



HAL
open science

Heterogeneity in mechanics of composite structures

Czeslaw Wozniak

► **To cite this version:**

| Czeslaw Wozniak. Heterogeneity in mechanics of composite structures. 2014. hal-00944255

HAL Id: hal-00944255

<https://hal.science/hal-00944255>

Preprint submitted on 10 Feb 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HETEROGENEITY IN MECHANICS OF COMPOSITE STRUCTURES

CZESLAW WOŹNIAK

Institute of Fundamental Technological Research, Warsaw

The aim of this contribution is to establish the theoretical basis for calculations and design of macroscopically heterogeneous composite structures. The "apparent" properties of such structures depend on position in the body. The considerations are carried out for deterministic description of linearly elastic composites with perfect bonding between material constituents.

1. Introduction

Composites on a microscopic scale (but large enough to apply continuum mechanics) are aggregates comprising two or more distinctly different homogeneous materials which on a macroscopic scale form together a new medium with "apparent" properties different from those of the individual constituents. Thus by the definition every composite is a micro-heterogeneous material continuum endowed with certain macro-properties. If the macro-properties are constant throughout the whole structure then the composite under consideration is said to be macroscopically homogeneous. However, in modern engineering systems we also encounter composite structural elements in which macro-properties are functions of position in the body. This fact leads to a natural definition of macro-heterogeneous composite structures. They are structures that are not made of one standardized composite material but possess different desired macro-properties in different parts. The macro-heterogeneity of the composite structure can be established by

- the curvilinear shape of structural elements such as curved fibrous rods and/or laminated shells,
- the desired geometrical arrangement of the reinforcement (direction of fibers) in the vicinity of holes and notches of structural elements,
- the variable volume percentage of constituents due to the desired physical properties in different parts of the structure,

- the complying with the cost constraints by replacing a high-cost fibers or particles in a common matrix by a low-cost ones in the lower stressed regions of the structure.

Generally speaking, the macro-heterogeneity of a composite structure allows one to achieve better design specification of this structure than that for a macro-homogeneous (standardized) composite. This is a global hybridization philosophy the aim of which is the conjunction of the local macro-properties of the composite material with the global design of the specific structural element, Thompson [7].

The list of papers and textbooks on micro-macro mechanics of composites is very extensive and will be not discussed here. The theoretical considerations are mainly devoted to the macro-homogeneous structures and homogenization theories (cf Jones [3], Christensen [2], Sanchez Palencia [6]). The aim of this contribution is to propose a general method of mathematical modelling for macro-heterogeneous elastic composite structures. The main attention is focused on the formulation of engineering models for such structures which can constitute a theoretical basis for both the design processes and numerical calculations. The proposed approach takes into account certain ideas of the non-standard method of modelling (cf Woźniak [8]), but emphasis is placed on some heuristic assumptions rather than on rigorous treatment through the non-standard analysis (cf Robinson [5]), or the asymptotic methods (cf Bensoussan et al. [1]). All considerations are carried out in the context of the linear elasticity theory, under assumption of the perfect bonding between material constituents of the composite and for the deterministic description of the spatial distributions of constituents. We start in section 2 with the preliminary concepts related to the macro-heterogeneous composites which enable us to develop the proposed method of modelling. Section 3 is devoted to the heuristic hypotheses which constitute the basis for modelling. In section 4 we derive the general governing equations for macro-heterogeneous composite structures. In the last section the obtained results are transformed to the form of an engineering theory which create a basis for calculations and design of macro-heterogeneous composites.

Notations. Throughout the paper subscripts i, j, k, l, m, n take the values 1, 2, 3 being related to the cartesian orthogonal coordinate system $0x_1x_2x_3$ in the referential space. The sub- and superscripts a, b run over $1, \dots, M$ if not stated otherwise and α, β take the values $1, \dots, N$. The summation convention holds for all aforementioned indices. Moreover, superscripts A, B take the values $1, \dots, R$ and a superscript s run over $1, \dots, S$. The points in the referential space are denoted by $\mathbf{x} \equiv (x_i)$, $\mathbf{r} \equiv (r_i)$, $\mathbf{p}^E \equiv (p_i^E)$, $E = 1, \dots, P$. Symbol τ stands for the time coordinate, $\tau \in [\tau_0, \tau_f]$. By $f(\cdot)|_{\Pi}$ we denote the restriction of an arbitrary function $f(\cdot)$ to the subdomain Π . Function $f(\cdot)$, defined on a certain subdomain of \mathcal{R}^3 , after translation of \mathcal{R}^3 by the vector $-\mathbf{r}$, is denoted by $f(\mathbf{r} + \bullet)$. Symbol \cong has to be read "is a close approximation of"; it is assumed

that from the numerical viewpoint two mathematical entities φ and ψ related by $\varphi \cong \psi$ can be treated, with a sufficient accuracy, as identical. If φ and ψ are differentiable functions then we assume that close approximation $\varphi \cong \psi$ also implies the close approximation of all derivatives of these fields. Moreover, if ψ is an argument of a certain function $h(\cdot)$, then $\varphi \cong \psi$ implies $h(\varphi) \cong h(\psi)$.

2. Preliminaries

From the formal point of view by a composite structure we shall mean the triple $S \equiv (\Omega, \rho(\cdot), a_{ijkl}(\cdot))$ where Ω is a regular region in the referential space occupied by the structure in its undeformed and unstressed state, $\rho(\cdot)$ is the mass density scalar field and $a_{ijkl}(\cdot)$ is the elastic modulae tensor field. The fields $\rho(\cdot)$, $a_{ijkl}(\cdot)$ are defined almost everywhere on Ω and assumed to be piecewise constant (constant in all regions occupied by an arbitrary but fixed constituent of the composite); as a rule they are highly oscillating fields. For the sake of simplicity by the structure S (or by a structure without the specification) we shall understand a material body endowed with the composite structure S .

A base for the analysis of macro-heterogeneous structures creates a decomposition of Ω into disjointed volume elements Π^A called mezzo volume elements (m.v.e.): $\Omega = \bigcup \Pi^A$, $A = 1, \dots, R$. Every m.v.e. should be a small piece of a structure S in which all micro heterogeneities can be detected but large enough to represent the "apparent" material properties of the structure in the vicinity of m.v.e. To this end we have to assume that the adjacent m.v.e. (except those possibly separated by an interfacial singular surface in Ω) have, roughly speaking, almost identical distribution of material constituents. It has to be emphasized that the remote m.v.e. can have distinctly different distribution of constituents. Such situations are typical for many structural elements encountered in engineering and industry, such as fibrous curved rods or laminated shells. Let r^A be a geometrical center of Π^A and define $\Pi(r^A) \equiv \Pi^A$, $A = 1, \dots, R$. Setting $V(r^A) \equiv \Pi(r^A) - r^A$ we shall represent the distribution of material properties in every m.v.e. $\Pi(r^A)$ by means of functions $\rho(r^A + y)$, $a_{ijkl}(r^A + y)$ defined for almost every $y \in V(r^A)$; the above functions will be denoted by $\rho(r^A + \bullet)|V(r^A)$, $a_{ijkl}(r^A + \bullet)|V(r^A)$, respectively (cf Notations).

The first step in the proposed line of modelling is to construct a discretized model of every $V(r^A)$, $A = 1, \dots, R$. To this end we approximate $V(r^A)$ by $V^1(r^A)$, and we assume that $\bar{V}^1(r^A)$ is a sum of certain disjointed mezzo-finite elements and their boundaries. Setting $\Pi^1(r^A) \equiv V^1(r^A) + r^A$ and $\bar{\Omega}^1 \equiv \bigcup \bar{\Pi}^1(r^A)$, $A = 1, \dots, R$ we also assume that $\bar{\Omega}^1 \cong \Omega$. At the same time we shall approximate functions $\rho(r^A + \bullet)|V(r^A)$, $a_{ijkl}(r^A + \bullet)|V(r^A)$ by functions $\rho^1(r^A + \bullet)|V^1(r^A)$,

$a_{ijkl}^1(\mathbf{r}^A + \bullet)|V^1(\mathbf{r}^A)$, respectively, which are constant in every mezzo-finite element of $V^1(\mathbf{r}^A)$. This procedure will be referred to as the mezzo-discretization. In order to simplify the subsequent considerations we shall restrict ourselves to the tetrahedron or triangle finite elements and denote by $\mathbf{y}^a(\mathbf{r}^A)$, $a = 1, \dots, M$, the nodes of the triangulation in $V^1(\mathbf{r}^A)$ and by $\mathbf{y}^a(\mathbf{r}^A)$, $a = M + 1, \dots, M + T$, the nodes on the boundary $\partial V^1(\mathbf{r}^A)$. Moreover, for every pair of adjacent m.v.e. $\Pi(\mathbf{r}^A)$, $\Pi(\mathbf{r}^B)$ (such that $\partial\Pi(\mathbf{r}^A) \cap \partial\Pi(\mathbf{r}^B) \neq \emptyset$ is not an element of any singular surface in Ω) we have $\mathbf{y}^a(\mathbf{r}^A) \cong \mathbf{y}^a(\mathbf{r}^B)$ for $a = 1, \dots, M + T$.

Let l be the largest characteristic length dimension of m.v.e. $\Pi(\mathbf{r}^A)$, $A = 1, \dots, R$, and $B(\mathbf{r}^A, l)$ stands for a ball in \mathcal{R}^3 with center \mathbf{r}^A and the radius l . The next step in the proposed line of modelling is a formal smoothing operation for these mathematical entities introduced above which are defined on the set of points \mathbf{r}^A , $A = 1, \dots, R$. To this end for $a = 1, \dots, M + T$, let be known a certain interpolation of the discrete function $\mathbf{y}^a(\mathbf{r}^1), \mathbf{y}^a(\mathbf{r}^2), \dots, \mathbf{y}^a(\mathbf{r}^R)$ by a differentiable function $\mathbf{y}^a(\bullet)$ defined on Ω and satisfying the close approximation conditions $\mathbf{y}^a(\mathbf{x}) \cong \mathbf{y}^a(\mathbf{r}^A)$ for every $\mathbf{x} \in B(\mathbf{r}^A, l)$, $A = 1, \dots, R$. Let us observe that, for an arbitrary but fixed $\mathbf{r} \in \Pi$, vectors $\mathbf{y}^a(\mathbf{r})$, $a = 1, \dots, M + T$, uniquely determine the discretized region $V^1(\mathbf{r})$ being treated as the triangulation nodes; if $\mathbf{r} \cong \mathbf{r}^A$ for some $A = 1, \dots, R$, then this discretization has to be a close approximation of the aforementioned discretization of $V^1(\mathbf{r}^A)$. Hence we obtain the piecewise constant functions $\rho^1(\mathbf{r} + \mathbf{y})$, $a_{ijkl}^1(\mathbf{r} + \mathbf{y})$ defined for almost every (a.e.) $\mathbf{y} \in V^1(\mathbf{r})$, which constitute the close approximation of the introduced above functions $\rho^1(\mathbf{r}^A + \mathbf{y})$, $a^1(\mathbf{r}^A + \mathbf{y})$ defined for a.e. $\mathbf{y} \in V^1(\mathbf{r}^A)$, respectively, provided that $\mathbf{r} \in B(\mathbf{r}^A, l)$.

The last step of these preliminary considerations leads to a family of certain auxiliary composite structures which will be called fine composite structures. To define such structures we introduce a positive variable parameter ε , sufficiently small compared to 1, $0 < \varepsilon \ll 1$. It is assumed that all terms of an order ε can be neglected in the subsequent analysis. Variable parameter ε does not possess any direct physical meaning but will play a role of a certain modelling micro-parameter.

Remark 1. Using the nonstandard analysis (cf Robinson [5]), as a mathematical tool of modelling we postulate that ε is not a variable parameter but an infinitely small constant (cf Woźniak [8]). In the rigorous asymptotic treatment (cf Bensoussan [1]), we have to assume that $\varepsilon \searrow 0$.

Let us assign to every ε , $0 < \varepsilon \ll 1$, a certain smoothing operation leading for every $\mathbf{r} \in \Omega$ to a discretized set $V^1(\mathbf{r})$ and to piecewise constant functions $\rho^1(\mathbf{r} + \bullet)$, $a_{ijkl}^1(\mathbf{r} + \bullet)$ defined almost everywhere on $V^1(\mathbf{r})$; for the sake of simplicity the dependence of $V^1(\mathbf{r})$, $\rho^1(\mathbf{r} + \bullet)$, $a_{ijkl}^1(\mathbf{r} + \bullet)$, $\mathbf{r} \in \Omega$, on ε is not indicated explicitly but has to be remembered. Now for an arbitrary but fixed ε we define

$V^\varepsilon(\tau) \equiv \varepsilon V^1(\tau)$, $\Pi^\varepsilon(\tau) \equiv V^\varepsilon(\tau) + \tau$, $\tau \in \Omega$. Let L^ε be a discrete set of points in Ω such that $\Pi^\varepsilon(\tau)$, $\tau \in L^\varepsilon$, are disjoint volume elements and $\bar{\Omega}^\varepsilon \equiv \bigcup \Pi^\varepsilon(\tau)$, $\tau \in L^\varepsilon$, is a closure of a regular region Ω^ε which constitutes a close approximation of $\bar{\Omega}$. These requirements impose the extra conditions on the smoothing operation assigned to the parameter ε . Setting

$$\begin{aligned} \rho^\varepsilon(\tau + \varepsilon \mathbf{y}) &\equiv \rho^1(\tau + \mathbf{y}) \\ a_{ijkl}^\varepsilon(\tau + \varepsilon \mathbf{y}) &\equiv a_{ijkl}^1(\tau + \mathbf{y}) \end{aligned} \quad (2.1)$$

for a.e. $\mathbf{y} \in V^1(\tau) \quad \tau \in L^\varepsilon$

we arrive finally at the family of fields $\rho^\varepsilon(\bullet)$, $a_{ijkl}^\varepsilon(\bullet)$ defined a.e. on the region Ω^ε . Hence every triple

$$S_\varepsilon \equiv (\Omega^\varepsilon, \rho^\varepsilon(\bullet), a_{ijkl}^\varepsilon(\bullet)) \quad 0 < \varepsilon < 1 \quad (2.2)$$

represents, from a purely formal point of view, a certain composite structure which will be referred to as the fine (or micro) composite structure.

Remark 2. The form of the fine composite structures S_ε , $0 < \varepsilon < 1$, depends on the postulated smooth interpolations $\mathbf{y}^a(\mathbf{x})$, $\mathbf{x} \in \Omega$, of the discrete functions $\mathbf{y}^a(\tau^A)$, $A = 1, \dots, R$, for $a = 1, \dots, M + T$; assigned to every ε , $0 < \varepsilon < 1$. For values of ε small enough this dependence is irrelevant provided that the close approximation conditions $\mathbf{y}^a(\mathbf{x}) \cong \mathbf{y}^a(\tau^A)$, $\mathbf{x} \in B(\tau^A, i)$, hold for every $A = 1, \dots, R$ and every $a = 1, \dots, M + T$. By using the nonstandard analysis approach, where ε is an infinitely small constant, it can be easily shown that $\bar{\Omega}^\varepsilon \equiv \bigcup \Pi^\varepsilon(\tau)$, $\tau \in L^\varepsilon$, is a fine partition of Ω into infinite number of infinitely small volume elements $\Pi^\varepsilon(\tau)$, $\tau \in L^\varepsilon$, cf Robinson 1966.

The formally introduced family of fine composite structures S_ε , $0 < \varepsilon < 1$, will play a crucial role in the line of reasoning leading to engineering macro-models for the structure S under consideration. It has to be emphasized that the macro-models of S have a physical sense only if the natural decomposition of Ω into m.v.e. Π_A , $A = 1, \dots, R$ is, roughly speaking, "fine enough". This requirement involves not only the shape of Ω and the value of the maximum characteristic length dimension l of m.v.e., but also the characteristic length dimensions related to the boundary and initial conditions for the structure S under consideration. In order to specify them we denote by $u_i^\Delta(\bullet, \tau)$ the boundary displacement fields defined on the part $\bar{\Delta}$ of $\partial\Omega$ (where $\text{mes}\Delta > 0$), by $t_i^\Gamma(\bullet, \tau)$ the boundary loading fields defined a.e. on $\Gamma \equiv \partial\Omega \setminus \bar{\Delta}$ and by $u_i^\Omega(\bullet)$, $v_i^\Omega(\bullet)$ the initial displacement and velocity fields, respectively, defined on Ω . We shall restrict ourselves to problems in which the boundary and initial conditions are uniquely determined by

the following three-tuple of external agencies

$$\left([\tau_0, \tau_f] \ni \tau \rightarrow \left(u_i^\Delta(\bullet, \tau), t_i^\Gamma(\bullet, \tau) \right), u_i^D(\bullet), v_i^D(\bullet) \right) \quad \bar{\Delta} \cup \bar{\Gamma} = \partial\Omega \quad (2.3)$$

which is assumed to be known a priori. To simplify the subsequent investigations we shall also assume that the body forces acting at the structure are described by the known constant vector b_i . The minimum characteristic length dimension related to (2.3) (i.e. to the geometry of Ω , Δ and Γ , to the space and time distribution of boundary loadings and displacements as well as to the space distribution of initial displacements and velocities) will be denoted by L . In the sequel we shall restrict ourselves to the physical situations in which the ratio $\lambda \equiv l/L$ is sufficiently small compared to 1, $\lambda \ll 1$. In this case the external agencies (2.3) will be referred to as admissible for the structure S under consideration.

By the response of the structure S to the agencies (2.3) we shall mean the displacement and stress fields $u_i(\bullet, \tau)$, $\sigma_{ij}(\bullet, \tau)$, respectively, which have to satisfy the known boundary and initial conditions determined by (2.3). The response of an arbitrary structure S_ε to the same external agencies (2.3) will be denoted by $u_i^\varepsilon(\bullet, \tau)$, $\sigma_{ij}^\varepsilon(\bullet, \tau)$, $\tau \in [\tau_0, \tau_f]$.

3. Basic assumptions and modelling hypotheses

The basis for the mathematical modelling of macro-heterogeneous composite structures will be given below in the form of three heuristic hypotheses. The first and second of them are related to the expected behaviour of the formally introduced fine composite structures S_ε , $0 < \varepsilon \ll 1$. The third hypothesis interrelates the behaviour of the composite structure S under consideration with those of the fine composite structures S_ε . The considerations will be carried out for an arbitrary small enough but fixed ε , $0 < \varepsilon \ll 1$.

In order to formulate the first hypothesis we introduce the family of functions $\chi_a(\mathbf{r}; \bullet)$, $\mathbf{r} \in L^\varepsilon$, defined and continuous on $V^1(\mathbf{r})$, linear in every (tetrahedral or triangle) mezzo-finite element of $V^1(\mathbf{r})$ and such that

$$\chi_a(\mathbf{r}; \mathbf{y}^b) = \lambda \delta_a^b \quad (3.1)$$

where λ is a small parameter introduced at the end of section 2. Every $\chi_a(\mathbf{r}; \bullet)$ will be called the shape function related to the point \mathbf{r} of L^ε . The hypothesis formulated below is based on the physical supposition that for every fine composite structure S_ε and every $\tau \in [\tau_0, \tau_f]$, the displacement field $u_i^\varepsilon(\bullet, \tau)$ is the sum of a mean and a highly oscillating part, where the oscillating part of displacements is due to the highly oscillating material properties of the fine composite structure. From the formal point of view this hypothesis impose certain constraints on the displacement fields $u_i^\varepsilon(\bullet, \tau)$ and will be referred to as

Fine constraint hypothesis. Displacement fields $u_i^e(\mathbf{x}, \tau)$, $\tau \in [\tau_0, \tau_f]$, for an arbitrary structure S_ε , $0 < \varepsilon \ll 1$, can be assumed in the form

$$u_i^e(\mathbf{x}, \tau) = U_i(\mathbf{x}, \tau) + \varepsilon \chi_a \left(\mathbf{r}; \frac{\mathbf{x} - \mathbf{r}}{\varepsilon} \right) Q_i^a(\mathbf{x}, \tau) \quad (3.2)$$

$$\mathbf{x} \in \Pi^\varepsilon(\mathbf{r}) \quad \mathbf{r} \in L^\varepsilon$$

where $U_i(\mathbf{x}, \tau)$, $Q_i^a(\mathbf{x}, \tau)$ are arbitrary vector fields defined and continuous in $\bar{\Omega}$, differentiable almost everywhere in Ω and independent of ε .

Remark. Notice that in Eq (3.2) we deal with displacements depending on three independent variables: \mathbf{x} , τ and ε ; that is why the right-hand side of Eq (3.2) determines a certain subclass of the displacement fields.

Functions $U_i(\mathbf{x}, \tau)$ will be called the macro-displacement fields and are assumed to satisfy the known invertibility conditions $\det(\delta_{ij} + U_{i,j}(\mathbf{x}, \tau)) > 0$. Functions $Q_i^a(\mathbf{x}, \tau)$, $a = 1, \dots, M$, will be called the corrector fields. At the end of section 4 it will be shown that values of all corrector fields for homogeneous structures are equal to zero.

In order to formulate the second hypothesis let us denote by $t_i^e(\mathbf{x}, \tau) = \sigma_{ij}^e(\mathbf{x}, \tau) n_j(\mathbf{x})$, $\mathbf{x} \in \partial\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$, the interactions on the interfaces between adjacent parts of volume elements $\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$. This hypothesis is related to the fine constraint hypothesis which implies that the displacements $u_i^e(\mathbf{x}, \tau)$ on the interfaces $\partial\Pi^\varepsilon(\mathbf{r})$ are independent of ε (notice that $u_i^e(\mathbf{x}, \tau) = U_i(\mathbf{x}, \tau)$ for every $\mathbf{x} \in \partial\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$). Hence we suppose that also the tractions $t_i^e(\mathbf{x}, \tau)$, $\mathbf{x} \in \partial\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$, can be assumed as independent of ε .

Fine interaction hypothesis. Interactions $t_i^e(\mathbf{x}, \tau)$, $\tau \in [\tau_0, \tau_f]$, on the interfaces of volume elements $\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$, for an arbitrary structure S_ε , $0 < \varepsilon \ll 1$, can be assumed in the form

$$t_i^e(\mathbf{x}, \tau) = T_{ij}(\mathbf{x}, \tau) n_j(\mathbf{x}) \quad \mathbf{x} \in \partial\Pi^\varepsilon(\mathbf{r}) \quad \mathbf{r} \in L^\varepsilon \quad (3.3)$$

where $T_{ij}(\mathbf{x}, \tau)$ are tensor fields defined a.e. on Ω and independent of ε and $n_j(\mathbf{x})$ is the unit vector normal to $\partial\Pi^\varepsilon(\mathbf{r})$.

Functions $T_{ij}(\mathbf{x}, \tau)$ will be called macro-stress fields. The fine interaction hypothesis imply that $\sigma_{ij}^e(\mathbf{x}, \tau) n_j(\mathbf{x}) = T_{ij}(\mathbf{x}, \tau) n_j(\mathbf{x})$, $\mathbf{x} \in \partial\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$.

The last hypothesis also takes into account the form of the displacement fields $u_i^e(\mathbf{x}, \tau)$ restricted by Eq (3.2) but is related to the kinematics of the structure S under consideration. In order to formulate this hypothesis let us observe that Eq

(3.2) can be written down in the form

$$u_i^\varepsilon(\mathbf{r} + \mathbf{y}, \tau) = U_i^\varepsilon(\mathbf{r} + \mathbf{y}, \tau) + \varepsilon \chi_{\alpha} \left(\mathbf{r}, \frac{\mathbf{y}}{\varepsilon} \right) Q_i^\alpha(\mathbf{r} + \mathbf{y}, \tau) \quad (3.4)$$

$$\mathbf{y} \in V^\varepsilon(\mathbf{r}) \quad \mathbf{r} \in L^\varepsilon \quad \tau \in [\tau_0, \tau_f]$$

Let us also introduce the fields $u_i^1(\bullet, \tau)$, defined and continuous on $\bar{\Omega}$ and differentiable a.e. in Ω , given by

$$u_i^1(\mathbf{r}^A + \mathbf{y}, \tau) \equiv U_i(\mathbf{r}^A + \mathbf{y}, \tau) + \chi_{\alpha}(\mathbf{r}^A; \mathbf{y}) Q_i^\alpha(\mathbf{r}^A + \mathbf{y}, \tau) \quad (3.5)$$

$$\mathbf{y} \in V(\mathbf{r}^A) \quad A = 1, \dots, R \quad \tau \in [\tau_0, \tau_f]$$

It can be seen that, roughly speaking, fields $u_i^1(\bullet, \tau)$ are obtained from $u_i^\varepsilon(\bullet, \tau)$ by the rescaling of the amplitude $1/\varepsilon$. The hypothesis we are now to formulate is based on the supposition that for small values of the parameter λ , kinematics of the structure S in every m.v.e. $\Pi(\mathbf{r}^A)$, $A = 1, \dots, R$, is similar to that of structure S_ε , $0 < \varepsilon \ll 1$, in $\Pi^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$ after suitable rescaling and provided that $\mathbf{r} \cong \mathbf{r}^A$. Hence we arrive at

Mezzo-micro interrelation hypothesis. If the external agencies (2.3) are admissible for the structure S then every displacement field $u_i(\bullet, \tau)$, $\tau \in [\tau_0, \tau_f]$ in this structure is a close approximation of the field $u_i^1(\bullet, \tau)$

$$u_i(\bullet, \tau) \cong u_i^1(\bullet, \tau) \quad \tau \in [\tau_0, \tau_f] \quad (3.6)$$

where the right-hand side of Eq (3.6) is determined by the definition (3.5).

Let us define

$$e_{ij}^1(\mathbf{r}^A + \mathbf{y}, \tau) \equiv u_{(i,j)}^1(\mathbf{r}^A + \mathbf{y}, \tau)$$

$$\sigma_{ij}^1(\mathbf{r}^A + \mathbf{y}, \tau) \equiv a_{ijkl}(\mathbf{r}^A + \mathbf{y}) e_{kl}^1(\mathbf{r}^A + \mathbf{y}, \tau) \quad (3.7)$$

$$\mathbf{y} \in V(\mathbf{r}^A) \quad A = 1, \dots, R \quad \tau \in [\tau_0, \tau_f]$$

Then the relation (3.6) also implies (cf Notations)

$$e_{ij}(\bullet, \tau) \cong e_{ij}^1(\bullet, \tau) \quad \sigma_{ij}(\bullet, \tau) \cong \sigma_{ij}^1(\bullet, \tau) \quad \tau \in [\tau_0, \tau_f] \quad (3.8)$$

Bearing in mind Eq (3.5) we conclude that the displacements, strain and stresses for the structure S under consideration can be evaluated on the basis of macro displacement fields $U_i(\bullet, \tau)$ and corrector fields $Q_i^\alpha(\bullet, \tau)$. We tacitly assume that the mezzo-discretization of every $V(\mathbf{r}^A)$, $A = 1, \dots, R$, is properly realized and the external agencies (2.3) are admissible for the structure. The last requirement is satisfied in most composite structural problems met in engineering and industry, where the parameter λ , from a numerical point of view, is negligibly small compared to 1.

Taking into account the close approximation formulae (3.6), (3.8) and definitions (3.5), (3.7) it is easy to see that the analysis of a macro-heterogeneous composite structure S can be reduced to the analysis of fine composite structures S_ε , $0 < \varepsilon \ll 1$. The latter will be based on

(i) Fine constraint and fine interaction hypotheses represented by Eqs.(3.2) and (3.3), respectively.

(ii) Principle of virtual work in the form of relations

$$\int_{V^1(\mathbf{r})} \sigma_{ij}^e(\mathbf{r} + \varepsilon \mathbf{y}, \tau) \bar{u}_{ij}^e(\mathbf{r} + \varepsilon \mathbf{y}) dv(\mathbf{y}) = \oint_{\partial V^1(\mathbf{r})} t_i^e(\mathbf{r} + \varepsilon \mathbf{y}, \tau) \bar{u}_i^e(\mathbf{r} + \varepsilon \mathbf{y}) da(\mathbf{y}) + \int_{V^1(\mathbf{r})} \rho^e(\mathbf{r} + \varepsilon \mathbf{y}) [b_i - \bar{u}_i^e(\mathbf{r} + \varepsilon \mathbf{y}, \tau)] \bar{u}_i^e(\mathbf{r} + \varepsilon \mathbf{y}) dv(\mathbf{y}) \quad (4.1)$$

$$\mathbf{r} \in L^e \quad \mathbf{x} \equiv \mathbf{r} + \varepsilon \mathbf{y} \in \Omega^e \quad \tau \in [\tau_0, \tau_f]$$

which have to hold for every virtual displacement field

$$\bar{u}_i^e(\mathbf{x}) = \bar{U}_i(\mathbf{x}) + \varepsilon \chi_{\alpha} \left(\tau, \frac{\mathbf{x} - \mathbf{r}}{\varepsilon} \right) \bar{Q}_i^e(\mathbf{x}) \quad (4.2)$$

$$\mathbf{x} \in \Omega^e \quad \mathbf{x} - \mathbf{r} \in V^e(\mathbf{r}) \quad \mathbf{r} \in L^e$$

and where $\bar{U}_i(\bullet)$, $\bar{Q}_i^e(\bullet)$ are arbitrary independent differentiable fields defined on Ω^e (except possibly some singular surfaces in Ω^e).

(iii) Stress-strain and strain-displacement relations given by

$$\sigma_{ij}^e(\mathbf{x}, \tau) = a_{ijkl}^e(\mathbf{x}) e_{kl}^e(\mathbf{x}, \tau) \quad \mathbf{x} \in \Omega^e \quad \tau \in [\tau_0, \tau_f] \quad (4.3)$$

and

$$e_{kl}^e(\mathbf{x}, \tau) = u_{(k,l)}^e(\mathbf{x}, \tau) \quad \mathbf{x} \in \Omega^e \quad \tau \in [\tau_0, \tau_f] \quad (4.4)$$

respectively.

The aforementioned relations have to hold for every parameter ε , small enough $0 < \varepsilon \ll 1$ (for an infinitely small positive constant in the sense of the nonstandard analysis or for $\varepsilon \searrow 0$ in the asymptotic approach). Hence terms of an order ε have to be neglected in all resulting relations.

Remark. Notice that the first partial derivatives $\chi_{\alpha,i} \left((\mathbf{x} - \mathbf{r})/\varepsilon \right)$ of functions $\varepsilon \chi_{\alpha} \left((\mathbf{x} - \mathbf{r})/\varepsilon \right) \in O(\varepsilon)$ are not of an order ε and hence the terms involving ε in Eqs (3.2), (4.2) cannot be rejected in the course of the analysis.

Setting $\tilde{U}_{ij}(\mathbf{r}) \equiv \tilde{U}_{ij}(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$, from Eq (4.2) we obtain

$$\begin{aligned} \tilde{u}_i^\varepsilon(\mathbf{r} + \varepsilon \mathbf{y}) &= \tilde{U}_i(\mathbf{r}) + \tilde{U}_{ij}(\mathbf{r}) y_j + O_i(\varepsilon) \\ \tilde{u}_{i,j}^\varepsilon(\mathbf{r} + \varepsilon \mathbf{y}) &= \tilde{U}_{ij}(\mathbf{r}) + \chi_{a,j}(\mathbf{y}) \tilde{Q}_i^a(\mathbf{r}) + O_{ij}(\varepsilon) \end{aligned} \quad (4.5)$$

$\mathbf{r} \in L^\varepsilon \qquad \mathbf{y} \in V^1(\mathbf{r}) \qquad \mathbf{x} = \mathbf{r} + \varepsilon \mathbf{y} \in \Omega$

where $\tilde{U}_i(\mathbf{r})$, $\tilde{U}_{ij}(\mathbf{r})$, $\tilde{Q}_i^a(\mathbf{r})$ have to be treated as arbitrary constants.

Similarly from Eq (3.3) we obtain

$$\begin{aligned} t_i^\varepsilon(\mathbf{r} + \varepsilon \mathbf{y}, \tau) &= [T_{ij}(\mathbf{r}, \tau) + \varepsilon T_{ijk}(\mathbf{r}, \tau) y_k] n_j(\mathbf{r} + \varepsilon \mathbf{y}) + o_i(\varepsilon) \\ \mathbf{y} \in \partial V^1(\mathbf{r}) \quad \mathbf{r} \in L^\varepsilon \quad \mathbf{r} + \varepsilon \mathbf{y} \in \Omega^\varepsilon \end{aligned} \quad (4.6)$$

Combining Eqs (4.3), (4.4), (3.2) let us substitute the right-hand side of the resulting formulae for $\sigma_{ij}^\varepsilon(\mathbf{r} + \varepsilon \mathbf{y}, \tau)$ into Eq (4.1). Let us also substitute the right-hand sides of Eqs (4.5), (4.6) into Eq (4.1). This way, after neglecting all terms of an order ε , we arrive at the variational condition which has to hold for every $\tilde{U}_i(\mathbf{r})$, $\tilde{U}_{ij}(\mathbf{r})$ and $\tilde{Q}_i^a(\mathbf{r})$. Setting aside rather long manipulations we jump to the final result. To this end we define averages

$$\langle f \rangle(\mathbf{r}) \equiv \frac{1}{\text{vol} V^1(\mathbf{r})} \int_{V^1(\mathbf{r})} f(\mathbf{r} + \mathbf{y}) d\mathbf{v}(\mathbf{y}) \quad \mathbf{r} \in \Omega \quad (4.7)$$

where $f(\mathbf{r} + \bullet)$ is an arbitrary integrable function defined almost everywhere on $V^1(\mathbf{r})$. Using Eq (4.6) it can be proved that the resulting relations can be represented in the form of the following differential field equations

$$T_{ij,j}(\mathbf{x}, \tau) + \langle \rho^1 \rangle(\mathbf{x}) b_i = \langle \rho^1 \rangle(\mathbf{x}) \tilde{U}_i(\mathbf{x}, \tau) \quad (4.8)$$

$$E_{ij}(\mathbf{x}, \tau) = U_{(ij)}(\mathbf{x}, \tau)$$

and the linear algebraic equations

$$T_{ij}(\mathbf{x}, \tau) = \langle a_{ijkl}^1 \rangle(\mathbf{x}) E_{kl}(\mathbf{x}, \tau) + \langle a_{ijkl}^1 \chi_{a,k} \rangle(\mathbf{x}) Q_l^a(\mathbf{x}, \tau) \quad (4.9)$$

$$\langle a_{ijkl}^1 \chi_{a,j} \rangle(\mathbf{x}) E_{kl}(\mathbf{x}, \tau) + \langle a_{ijkl}^1 \chi_{a,j} \chi_{b,k} \rangle(\mathbf{x}) Q_l^b(\mathbf{x}, \tau) = 0$$

which have to hold for every $\mathbf{x} \in \Omega$ (except possibly at some singular points, lines and surfaces in Ω) and for every $\tau \in [\tau_0, \tau_f]$. The symmetric tensor $E_{ij}(\mathbf{x}, \tau)$ introduced here will be called the macro-strain tensor. The average $\langle \rho^1 \rangle(\mathbf{x})$ will be called the macro mass density and the averages $\langle a_{ijkl} \rangle(\mathbf{x})$, $\langle a_{ijkl} \chi_{a,k} \rangle(\mathbf{x})$, $\langle a_{ijkl} \chi_{a,j} \chi_{b,k} \rangle(\mathbf{x})$ will be referred to as the macro elastic modulae. From the known boundary and initial conditions for structures S_ε , $0 < \varepsilon \ll 1$, determined

by the external agencies (2.3), using Eqs (3.2), (3.3) and neglecting terms of an order ε , we arrive at the following boundary and initial conditions for the macro-stresses and the macro-displacements

$$\begin{aligned} T_{ij}(\mathbf{x}, \tau) n_j(\mathbf{x}) &= t_i^\Gamma(\mathbf{x}, \tau) & \text{for a.e. } \mathbf{x} \in \Gamma \\ U_i(\mathbf{x}, \tau) &= u_i^\Delta(\mathbf{x}, \tau) & \mathbf{x} \in \Delta \quad \tau \in [\tau_0, \tau_f] \\ U_i(\mathbf{x}, \tau_0) &= u_i^\Omega(\mathbf{x}) & \dot{U}_i(\mathbf{x}, \tau_0) = v_i^\Omega(\mathbf{x}) \quad \mathbf{x} \in \Omega \end{aligned} \quad (4.10)$$

we have tacitly assumed here that the part Γ of the boundary $\text{partial}\Omega$ does not intersect interiors of regions $\Pi^\varepsilon(\mathbf{r}) \equiv \mathbf{r} + V^\varepsilon(\mathbf{r})$, $\mathbf{r} \in L^\varepsilon$.

Denote by $[f](\mathbf{x})n_i(\mathbf{x})$ a jump of a function $f(\bullet)$ across a surface element oriented by the unit normal $n_i(\mathbf{x})$ assigned to a point \mathbf{x} . It is easy to show that the macro-stress tensor has to satisfy conditions $[T_{ij}](\mathbf{x}, \tau)n_j(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Omega$; hence from Eq (3.3) it follows that the continuity of stresses $\sigma_{ij}^\varepsilon(\mathbf{x}, \tau)n_j(\mathbf{x})$ on the interfaces between adjacent cells $\Pi^\varepsilon(\rho)$, $\rho \in L^\varepsilon$, is satisfied. From Eqs (4.8) we obtain that the macro-stress tensor is symmetric: $T_{ij}(\mathbf{x}, \tau) = T_{ji}(\mathbf{x}, \tau)$.

Eqs (4.8), (4.9) represent what will be called the macro model of the structure S under considerations. Eqs (4.8) will be referred to as the macro equations of motion and macro-strain/displacement relations; their form coincides with that of the well known relations of continuum mechanics for small strains. Eqs (4.9) will be called macro-constitutive relations (for linear-elastic media); their form involves correctors $Q_i^a(\mathbf{x}, \tau)$, $a = 1, \dots, m$, which are the extra unknowns governed by the linear algebraic systems of equations (4.9)₂. It can be proved that under the well known conditions imposed on elastic modulae tensor fields $a_{ijkl}(\bullet)$, every linear mapping $\mathcal{R}^{3m} \rightarrow \mathcal{R}^{3m}$ given by $Q_{ai} = \langle a_{ijkl}^1 \chi_{a,j} \chi_{b,l} \rangle(\mathbf{x}) Q_b^k$ is invertible. Hence, from a purely formal point of view, correctors $Q_i^a(\mathbf{x}, \tau)$ could be eliminated from Eq (4.9). Notice, that by means of $\langle \chi_{a,j} \rangle(\mathbf{x}) \equiv 0$, for micro-homogeneous structures (a_{ijkl}^1 are constant) we obtain $\langle a_{ijkl}^1 \chi_{a,j} \rangle(\mathbf{x}) \equiv 0$; hence Eqs (4.9)₂, due to the invertibility of $\langle a_{ijkl}^1 \chi_{a,j} \chi_{b,l} \rangle(\mathbf{x})$, imply that all correctors are equal to zero: $Q_i^a(\mathbf{x}, \tau) = 0$, $\mathbf{x} \in \Omega$, $\tau \in [\tau_0, \tau_f]$; this is a trivial case which reduces Eqs (4.8), (4.9) to the well known equations of the linear elasticity theory. For macro-homogeneous (periodic) composite structures all macro-elastic modulae in Eqs (4.9)₂ are constant; in this case we arrive at the results given by Woźniak [8], Matysiak and Woźniak [4] and explored in a series of papers.

It can be proved that, under the known regularity conditions, the boundary value problem described by Eqs (4.8), (4.9), (4.10) has the unique solution $[\tau_0, \tau_f] \ni \tau \rightarrow (U_i(\bullet, \tau), Q_i^a(\bullet, \tau))$. Suppose that this solution has been calculated. Then using equations Eqs (3.5), (3.7) we can uniquely determine fields $u_i^1(\bullet, \tau)$, $e_{ij}^1(\bullet, \tau)$, $\sigma_{ij}^1(\bullet, \tau)$. After that the close approximation formulae (3.6), (3.8) enable us to evaluate displacement $u_i(\bullet, \tau)$, strain $e_{ij}(\bullet, \tau)$ and stress $\sigma_{ij}(\bullet, \tau)$ fields in the structure S under consideration. Thus the derived macro-model of the structure S plays the fundamental role in the mathematical modelling of

macro-heterogeneous composite structures. On the basis of this model we shall formulate in the subsequent section an engineering theory for the structures under consideration.

5. Results

The macro heterogeneity of composite structures implies that the averaged mass density in Eqs (4.8) and the macro-elastic modulae in Eqs (4.9) are functions of the position. From a calculational point of view the exact (analytical) form of these functions cannot be derived directly from their definitions. In order to obtain from Eqs (4.8), (4.9) a basis for numerical analysis we have previously to remove this drawback. To this end we introduce a set of points \mathbf{p}^E , $E = 1, \dots, P$, in Ω which are the geometric centers of certain "representative" m.v.e. Hence $\mathbf{p}^1, \dots, \mathbf{p}^P$ is a subset of $\mathbf{r}^1, \dots, \mathbf{r}^R$ where P is small compared to R . Then we construct scalar functions $\pi_E(\bullet)$, defined and continuous on $\bar{\Omega}$, such that

$$\pi_E(\mathbf{p}^F) = \delta_E^F \quad E, F \in 1, \dots, P$$

which will play a role of smooth interpolation functions. We shall assume that the postulated a priori form of functions $\pi_E(\bullet)$, $E = 1, \dots, P$, leads to the following close approximation formulae

$$\begin{aligned} \langle \rho^1 \rangle(\mathbf{x}) &\cong \sum_{E=1}^P \pi_E(\mathbf{x}) \langle \rho^1 \rangle(\mathbf{p}^E) \\ \langle a_{ijkl}^1 \rangle(\mathbf{x}) &\cong \sum_{E=1}^P \pi_E(\mathbf{x}) \langle a_{ijkl}^1 \rangle(\mathbf{p}^E) \\ \langle a_{ijkl}^1 \chi_{a,k} \rangle(\mathbf{x}) &\cong \sum_{E=1}^P \pi_E(\mathbf{x}) \langle a_{ijkl}^1 \chi_{a,k} \rangle(\mathbf{p}^E) \\ \langle a_{ijkl}^1 \chi_{a,j} \chi_{b,l} \rangle(\mathbf{x}) &\cong \sum_{E=1}^P \pi_E(\mathbf{x}) \langle a_{ijkl}^1 \chi_{a,j} \chi_{b,l} \rangle(\mathbf{p}^E) \end{aligned} \quad (5.1)$$

Thus the right-hand sides of the above formulae can be effectively used to determine macro-properties of the macro-heterogeneous composite structure under consideration.

Let us decompose region Ω into disjointed regions Ω^s , $s = 1, \dots, S$, $\bar{\Omega} = \bigcup \bar{\Omega}^s$, such that the interfaces between these regions do not intersect any m.v.e. $\Pi(\mathbf{r}^A)$, $A = 1, \dots, R$. Every Ω^s will be referred to as the macro-element of the structure. The calculational form of the macro-model of composites will be based on the extra

assumption that the fields $Q_i^s(\mathbf{s}, \tau)$ of correctors are constant in every macro-element Ω^s

$$Q_i^s(\mathbf{x}, \tau) = (Q_i^s)^s(\tau) \quad \text{for every } \mathbf{x} \in \Omega^s \quad s = 1, \dots, S \quad (5.2)$$

It has to be emphasized that this assumption does not violate the continuity of the displacement fields $\mathbf{u}_i(\mathbf{s}, \tau)$ (cf Eq (3.5), where $\chi_{\mathbf{s}}(\mathbf{r}^A, \mathbf{y}) = \mathbf{0}$ for every $\mathbf{y} \in \partial V(\mathbf{r}^A)$). Let us also introduce the finite element approximation for the macro-displacement fields (macro discretization procedure) by means of

$$U_i(\mathbf{x}, \tau) = N_{\alpha}(\mathbf{x}) U_i^{\alpha}(\tau) \quad \mathbf{x} \in \Omega \quad (5.3)$$

where $N_{\alpha}(\mathbf{s})$, $\alpha = 1, \dots, N$, are the known shape functions defined and continuous on Ω , assigned to the nodal points $\mathbf{x}^{\alpha} \in \Omega$ of the discretization lattice and hence satisfying conditions $N_{\alpha}(\mathbf{x}^{\beta}) = \delta_{\alpha}^{\beta}$. It follows that $U_i^{\alpha}(\tau) = U_i(\mathbf{x}^{\alpha}, \tau)$, $\alpha = 1, \dots, N$, are displacements at the nodal points.

Remark. The finite elements of the above discretization approach, as a rule, do not coincide with the macro elements Ω^s , $s = 1, \dots, S$, in Eq (5.2).

The well known finite element procedure results in the following system of equations

$$\begin{aligned} & \left[\int_{\Omega} \langle a_{ijkl}^1 \rangle N_{\alpha,j} N_{\beta,k} dv \right] U_i^{\beta}(\tau) + \left[\int_{\Omega} \langle \rho^1 \rangle N_{\alpha} N_{\beta} dv \right] \ddot{U}_i^{\beta}(\tau) + \\ & + \sum_{s=1}^S \left[\int_{\Omega^s} \langle a_{ijkl}^1 \chi_{\alpha,k} \rangle N_{\alpha,j} dv \right] (Q_i^s)^s(\tau) = \int_{\Gamma} t_i^{\Gamma} N_{\alpha} d\mathbf{a} + \int_{\Omega} \langle \rho^1 \rangle N_{\alpha} dv b_i; \end{aligned} \quad (5.4)$$

$$\begin{aligned} & \left[\int_{\Omega^s} \langle a_{ijkl}^1 \chi_{\alpha,j} \chi_{\beta,k} \rangle dv \right] (Q_i^s)^s(\tau) + \left[\int_{\Omega^s} \langle a_{ijkl}^1 \chi_{\alpha,j} \rangle N_{\alpha,k} dv \right] U_i^{\alpha}(\tau) = 0 \\ & s = 1, \dots, S \quad \tau \in [\tau_0, \tau_f] \end{aligned}$$

All terms in the parenthesis are the known constant coefficients. Rearranging those coefficients to the matrix form and representing $U_i^{\alpha}(\tau)$, $(Q_i^s)^s(\tau)$, $s = 1, \dots, S$, by the column vectors $\mathbf{U}(\tau)$, $\mathbf{Q}^s(\tau)$, respectively, we shall transform Eqs (5.4) and obtain finally

$$\begin{aligned} & \mathbf{K}\mathbf{U}(\tau) + \mathbf{M}\ddot{\mathbf{U}}(\tau) + \mathbf{H}^1\mathbf{Q}^1(\tau) + \dots + \mathbf{H}^S\mathbf{Q}^S(\tau) = \mathbf{R}(\tau) \\ & \mathbf{L}^s\mathbf{Q}^s(\tau) + \mathbf{H}^{sT}\mathbf{U}(\tau) = \mathbf{0} \\ & s = 1, \dots, S \quad \tau \in [\tau_0, \tau_f] \end{aligned} \quad (5.5)$$

It is easy to see that \mathbf{K} and \mathbf{M} are the global stiffness and mass $3N \times 3N$ matrices, respectively, $\mathbf{U}(\tau)$ is the $3N$ column vector of nodal macro-displacements $U_i^{\alpha}(\tau)$

and $R(\tau)$ is the $3N$ column vector of external forces. The heterogeneity characteristics of the composite structure are contained in the non singular macro-stiffness $3M \times 3M$ matrices L^s , $s = 1, \dots, S$, and in the macro-heterogeneity $3N \times 3M$ matrices H^s , $s = 1, \dots, S$ (for homogeneous structures every H^s is a zero matrix, cf section 4). At last $Q^s(\tau)$ are $3M$ column vectors of correctors $(Q_i^s)^s(\tau)$ assigned to the pertinent macro elements Ω^s , $s = 1, \dots, S$.

Eqs (5.5), (which are a matrix form of Eqs (5.4)), together with the initial conditions $U(\tau_0) = U_0$, $\dot{U}(\tau_0) = V_0$ (where U_0 , V_0 have to be consistent with Eqs (4.10)₃) and Eqs (3.5) ÷ (3.8) represent an engineering model of macro-heterogeneous composite structures. The accuracy of this model depends on the choice of discretizations and interpolations leading to Eqs (5.5) both on the mezzo- and macro-levels. In order to formulate Eqs (5.5) for the composite structure $S = (\Omega, \rho(\bullet), a_{ijkl}(\bullet))$ under consideration we have to fulfill the following requirements

- To establish a system of the "representative" m.v.e. $\Pi(\mathcal{P}^E)$, $E = 1, \dots, P$ and a system of macro elements Ω^s , $s = 1, \dots, S$.
- To discretize every m.v.e. $\Pi(\mathcal{P}^E)$, $E = 1, \dots, P$, and hence to determine functions $\rho^1(\mathcal{P}^E + \bullet)$, $a_{ijkl}^1(\mathcal{P}^E + \bullet)$ defined almost everywhere on $V^1(\mathcal{P}^E)$ (mezzo-discretization procedure).
- To introduce the interpolation functions $\pi_E(\bullet)$, $E = 1, \dots, P$.
- To introduce the finite element approximation of the region Ω and establish the shape functions $N_\alpha(\bullet)$ (macro-discretization procedure).
- Using the close approximation formulae (5.1) to calculate all integrals in Eqs (5.4) and then to represent the obtained results in the final matrix form (5.5). If the solution $U_i^s(\tau)$, $(Q_i^s)^s(\tau)$ to the initial value problem for Eqs (5.5) is obtained then we have
 - to determine the macro displacement fields $U_i(\bullet, \tau)$ and corrector fields $Q_i^s(\bullet, \tau)$, on the basis of Eqs (5.3) and (5.2), respectively,
 - using Eqs (3.5), (3.7), and remembering formulae (3.6), (3.8), to evaluate the displacements, strains and stresses in particular m.v.e. of the macro-heterogeneous composite structure under consideration.

The above procedure is characteristic for calculational research of the known composite structures. More involved are problems related to the design of macro-heterogeneous composite structures. In these problems we do not deal with one standardized composite material and desired macro-heterogeneity allows better tailoring of both micro- and macro-properties of a composite in order to derive better design specification of the structure. The results given above also can

constitute the theoretical background for the numerical analysis of such problems. More detailed investigations related to the applications and verification of the proposed engineering approach to macro-heterogeneous composite structures will be presented separately.

References

1. BENSOUSSAN A., LIONS J.L., PAPANICOLAOU G., 1980, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam
2. CHRISTENSEN R.M., 1979, *Mechanics of Composite Materials*, J.Wiley, New York
3. JONES R., 1975, *Mechanics of Composite Materials*, Mc Graw Hill, New York
4. MATYSIAK S., WOŹNIAK CZ., 1987, *Micromorphic effects in a modelling of periodic multilayered elastic composites*, Int.J.Engng Sci., 25, 549-559
5. ROBINSON A., *Nonstandard Analysis, 1966*, North-Holland, Amsterdam
6. SANCHEZ PALENCIA E., 1980, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, 127, Springer, Berlin
7. THOMPSON B.S., 1987, *Composite laminate components for robotic and machine systems: Research issues in design*, Appl.Mech.Rev., 40, 1545-1552
8. WOŹNIAK CZ., 1987, *A nonstandard method of modelling of thermoelastic periodic composites*, Int.J.Engng Sci., 25, 483-498

Niejednorodność w mechanice kompozytów

Streszczenie

Celem pracy jest zbudowanie podstaw teorii pozwalającej na obliczanie i konstruowanie makroskopowo niejednorodnych kompozytów. Efektywne własności takich struktur zależą od umiejscowienia. Rozważono deterministyczny model liniowo-sprężystych kompozytów przy doskonałym sklejeniu poszczególnych składników materiałowych.

Manuscript received January 21, 1992; accepted for print February 21, 1992