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A lower bound on the barrier parameter of barriers on convex cones

Roland Hildebrand *

February 17, 2012

Abstract

Let $K \subset \mathbb{R}^n$ be a regular convex cone, let $e_1, \dots, e_n \in \partial K$ be linearly independent points on the boundary of a compact affine section of the cone, and let $x^* \in K^\circ$ be a point in the relative interior of this section. For $k = 1, \dots, n$, let l_k be the line through the points e_k and x^* , let y_k be the intersection point of l_k with ∂K opposite to e_k , and let z_k be the intersection point of l_k with the linear subspace spanned by all points e_l , $l = 1, \dots, n$ except e_k . We give a lower bound on the barrier parameter ν of logarithmically homogeneous self-concordant barriers $F : K^\circ \rightarrow \mathbb{R}$ on K in terms of the projective cross-ratios $q_k = (e_k, x^*; y_k, z_k)$. Previously known lower bounds by Nesterov and Nemirovski can be obtained from our result as a special case. As an application, we construct an optimal barrier for the epigraph of the $\|\cdot\|_\infty$ -norm in \mathbb{R}^n and compute lower bounds on the barrier parameter for the power cone and the epigraph of the $\|\cdot\|_p$ -norm in \mathbb{R}^2 .

1 Introduction

In modern convex optimization, interior point methods are the primary tool to solve conic programs. A central role in solution algorithms for conic programs over some regular (with nonempty interior, containing no lines) convex cone K is assigned to a smooth real-valued convex function $F : K^\circ \rightarrow \mathbb{R}$ on the interior of the cone, the *barrier*. In order to be useful for optimization, the barrier has to satisfy certain properties [4, Section 2.3]. The second and third derivative have to satisfy the self-concordance relation

$$F'''(x)[h, h, h] \leq 2(F''(x)[h, h])^{3/2} \quad \forall x \in K^\circ, h \in T_x K^\circ, \quad (1)$$

with h running through the tangent space at x . The function F has to tend to infinity as its argument tends to the boundary of the cone,

$$\lim_{x \rightarrow \partial K} F(x) = +\infty, \quad (2)$$

and it has to satisfy the logarithmic homogeneity condition

$$F(\alpha x) = -\nu \log \alpha + F(x) \quad \forall \alpha > 0, x \in K^\circ. \quad (3)$$

A smooth function $F : K^\circ \rightarrow \mathbb{R}$ satisfying conditions (1,2,3) is called a *logarithmically homogeneous self-concordant barrier* for the cone K . The real constant ν is called the *barrier parameter* of the barrier F .

The lower the barrier parameter of a barrier, the faster are the interior point algorithms based on this barrier. For conic optimization problems over a cone K , it is therefore desirable to have barriers on K with a barrier parameter as small as possible. We call a logarithmically homogeneous self-concordant barrier on K *optimal* if it has the lowest possible barrier parameter.

Optimality of a barrier F is proven by verifying properties (1,2,3) and showing that the barrier parameter ν of F is equal to a lower bound ν_* on this parameter for the given cone. For general cones, all lower bounds on the barrier parameter which are available today are based on a result of Nesterov

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and Nemirovski [4, Sect. 2.3.4]. Namely, if for some boundary point $z \in \partial K$ of the cone there exists a neighbourhood U of z and affine half-spaces $A_1, \dots, A_k \subset \mathbb{R}^n$ with $z \in \partial A_j$, $j = 1, \dots, k$, such that the normals to the half-spaces at z are linearly independent and the intersection $U \cap K$ equals the intersection $U \cap A_1 \cap \dots \cap A_k$ ¹, then a lower bound on the barrier parameter of any self-concordant barrier on K is given by $\nu_* = k$. Based on this result, Güler and Tunçel proved that the minimum over the Carathéodory numbers of all points in the interior of K also is a lower bound on the barrier parameter [2, Prop. 4.1]. In this way, the standard barriers for the symmetric cones used in linear, conic quadratic, and semi-definite programming are shown to be optimal. Optimal barriers can be constructed also for general homogeneous cones, with the barrier parameter equal to the rank of the cone [2, Theorem 4.1].

In this contribution, we provide a new lower bound on the barrier parameter of barriers on a general cone (Theorems 4.4, 5.4). For n -dimensional cones, this lower bound is contained in the interval $[2, n]$ (Corollary 3.2 and Theorem 5.7). From our result, a slightly stronger bound than that in [4, Sect. 2.3.4] follows as special case (Theorem 6.1). In contrast to the previously known bounds, our results are non-trivial also for "round" cones, i.e., cones with a smooth boundary. The bound is a function of the projective cross-ratio of geometric objects that are constructed from n boundary points of the cone in general position and an interior point. As an application, we compute lower bounds on the barrier parameter for the epigraph of the $\|\cdot\|_\infty$ -norm (Corollary 6.2), the power cone (Corollary 7.1), and the epigraph of the $\|\cdot\|_p$ -norm in \mathbb{R}^2 (Corollary 7.2).

The remainder of the paper is structured as follows. In the next section we introduce the projective cross-ratio and consider some of its elementary properties. In Section 3 we prove an auxiliary result, which essentially applies to 2-dimensional cones. In Section 4 we deduce the lower bound on the barrier parameter of barriers on general cones and in the subsequent section we investigate its properties. In Section 6 we deduce the bounds in [4, Sect. 2.3.4] from our result, and in the last section we apply our results to the power cone and the epigraph of the $\|\cdot\|_p$ -norm. Finally, in the appendix we construct an optimal barrier for the epigraph of the $\|\cdot\|_\infty$ -norm.

2 The projective cross-ratio

Let $x_1, x_2, x_3, x_4 \in \mathbb{R} \cup \{\infty\}$ be four distinct points in the 1-point compactification of the real line. The *projective cross-ratio* of the quadruple (x_1, x_2, x_3, x_4) is defined as the number

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} \in \mathbb{R},$$

where the differences containing the value ∞ are cancelled in the event that one of the points in the quadruple is ∞ . This function can be extended continuously to a $(\mathbb{R} \cup \{\infty\})$ -valued function on the set of quadruples of points in $\mathbb{R} \cup \{\infty\}$ of which no three coincide.

The projective cross-ratio is invariant under projective transformations of $\mathbb{R} \cup \{\infty\}$, and can hence also be considered as a $(\mathbb{R} \cup \{\infty\})$ -valued function on quadruples of points on the real projective line $\mathbb{R}P^1$. Alternatively, it can be considered as a $(\mathbb{R} \cup \{\infty\})$ -valued function on quadruples of 1-dimensional linear subspaces of \mathbb{R}^2 , as the set of such subspaces is isomorphic to $\mathbb{R}P^1$.

As can be easily checked, the projective cross-ratio possesses the symmetry

$$(x_1, x_2; x_3, x_4) = \frac{1}{(x_2, x_1; x_3, x_4)}, \quad (4)$$

with the values 0 and ∞ being considered as reciprocal.

In the next two sections we consider the projective cross-ratio as defined on quadruples of coplanar lines through the origin, while in the example sections it will be more convenient to consider projective cross-ratios of quadruples of collinear points.

¹Actually, in [4, Prop. 2.3.6] it is required that K is polyhedral, but this is not used in the proof.

3 2-dimensional linear sections

In this section we prove an auxiliary result which essentially provides a construction of the optimal barrier on a 2-dimensional convex cone under the condition that the direction of the gradient of the barrier at some interior point of the cone is fixed to some value.

Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a regular convex cone, $x^* \in K^\circ$ a point in the interior of K , l a line containing x^* and intersecting the boundary ∂K of the cone in the points e, y . Denote by L the 2-dimensional linear hull of l and by l_{x^*}, l_e, l_y the 1-dimensional linear subspaces spanned by the vectors x^*, e, y , respectively. Let $F : K^\circ \rightarrow \mathbb{R}$ be a logarithmically homogeneous self-concordant barrier on K with barrier parameter ν , and define $p^* = -F'(x^*)$. Note that p^* is a linear functional on \mathbb{R}^n , located in the interior of the dual cone K^* . The kernel of p^* is an $n - 1$ -dimensional linear subspace $L_{p^*} \subset \mathbb{R}^n$, which intersects K in the origin. Denote the 1-dimensional intersection of L_{p^*} with the 2-dimensional subspace L by l_{p^*} . Then $l_{x^*}, l_e, l_y, l_{p^*}$ are four mutually distinct coplanar lines, and we can define their projective cross-ratio $r = (l_e, l_y; l_{x^*}, l_{p^*})$. The arrangement of the lines implies that $-\infty < r < 0$.

Lemma 3.1. *Assume above notations. Then ν is bounded from below by $\nu_* = \frac{2}{1 + \frac{|r+1|}{r-1}} = 1 + \max(-r, -r^{-1})$.*

Proof. Let $K_L = K \cap L$ and let F_L be the restriction of F on the interior K_L° of K_L . Then F_L is a self-concordant barrier for the 2-dimensional cone K_L with the same barrier parameter ν as K . Moreover, the linear functional $p_L^* = -F_L'(x^*)$ is the restriction of the linear functional p^* on L , and hence its kernel coincides with l_{p^*} . The assertion of the lemma for the cone K thus reduces to the assertion for the cone K_L .

In order to avoid unnecessary notations, we shall assume without restriction of generality that $n = 2$ and hence $K = K_L$. Let $\gamma_* = \frac{\nu_* - 2}{\sqrt{\nu_* - 1}} = \frac{|r+1|}{\sqrt{-r}}$ and let $\lambda_\pm^* = -\frac{\gamma_*}{2} \pm \sqrt{\frac{\gamma_*^2}{4} + 1}$ be the roots of the quadratic equation $\lambda^2 + \gamma_*\lambda - 1 = 0$. Then $-r\gamma_*^2 = (r+1)^2$ and hence $r = -\frac{\gamma_*^2 + 2}{2} \pm \gamma_*\sqrt{\frac{\gamma_*^2}{4} + 1} = -(\lambda_\mp^*)^2$. Since $\gamma_* \geq 0$, we have $\lambda_-^* \leq -1$ and $0 < \lambda_+^* \leq 1$. Therefore $r = -(\lambda_-^*)^2$ if $r \leq -1$ and $r = -(\lambda_+^*)^2$ if $r \geq -1$. Note that in the former case we have $(l_y, l_e; l_{x^*}, l_{p^*}) = r^{-1} = -(\lambda_+^*)^2$ by (4).

Let us now introduce a coordinate system in \mathbb{R}^2 such that the lines l_{x^*}, l_{p^*} are parallel to the basis vectors $(1, 0)^T, (0, 1)^T$, respectively. Let further the lines l_e, l_y be parallel to the vectors $(1, \lambda_\pm^*)^T$, respectively, if $r \geq -1$, and to the vectors $(1, \lambda_\mp^*)^T$, respectively, if $r < -1$. This is always possible because

$$(\lambda_+^*, \lambda_-^*; 0, \infty) = \frac{(\lambda_+^* - 0)(\lambda_-^* - \infty)}{(\lambda_-^* - 0)(\lambda_+^* - \infty)} = \frac{\lambda_+^*}{\lambda_-^*} = -(\lambda_+^*)^2 = \begin{cases} (l_y, l_e; l_{x^*}, l_{p^*}), & r < -1; \\ (l_e, l_y; l_{x^*}, l_{p^*}), & r \geq -1, \end{cases}$$

and the projective cross-ratio of a quadruple of distinct lines through the origin of a plane is the only invariant of the quadruple with respect to the general linear group of this plane. Let us further scale the coordinates by a homothety such that $x^* = (1, 0)^T$. In this coordinate system the cone K is given by the set $\{x = (x_1, x_2)^T \mid x_1 \geq 0, \lambda_-^* x_1 \leq x_2 \leq \lambda_+^* x_1\}$, and $p^* = -F'(x^*) = (\nu, 0)^T$ due to the identity $\langle p^*, x^* \rangle = \nu$ [4, eq. (2.3.13)].

Consider the 1-dimensional Lie group $A(t) = \exp(t \cdot a) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ generated by the Lie algebra element $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let $x(t) = A(t)x^* = (1, t)^T$ and consider the scalar $f(t) = \nu^{-1}F(x(t))$. We shall now establish a differential inequality which is satisfied by the function $f(t)$. Formulas [4, eq. (2.3.12–14)] and its derivatives yield the relations $F'(x)[x] = -\nu$, $F''(x)[x, \cdot] = -F'(x)$, $F'''(x)[x, \cdot, \cdot] = -2F''(x)$. At $x(t) = A(t)x^*$ we thus have

$$\begin{aligned} F' &= \nu A^T(-t) \begin{pmatrix} -1 \\ \alpha \end{pmatrix}, \quad F'' = \nu A^T(-t) \begin{pmatrix} 1 & -\alpha \\ -\alpha & \beta \end{pmatrix} A(-t), \\ F'''[A(t)h, \cdot, \cdot] &= -2\nu A^T(-t) \left[\begin{pmatrix} 1 & -\alpha \\ -\alpha & \beta \end{pmatrix} h_1 + \begin{pmatrix} -\alpha & \beta \\ \beta & \mu \end{pmatrix} h_2 \right] A(-t) \end{aligned} \quad (5)$$

for some $\alpha, \beta, \mu \in \mathbb{R}$. Here $\beta > \alpha^2$, because $F'' \succ 0$, and $h = (h_1, h_2)^T \in \mathbb{R}^2$ is an arbitrary vector. Condition (1), applied to the vector $A(t)h$, then yields

$$-h_1^3 + 3\alpha h_1^2 h_2 - 3\beta h_1 h_2^2 - \mu h_2^3 \leq \sqrt{\nu}(h_1^2 - 2\alpha h_1 h_2 + \beta h_2^2)^{3/2}.$$

Setting $h = (\alpha - \sqrt{\frac{\beta - \alpha^2}{\nu - 1}}, 1)^T$, we obtain after some calculations

$$-(\alpha^3 + 3\alpha(\beta - \alpha^2) + \mu) \leq \gamma(\beta - \alpha^2)^{3/2}, \quad (6)$$

where $\gamma = \frac{\nu - 2}{\sqrt{\nu - 1}}$. Let us compute the derivatives of $f(t)$. By (5) and using the fact that $\dot{x} = (0, 1)^T$ is constant, we have

$$\dot{f} = \nu^{-1} F'(x)[ax] = \alpha, \quad \ddot{f} = \nu^{-1} F''(x)[ax, ax] = \beta, \quad f^{(3)} = \nu^{-1} F'''(x)[ax, ax, ax] = -2\mu.$$

Inserting this into (6), we get

$$4\dot{f}^3 - 6\dot{f}\ddot{f} + f^{(3)} \leq 2\gamma(\ddot{f} - \dot{f}^2)^{3/2}. \quad (7)$$

We would like to bound the function $f(t)$ by the solution of the differential equation which is obtained when one assumes equality in (7). This is accomplished by introducing the variables $q_{\pm} = -\frac{\sqrt{\dot{f} - \dot{f}^2}}{\lambda_{\mp}} - \dot{f}$, where $\lambda_{\pm} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + 1}$ are the roots of the quadratic equation $\lambda^2 + \gamma\lambda - 1 = 0$. These variables satisfy the relation $q_+ > q_-$ and \dot{f}, \ddot{f} can be recovered from them by the formula

$$\dot{f} = -\frac{\lambda_+}{\lambda_+ - \lambda_-} q_- + \frac{\lambda_-}{\lambda_+ - \lambda_-} q_+, \quad \ddot{f} = \frac{\lambda_+}{\lambda_+ - \lambda_-} q_-^2 - \frac{\lambda_-}{\lambda_+ - \lambda_-} q_+^2. \quad (8)$$

We have $\ddot{f} - \dot{f}^2 = \frac{(q_+ - q_-)^2}{\gamma^2 + 4}$, and hence $(\ddot{f} - \dot{f}^2)^{3/2} = \frac{(q_+ - q_-)^3}{(\gamma^2 + 4)^{3/2}}$. A simple calculus then shows that $2\gamma(\ddot{f} - \dot{f}^2)^{3/2} - 4\dot{f}^3 + 6\dot{f}\ddot{f} = -\frac{2\lambda_+}{\lambda_+ - \lambda_-} q_-^3 + \frac{2\lambda_-}{\lambda_+ - \lambda_-} q_+^3$ and hence by (7)

$$f^{(3)} \leq -\frac{2\lambda_+}{\lambda_+ - \lambda_-} q_-^3 + \frac{2\lambda_-}{\lambda_+ - \lambda_-} q_+^3. \quad (9)$$

Recall that $p^* = (\nu, 0)^T$ and hence $\dot{f}(0) = 0$. Hence by (8) we have

$$\lambda_+ q_-(0) = \lambda_- q_+(0). \quad (10)$$

Differentiating the first relation in (8) with respect to t and expressing \ddot{f} by the second relation, we obtain

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} (q_-^2 + \dot{q}_-) - \frac{\lambda_-}{\lambda_+ - \lambda_-} (q_+^2 + \dot{q}_+) = 0. \quad (11)$$

Differentiating the second relation in (8) and inserting into (9) yields the inequality

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} q_-(q_-^2 + \dot{q}_-) - \frac{\lambda_-}{\lambda_+ - \lambda_-} q_+(q_+^2 + \dot{q}_+) \leq 0. \quad (12)$$

Combining (11) with (12) yields

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} (q_- - q_+)(q_-^2 + \dot{q}_-) \leq 0, \quad \frac{\lambda_-}{\lambda_+ - \lambda_-} (q_- - q_+)(q_+^2 + \dot{q}_+) \leq 0,$$

which by the relation $q_+ > q_-$ gives

$$q_-^2 + \dot{q}_- \geq 0, \quad q_+^2 + \dot{q}_+ \leq 0. \quad (13)$$

The solution of the differential equation $q^2 + \dot{q} = 0$ is given by $q(t) = \frac{1}{t + q^{-1}(0)}$. The differential inequalities (13) then yield the bounds

$$q_-(t) \geq \frac{1}{t + q_-^{-1}(0)}, \quad t \geq 0; \quad q_+(t) \geq \frac{1}{t + q_+^{-1}(0)}, \quad t \leq 0. \quad (14)$$

By (10) and the condition $q_+ > q_-$ we have $q_-(0) < 0$ and $q_+(0) > 0$. Note that $\lim_{t \rightarrow \lambda_+^*} f(t) = +\infty$, hence by convexity of f we have $\lim_{t \rightarrow \lambda_+^*} \dot{f}(t) = +\infty$ and $\lim_{t \rightarrow \lambda_+^*} q_-(t) = -\infty$. The right-hand side in the first relation in (14) tends to $-\infty$ for $t \rightarrow -q_+^{-1}(0)$, and hence $\lambda_+^* \geq -q_+^{-1}(0)$. Likewise, $\lim_{t \rightarrow \lambda_-^*} f(t) = +\infty$, hence $\lim_{t \rightarrow \lambda_-^*} \dot{f}(t) = -\infty$ and $\lim_{t \rightarrow \lambda_-^*} q_+(t) = +\infty$. The right-hand side in the second relation in (14) tends to $+\infty$ for $t \rightarrow -q_+^{-1}(0)$, and hence $\lambda_-^* \geq -q_+^{-1}(0)$.

We thus get $\frac{\lambda_+^*}{\lambda_+} \geq -\frac{1}{\lambda_+ q_-(0)}$, $-\frac{\lambda_-^*}{\lambda_-} \geq \frac{1}{\lambda_- q_+(0)}$. Combining with (10), this yields $\frac{\lambda_+^*}{\lambda_+} \geq \frac{\lambda_-^*}{\lambda_-}$. Finally, inserting the values $\lambda_- = -\sqrt{\nu-1}$, $\lambda_+ = \frac{1}{\sqrt{\nu-1}}$, $\lambda_-^* = -\sqrt{\nu_*-1}$, $\lambda_+^* = \frac{1}{\sqrt{\nu_*-1}}$, we arrive at the desired conclusion. \square

Corollary 3.2. *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a regular convex cone, and $F : K^\circ \rightarrow \mathbb{R}$ a logarithmically homogeneous self-concordant barrier on K with barrier parameter ν . Then $\nu \geq 2$.* \square

If $n = 2$ and the bound on the barrier parameter given by Lemma 3.1 is saturated, then $\lambda_+^* = \lambda_+ = -q_+^{-1}(0)$ and $\lambda_-^* = \lambda_- = -q_+^{-1}(0)$. Inequalities (14) must be saturated too and $\dot{f}(t) = -\frac{\lambda_+}{\lambda_+ - \lambda_-} \frac{1}{t - \lambda_+} + \frac{\lambda_-}{\lambda_+ - \lambda_-} \frac{1}{t - \lambda_-}$. This can be integrated, yielding $f(t) = -\frac{\lambda_+}{\lambda_+ - \lambda_-} \log(t - \lambda_+) + \frac{\lambda_-}{\lambda_+ - \lambda_-} \log(t - \lambda_-) + \text{const}$, thus determining the barrier F up to an additive constant. It is not hard to check that F is invariant under the action of the Lie group $\exp(t \cdot a)$ generated by the Lie algebra element $a = \begin{pmatrix} 0 & 1 \\ 1 & -\gamma \end{pmatrix}$. This group acts transitively on the set of rays constituting the interior of the cone K , and hence the expression ν_* in Lemma 3.1 is independent of the choice of the interior point x^* in this case.

4 Main result

In this section a lower bound on the barrier parameter ν for a logarithmically homogeneous self-concordant barrier $F : K^\circ \rightarrow \mathbb{R}$ on a given regular convex cone $K \subset \mathbb{R}^n$ is obtained from Lemma 3.1 by the following consideration. Let $x^* \in K^\circ$ be an interior point of K , denote $p^* = -F'(x^*)$, and let L_1, \dots, L_m be 2-dimensional linear subspaces containing x^* . For fixed p^* and for each $i = 1, \dots, m$, application of Lemma 3.1 to the intersection $K_{L_i} = L_i \cap K$ gives rise to a lower bound $\nu_i^*(p^*)$ on ν , and hence $\nu \geq \max_{i=1, \dots, m} \nu_i^*(p^*)$. If m is smaller than the dimension n of the cone K , then for subspaces L_1, \dots, L_m in general position p^* can be chosen such that $\max_{i=1, \dots, m} \nu_i^*(p^*) = 2$, and no information is gained with respect to Corollary 3.2. If, however, $m \geq n$, then $\min_{p^*} \max_{i=1, \dots, m} \nu_i^*(p^*) > 2$ in general, except in the case that K is the Lorentz cone. The goal of this section is to solve the minimax problem $\min_{p^*} \max_{i=1, \dots, m} \nu_i^*(p^*)$ for the case $m = n$. We will work under the following assumption.

Assumption 4.1. *Let $K \subset \mathbb{R}^n$ be a regular convex cone, let $x^* = (x_1^*, \dots, x_n^*)^T \in K^\circ$, and let $L_1, \dots, L_n \subset \mathbb{R}^n$ be 2-dimensional linear subspaces containing x^* . Assume that there exist linearly independent vectors $e_i \in L_i \cap \partial K$, $i = 1, \dots, n$, and let $y_i \in L_i \cap \partial K$ be such that $L_i = \text{span}\{e_i, y_i\}$, $i = 1, \dots, n$. Let l_k be the line through the points e_k and y_k , and let z_k be the intersection point of l_k with the $(n-1)$ -dimensional linear subspace \tilde{L}_k spanned by all points e_l , $l = 1, \dots, n$ except e_k . Let $l_{e_k}, l_{y_k}, l_{z_k}$, and l_{x^*} be the linear hulls of e_k, y_k, z_k , and x^* , respectively. Then for each $k = 1, \dots, n$ the 4 lines $l_{e_k}, l_{y_k}, l_{z_k}, l_{x^*}$ are in L_k and hence coplanar, and no three of them are identical. Let $q_k = [l_{e_k}, l_{x^*}; l_{y_k}, l_{z_k}]$ be their projective cross-ratio.*

Now let $F : K^\circ \rightarrow \mathbb{R}$ be a logarithmically homogeneous self-concordant barrier with parameter ν , set $p^* = -F'(x^*)$, and let $l_{p_k^*}$ be the 1-dimensional intersection of the plane L_k with the kernel of the linear functional p^* . We shall express the projective cross-ratio $r_k = [l_{e_k}, l_{y_k}; l_{x^*}, l_{p_k^*}]$, which determines the bound $\nu_k^*(p^*)$ from Lemma 3.1, in terms of the projective cross-ratios q_k and the linear functional p^* .

Introduce a coordinate system in \mathbb{R}^n with basis vectors equal to e_k . Let p_k^* be the elements of p^* in these coordinates. Note that p^* is in the interior of the dual cone K^* , and hence $\langle p^*, e_k \rangle = p_k^* > 0$ for all $k = 1, \dots, n$.

In each of the planes L_k , consider the affine line \tilde{l}_k through x^* which is parallel to l_{e_k} . Introduce a real parameter λ on \tilde{l}_k , putting the number λ in correspondence with the point $x^* + \lambda e_k$. Then the intersection point of \tilde{l}_k with l_{e_k} is the infinitely remote point and can be assigned the parameter

$\lambda = +\infty$. The intersection point of \tilde{l}_k with l_{x^*} has parameter value $\lambda = 0$. The coordinate vector of the intersection point of \tilde{l}_k with l_{z_k} has a zero at its k -th entry and thus has parameter value $\lambda = -x_k^*$. Let λ_k be the parameter value corresponding to the intersection point of \tilde{l}_k with l_{y_k} . Since the segment of \tilde{l}_k corresponding to the parameter values $\lambda \in [0, \infty)$ lies in the interior of K , we have $\lambda_k < 0$. Finally, the intersection point of \tilde{l}_k with $l_{p_k^*}$ is given by the equation $\langle p^*, x^* + \lambda e_k \rangle = 0$, which yields the value $\hat{\lambda}_k = -\frac{\langle p^*, x^* \rangle}{p_k^*}$ for the parameter of this intersection point. Note that $\hat{\lambda}_k < \lambda_k$, because p^* has to be positive on y_k . We then obtain the projective cross-ratios

$$q_k = [l_{e_k}, l_{x^*}; l_{y_k}, l_{z_k}] = \frac{(\infty - \lambda_k)(0 - (-x_k^*))}{(\infty - (-x_k^*))(0 - \lambda_k)} = -\frac{x_k^*}{\lambda_k}, \quad (15)$$

$$r_k = [l_{e_k}, l_{y_k}; l_{x^*}, l_{p_k^*}] = \frac{(\infty - 0)(\lambda_k - \hat{\lambda}_k)}{(\infty - \hat{\lambda}_k)(\lambda_k - 0)} = 1 - \frac{\hat{\lambda}_k}{\lambda_k} = 1 + \frac{\langle p^*, x^* \rangle}{p_k^* \lambda_k}, \quad (16)$$

from which we get

$$\frac{r_k + 1}{r_k - 1} = 1 + \frac{2p_k^* \lambda_k}{\langle p^*, x^* \rangle}. \quad (17)$$

Lemma 4.2. *The relation $\sum_{k: q_k > 0} q_k > 1$ holds.*

Proof. Let $I_+ = \{k \mid q_k > 0\}$ and $Q_+ = \sum_{k \in I_+} q_k$.

If $I_+ = \emptyset$, then x^* is in the nonpositive orthant by (15). Thus $-x^*$ is also in K as a linear combination of the basis vectors e_k with nonnegative coefficients. This contradicts the regularity of K .

Hence $I_+ \neq \emptyset$. For every k , $x^* + \lambda_k e_k$ is a positive multiple of y_k and hence equals a nonzero vector in ∂K . Consider the point $s = \sum_{k \in I_+} q_k (x^* + \lambda_k e_k) = (Q_+ - 1)x^* + (x^* - \sum_{k \in I_+} x_k^* e_k)$, where the second relation comes from (15). As a linear combination of nonzero elements in K , with positive coefficients, the point s is also a nonzero vector in K . The vector $x^* - \sum_{k \in I_+} x_k^* e_k$ has only nonpositive components and is hence in $-K$. It follows that $(Q_+ - 1)x^* = s - (x^* - \sum_{k \in I_+} x_k^* e_k)$ is a nonzero vector in K , which implies $Q_+ > 1$ and proves the lemma. \square

By Lemma 3.1, the barrier parameter ν of the barrier F is bounded from below by the expression $\max_{k=1, \dots, n} \frac{2}{1 - \left| \frac{r_k + 1}{r_k - 1} \right|}$. This bound still depends on the negative gradient p^* of F at x^* . A lower bound on the barrier parameter of an arbitrary logarithmically homogeneous self-concordant barrier on K is then given by

$$\nu_* = \min_{p^*} \max_k \frac{2}{1 - \left| \frac{r_k + 1}{r_k - 1} \right|} = \frac{2}{1 - \min_{p^*} \max_k \left| \frac{r_k + 1}{r_k - 1} \right|}, \quad (18)$$

where p^* is subject to the constraints $\langle p^*, e_k \rangle > 0$, $\langle p^*, y_k \rangle > 0$ for all $k = 1, \dots, n$. By (17) this transforms into

$$\nu_* = \frac{2}{1 - \min_{p^*} \max_k \left| 1 + \frac{2p_k^* \lambda_k}{\langle p^*, x^* \rangle} \right|}, \quad (19)$$

where the components of p^* have to satisfy the requirements $p_k^* > 0$ and $-\frac{\langle p^*, x^* \rangle}{p_k^*} = \hat{\lambda}_k < \lambda_k$. Equivalently, $0 < \frac{p_k^*}{\langle p^*, x^* \rangle} < -\frac{1}{\lambda_k}$.

Introduce variables $\alpha_k = 1 + \frac{2p_k^* \lambda_k}{\langle p^*, x^* \rangle} \in (-1, 1)$. These variables have to satisfy the additional requirement $\sum_{k=1}^n \frac{\alpha_k - 1}{2\lambda_k} x_k^* = 1$, which by (15) is equivalent to $\sum_{k=1}^n \frac{1 - \alpha_k}{2} q_k = 1$. We shall now solve the minimax problem

$$\min \left\{ \max_k |\alpha_k| \mid \alpha_k \in (-1, 1), \sum_{k=1}^n \frac{1 - \alpha_k}{2} q_k = 1 \right\}. \quad (20)$$

Lemma 4.3. *The value of problem (20) is given by $\frac{|\sum_k q_k - 2|}{\sum_k |q_k|} < 1$.*

Proof. First note that $\sum_k |q_k| \geq \sum_k q_k$ and hence $\sum_k q_k - 2 < \sum_k |q_k|$. On the other hand, by Lemma 4.2 we have $\sum_k (q_k + |q_k|) > 2$ and hence $2 - \sum_k q_k < \sum_k |q_k|$. It follows that $\frac{|\sum_k q_k - 2|}{\sum_k |q_k|} \in [0, 1)$. It is easily checked that this value is attained by the solution $\alpha_k = \alpha_k^* = \frac{\sum_l q_l - 2}{\sum_l |q_l|} \operatorname{sgn} q_k \in (-1, 1)$.

On the other hand, every feasible vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ has to satisfy the constraint $\sum_k \alpha_k q_k = \sum_k q_k - 2$. It follows that $(\max_k |\alpha_k|) \sum_k |q_k| \geq \sum_k |\alpha_k| |q_k| \geq |\sum_k q_k - 2|$, which yields $\max_k |\alpha_k| \geq \frac{|\sum_k q_k - 2|}{\sum_k |q_k|}$ and thus proves optimality of the solution $\alpha_k = \alpha_k^*$. \square

Inserting the optimal value of (20) into (19), we obtain the following theorem.

Theorem 4.4. *Under Assumption 4.1, the barrier parameter ν of any logarithmically homogeneous self-concordant barrier F on the cone K is bounded from below by the quantity*

$$\nu_* = \frac{2}{1 - \frac{|\sum_k q_k - 2|}{\sum_k |q_k|}}.$$

5 Properties of the lower bound

In this section we will consider the bound given by Theorem 4.4 in more detail.

Lemma 5.1. *Assume the conditions of Assumption 4.1. Let $I \subset \{1, \dots, n\}$ be such that the vectors in the set $\{y_k \mid k \in I\} \cup \{e_k \mid k \notin I\}$ are linearly independent. Then the bound in Theorem 4.4 is invariant under an interchange of the points e_k and y_k for all $k \in I$.*

Proof. The exchange of y_k and e_k leaves the subspace L_k and the line $l_{p_k^*}$ invariant and hence by (4) leads to the transformation $r_k \mapsto r_k^{-1}$ in (16). This in turn leads to the transformation $\frac{r_k+1}{r_k-1} \mapsto -\frac{r_k+1}{r_k-1}$. The set of constraints $\langle p^*, e_k \rangle > 0$, $\langle p^*, y_k \rangle > 0$ is also left invariant, and hence by (18) the value of ν_* is left unchanged. \square

Remark 5.2. The assertion of Lemma 5.1 cannot be easily inferred directly from the explicit expression of the bound ν_* in Theorem 4.4, because the exchange of e_k and y_k for one index k changes the lines l_{z_l} for all $l \neq k$ and hence all projective cross-ratios q_l in (15) are changed.

Lemma 5.3. *Assume the conditions of Assumption 4.1. There exists a subset $I \subset \{1, \dots, n\}$ of indices with complement \bar{I} such that the set $\{e_k \mid k \in \bar{I}\} \cup \{y_k \mid k \in I\}$ is linearly independent, and x^* is a linear combination of the vectors in this set with nonnegative coefficients.*

Proof. The assertion of the lemma will follow from the statement that we can render all entries of x^* nonnegative by exchanging the vectors e_k and y_k for a number of indices k and adapting the coordinate system accordingly. Let us prove this statement.

Suppose there exists an index k such that $x_k^* < 0$. Recall that z_k is the unique point on l_k situated in the linear subspace \hat{L}_k spanned by all e_l except e_k . Therefore, linear dependence of the vectors $e_1, \dots, e_{k-1}, y_k, e_{k+1}, \dots, e_n$ is equivalent to the relation $z_k = y_k$. Since $x_k^* < 0$, the line l_{x^*} intersects l_k in a point opposite to e_k with respect to z_k . But y_k is opposite to e_k with respect to l_{x^*} in L_k , hence $z_k \neq y_k$. This proves that we can exchange the roles of e_k and y_k without violating the condition of linear independence of the points e_l .

Assume without restriction of generality that x^* is situated on the line segment between e_k and y_k . This is equivalent to multiplication of x^* by a positive constant and does not change the signs of the entries x_k^* . We have the explicit expression $z_k = \frac{x^* - x_k^* e_k}{1 - x_k^*}$, deriving from the condition that z_k is the affine combination of e_k and x^* whose k -th entry vanishes. On the other hand, we have $y_k = \frac{x^* + \lambda_k e_k}{1 + \lambda_k}$, deriving from the condition that y_k is the affine combination of e_k and x^* which is a multiple of $x^* + \lambda_k e_k$. It follows that

$$x^* = \frac{x_k^*(1 + \lambda_k)}{\lambda_k + x_k^*} y_k + \frac{\lambda_k(1 - x_k^*)}{\lambda_k + x_k^*} z_k = \frac{x_k^*(1 + \lambda_k)}{\lambda_k + x_k^*} y_k + \frac{\lambda_k}{\lambda_k + x_k^*} (x^* - x_k^* e_k). \quad (21)$$

Note that $x^* - x_k^* e_k$ is a linear combination of the points $e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n$, with coefficients being equal to the corresponding entries of x^* . From (21) it then follows that the coefficients \tilde{x}_l^* of x^* , when expressed as a linear combination of the vectors $e_1, \dots, e_{k-1}, y_k, e_{k+1}, \dots, e_n$, are given by

$$\tilde{x}_k^* = \frac{x_k^*(1 + \lambda_k)}{\lambda_k + x_k^*}, \quad \tilde{x}_l^* = \frac{\lambda_k}{\lambda_k + x_k^*} x_l^*, \quad l \neq k.$$

Since x^* is situated between y_k and z_k on the line l_k and is different from these points, we have by (21) that $\frac{x_k^*(1 + \lambda_k)}{\lambda_k + x_k^*} > 0$, $\frac{\lambda_k(1 - x_k^*)}{\lambda_k + x_k^*} > 0$, and hence also $\frac{\lambda_k}{\lambda_k + x_k^*} > 0$. Therefore the sign of \tilde{x}_l^* equals that of x_l^* for $l \neq k$, while \tilde{x}_k^* is positive.

As a consequence, exchanging e_k and y_k has led to a decrease in the number of negative entries of x^* by one. Repeating this process, we can eliminate all negative entries of x^* . \square

Theorem 5.4. *In addition to Assumption 4.1, suppose that x^* is contained in the simplicial cone generated by the vectors e_k . Then the lower bound in Theorem 4.4 is given by*

$$\nu_* = \begin{cases} \frac{\sum_k q_k}{\sum_k q_k - 1}, & \sum_k q_k \leq 2; \\ \sum_k q_k, & \sum_k q_k \geq 2. \end{cases} \quad (22)$$

Proof. Choose a coordinate system in \mathbb{R}^n with basis vectors e_k , $k = 1, \dots, n$. Then by assumption of the theorem all entries of x^* are nonnegative, and by (15) all projective cross-ratios q_k are nonnegative. Then the bound in Theorem 4.4 simplifies to (22). \square

We now consider the situation when $q_k = 0$ for some k . In this case $x^* = z_k$ is contained in the $(n - 1)$ -dimensional linear subspace $\hat{L}_k \subset \mathbb{R}^n$ spanned by all e_l except e_k . For every $l \neq k$, since both e_l, x^* are in \hat{L}_k , we also have $y_l \in \hat{L}_k$. We can then apply the construction of the previous section to the $n - 1$ 2-dimensional subspaces $L_1, \dots, L_{k-1}, L_{k+1}, \dots, L_n$ of the $n - 1$ -dimensional cone $\tilde{K} = K \cap \hat{L}_k$. Since all q_l , $l \neq k$, retain their values in the lower-dimensional space, and $\sum_{l=1}^n q_l = \sum_{l \neq k} q_l$, $\sum_{l=1}^n |q_l| = \sum_{l \neq k} |q_l|$ by $q_k = 0$, Theorem 4.4 will yield the same bound ν_* for the cone \tilde{K} as it has for the cone K . In other words, in the case $q_k = 0$ the bound given by Theorem 4.4 for the cone K is essentially a consequence of a similar bound for the cone \tilde{K} , which is a linear section of K with codimension 1.

Lemma 5.5. *Assume the conditions of Theorem 5.4. Then for all $k = 1, \dots, n$ we have $q_k \leq 1$, and for all index sets $I \subset \{1, \dots, n\}$ of cardinality $n - 1$ we have $\sum_{k \in I} q_k \geq 1$.*

Proof. Assume the notations of the previous section. Since the simplicial cone K_S generated by the vectors e_k is contained in K , the interval $I_S = l_k \cap K_S$ is contained in the interval $I_K = l_k \cap K$ for every $k = 1, \dots, n$. Note that y_k is the endpoint of I_K opposite to e_k , while z_k is the endpoint of I_S opposite to e_k . Hence y_k is situated on l_k opposite to e_k with respect to z_k (but it may coincide with z_k). Recall that the parameter of the intersection point of l_{e_k} with \tilde{l}_k was $\lambda = +\infty$, the intersection point of l_{z_k} with \tilde{l}_k had parameter $\lambda = -x_k^*$, and the intersection point of l_{y_k} with \tilde{l}_k had parameter $\lambda = \lambda_k$. Therefore $\lambda_k \leq -x_k^*$, which by (15) yields $q_k \leq 1$.

Let us prove the second part by contradiction. Suppose there exists a subset $I \subset \{1, \dots, n\}$ of $n - 1$ indices such that $\sum_{k \in I} q_k < 1$. For $l \in I$, define $a_l = \frac{q_l}{1 - \sum_{k \in I} q_k} \geq 0$. We then have

$$x^* + \sum_{l \in I} a_l(1 + \lambda_l)y_l = x^* + \sum_{l \in I} \frac{q_l}{1 - \sum_{k \in I} q_k} (x^* + \lambda_l e_l) = \frac{1}{1 - \sum_{k \in I} q_k} \left(x^* - \sum_{l \in I} x_l^* e_l \right),$$

where for the second equality we used (15). The leftmost side is the sum of an interior point of K and boundary points of K and is hence also an interior point of K . On the other hand, the rightmost side is proportional to $e_{\hat{k}}$, where \hat{k} is the index missing in I , and is hence a boundary point of K . This leads to a contradiction, thus proving the lemma. \square

Corollary 5.6. *Assume the conditions of Theorem 5.4. Then the bound given in Theorem 4.4 satisfies $\nu_* \leq n$. The equality $\nu_* = n$ holds if and only if K is either a simplicial cone with generators e_1, \dots, e_n , or a simplicial cone with generators y_1, \dots, y_n .*

Proof. By (22) the first assertion of the corollary is equivalent to the inequalities

$$\frac{n}{n-1} \leq \sum_k q_k \leq n.$$

These can be obtained by summing the inequalities $q_k \leq 1$, $\sum_{k \in I} q_k \geq 1$ from Lemma 5.5 over all indices k or all index sets I of cardinality $n-1$, respectively.

Assume now that $\nu_* = n$. By (22) we then have either $\sum_k q_k = \frac{n}{n-1}$ or $\sum_k q_k = n$.

Let us consider the first case. By Lemma 5.5 we have $q_k = \frac{1}{n-1}$ for all k , and hence $\lambda_k = -(n-1)x_k^*$. Note that y_k is a positive multiple of the point $x^* + \lambda_k e_k$, hence $y_k = \beta_k(x^* - (n-1)x_k^* e_k)$, $\beta_k > 0$ for all k . Let $\hat{k} \in \{1, \dots, n\}$ and $I = \{1, \dots, n\} \setminus \{\hat{k}\}$. Then we have

$$\sum_{k \in I} \beta_k^{-1} y_k = \sum_{k \in I} (x^* - (n-1)x_k^* e_k) = (n-1)x_{\hat{k}}^* e_{\hat{k}},$$

and $e_{\hat{k}}$ is contained in the relative interior of the convex cone generated by the set $\{y_k \mid k \in I\}$. But $e_{\hat{k}} \in \partial K$, which implies that this cone is entirely contained in ∂K . Repeating this argument for all \hat{k} , we see that the boundary of the simplicial cone generated by the y_k is contained in ∂K . On the other hand, $\sum_{k=1}^n \beta_k^{-1} y_k = \sum_{k=1}^n (x^* - (n-1)x_k^* e_k) = x^* \in K^\circ$, which implies that the points y_k are linearly independent and that K equals the cone generated by the y_k .

We now pass to the case $\sum_k q_k = n$. By Lemma 5.5 we have $q_k = 1$ for all k . This is equivalent to the relations $y_k = z_k$, and y_k is contained in the cone generated by the set $\{e_l \mid l \neq k\}$ for all k . Moreover, it follows that $z_k \in \partial K$ and hence $z_k \neq x^*$, which implies $x_k^* > 0$ for all k . By $\lambda_k = -x_k^*$ we then have that y_k is a positive multiple of $x^* - x_k^* e_k = \sum_{l \neq k} x_l^* e_l$, and y_k is in the relative interior of the cone generated by the set $\{e_l \mid l \neq k\}$. As in the previous paragraph, this whole cone must then be in ∂K for all k , and ∂K contains the boundary of the simplicial cone generated by the e_k . Thus K equals this simplicial cone. This proves one direction of the equivalence asserted in the second part of the corollary.

Let us prove the opposite direction. Assume that K is a simplicial cone generated by either the vectors e_k or the vectors y_k . By possibly exchanging the roles of e_k and y_k for all k , we by virtue of Lemma 5.1 can assume that K is generated by the e_k . The line l_k intersects ∂K in e_k and in the face opposite to e_k , which implies that the second intersection point y_k coincides with z_k . But then $\lambda_k = -x_k^*$ and $q_k = 1$ for all k , which by (22) yields the assertion to be proven. \square

Theorem 5.7. *Assume the conditions of Assumption 4.1. The lower bound ν_* given in Theorem 4.4 cannot exceed the dimension n of the cone K . The equality $\nu_* = n$ holds if and only if K is a simplicial cone with generators $\tilde{e}_1, \dots, \tilde{e}_n$ such that $\tilde{e}_k \in \{e_k, y_k\}$ for every $k = 1, \dots, n$.*

Proof. The theorem is a consequence of Lemmas 5.1, 5.3, and Corollary 5.6. \square

6 Relation with Nesterovs and Nemirovskis bound

In this section we deduce the lower bound of Nesterov and Nemirovski, applied to convex cones, from our result. Proposition 2.3.6 in [4] states that if $C \subset \mathbb{R}^n$ is a convex polytope and $e_n \in \partial C$ is a boundary point belonging to exactly k $(n-1)$ -dimensional faces of C , such that the normal vectors to these faces are linearly independent, then k is a lower bound on the barrier parameter of any self-concordant barrier on C . We are interested in the situation when C is a cone and prove the following slightly stronger result for this case.

Theorem 6.1. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a regular polyhedral convex cone, let $e_{k+1} \in \partial K$ be a nonzero boundary point belonging to exactly k $(n-1)$ -dimensional faces of K , $2 \leq k \leq n-1$, such that the normals to these faces are linearly independent. Then $\nu_* = k+1$ is a lower bound on the barrier parameter of any logarithmically homogeneous self-concordant barrier on K .*

Proof. By an appropriate choice of coordinates, we can assume that $e_{k+1} = (1, 0, \dots, 0)^T$, the set $\{x = (x_0, \dots, x_{n-1})^T \in K \mid x_0 = 1\}$ is a compact section of K , and there exists a neighbourhood U of e_{k+1} such that $x \in U \cap K$ if and only if $x \in U$ and $x_j \leq 0$, $j = 1, \dots, k$. Let us further assume without restriction of generality that $k = n - 1$, otherwise we replace the cone K by the $(k + 1)$ -dimensional intersection $K \cap L$, where $L \subset \mathbb{R}^n$ is the $(k + 1)$ -dimensional linear subspace determined by the equations $x_{k+1} = \dots = x_{n-1} = 0$.

Set $x^* = (1, -\varepsilon, \dots, -\varepsilon)^T$, $e_j = (1, -\varepsilon, \dots, -\varepsilon, 0, -\varepsilon, \dots, -\varepsilon)^T$, $j = 1, \dots, n - 1$, where the zero is located at position j . For small enough $\varepsilon > 0$ we then have $x^* \in K^\circ$, $e_j \in \partial K$, $j = 1, \dots, n$. Let further l_j be the line through x^* and e_j and let y_j be the intersection point of l_j with ∂K opposite to e_j , $j = 1, \dots, n$. We then have $y_j = (1, -\varepsilon, \dots, -\varepsilon, -\alpha_j, -\varepsilon, \dots, -\varepsilon)^T$ for $j = 1, \dots, n - 1$, $y_n = (1, -\alpha_n, \dots, -\alpha_n)^T$, and $\lim_{\varepsilon \rightarrow 0} \alpha_j > 0$ for all $j = 1, \dots, n$. For $i \in \{1, \dots, n\}$, denote by $\hat{L}_i \subset \mathbb{R}^n$ the $(n - 1)$ -dimensional linear subspace spanned by all e_j except e_i , and let z_i be the intersection point of \hat{L}_i with the line l_i . It is not hard to check that $z_i = (1, -\varepsilon, \dots, -\varepsilon, -\beta_i, -\varepsilon, \dots, -\varepsilon)^T$, $i = 1, \dots, n - 1$, $z_n = (1, -\beta_n, \dots, -\beta_n)^T$ with $\beta_i = \varepsilon \frac{n-2}{n-3}$, $i = 1, \dots, n - 1$, and $\beta_n = \varepsilon \frac{n-2}{n-1}$. For $n = 3$ the line l_i does not intersect L_i , $i = 1, 2$, and we set $\beta_1 = \beta_2 = \infty$ in this case.

The projective cross-ratios (15) are then given by

$$q_j = [0, -\varepsilon; -\alpha_j, -\varepsilon \frac{n-2}{n-3}] = \frac{\alpha_j}{(n-2)(\alpha_j - \varepsilon)}, \quad j = 1, \dots, n-1;$$

$$q_n = [0, -\varepsilon; -\alpha_n, -\varepsilon \frac{n-2}{n-1}] = \frac{-\alpha_n}{(n-2)(\alpha_n - \varepsilon)}.$$

Theorem 4.4 then yields the lower bound

$$\nu_* = \frac{2}{1 - \frac{|\sum_{j=1}^{n-1} \frac{\alpha_j}{\alpha_j - \varepsilon} - \frac{\alpha_n}{\alpha_n - \varepsilon} - 2(n-2)|}{\sum_{j=1}^n \frac{\alpha_j}{\alpha_j - \varepsilon}}}$$

on the barrier parameter of any logarithmically homogeneous self-concordant barrier on K . The relation $\lim_{\varepsilon \rightarrow 0} \nu_* = n$ completes the proof. \square

Corollary 6.2. *Let $n \geq 3$. The epigraph of the $\|\cdot\|_\infty$ -norm in \mathbb{R}^{n-1} ,*

$$K_{n,\infty} = \{(x_0, \dots, x_{n-1})^T \mid x_0 \geq |x_k| \ \forall k = 1, \dots, n-1\} \subset \mathbb{R}^n, \quad (23)$$

cannot have a logarithmically homogeneous self-concordant barrier with barrier parameter less than $\nu_ = n$.* \square

An optimal barrier for the cone $K_{n,\infty}$ is given by

$$F(x_0, \dots, x_{n-1}) = -\sum_{k=1}^{n-1} \log(x_0^2 - x_k^2) + (n-2) \log x_0. \quad (24)$$

For a proof see the appendix.

Note that since every convex quadrangle is projectively equivalent to a square, the barrier (24) for $n = 3$ yields an optimal barrier also for an arbitrary regular polyhedral cone $K \subset \mathbb{R}^3$ which is generated by 4 extreme rays.

7 Further examples

7.1 Power cone

The power cone is a 3-dimensional regular convex cone defined by

$$K_p = \{(u, v, w)^T \mid u \geq 0, v \geq 0, u^{1/p} v^{1/q} \geq |w|\}, \quad (25)$$

where $p \in (2, \infty)$ is a parameter and $\frac{1}{p} + \frac{1}{q} = 1$. In [3] Nesterov proposed a logarithmically homogeneous self-concordant barrier with barrier parameter $\nu = 4$ for this cone. In [1] a logarithmically homogeneous

self-concordant barrier with barrier parameter $\nu = 3$ was proposed and a logarithmically homogeneous function with homogeneity parameter $\nu = 3 - \frac{2}{p}$ was conjectured to be self-concordant. We shall now give a lower bound on the barrier parameter of any logarithmically homogeneous self-concordant barrier on K_p .

Consider the compact section $D = \{(u, v, w)^T \in K_p \mid u + v = 1\}$ of K_p . Introducing the variable $\rho = u - v \in [-1, 1]$, we can parameterize D by $(\rho, w)^T$. Inserting the relations $u = \frac{1+\rho}{2}$, $v = \frac{1-\rho}{2}$ into the inequality defining the cone K_p , we see that D is given by the set $\{(\rho, w)^T \mid |\rho| \leq 1, (1+\rho)^{1/p}(1-\rho)^{1/q} \geq 2|w|\}$.

Let now $\gamma > 0$ be the unique positive root of the transcendent equation

$$q(1 + \gamma^{-1/q}) = p(1 + \gamma^{-1/p}). \quad (26)$$

Choose $\rho_2, \rho_3 \in (-1, 1)$ such that $\rho_2 < \rho_3$ and $\gamma = \frac{(1-\rho_3)(1+\rho_2)}{(1+\rho_3)(1-\rho_2)}$. Then ρ_3 can be expressed as a function of ρ_2 by $\rho_3 = \frac{1+\rho_2-\gamma(1-\rho_2)}{1+\rho_2+\gamma(1-\rho_2)}$. Define further $w_k = \frac{1}{2}(1 + \rho_k)^{1/p}(1 - \rho_k)^{1/q}$, $k = 2, 3$.

Set $e_1 = (1, 0)^T$, $y_1 = (-1, 0)^T$, $e_2 = (\rho_2, w_2)^T$, $y_2 = (\rho_3, -w_3)^T$, $e_3 = (\rho_2, -w_2)^T$, $y_3 = (\rho_3, w_3)^T$. All these 6 points are located on the boundary of D , and the lines l_k through e_k, y_k , $k = 1, 2, 3$, intersect in the common point $x^* = (\frac{\rho_2 w_3 + \rho_3 w_2}{w_2 + w_3}, 0)^T$, which lies in the interior of the triangle formed by the e_k . The line l_1 intersects the line through e_2, e_3 in the point $z_1 = (\rho_2, 0)^T$, while l_3 intersects the line through e_1, e_2 in the point

$$z_3 = \frac{1}{(1 - \rho_2)(w_2 + w_3) + w_2(\rho_3 - \rho_2)} \begin{pmatrix} \rho_2 w_3(1 - \rho_2) + w_2(2\rho_3 - \rho_2\rho_3 - \rho_2) \\ w_2(w_2(1 - \rho_3) + w_3(1 - \rho_2)) \end{pmatrix}.$$

From this we readily compute the projective cross-ratios

$$q_1 = \frac{2w_2(\rho_3 - \rho_2)}{(1 - \rho_2)(w_2(1 + \rho_3) + w_3(1 + \rho_2))}, \quad (27)$$

$$q_2 = q_3 = \frac{w_2(1 - \rho_3) + w_3(1 - \rho_2)}{2w_3(1 - \rho_2)}, \quad (28)$$

and hence

$$\sum_{k=1}^3 q_k - 1 = \frac{w_2(1 + \rho_3)(w_2(1 - \rho_3) + w_3(1 - \rho_2))}{w_3(1 - \rho_2)(w_2(1 + \rho_3) + w_3(1 + \rho_2))} = \frac{1 + \gamma^{1/p}}{1 + \gamma^{1/q}} > 1.$$

By (22) we then obtain

$$\nu_* = q_1 + q_2 + q_3 = 1 + \frac{1 + \gamma^{1/p}}{1 + \gamma^{1/q}}. \quad (29)$$

Corollary 7.1. *Let $K_p \subset \mathbb{R}^3$ be the power cone given by (25) with parameter $p \geq (2, +\infty)$. If F is a logarithmically homogeneous self-concordant barrier on K_p , then its barrier parameter satisfies the inequality $\nu \geq 1 + \frac{1+\gamma^{1/p}}{1+\gamma^{1/q}}$, where γ is given by (26) and $\frac{1}{p} + \frac{1}{q} = 1$. \square*

In Fig. 1 the lower bound ν_* as a function of p is depicted along with the barrier parameters of the barriers proposed in [3] and [1]².

7.2 Epigraph of the $\|\cdot\|_p$ norm

Next we consider the epigraph of the $\|\cdot\|_p$ -norm in \mathbb{R}^2 for $1 \leq p \leq \infty$, i.e., the 3-dimensional cone

$$K_{3,p} = \{(x_0, x_1, x_2)^T \mid x_0 \geq (|x_1|^p + |x_2|^p)^{1/p}\}. \quad (30)$$

In [3] Nesterov proposed a method to construct a logarithmically homogeneous self-concordant barrier with barrier parameter $\nu = 2\tilde{\nu}$ for this cone if a logarithmically homogeneous self-concordant barrier for the corresponding power cone K_p is available which has barrier parameter $\tilde{\nu}$. In [1] the universal barrier [4, Sect. 2.5] for $K_{3,p}$ was computed and a barrier parameter $\nu = \frac{3p}{p+1}$ ($p \geq 2$) for this barrier

²As noted by an anonymous referee, expression (29) can be rewritten as $2 + (1 - q/p)\gamma^{1/p}$.

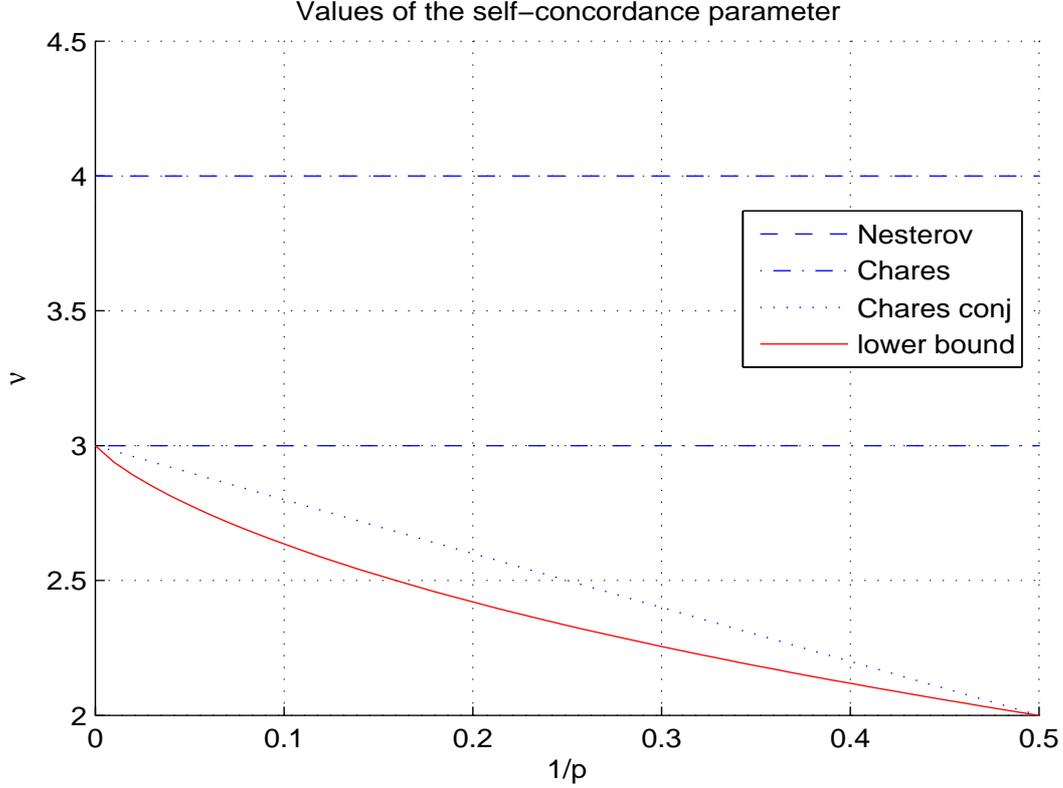


Figure 1: Barrier parameters of barriers for the power cone and lower bound

was conjectured on the basis of randomized numerical experiments. We shall now give a lower bound on the barrier parameter for any logarithmically homogeneous self-concordant barrier on $K_{3,p}$.

For $p = 2$, $K_{3,p}$ is the 3-dimensional Lorentz cone, whose optimal barrier parameter is 2. For $p = 1$ and $p = \infty$ $K_{3,p}$ is a polyhedral cone with 4 extreme rays. This case was considered in the previous section, where it was established that the optimal barrier parameter equals 3.

Let us consider the case $2 < p < \infty$. Let $\gamma \in (0, 1)$ be the unique solution of the equation $(1 - \frac{2}{p})\gamma^{1-1/p} + (1 - \frac{1}{p})\gamma^{1-2/p} - \frac{1}{p} = 0$. Note that this definition coincides with (26). For $\delta \in (0, 1)$, set

$$\rho_2 = 1 - \delta, \quad \rho_3 = 1 - \gamma\delta, \quad w_2 = (1 - \rho_2^p)^{1/p}, \quad w_3 = (1 - \rho_3^p)^{1/p}. \quad (31)$$

Consider the compact section $D = \{(\rho, w)^T \mid (1, \rho, w)^T \in K_{3,p}\}$ of the cone $K_{3,p}$ and let $e_1, e_2, e_3 \in \partial D$, $x^* \in D^\circ$ be as in the previous subsection. Then the projective cross-ratios q_k will be given by the same formulas (27), and

$$q_1 + q_2 + q_3 = 2 + \frac{w_2^2(1 - \rho_3^2) - w_3^2(1 - \rho_2^2)}{w_3(1 - \rho_2)(w_3(1 + \rho_2) + w_2(1 + \rho_3))}. \quad (32)$$

Now note that $\frac{1-x^2}{(1-x^p)^{2/p}}$ is a strictly monotonely decreasing function for $x \in (0, 1)$, hence $\frac{1-\rho_3^2}{w_3^2} < \frac{1-\rho_2^2}{w_2^2}$ and $q_1 + q_2 + q_3 < 2$. By (22) we then get the lower bound

$$\nu_* = \frac{q_1 + q_2 + q_3}{q_1 + q_2 + q_3 - 1} = 2 + \frac{w_2^2(1 - \rho_2^2) - w_3^2(1 - \rho_3^2)}{w_2(1 + \rho_3)[w_3(1 - \rho_2) + w_2(1 - \rho_3)]} = 2 + \frac{(w_3/w_2)^2(2 - \delta) - \gamma(2 - \gamma\delta)}{(2 - \gamma\delta)(w_3/w_2 + \gamma)}.$$

We have

$$\lim_{\delta \rightarrow 0} \frac{w_3}{w_2} = \left(\lim_{\delta \rightarrow 0} \frac{1 - (1 - \gamma\delta)^p}{1 - (1 - \delta)^p} \right)^{1/p} = \left(\lim_{\delta \rightarrow 0} \frac{\gamma(1 - \gamma\delta)^{p-1}}{(1 - \delta)^{p-1}} \right)^{1/p} = \gamma^{1/p}, \quad (33)$$

and therefore

$$\lim_{\delta \rightarrow 0} \nu_* = 2 + \frac{\gamma^{2/p} - \gamma}{\gamma^{1/p} + \gamma} = 2 + \frac{\gamma^{1/p} - \gamma^{1-1/p}}{1 + \gamma^{1-1/p}},$$

which, remarkably, coincides with (29).

Let us now consider the case $1 < p < 2$. Let $\gamma \in (0, 1)$ be the unique solution of the equation $(\frac{2}{p} - 1)\gamma^{1/p} + \frac{1}{p}\gamma^{2/p-1} - (1 - \frac{1}{p}) = 0$. For $\delta \in (0, 1)$, define ρ_2, ρ_3, w_2, w_3 by (31) and let $e_1, e_2, e_3 \in \partial D$, $x^* \in D^\circ$ be again as in the previous subsection. Then we again obtain (32), but $\frac{1-x^2}{(1-x^p)^{2/p}}$ is now a strictly monotonely increasing function for $x \in (0, 1)$. Hence $\frac{1-\rho_3^2}{w_3^2} > \frac{1-\rho_2^2}{w_2^2}$ and $q_1 + q_2 + q_3 > 2$. By (22) we then get the lower bound

$$\nu_* = q_1 + q_2 + q_3 = 2 + \frac{(w_2/w_3)^2(2 - \gamma\delta)\gamma - (2 - \delta)}{(2 - \delta) + w_2/w_3(2 - \gamma\delta)}.$$

As in (33) we get $\lim_{\delta \rightarrow 0} \frac{w_2}{w_3} = \gamma^{-1/p}$, which yields

$$\lim_{\delta \rightarrow 0} \nu_* = 2 + \frac{\gamma^{1-2/p} - 1}{1 + \gamma^{-1/p}} = 2 + \frac{\gamma^{1-1/p} - \gamma^{1/p}}{1 + \gamma^{1/p}}.$$

Note that this again coincides with (29), but with p and q interchanged.

Combining these results, we get the following lower bound.

Corollary 7.2. *Let $K_{3,p} \subset \mathbb{R}^3$ be the epigraph of the $\|\cdot\|_p$ -norm given by (30) with parameter $p \in [1, \infty]$. Set $c = \min(\frac{1}{p}, 1 - \frac{1}{p}) \in [0, \frac{1}{2}]$ and let $\gamma \in [0, 1)$ be a solution of the equation $(1 - 2c)\gamma^{1-c} + (1 - c)\gamma^{1-2c} - c = 0$. If F is a logarithmically homogeneous self-concordant barrier on $K_{3,p}$, then its barrier parameter satisfies the inequality $\nu \geq 1 + \frac{1+\gamma^c}{1+\gamma^{1-c}}$. \square*

As already noted, as functions of p the lower bounds given in Corollaries 7.1 and 7.2 coincide.

Remark 7.3. The choices of the points e_k and x^* in Subsections 7.1 and 7.2 are optimal, i.e., the bounds in Corollaries 7.1 and 7.2 are the best possible which can be obtained with our method.

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A Self-concordance of function (24)

In this section it is proven that the function F given by (24) is a self-concordant barrier for the cone $K_{n,\infty}$ defined in (23). The proof is due to an anonymous reviewer. It replaces the much more complicated proof from the first version of the paper.

Consider the positive orthant \mathbb{R}_+^{n-1} with its optimal barrier $F_{\mathbb{R}_+^{n-1}}(y) = -\sum_{k=1}^{n-1} \log y_k$. As a homogeneous cone, it has rank $n - 1$. From this cone we construct [2, Sect. 3] (see also [4, pp. 165–166]) a homogeneous cone of rank n and dimension $2n - 1$,

$$SC(\mathbb{R}_+^{n-1}, B) = \{(y, x, x_0) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \mid x_0 \geq 0, x_0 y - B(x) \in \mathbb{R}_+^{n-1}\},$$

where $B : \mathbb{R}^{n-1} \rightarrow \mathbb{R}_+^{n-1}$ is a symmetric vector-valued bilinear form given by $B(x) = (x_1^2, \dots, x_{n-1}^2)^T$. An optimal barrier for the cone $SC(\mathbb{R}_+^{n-1}, B)$ with barrier parameter n is given by [2, Theorem 4.1]

$$F_{SC(\mathbb{R}_+^{n-1}, B)}(y, x, x_0) = F_{\mathbb{R}_+^{n-1}}(y - x_0^{-1}B(x)) - \log x_0 = - \sum_{k=1}^{n-1} \log(x_0 y_k - x_k^2) + (n-2) \log x_0.$$

Now the intersection of the cone $SC(\mathbb{R}_+^{n-1}, B)$ with the linear subspace given by $x_0 = y_1 = \dots = y_{n-1}$ is isomorphic to the cone $K_{n, \infty}$, and the restriction of the barrier $F_{SC(\mathbb{R}_+^{n-1}, B)}$ to this intersection induces the function F on $K_{n, \infty}$. This proves that F is actually a self-concordant barrier.