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# Micro-macro dynamics of periodic material structures

Cz. Woźniak

*Institute of Fundamental Technological Research PAN, Warsaw, Poland*

**ABSTRACT:** In this contribution we propose a new non-asymptotic method of macro-modelling of periodic materials and structures. The obtained models can be applied to the analysis of vibration and wave propagation problems with the wavelength of an order of a periodicity cell dimension. The consideration are restricted to the linear elastic materials and the small deformation gradient theory.

## 1 STATEMENT OF THE PROBLEM

Asymptotic methods of macro-modelling for micro-periodic structures lead to macro-homogeneous media with the constant (averaged) mass density (cf. Bachvalov & Panasenko (1984) and Bensoussan et al. (1980)). Hence, asymptotic equations are not able to describe dispersion effects due to the micro-heterogeneity of composites and can be applied solely to problems in which the time-dependent excitations of the structure produce wavelength much larger than the maximum length dimension of a periodicity cell (a long-wave approximation). The aim of this contribution is to propose a new non-asymptotic method of macro-modelling for micro-heterogeneous composite structures. The obtained equations of micro-macro dynamics can be applied to vibration and wave propagation problems with the wavelength of an order of a cell length dimension (a short-wave approximation).

## 2 BASIC ASSUMPTIONS

The subject of the analysis is a linear elastic micro-periodic composite body, which in its initial natural state occupies a region  $\Omega$  in a 3-space parametrized by cartesian orthogonal coordinates  $x_i$  with a orthonormal basis  $e_i$  ( $i, j, k, l$  run over 1, 2, 3; summation convention holds). The properties of this body are determined by a mass density  $\rho(\cdot)$  and the tensor of elastic modulae  $A_{ijkl}(\cdot)$  which are  $V$ -periodic functions, where

$$V \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$$

is a certain representative volume element (r.v.e.). Periods  $l_1, l_2, l_3$  are sufficiently small related to the smallest characteristic length dimension of  $\Omega$ .

In order to formulate basic hypotheses leading to macro-models of the micro-periodic body, we shall introduce two auxiliary concepts. The first of them is that of a  $V$ -macro function. It is a continuous function  $F(\cdot)$  defined on  $\Omega$  which for every  $x, z \in \Omega$  and  $z - x \in V$ , satisfies condition  $F(x) \equiv F(z)$ . Generally speaking  $V$ -macro functions describe the macroscopic behavior of the body. The phenomena related to heterogeneous micro-periodic material structure, from the qualitative point of view, will be described by means of independent functions  $h_a(\cdot)$ ,  $a=1, \dots, n$ , which are continuous in  $\Omega$  and satisfying conditions:  $h_a(x) = h_a(x + l_i e_i)$ ,  $h_a(x) = -h_a(x + l_i e_i / 2)$  for every  $x, x + l_i e_i \in \Omega$ ,  $i=1, 2, 3$  (no summation with respect to "i"). Moreover, we assume that  $\langle \rho h_a \rangle = 0$ , where  $\langle \cdot \rangle$  is the known averaging operator defined by

$$\langle f \rangle(z) = \frac{1}{|V|} \int_V f(z+y) dv(y), \quad dv(y) \equiv dy_1 dy_2 dy_3$$

for an arbitrary integrable function  $f(\cdot)$ . Functions  $h_a(\cdot)$  will be called micro-oscillatory shape functions. The choice of shape functions has to be postulated a priori in every special problem and depends on the character of micro oscillations which we

are going to analyze.

The proposed method of macro-modelling is based on the assumption that displacement fields  $u_i(\cdot, \tau)$  at an arbitrary time instant  $\tau$  can be expected in the form (indices  $a, b$  run over  $1, \dots, n$ , summation convention holds):

$$(1) \quad u_i(x, \tau) = U_i(x, \tau) + h_a(x) D_i^a(x, \tau), \quad x \in \Omega,$$

where  $U_i(\cdot, \tau)$ ,  $D_i^a(\cdot, \tau)$  are arbitrary V-macro functions together with their first and second order derivatives. The V-macro fields  $U_i(\cdot, \tau)$  are called macro-displacements and  $h_a D_i^a(\cdot, \tau)$  are oscillations due to the micro-periodic structure of the body. The V-macro functions  $D_i^a(\cdot, \tau)$  constitute the quantitative characteristics of micro-oscillations and will be called corrector fields.

The second assumption of the proposed method of macro-modelling takes into account the micro-oscillatory character of shape functions  $h_a(\cdot)$ ; we shall assume that in formulas for material derivatives of  $h_a D_i^a(\cdot, \tau)$  terms involving  $h_a(\cdot)$  can be neglected as small compared to terms involving derivatives  $\nabla h_a(\cdot)$ . Hence, the approximation

$$(2) \quad u_{i,j}(x, \tau) \cong U_{i,j}(x, \tau) + h_{a,j}(x) D_i^a(x, \tau)$$

will be used in the subsequent analysis.

### 3 MACRO-MODELLING

The governing equations of the proposed micro-macro elastodynamics will be derived by applying assumptions of Sect. 2 to the well known action functional

$$\mathcal{A} = \int_{\Omega} \left[ \frac{1}{2} \rho(x) \dot{u}_i \dot{u}_i - \frac{1}{2} A_{ijkl}(x) u_{i,j} u_{k,i} + \right. \\ \left. + \rho(x) b_i u_i \right] dv(x), \quad dv(x) \equiv dx_1 dx_2 dx_3, \quad (3)$$

where  $b_i$  are constant body forces. Substituting Eqs (1), (2) into (3), bearing in mind V-macro property of macro-displacements  $U_i(\cdot, \tau)$ , micro-correctors  $D_i^a(\cdot, \tau)$  and of their derivatives, and using condition  $\langle \rho h_a \rangle = 0$ , after some manipulations we obtain the approximation  $\mathcal{A} \cong \mathcal{A}_0$ , where

$$\mathcal{A}_0 = \int_{\Omega} \left[ \frac{1}{2} \langle \rho \rangle \dot{U}_i \dot{U}_i + \frac{1}{2} \langle \rho h_a h_b \rangle D_i^a D_i^b - \right. \\ \left. - \frac{1}{2} \langle A_{ijkl} \rangle U_{i,j} U_{k,i} - \langle A_{ijkl} h_{a,j} \rangle U_{k,i} D_i^a - \right. \\ \left. - \frac{1}{2} \langle A_{ijkl} h_{a,j} h_{b,i} \rangle D_i^a D_i^b + \langle \rho \rangle b_i U_i \right] dv(x). \quad (4)$$

Due to V-periodicity of  $\rho(\cdot)$ ,  $A_{ijkl}(\cdot)$ ,  $h_a(\cdot)$  all averages in (4) are constants which characterize the material and inertial properties of a dynamic system determined by action functional  $\mathcal{A}_0$ . This dynamic system represents the macro-model of micro-heterogeneous periodic body. Lagrange equations for  $\mathcal{A}_0$  read

$$\langle A_{ijkl} \rangle U_{k,j,i} + \langle A_{ijkl} h_{b,i} \rangle D_k^b + \langle \rho \rangle b_i = \\ = \langle \rho \rangle \ddot{U}_i; \\ (5) \quad \langle \rho h_a h_b \rangle D_i^b + \langle A_{ijkl} h_{a,j} h_{b,i} \rangle D_i^b = \\ = - \langle A_{ijkl} h_{a,j} \rangle U_{k,i}.$$

Eqs (5) constitute the governing equations of linear macro-elastodynamics for micro-heterogeneous periodic bodies. The above equations have to be considered together with the boundary and initial conditions for  $U_i(\cdot)$  and initial conditions for  $D_i^a(\cdot)$  in the form consistent with Eqs (1) and (2). It has to be emphasized that the inertial properties of the obtained macro-model are described not only by an averaged mass density  $\langle \rho \rangle$  but also by modulae  $\langle \rho h_a h_b \rangle$  which depend on the length dimensions of the r.v.e. Due to this fact the scale and dispersion effects related to micro-heterogeneous material structure of the body can be investigated. Averages  $\langle \rho h_a h_b \rangle$  will be called micro-inertial modulae. Let us observe that for homogeneous materials

$$\langle A_{ijkl} h_{a,j} \rangle = A_{ijkl} \langle h_{a,j} \rangle = 0$$

and hence, under initial conditions

$$D_i^a(x, 0) = \dot{D}_i^a(x, 0) = 0, \quad x \in \Omega,$$

the second from Eqs (5) has only trivial solution  $D_i^a \equiv 0$ , and the first from Eqs (5) reduces to the well known equations of the linear elastodynamics. Thus we conclude that the correctors  $D_i^a$  describe the effect of micro-heterogeneity on macro behavior of the composite.

It has to be remembered that solutions to Eqs (5) have a physical sense only if  $U_i(\cdot, \tau)$ ,  $D_i^a(\cdot, \tau)$  are V-macro functions for every  $\tau$ .

In the asymptotic case term  $\langle \rho h_a h_b \rangle D_i^b$  drops out from Eqs (5) and we arrive at the macro-model proposed by Woźniak (1987).

### 4 EXAMPLE

We shall apply Eqs (5) to the problem of a straight micro-periodic bar treated as an uniaxial structure. The r.v.e.  $V$  is now reduced to the straight-line segment  $(-1/2,$

1/2) of the x-axis,  $x \equiv x_1$ . We assume that the Young modulus  $E(x)$  and mass density  $\rho(x)$  are equal to  $E_1, \rho_1$  on  $(-a/2, a/2)$  and  $E_2, \rho_2$  on  $(-1/2, 1/2) \setminus (-a/2, a/2)$ , where  $0 < a < 1$ . In this case we introduce only one (continuous and periodic) shape function  $h(\cdot)$ ,  $h(x) = h(x+1)$ ,  $x \in \mathbb{R}$ , which is piecewise linear and takes the values  $h(-1/2) = h(0) = h(1/2) = 0$ ,  $h(-a/2) = 1$ ,  $h(a/2) = -1$ . After neglecting body forces and setting  $(\cdot)' \equiv \partial(\cdot)/\partial x_1$ , Eqs (5) in this special case yield

$$\begin{aligned} \langle E \rangle U''(x, \tau) + \langle E h' \rangle D'(x, \tau) &= \langle \rho \rangle \ddot{U}(x, \tau), \\ \langle \rho h h \rangle \ddot{D}(x, \tau) + \langle E h' h' \rangle D(x, \tau) &= -\langle E h' \rangle U'(x, \tau), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \langle \rho \rangle &= [\rho_1 a + \rho_2 (1-a)]/1, \quad \langle E \rangle = [E_1 a + E_2 (1-a)]/1, \\ \langle E h' \rangle &= 2(E_1 - E_2), \quad \langle E h' h' \rangle = 4l(E_1/a + E_2/(1-a)), \\ \langle \rho h h \rangle &= l^2 \langle \rho \rangle / 3. \end{aligned}$$

Let us define

$$\mu^2 \equiv \langle E h' h' \rangle / \langle \rho h h \rangle, \quad E^{eff} \equiv \langle E \rangle - \langle E h' \rangle^2 / \langle E h' h' \rangle$$

and assume

$$U(x, \tau) = U_0(x, \tau) \exp(i\omega\tau),$$

$$D(x, \tau) = D_0(x, \tau) \exp(i\omega\tau).$$

The analysis of free vibrations leads to the conclusion that: (i) if

$$(\omega/\mu)^2 < E^{eff} / \langle E \rangle \quad \text{or} \quad (\omega/\mu)^2 > 1$$

then there exist sinusoidal vibrations

$$U_0(x) = A \cos kx, \quad D_0(x) = B \sin kx,$$

$$(ii) \quad \text{if} \quad E^{eff} / \langle E \rangle < (\omega/\mu)^2 < 1$$

then there exist exponential vibrations

$$U_0(x) = A \cosh kx, \quad D_0(x) = B \sinh kx,$$

$$(iii) \quad \text{if} \quad (\omega/\mu)^2 = 1 \quad \text{or} \quad (\omega/\mu)^2 = E^{eff} / \langle E \rangle$$

then we arrive at degenerated or trivial case, respectively. This classification holds for  $\langle E h' \rangle \neq 0$ ; if  $\langle E h' \rangle = 0$  (i.e. for a homogeneous bar) then only sinusoidal vibrations are possible.

In the case (i) (for sinusoidal vibrations),  $U_0(\cdot)$ ,  $D_0(\cdot)$  are V-macro fields only if  $lk \ll 1$ . Treating  $lk$  as a small parameter it can be shown that

$$(7) \quad \omega^2 = \frac{E^{eff}}{\langle \rho \rangle} k^2 \left[ 1 - \frac{1}{3} (lk)^2 \frac{\langle E h' \rangle^2}{\langle E h' h' \rangle^2} \right] + o(l^2 k^2).$$

The second term on the right hand side of Eq. (7) describe the dispersion effect due to micro-heterogeneous structure of the bar.

In the case (ii) (for exponential vibrations), we can prove that

$$(8) \quad \omega^2 = \frac{\langle E \rangle}{\langle \rho \rangle} k^2 \left[ 1 - \frac{E^{eff}}{\langle E \rangle} \right] (\omega/\mu)^2 + \omega^2 (\omega/\mu)^2.$$

Eq. (8) has a physical sense only for micro-heterogeneous bar because in the case of homogeneity  $E^{eff} / \langle E \rangle = 1$  and there no exponential vibrations.

The detailed analysis of this problem will be given in a forthcoming paper.

## CONCLUSIONS

The characteristic feature of the proposed approach is that the inertial properties of the obtained macro-model given by Eqs (5) are described not only by an averaged mass density  $\langle \rho \rangle$  but also by micro-modulae  $\langle \rho h h \rangle$  depending on length dimensions of r.v.e. Hence, the equations of micro-macro dynamics makes it possible to investigate the scale and dispersion effects due to micro-heterogeneity of the body. The advantage of the proposed approach is a relatively simple form of Eqs (5); from the illustrative example given above it follows that the general theory can be successfully applied to the analysis of engineering problems. The main drawback of the proposed method of macro-modelling lies in an unprecise choice of shape functions which is often based on the intuition of the researcher.

Nonlinear treatment of micro-macro dynamics and its applications has been also analyzed and will be presented separately.

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