbol{Z})$, $\boldsymbol{Z} \in \Lambda$ is a periodicity cell of a body under consideration and $V_R$ can be treated as a region occupied by a certain representative volume element (r.v.e.) of the body. Moreover, we assume that $l \equiv \max\{l_1, l_2, l_3\}$ is sufficiently small compared to the smallest characteristic length dimension of $\Omega_R$; a body under consideration will be referred to as a micro-periodic body.

In order to formulate the basic hypothesis for the macro-models of the micro-periodic body, we shall introduce two leading concepts.

A continuous function $F(\cdot)$ defined on $\Omega_R$ will be called *$V_R$-macro function* if $F(\boldsymbol{X}) \approx F(\boldsymbol{Z})$ for every $\boldsymbol{X}, \boldsymbol{Z} \in \Omega_R$ such that $\boldsymbol{Z} - \boldsymbol{X} \in V_R$. Roughly speaking, using $V_R$-macro functions we shall describe the macroscopic behavior of the body.

By $h_a(\cdot)$, $a = 1, 2, \ldots, n$ we shall denote a sequence of continuous independent functions satisfying conditions: $h_a(\boldsymbol{X}) = h_a(\boldsymbol{X} + l_i\boldsymbol{e}_i)$, $h_a(\boldsymbol{X}) = -h_a(\boldsymbol{X} + l_i\boldsymbol{e}_i/2)$ for $i = 1, 2, 3$ (no summation with respect to "$i$") for every $\boldsymbol{X} \in \Omega_R$ and $\boldsymbol{X} + l_i\boldsymbol{e}_i \in \Omega_R$. We shall also assume that $\langle \rho_R h_a \rangle = 0$. Functions $h_a(\cdot)$ will be called *micro-oscillatory shape functions*. The shape functions have to be postulated *a priori* in every problem under consideration and play a role similar to that of shape functions used in the known finite element method. Generally speaking, functions $h_a(\cdot)$ describe from the qualitative point of view the character of micro-oscillations produced by a micro-heterogeneous structure of a body.

**3. Macro-modelling approach.** Using aforementioned concepts we shall formulate the basic hypothesis of the proposed approach. We denote by $x_i = p_i(\boldsymbol{X}, \tau)$, $\boldsymbol{X} \in \Omega_R$, a position of a point $\boldsymbol{X}$ of the body at a time instant $\tau$, $\tau \in [\tau_0, \tau_f]$, and assume that every deformation $p_i(\cdot, \tau)$ can

be represented as a superimposition of micro-oscillations (due to a micro-heterogeneous structure of a body) on a certain macro-deformation.

M a c r o - m o d e l l i n g h y p o t h e s i s (MMH). A motion of a micro-periodic body under consideration can be described by the formulae

$$p_i(\boldsymbol{X},\tau) = p_i(\boldsymbol{X},\tau_0) + U_i(\boldsymbol{X},\tau) + h_a(\boldsymbol{X})Q_i^a(\boldsymbol{X},\tau), \tag{1}$$
$$\boldsymbol{X} \in \Omega_R, \quad \tau \in [\tau_0,\tau_f],$$

where $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are $V_R$-macro functions together with their material and time derivatives of the first and second order.

Functions $U_i(\cdot,\tau)$ and $Q_i^a(\cdot,\tau)$ are called *macro-displacements* and *correctors*, respectively. It can be seen that correctors $Q_i^a(\cdot,\tau)$ describe micro-oscillations from the quantitative point of view (the known functions $h_a(\cdot)$ characterize these oscillations from the qualitative point of view).

The governing equations of the proposed macro-elastodynamics will be derived applying MMH to the action functional for a hyperelastic body under consideration

$$\mathcal{A} = \int_{\Omega_R} \Big[\frac{1}{2}\rho_R(\boldsymbol{X})\dot{p}_i(\boldsymbol{X},\tau)\dot{p}_i(\boldsymbol{X},\tau) - \varepsilon_R(\boldsymbol{X},\nabla\boldsymbol{p}) \tag{2}$$
$$+ \rho_R(\boldsymbol{X})b_i(p_i(\boldsymbol{X},\tau) - p_i(\boldsymbol{X},\tau_0))\Big]dv_R(\boldsymbol{X}),$$

where $b_i$ are constant body forces. In the sequel we assume that $\nabla\boldsymbol{p}(\cdot,\tau_0)$ is a $V_R$-macro function and denote $\boldsymbol{Q} \equiv (\boldsymbol{Q}^1,\ldots,\boldsymbol{Q}^n)$. Substituting the right-hand side of (1) into (2), taking into account that $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are $V_R$-macro functions together with their derivatives and condition $\langle\rho_R h_\alpha\rangle = 0$, we obtain

$$\begin{aligned}
\mathcal{A} \approx \sum_{Z\in\Lambda}\int_{V(Z)} \Big[&\frac{1}{2}\rho_R(\boldsymbol{X})(\dot{U}_i(\boldsymbol{Z},\tau) + h_a(\boldsymbol{X})\dot{Q}_i^a(\boldsymbol{Z},\tau))(\dot{U}_i(\boldsymbol{Z},\tau)\\
&+ h_b(\boldsymbol{X})\dot{Q}_i^b(\boldsymbol{Z},\tau)) - \varepsilon_R(\boldsymbol{X},\nabla\boldsymbol{p}(\boldsymbol{Z},\tau_0)\\
&+ \nabla\boldsymbol{U}(\boldsymbol{Z},\tau) + \nabla h_a(\boldsymbol{X})\otimes\boldsymbol{Q}^a(\boldsymbol{Z},\tau) + h_a(\boldsymbol{X})\nabla\boldsymbol{Q}^a(\boldsymbol{Z},\tau))\\
&+ \rho_R(\boldsymbol{X})b_i(U_i(\boldsymbol{Z},\tau) + h_a(\boldsymbol{X})Q_i^a(\boldsymbol{Z},\tau))\Big]dv_R(\boldsymbol{X})\\
= \sum_{Z\in\Lambda}\Big[&\frac{1}{2}\langle\rho_R\rangle\dot{U}_i(\boldsymbol{Z},\tau)\dot{U}_i(\boldsymbol{Z},\tau) + \frac{1}{2}\langle\rho_R h_a h_b\rangle\dot{Q}_i^a(\boldsymbol{Z},\tau)\dot{Q}_i^b(\boldsymbol{Z},\tau)\\
&- \langle\varepsilon_R\rangle(\boldsymbol{Z},\nabla\boldsymbol{U}(\boldsymbol{Z},\tau),\boldsymbol{Q}(\boldsymbol{Z},\tau),\nabla\boldsymbol{Q}(\boldsymbol{Z},\tau)) + \langle\rho_R\rangle b_i U_i(\boldsymbol{Z},\tau)\Big]\\
\approx \int_{\Omega_R}\Big[&\frac{1}{2}\langle\rho_R\rangle\dot{U}_i(\boldsymbol{X},\tau)\dot{U}_i(\boldsymbol{X},\tau) + \frac{1}{2}\langle\rho_R h_a h_b\rangle\dot{Q}_i^a(\boldsymbol{X},\tau)\dot{Q}_i^b(\boldsymbol{X},\tau)\\
&-\langle\varepsilon_R\rangle(\boldsymbol{X},\nabla\boldsymbol{U}(\boldsymbol{X},\tau),\boldsymbol{Q}(\boldsymbol{X},\tau),\nabla\boldsymbol{Q}(\boldsymbol{X},\tau)) + \langle\rho_R\rangle b_i U_i(\boldsymbol{X},\tau)\Big]dv_R(\boldsymbol{X}).
\end{aligned}$$

On the basis of above approximation we shall postulate that under MMH the dynamics of the micro-periodic body can be described by the action functional

$$(3)\qquad \tilde{\mathcal{A}} = \int\limits_{\Omega_R} \Big[\frac{1}{2}\langle\rho_R\rangle\dot{U}_i\dot{U}_i + \frac{1}{2}\langle\rho_R h_a h_b\rangle\dot{Q}_i^a\dot{Q}_i^b - \langle\varepsilon_R\rangle(\boldsymbol{X},\nabla\boldsymbol{U},\boldsymbol{Q},\nabla\boldsymbol{Q}) + \langle\rho_R\rangle b_i U_i\Big]dv_R.$$

Functional (3) represents the dynamic system treated as a macro-model of micro-periodic bodies under consideration.

**4. Results.** Taking into account that

$$\frac{\partial\langle\varepsilon_R\rangle}{\partial Q_{i'\alpha}^a} \approx \frac{\partial\langle\varepsilon_R h_a\rangle}{\partial U_{i'\alpha}}, \qquad \frac{\partial\langle\varepsilon_R\rangle}{\partial Q_i^a} \approx \frac{\partial\langle\varepsilon_R h_{a'\alpha}\rangle}{\partial U_{i'\alpha}}$$

and denoting

$$(4)\qquad \begin{aligned} T_R^{i\alpha} &= T_R^{i\alpha}(\boldsymbol{X},\nabla\boldsymbol{U} + \nabla h_a\otimes\boldsymbol{Q}^a + h_a\nabla\boldsymbol{Q}^a) \equiv \frac{\partial\varepsilon_R}{\partial U_{i'\alpha}},\\ A_R^{i\alpha j\beta} &= A_R^{i\alpha j\beta}(\boldsymbol{X},\nabla\boldsymbol{U} + \nabla h_a\otimes\boldsymbol{Q}^a + h_a\nabla\boldsymbol{Q}^a) \equiv \frac{\partial^2\varepsilon_R}{\partial U_{i'\alpha}\partial U_{j'\beta}}, \end{aligned}$$

after some manipulations we obtain the Lagrange equations for the functional (3),

$$(5)\qquad \begin{aligned} &\langle A_R^{i\alpha j\beta}\rangle U_{j'\alpha\beta} + \langle A_R^{i\alpha j\beta}h_{a'\beta}\rangle Q_{j'\alpha}^a + \langle A_R^{i\alpha j\beta}h_a\rangle Q_{j'\alpha\beta}^a\\ &\qquad + \langle T_R^{i\alpha}\rangle_{,\alpha} + \langle\rho_R\rangle b_i = \langle\rho_R\rangle\ddot{U}_i,\\ &\langle A_R^{i\alpha j\beta}h_a h_b\rangle Q_{j'\alpha\beta}^b + \langle A_R^{i\alpha j\beta}h_a h_{b'\beta}\rangle Q_{j'\alpha}^b + \langle A_R^{i\alpha j\beta}h_a\rangle U_{j'\alpha\beta}\\ &\qquad - \langle T_R^{i\alpha}h_a\rangle_{,\alpha} - \langle T_R^{i\alpha}h_{a'\alpha}\rangle = \langle\rho_R h_a h_b\rangle\ddot{Q}_i^b. \end{aligned}$$

The averages $\langle\cdot\rangle$ in Eqs (5) involving $A_R^{i\alpha j\beta}$ or $T_R^{i\alpha}$ depend on: $\boldsymbol{X}$, $\nabla\boldsymbol{U}(\boldsymbol{X},\tau)$, $\boldsymbol{Q}(\boldsymbol{X},\tau)$, $\nabla\boldsymbol{Q}(\boldsymbol{X},\tau)$. Equations (5) are governing ones of nonlinear macro-elastodynamics for micro-periodic heterogeneous composites. The basic kinematic unknowns are macro-displacements $U_i(\cdot,\tau)$ and correctors $Q_i^a(\cdot,\tau)$, $\tau\in[\tau_0,\tau_f]$. It has to be emphasized that solutions to Eqs (5) have physical sense only if $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ and their material and time derivatives for every $\tau\in[\tau_0,\tau_f]$ are $V_R$-macro functions.

For homogeneous bodies $T_R^{i\alpha}(\boldsymbol{X},\nabla\boldsymbol{U}(\boldsymbol{X},\tau) + \nabla h_a(\boldsymbol{X})\otimes\boldsymbol{Q}^a(\boldsymbol{X},\tau) + h_a(\boldsymbol{X})\nabla\boldsymbol{Q}^a(\boldsymbol{X},\tau)) \approx T_R^{i\alpha}(\boldsymbol{Z},\nabla\boldsymbol{U}(\boldsymbol{Z},\tau) + \nabla h_a(\boldsymbol{X})\otimes\boldsymbol{Q}^a(\boldsymbol{Z},\tau) + h_a(\boldsymbol{X})\nabla\boldsymbol{Q}^a(\boldsymbol{Z},\tau))$ for every $X\in V(\boldsymbol{Z})\cap\Omega_R$, $\boldsymbol{Z}\in\Omega_R$. It follows that for $\boldsymbol{Q}^a\equiv 0$ we obtain $\langle T_R^{i\alpha}h_{a'\alpha}\rangle \approx T_R^{i\alpha}\langle h_{a'\alpha}\rangle = 0$ and similarly we obtain $\langle A_R^{i\alpha j\beta}h_a\rangle \approx A_R^{i\alpha j\beta}\langle h_a\rangle = 0$. Hence, Eq. (5.2) for homogeneous bodies has

the solution $\boldsymbol{Q}^a \equiv 0$ and Eq. (5.1) reduces to the well-known equation of motion of nonlinear elastodynamics.

For heterogeneous bodies the correctors are not equal to zero because $\langle T_R^{i\alpha} h_{a'\alpha} \rangle \neq 0$. We assume that the correctors are only due to the heterogeneous structure of the body and to initial conditions for $\boldsymbol{Q}^a$, being independent of the boundary conditions. To this end we shall apply Eqs (5) solely to problems for unbounded media assuming that Eqs (5) hold in the whole reference space (except at some singular surfaces, lines or points). Under this assumption we neglect the effect of boundary conditions on the corrector fields. In order to pass to the boundary value problems (for bounded microperiodic bodies) we shall introduce the extra assumption.

Strain energy approximation (SEA). In the argument $\nabla \boldsymbol{p}(\boldsymbol{X}, \tau) = \nabla \boldsymbol{p}(\boldsymbol{X}, \tau_0) + \nabla \boldsymbol{U}(\boldsymbol{X}, \tau) + \nabla h_a(\boldsymbol{X}) \otimes \boldsymbol{Q}^a(\boldsymbol{X}, \tau) + h_a(\boldsymbol{X}) \cdot \nabla \boldsymbol{Q}^a(\boldsymbol{X}, \tau)$ of the strain energy density $\varepsilon_R(X, \nabla \boldsymbol{p})$, the terms involving $h_a$ are small compared to the ones depending on $\nabla h_a$ and can be neglected. Hence, we assume that $\varepsilon_R = \varepsilon_R(\boldsymbol{X}, \nabla \boldsymbol{p}(\boldsymbol{X}, \tau_0) + \nabla \boldsymbol{U}(\boldsymbol{X}, \tau) + \nabla h_a(\boldsymbol{X}) \otimes \boldsymbol{Q}^a(\boldsymbol{X}, \tau))$.

The aforementioned assumption is based on the highly oscillating character of shape functions $h_a(\cdot)$, the values of which are of order $\mathcal{O}(l)$, while their derivatives are independent of $l$. Hence, for sufficiently small $l$, the values of $h_a(\cdot)$ can be treated as small parameters in formulas involving $\varepsilon_R$.

Taking into account SEA, we neglect terms involving $h_a$ on the left-hand sides of Eqs (5). Thus, the obtained macro-model of micro-periodic bodies under consideration will be governed by

$$
\begin{aligned}
&\langle A_R^{i\alpha j\beta} \rangle U_{j'\alpha\beta} + \langle A_R^{i\alpha j\beta} h_{a'\beta} \rangle Q^a_{j'\alpha} + \langle T_R^{i\alpha} \rangle_{'\alpha} + \langle \rho_R \rangle b_i = \langle \rho_R \rangle \ddot{U}_i, \\
&- \langle T_R^{i\alpha} h_{a'\alpha} \rangle = \langle \rho_R h_a h_b \rangle \ddot{Q}^b_i,
\end{aligned} \tag{6}
$$

with denotations of (4). The averages $\langle \cdot \rangle$ in Eqs (6) involving $A_R^{i\alpha j\beta}$ or $T_R^{i\alpha}$ depend on : $\boldsymbol{X}$, $\nabla \boldsymbol{U}(\boldsymbol{X}, \tau)$, $\boldsymbol{Q}(\boldsymbol{X}, \tau)$. Equations (6) have to be considered together with boundary and initial conditions for macro-displacements $U_i(\cdot)$ and initial conditions for correctors $Q_i^a(\cdot)$, no boundary conditions for correctors $Q_i^a(\cdot)$ are required. It has to be remembered that solutions to the boundary value problems for Eqs (6) have physical sense only if $U_i(\cdot, \tau)$ and $Q_i^a(\cdot, \tau)$, $\tau \in [\tau_0, \tau_f]$ together with their derivatives are $V_R$-macro functions. Let us observe that Eqs (6) have the relatively simple form which do not involve material derivatives of correctors. In this case the micro-oscillations $h_a Q_i^a$ depend exclusively on heterogeneous structure of the body and initial conditions for $Q_i^a$. The method of macro-modelling based on both MMH and SEA can be referred to as the semi-asymptotic one. In the case of asymptotic macro-modelling (based on assumption that $l \to 0$), Eq. (6.2) reduces

to the form $\langle T_R^{i\alpha} h_{a'\alpha} \rangle = 0$; this condition represents the system of algebraic equations for correctors $Q_i^a(\cdot)$. This special model has been proposed in [3] and developed in a series of papers, e.g. [4–7].

**5. Concluding remarks.** It has to be emphasized that averages $\langle \cdot \rangle$ in Eqs (5), (6) which depend on $A_R^{i\alpha j\beta}$, $T_R^{i\alpha}$ are $V_R$-macro functions of $\boldsymbol{X}$, $\boldsymbol{X} \in \Omega_R$, and $\langle \rho_R \rangle$, $\langle \rho_R h_a h_b \rangle$ are constants. Thus, the aforementioned equations do not involve in an explicit form any highly oscillating function (like $\rho_R(\cdot), \varepsilon_R(\cdot, \boldsymbol{F})$). Hence, the obtained macro-models constitute a proper tool for numerical analysis of engineering problems for micro-periodic elastic composites.

The characteristic feature of the obtained equations of macro-elastodynamics (Eqs (5) for unbounded media as well as Eqs (6) with the pertinent boundary conditions) is that inertial properties of a body are described not only by the averaged mass density $\langle \rho_R \rangle$ but also by the micro-module $\langle \rho_R h_a h_b \rangle$. These micro-modules depend on length dimensions of r.v.e.; hence, the proposed elastodynamics of micro-periodic composites makes it possible to investigate the scale and dispersion effects on the macro-behavior of the body. The main drawback of the proposed method of macro-modelling is an unprecise choice of shape functions $h_a(\cdot)$ which is often based on the intuition of the researcher. Examples of application of Eqs (5), (6) are given in [8]. The incremental and linear form of the proposed macro-elastodynamics will be presented in separate papers.

This research was supported by Grant 3 3310 92 03 of the State Committee for Scientific Research.

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