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MECHANICS OF THE DISCRETE AND CONTINUOUS SYSTEMS
IN THE LIGHT OF THE NON-STANDARD ANALYSIS

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1. INTRODUCTION

Within the known treatment of Newtonian Mechanics, i.e., when the classical analysis is the mathematical tool of mechanics, the real bodies are represented either by the mass-point discrete systems or by the material continua. Thus the discrete mechanics and the continuum mechanics constitute two separate branches within classical formulation of Newtonian mechanics. In the contribution we are to show that if the foundations of mechanics are expressed in terms of the non-standard analysis then the real bodies can be represented exclusively by the mass-point systems and we can confine ourselves to the discrete (but "non-standard") mechanics only. All classical material systems (including material continuum) can be obtained by introducing different topologies to the non-standard metric space in which the fixed non-standard mass-point system is embedded. The reader is assumed to be familiar with the basic concepts of the non-standard analysis given in [1-4].

We start with the mathematical concepts which will be needed in what follows; more detailed analysis of these concepts can be found in [5]. We also use the well known denotations of the non-standard analysis.

An arbitrary subset A of *R , such that $\alpha a \in A$ for every $\alpha \in R$ and for a fixed $a \in {}^*R$, will be called A -system (Archimedean system in *R).

Two A -systems, A_1, A_2 , are said to be close if there exists one-to-one mapping $f : A_1 \rightarrow A_2$, defined by $a_2 = (1+\varepsilon)a_1$, $a_1 \in A_1, a_2 \in A_2$, where ε is an arbitrary infinitesimal number. Such relation of the proximity in the set \mathcal{A} of all A -Systems is the relation of equivalence. The corresponding quotient structure Σ has elements denoted by $\sigma, \sigma \in \Sigma$, which will be called σ -classes. In Σ we shall introduce the relation of order, putting $\sigma_1 < \sigma_2$ iff for some $a_1 \in R_{\sigma_1}^+, a_2 \in R_{\sigma_2}^+, R_{\sigma}^+ \equiv \bigcup_{A \in \sigma} A \cap {}^*R^+$, we have $a_1 < a_2$.

The number $r \in {}^*R$ will be called σ -finite if there exists the finite number α such that $r = \alpha a$ for some $a \in R_{\sigma}$. The set of all σ -finite numbers will be denoted by F_{σ} .

Let r be a σ -finite number, $r \in F_{\sigma}$, and $A \in \sigma$. Then there exists uniquely defined number of the A -system, denoted by A_r , given by ${}^A_r = {}^o\alpha a$, $a \in A$, where $r = \alpha a$ and ${}^o\alpha$ is the standard part of the finite number α . The number A_r is said to be the A -standard part of the σ -finite number r . If $r \in A$ then ${}^A_r = r$.

Let $({}^*T, \rho)$ be the fixed non-standard metric space with the distance function $\rho = \rho(p, q); p, q \in {}^*T$. Let an arbitrary but fixed A -system also be given. The set of points $q \in {}^*T$, such that ${}^A_{\rho(p, q)} < {}^A_r$, where p is the fixed point in *T , will be called S_A -ball with the centre p and a radius $r = {}^A_r$. The S_A -ball will be denoted by $S_A(p, r)$. If $A \in \sigma$ then the set of all S_A -balls constitutes the basis for a certain topology in *T , which will be called S_{σ} -topology. It can be proved that the sets of S_{A_1} -balls

and S_{A_2} -balls are two bases for one S_σ -topology in $*T$, provided that $A_1 \in \sigma$ and $A_2 \in \sigma$. If $R \in \sigma$ then S_σ -topology coincides with the well known S -topology in $*T$, cf. [1] (p.107). Following [1] (cf. Chapter IV) we can now define the concepts of S_σ -open set, S_σ -interior, S_σ -boundary and S_σ -closure. Also using the procedure analogous to that given in [1] we can formulate many theorems concerning all S_σ -concepts. It can be proved that if $\sigma_1 < \sigma_2$ then S_{σ_1} -topology is finer than S_{σ_2} -topology. Thus we have arrived at the system $H = (S_\sigma)_{\sigma \in \Sigma}$ of topologies in an arbitrary non-standard metric space $(*T, \rho)$. This system will be called the hierarchy of topologies H in $(*T, \rho)$ and plays an important role in the non-standard approach to the Newtonian mechanics.

Some lemmas and theorems connected with the concepts given above as well as the more detailed analysis of these concepts can be found in [5].

3. ON THE NON-STANDARD FORMULATION OF DISCRETE MECHANICS

To simplify the analysis let us assume that in the Galilean space-time there is given one fixed inertial coordinate system and let us denote by $E = R^3$ the corresponding physical space. Then in the formulation of Newtonian discrete mechanics we shall deal with only two kinds of individuals: material points and real numbers. We shall assume that the set U of all material points is infinite but countable. Any finite subset of U will be called the material system.

To develop non-standard theory of the foregoing material systems (non-standard particle or discrete mechanics) we have to assume that U and the set R of real numbers are embedded in certain full structure $M = (A_\tau)_{\tau \in T}$, i.e., that $U \subset A_0$ and $R \subset A_0$. On passing to an enlargement $*M = (*B_\tau)_{\tau \in T}$, $*B_0 = *A_0$, of the full structure M we have $*U \subset *A_0$ and $*R \subset *A_0$. Any Q -finite subset of $*U$ (the subset $\{P_1, \dots, P_n\}$, where $n \in *N$ is a fixed natural number which may be finite or infinite, [1]) will be also called the

material system.

Let L be the formal language such that $K_0(M, L)$ is the set of all sentences which hold in M . Moreover, let $\mathcal{D}_0, \mathcal{D}_0 \subset K_0(M, L)$, be the set of sentences which characterize the classical Newtonian discrete mechanics. By the definition of $*M$, all sentences of K_0 (and these of \mathcal{D}_0) are true in $*M$. Roughly speaking, every notion of Newtonian discrete mechanics which is meaningful for U and R is also meaningful for $*U$ and $*R$, and every statement of Newtonian discrete mechanics which is meaningful and true for U and R is also meaningful and true for $*U$ and $*R$ (provided that only internal relations of $*M$ are taken into account). Thus we have arrived to what will be called the non-standard model of discrete mechanics. Within this model we deal with the configurations of points (elements of $*U$) not in the physical space E but in the extended physical space $*E = (*R)^3$, with the masses of points which can be infinitely small or infinite, with the interactions which can also be infinitely small or infinite, etc. It must be stressed, however, that not all that can be said about the non-standard model of mechanics can be deduced trivially from the known formulation of mechanics. For example, the non-standard material systems $\{P_1, \dots, P_n\}$, where n is an arbitrary unfinite natural number, have no counterparts in the known formulation of discrete mechanics.

4. PHYSICAL INTERPRETATIONS OF NON-STANDARD MATERIAL SYSTEMS

Let $\mathcal{B} = \{P_1, \dots, P_n\}$ be an arbitrary but fixed non-standard material system. Each one-to-one mapping $\kappa : \mathcal{B} \rightarrow *E$ such that $\kappa(\mathcal{B})$ is the internal set in $*E$, will be called the configuration of \mathcal{B} . We shall see that the set $\kappa(\mathcal{B})$ has different physical interpretations, i.e., it can be interpreted as the discrete system, as the continuous body and even as one mass point. These interpretations will depend on the kind of the S_σ -topology taken from the hierarchy $H = (S_\sigma)_{\sigma \in \Sigma}$. Many other interpretations of $\kappa(\mathcal{B})$ can be also taken

into account; it follows from the fact that the hierarchy H is unfinite.

Let us denote by $\mu_\sigma(p)$ the set of points q in $*E$ such that $\rho(p,q)$ is σ -infinitesimal, i.e., has the form εa for some $a \in R$ and for an arbitrary infinitesimal number ε . The set $\mu_\sigma(p)$ will be called the σ -monad of the point p , $p \in *E$. Analogously, the set $G_\sigma(p)$ of points q in $*E$ such that $\rho(p,q)$ is σ -finite will be called the σ -galaxy, [5]. The physical interpretations of $\kappa(B)$, depending on S_σ , $\sigma \in \Sigma$, can be realized in certain spaces E_σ . To introduce these spaces let us define the relation of equivalence \sim_{in} in $*E$ by the condition: $p \sim q$ iff $q \in \mu_\sigma(p)$. This relation (defined for every $\sigma \in \Sigma$) gives rise to a quotient structure E_σ in which points $x \in E_\sigma$ are σ -monads of $*E$. If we introduce the projection $\psi_\sigma: *E \rightarrow E_\sigma$ (i.e., $x = \psi_\sigma(p)$) then the open sets of E_σ are images of the open sets of $*E$ under ψ_σ . We can also observe that if the points $p, q \in *E$ belong to the one σ -galaxy, $q \in G_\sigma(p)$, then we can define the distance function $\delta_A(x, y)$ in E_σ putting $\delta_A(x, y) \equiv \delta_\rho^A(p, q)$, where $x = \psi_\sigma(p)$, $y = \psi_\sigma(q)$ and A is the fixed A -system belonging to σ . The galaxies of E_σ are the images of the σ -galaxies of $*E$ under ψ_σ . The metric $\delta_A(x, y)$, $A \in \sigma$, is defined in each galaxy of E_σ .

Now it can be easily observed that one fixed set $\kappa(B)$ in $*E$ can have different images in the different spaces E_σ , $\sigma \in \Sigma$, given by the sets $\psi_\sigma(\kappa(B))$, $\psi_\sigma(\kappa(B)) \subset E_\sigma$. Let $\sigma_1 > \sigma_2 > \sigma_3$; then S_{σ_1} -topology is finer than S_{σ_2} -topology and S_{σ_2} -topology is finer than S_{σ_3} -topology. In this case the sets $\psi_{\sigma_1}(\kappa(B))$, $\psi_{\sigma_2}(\kappa(B))$, $\psi_{\sigma_3}(\kappa(B))$ can represent, for example, the finite system of points in E_{σ_1} , the open region in E_{σ_2} , and one point in E_{σ_3} , respectively. Putting $A_1 \in \sigma_1$, $A_2 \in \sigma_2$, $A_3 \in \sigma_3$ and $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, where A_1, A_2, A_3 are the fixed A -systems and a_1, a_2, a_3 are their fixed elements, we can interpret a_1 as the interatomic distance, a_2 as the length dimension of the whole real body ($a_1 = \varepsilon a_2$, ε being the positive infinitesimal number) and a_3 as distance which is

infinite with respect to the dimensions of the body (i.e., $a_2 = \epsilon a_3$, ϵ being also the positive infinitesimal number). Roughly speaking, finer S_σ -topology in $*E$ is, a finer description of the body under consideration can be realized. On the other hand, in such "fine" description of the real body the points of $\psi_\sigma(\kappa(\mathcal{B}))$ can be situated in the different galaxies. It means that we are not able to describe the whole body (in the fixed configuration) represented in the suitable space E_σ in one metric only, because the distance function $\delta_A(x,y)$ is defined in each galaxy of E_σ independently. Such a situation has also the clear physical interpretation because if the insight into the structure of the real body is deeper then we have to confine ourselves to the analysis of the smaller parts of this body.

The foregoing consideration was connected exclusively with the problem of configuration of the non-standard material system. However, analogous investigations can also be carried on with respect to the problem of the motion of the non-standard material system, the interactions within this system, different analytical form of the laws of motion depending on the suitable S_σ -topology in $*E$, etc. Some from these problems are still open; the particulars can be found in [5].

5. FINAL REMARKS

To summarize the foregoing considerations let us compare the "classical" and "non-standard" concepts of material systems in Newtonian mechanics. In the classical treatment:

1. One real body can be represented in mechanics by the different material systems such as: finite or countable set of mass-points, set of three dimensional elementary manifolds, one differentiable manifold, one mass-point, etc.

2. The governing relations of mechanics have to be formulated independently for the different classes of material systems. We have to deal, at the least, with the particle or

discrete mechanics and with the continuum mechanics though in both cases the same "real" body can be concerned.

3. In continuum mechanics we have to deal with certain unreasonable (from the point of view of physics) features of the concept of material continuum (with the material elements, mass of which is equal to zero, etc.).

In the non-standard approach to the foundations of Newtonian mechanics:

1. We can confine ourselves to the one model of the real body (to the one material system only) represented by the Q-finite (but not absolutely finite, cf. [1] (p.123)) system of mass-points.

2. The foundations of Newtonian mechanics can be formulated entirely in terms of the "mass-points" mechanics, however in the non-standard form.

3. All known "standard" models of the real body in Newtonian mechanics (material continuum, countable set of mass-points, one material point etc.) are interpreted exclusively as the different "insights" into the structure of one fixed discrete non-standard material system (non-standard mass-point system). Every such "insight" is realized by applying the suitable S_{σ} -topology to the metric space in which the non-standard material system is embedded. In particular, the known material continuum can be obtained as the result of the suitable projection (cf. Sec. 4) of the non-standard discrete material system. Roughly speaking, the material "standard" continuum is equivalent to the non-standard mass-point system in which the masses of points are infinitesimal and the interparticle distances are also infinitesimal.

Let us also observe that the non-standard approach to Newtonian mechanics in which we deal with one model of the real body (one non-standard mass-point system) and with the different levels on which this model can be analysed (with the different

S_σ -topologies in the extended physical space) is not only simpler than the known approach but also has the clear physical interpretations. Namely to every level of analysis (atomic, microscopic, macroscopic) is assigned the suitable non-standard topology. It must be also stressed that the non-standard approach to Newtonian mechanics is not only restricted to the foundations of mechanics but can also be applied to the different more special problems of mechanics; generally speaking it leads to the larger class of the material systems than the classic approach, where the "standard" mathematical analysis constitutes our basic tool.

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