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# Composite shells with interlaminar imperfections\*

Cz. Woźniak and M. Woźniak, Warszawa

**Summary:** In this contribution the effect of interlaminar initial imperfections on a composite shell behavior is investigated. The constitutive equations for shells with initial interlaminar bonding imperfections are obtained.

## Verbundstoffschalen mit Zwischenschichtdefekten

**Übersicht:** In diesem Beitrag wird der Einfluß von anfänglichen Zwischenschichtdefekten auf das Verhalten einer Verbundstoffschale untersucht und die Materialgesetze für solche Schalen werden aufgestellt.

## 1 Introduction

In most engineering problems a composite shell is designed as a thin laminated structure in which every lamina is made of a certain standardized anisotropic macro-homogeneous material. Under assumption of the perfect bonding between the laminae different sandwich-type shell theories have been taken as a basis for the analysis of a structural shell element; as a classical example we can mention here the approach proposed in [1]. The objective of this contribution is to propose a certain unified approach to the formation of 2D-theories for laminated linear elastic shells with initial imperfections in the interlaminar bonding. This problem, according to the authors knowledge, has not been yet analyzed in the recent literature on the laminated shell theories. The proposed 2D-shell theory takes also into account the effect of the interlaminar strain discontinuities on the shell behavior. The obtained result allow to achieve a better shell design specification than that neglecting initial bonding imperfections and interlaminar strain discontinuities.

## 2 Preliminaries

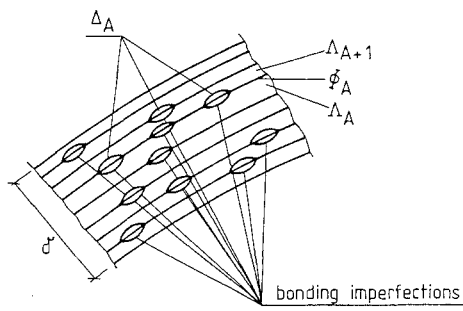
The fragment of the laminated shell under consideration is shown in Fig. 1. The region  $\Omega$  in 3-space occupied by undeformed shell is assumed to be parameterized by the system of normal coordinates  $\theta^1, \theta^2, \zeta$ , where  $\theta \equiv (\theta^1, \theta^2) \in \bar{I}$  are coordinates on the shell midsurface ( $\bar{I}$  is a regular region on  $\mathbb{R}^2$ ) and  $\zeta \in [-\delta/2, \delta/2]$ ,  $\delta$  being the shell thickness. The shell is made of  $M + 1$  laminae  $A_A$ ,  $A = 1, \dots, M + 1$ ; we also define  $A \equiv \cup A_A$ . By  $\Phi_A$ ,  $A = 1, \dots, M$  we denote the surfaces separating  $A$ -th and  $A + 1$ -th laminae, the part of  $\Phi_A$  consisting of many small initial bonding imperfections will be denoted by  $\Delta_A$ . By  $\chi_A$  we define a characteristic function of  $\Delta_A$  as a subset of  $\Phi_A$ . The thickness of an arbitrary  $A$ -th lamina is constant and will be denoted by  $\delta_A$ ,  $A = 1, \dots, M + 1$ .

We assume that:

- (i) the material of every lamina is anisotropic macro-homogeneous and linear elastic; every plane tangent to a coordinate surface  $\zeta = \text{const}$  is an elastic symmetry plane;
- (ii) there is an unilateral contact without a friction across every interlaminar bonding imperfection as well as the perfect bonding on the remaining part  $\Phi_A \setminus \Delta_A$ ,  $A = 1, \dots, M$ ;

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**Fig. 1.** Fragment of a laminated shell

(iii) the maximum characteristic length dimension of every single imperfection is small compared to the shell thickness  $\delta$ ;

(iv) the distribution of imperfections on every interface  $\Phi_A$  is random;

(v) the problem can be considered within the theory of small displacement gradients;

without the loss of generality we neglect the body forces, restrict ourselves to static problems and assume that the shell is clamped on the edges.

It can be observed that the condition of unilateral contact between laminae across imperfections implies a nonlinear behavior of a shell.

Introducing 2D-theory for shells under consideration we shall deal with the following modeling problems:

(i) how to include the possible displacement jumps across imperfections into a formal structure of 2D-theory for laminated shells;

(ii) how to describe the strain jump discontinuities across interfaces  $\Phi_A$  within 2D-shell theory;

(iii) how to represent the random distribution of interlaminar imperfections within the framework of the deterministic 2D-theory;

(iv) how to obtain an effective form of the elasticity tensors for laminated shells with initial imperfections.

The approach starts in Sect. 3 with the well known 3D-formulation of the problem. In order to pass to 2D-shell theory a new kinematic shell hypothesis will be formulated in Sect. 4. In Sect. 5 an averaged deterministic approximation for a random distribution of interlaminar imperfections is proposed. Under some extra hypothesis a certain effective form of shell constitutive equations is derived in Sect. 6. The discussion of the obtained result in Sect. 7 ends the paper.

## 2.1 Denotations

The sub- and superscripts  $i, j, \dots$  run over sequence 1, 2, 3 and are related to the normal coordinate system in the shell  $\theta^1, \theta^2, \theta^3$  (where  $\theta^3 \equiv \zeta$ ); the sub- and superscripts  $\alpha, \beta, \dots$  run over 1, 2 and are related to the coordinates  $\theta^1, \theta^2$ . Index  $a$  runs over 1,  $\dots, m$ . The summation convention holds with respect to all aforementioned indexes. A single vertical line  $|_i$  stands for the covariant derivative in the coordinate system  $\theta^1, \theta^2, \theta^3$  in the region occupied by the undeformed shell. A double vertical line  $\|_\alpha$  denotes the covariant derivative on the shell midsurface. Indexes  $A, \dots$  run over 1,  $\dots, M$  unless otherwise stated. We also denote  $f_{,3} \equiv \partial f / \partial \theta_3 = \partial f / \partial \zeta$  for an arbitrary differentiable function  $f$  which depend on  $\zeta$ . By  $\nabla\psi$  we denote a gradient of an arbitrary differentiable vector field  $\psi(\cdot)$  and define the symmetrized part of this gradient by  $\nabla(\psi) \equiv \frac{1}{2} (\nabla\psi + (\nabla\psi)^T)$ .

## 3 3D-formulation

By  $\mathbf{T}(\cdot)$ ,  $\mathbf{u}(\cdot)$  we denote stress tensor and displacement vector fields, respectively defined in  $\mathcal{A} \equiv \Omega \setminus \cup \Phi_A$ . The possible jump of  $\mathbf{u}$  across  $\Phi_A$  will be denoted by

$$[\mathbf{u}]_A \equiv \lim_{\zeta \searrow \zeta_A} \mathbf{u} - \lim_{\zeta \nearrow \zeta_A} \mathbf{u}$$

where the coordinate surface  $\zeta = \zeta_A$  coincides with the interface  $\Phi_A$ . The interlaminar stress vector field on  $\Phi_A$  will be denoted by  $\mathbf{t}_A(\cdot)$ :  $\mathbf{t}_A = \lim_{\zeta \nearrow \zeta_A} \mathbf{T}\mathbf{n}$ , where  $\mathbf{n}$  is a unit normal to a parametric surface

$\zeta = \text{const}$  at an arbitrary point of  $\Omega$ . We also denote  $t_A^3 \equiv \mathbf{t}_A \cdot \mathbf{n}$ ,  $\mathbf{t}_A^S = \mathbf{t}_A - \mathbf{n}t_A^3$  as a normal and a tangent (shear) interlaminar stress, respectively.

The governing relations for the shell under consideration are given by:

(i) the principle of virtual work

$$\int_{\Omega} \text{tr}(\mathbf{T}\check{\mathbf{E}}) dv + \sum_{A=1}^M \int_{\Phi_A} \mathbf{t}_A \cdot [\check{\mathbf{u}}]_A da = \int_{\Gamma} \mathbf{p} \cdot \check{\mathbf{u}} da, \quad \text{for every } \check{\mathbf{u}}, \check{\mathbf{u}}|_{\partial\Omega \setminus \Gamma} = 0, \quad \check{\mathbf{E}} \equiv \nabla(\check{\mathbf{u}}), \quad (1)$$

where  $\Gamma$  stands for the union of boundary surfaces  $\zeta = \zeta_0$ ,  $\zeta = \zeta_{M+1}$ , and  $\mathbf{p}$  are the boundary loadings on  $\Gamma$ ;

(ii) the stress-strain and the strain-displacement relations

$$\mathbf{T} = \mathbf{C}[\mathbf{E}], \quad \mathbf{E} = \nabla(\mathbf{u}) \quad \text{in } A, \quad (2)$$

where  $\mathbf{C}$  is a tensor of elastic moduli;

(iii) interlaminar conditions

$$\mathbf{t}_A^S = 0 \quad \text{on } \Delta_A, \quad [\mathbf{u}]_A = 0 \quad \text{on } \Phi_A \setminus \Delta_A, \quad A = 1, \dots, M, \quad (3)$$

$$\int_{\Delta_A} (\check{v} - [u_3]_A) t_A^3 da \leq 0, \quad \forall \check{v} \geq 0, \quad [u_3]_A \geq 0 \quad \text{on } \Delta_A, \quad A = 1, \dots, M. \quad (4)$$

For the sake of simplicity we assume that the displacements on the shell edges  $\partial\Omega \setminus \Gamma$  are assumed to be known. The variational inequalities (4) together with (3)<sub>1</sub> describe the unilateral contact across imperfections. (1)–(4) lead to the well known Signorini problem of the linear elasticity theory for the displacement field  $\mathbf{u}(\cdot)$ , [2]. Until  $\Delta_A$  are not known *a priori* then the solution to this problem can not be obtained. Hence, the aforementioned relations will be used only as a starting point for further considerations.

#### 4 Formation of 2D-theories

The passage from 3D-formulation, given by (1)–(4) to the 2D-shell theory will be based on two hypothesis. In order to formulate the first of them we introduce two kinds of what are called *shape functions*  $h_A(\cdot)$  and  $i_A(\cdot)$ ,  $A = 1, \dots, M$ , which depend on  $\zeta$ ; their diagrams together with the diagrams of their first derivatives are shown in Fig. 2. Functions  $h_A(\cdot)$  will be called *strain jump shape functions* and  $i_A(\cdot)$  are called *displacement jump shape functions*. For the periodic laminated structures the aforementioned shape functions have a sense similar to that introduced in [3]. By  $\alpha_a(\zeta)$ ,  $\zeta \in [-\delta/2, \delta/2]$ ,

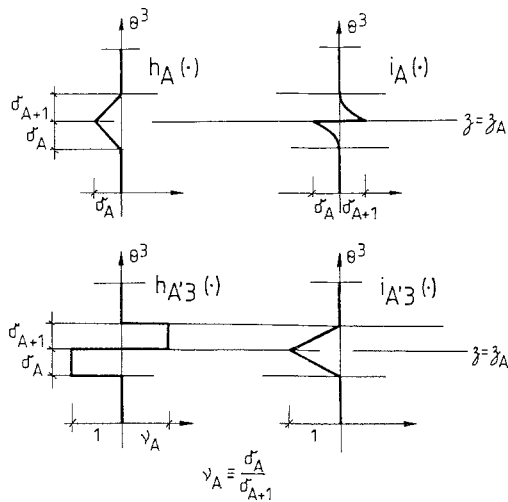


Fig. 2. Shape functions and their derivatives

$\alpha_a(0) = 0$ ,  $a = 1, \dots, m$  we also introduce functions such that  $u_i(\boldsymbol{\theta}, \zeta) = w_i(\boldsymbol{\theta}) + \alpha_a(\zeta) d_i^a(\boldsymbol{\theta})$  is a kinematic hypothesis related to a  $3 + 3m$  parameter shell theory in which  $w_i(\boldsymbol{\theta})$  are displacements of the shell midsurfaces and  $d_i^a(\boldsymbol{\theta})$  are certain “higher order displacements”. For example setting  $m = 1$  and  $\alpha_1(\zeta) = \zeta$  we arrive at the known six-parameter shell theory. In the kinematic hypothesis formulated below we shall introduce, independently of  $w_i(\cdot)$ ,  $d_i^a(\cdot)$ , extra unknown vector fields  $q_i^A(\cdot)$ ,  $j_i^A(\cdot)$  defined on  $\Phi_A$  which will be referred to as strain jump correctors and displacement jump correctors, respectively. Moreover, by  $\mu_j^i(\boldsymbol{\theta}, \zeta)$  we denote the known shell shifters given by  $\mu_\beta^\alpha \equiv \delta_\beta^\alpha - \zeta b_\beta^\alpha(\boldsymbol{\theta})$ ,  $\mu_3^\alpha = \mu_\alpha^3 = 0$ ,  $\mu_3^3 = 1$ , where  $b_\beta^\alpha(\zeta)$  are components of the second fundamental tensor of the shell midsurface.

Under the aforementioned denotations we formulate the following:

1. *Extended Kinematic Hypothesis (EKH)*. Every displacement field  $u_i(\cdot)$  in the laminated shell with initial imperfections can be assumed in the form

$$u_i(\boldsymbol{\theta}, \zeta) = \mu_i^j(\boldsymbol{\theta}, \zeta) \left[ w_j(\boldsymbol{\theta}) + \sum_{a=1}^m \alpha_a(\zeta) d_j^a(\boldsymbol{\theta}) + \sum_{A=1}^M h_A(\zeta) q_j^A(\boldsymbol{\theta}) + \sum_{A=1}^M i_A(\zeta) j_j^A(\boldsymbol{\theta}) \right],$$

$$\boldsymbol{\theta} \equiv (\theta^1, \theta^2) \in \Pi, \quad \zeta \in (-\delta/2, \delta/2). \quad (5)$$

The meaning of terms in (5) involving extra unknowns  $q_i^A(\cdot)$  and  $j_i^A(\cdot)$  is related to the laminated structure of the shell and to the presence of interlaminar imperfections.

Let  $\delta_A$  be the thickness of  $A$ -th lamina,  $A = 1, \dots, M + 1$ , and  $L$  be the minimum characteristic length dimension of the shell midsurface. Setting  $\varepsilon \equiv \max \delta_A/L$ , treat  $\varepsilon$  as a small parameter, we shall postulate the following:

2. *Thin Lamina Assumption (TLA)*. Terms of an order  $O(\varepsilon)$  will be neglected compared to terms not involving  $\varepsilon$ . Applying the aforementioned assumptions, after some manipulations, we derive from (1)–(4), the 2D-formulation of the shell problem under consideration. This formulation consists of:

(i) field equations:

$$\begin{aligned} H^{\beta\alpha}{}_{\parallel\beta} - b_\beta^\alpha H^{\beta3} + p^\alpha &= 0, & H^{\beta3}{}_{\parallel\beta} + b_{\alpha\beta} H^{\alpha\beta} + p^3 &= 0, \\ M_a^{\beta\alpha}{}_{\parallel\beta} - b_\beta^\alpha M_a^{\beta3} - s_a^\alpha + m_a^\alpha &= 0, & M_a^{\beta3}{}_{\parallel\beta} + b_{\alpha\beta} M_a^{\alpha\beta} - s_a^3 + m_a^3 &= 0, \end{aligned} \quad (6)$$

$$a = 1, \dots, m; \quad \alpha, \beta = 1, 2,$$

where

$$\begin{aligned} H^{\beta\alpha} &\equiv \int_{-\delta/2}^{\delta/2} T^{\beta\gamma} \mu_\gamma^\alpha \mu \, d\zeta, & H^{\beta3} &\equiv \int_{-\delta/2}^{\delta/2} T^{\beta3} \mu \, d\zeta, & \alpha, \beta, \gamma &= 1, 2, \\ M_a^{\beta\alpha} &\equiv \int_{-\delta/2}^{\delta/2} \alpha_a T^{\beta\gamma} \mu_\gamma^\alpha \mu \, d\zeta, & M_a^{\beta3} &\equiv \int_{-\delta/2}^{\delta/2} \alpha_a T^{\beta3} \mu \, d\zeta, \\ s_a^i &\equiv \int_{-\delta/2}^{\delta/2} \alpha_{a,3} T^{i3} \mu \, d\zeta, & p^i &\equiv p_{(+)}^i \mu_j^i(\delta/2) + p_{(-)}^i \mu_j^i(-\delta/2) \\ m_a^i &\equiv p_{(+)}^i \alpha_a \mu_j^i(\delta/2) + p_{(-)}^i \alpha_a \mu_j^i(-\delta/2), & i, j &= 1, 2, 3, \end{aligned} \quad (7)$$

where  $\mu$  is the surface invariant and  $p_{(+)}$ ,  $p_{(-)}$  are the loadings applied at the upper and lower shell boundaries, respectively. It can be observed that (6) have a form similar to that of the known  $3 + 3m$  parameter shell theory,

(ii) constitutive equations.

The components  $T^{ij}$  of the stress tensor  $\mathbf{T}$  in definitions (7) are determined by the constitutive relations:

$$T^{ij} = C^{ijkl} \left( w_{(k|l)} + \sum_{a=1}^m \alpha_a d_{(k|l)}^a \right) + \sum_{a=1}^m C^{ijk3} \alpha_{a,3} d_k^a + \sum_{A=1}^M C^{ijk3} (h_{A,3} q_k^A + \chi_A i_{A,3} j_k^A) \quad \text{in } \mathcal{A}. \quad (8)$$

(iii) interlaminar stress continuity conditions:

$$[T^{i3}]_A = 0 \quad \text{in } \Phi_A, \quad A = 1, \dots, M, \quad (9)$$

where  $[T^{i3}]_A \equiv \lim_{\zeta \searrow \zeta_{A+}} T^{i3} - \lim_{\zeta \nearrow \zeta_{A-}} T^{i3}$ .

(iv) interlaminar imperfections conditions:

$$\begin{aligned} \chi_A t_A^\alpha &= 0, \quad (1 - \chi_A) j_i^A = 0, \\ \int_{\Phi_A} (v - j_3^A) \chi_A t_A^3 da &\leq 0 \quad \forall v \geq 0, \quad j_3^A \geq 0 \quad \text{in } \Phi_A, \quad A = 1, \dots, M, \end{aligned} \quad (10)$$

where  $t_i^A \equiv \lim_{\zeta \nearrow \zeta_{A-}} T^{i3}$ .

Substituting the *RHS* of (8) into (7) and then substituting the *RHS* of the obtained formulae into (6) we arrive at the system of equations which together with (9), (10) constitutes the system of governing relations for laminated shells with initial interlaminar imperfections. Because the exact distributions of imperfections  $\chi_A$  are not known *a priori* then (6)–(10) do not lead to any well stated boundary value problem for  $w_i(\cdot)$ ,  $d_i^a(\cdot)$ ,  $q_i^A(\cdot)$  and  $j_i^A(\cdot)$ . That is why we shall pass below to a certain averaged 2D-formulation of the shell problem.

## 5 Passage to averaged 2D-theories

Let  $B(\mathbf{x}, \delta)$  stand for a ball with a center  $\mathbf{x} \equiv (\boldsymbol{\theta}, \zeta)$  and a radius  $\delta$ . Define  $B_A(\mathbf{x}) \equiv B_A(\mathbf{x}, \delta) \cap \Phi_A$ . For an arbitrary integrable field  $\psi(\cdot)$  defined almost everywhere on  $\Phi_A$  we shall introduce its average:

$$\langle \psi \rangle_A(\mathbf{x}) \equiv \frac{1}{|B_A(\mathbf{x})|} \int_{B_A(\mathbf{x})} \psi(\mathbf{y}) da(\mathbf{y}).$$

The passage to the 2D-theories with random distribution of imperfections will be based on two hypothesis:

1. *Imperfection Density Hypothesis (IDH)*. There exist constants  $\tilde{\chi}_A, \tilde{\chi}_A \in [0, 1)$  such that  $\tilde{\chi}_A \cong \langle \chi_A \rangle_A(\mathbf{x})$  for every  $\mathbf{x} \in \Phi_A, A = 1, \dots, M$ . The constants  $\tilde{\chi}_A$  will be called imperfection densities on  $\Phi_A$  and are assumed to be known *a priori*.

2. *Imperfection Averaging Hypothesis (IAH)*. Every interface  $\Phi_A$  of a laminated shell with micro-imperfections can be modelled as an ideal mixture of delaminated part  $\Delta_A$  and perfectly bonded part  $\Phi_A \setminus \Delta_A$ . Taking into account (IAH) we also introduce the partial jump correctors  $\tilde{j}_i^A \cong \langle \chi_A j_i^A \rangle_A$  and partial tractions (partial interlaminar stresses)  $\tilde{t}_A^i \cong \langle \chi_A t_A^i \rangle_A$ . After averaging of the shell governing equations (6)–(10) (according to assumptions of the ideal mixture theory) we obtain:

(i) field equations in the form (6) with the denotations (7).

(ii) constitutive equations for stresses

$$T^{ij} = C^{ijkl} \left( w_{(k|l)} + \sum_{a=1}^m \alpha_a d_{(k|l)}^a \right) + \sum_{a=1}^m C^{ijk3} \alpha_{a,3} d_k^a + \sum_{A=1}^M C^{ijk3} (h_{A,3} q_k^A + i_{A,3} \tilde{j}_k^A) \quad \text{in } A. \quad (11)$$

(iii) interlaminar stress continuity conditions in the form (9).

(iv) constitutive equations for partial tractions:

$$\begin{aligned} \tilde{t}_A^i &= C_A^{i3k3} \tilde{j}_k^A + \tilde{\chi}_A \left[ C_A^{i3kl} \left( w_{(k|l)} + \sum_{a=1}^m \alpha_a^A d_{(k|l)}^a \right) \right. \\ &\quad \left. + \sum_{a=1}^m C_A^{i3k3} \alpha_{a,3}^A d_k^a + C_A^{i3k3} (v_{A-1} q_k^{A-1} - q_k^A) \right] \quad \text{in } \Phi_A, \end{aligned} \quad (12)$$

where  $C_A^{i3kl} \equiv \lim_{\zeta \nearrow \zeta_{A-}} C^{i3kl}(\zeta)$ ,  $\alpha_a^A \equiv \alpha_a(\zeta_A)$ ,  $v_A \equiv \delta_A / \delta_{A+1}$ .

(v) averaged interlaminar imperfection conditions:

$$\tilde{t}_A^\alpha = 0, \quad \int_{\Phi_A} (v - \tilde{j}_3^A) \tilde{t}_A^3 da \leq 0 \quad \forall v \geq 0, \quad \tilde{j}_3^A \geq 0 \quad \text{in } \Phi_A. \quad (13)$$

It will be shown in Sect. 6 that on the basis of (9), (11)–(13) a certain effective form of shell constitutive equations can be derived. This form does not involve unknown corrector fields  $q_i^A(\cdot), j_i^A(\cdot)$  and plays the role similar to that of the constitutive equations for  $3 + 3m$  parameter shell theory.

## 6 Effective form of constitutive equations

In this section we are to show that under certain extra assumptions it is possible to obtain an interrelation of the form

$$\mathbf{T} = \mathbf{C}^{\text{eff}}[\nabla(\mathbf{w}), \nabla(\mathbf{d}), \mathbf{d}] \quad (14)$$

where  $\mathbf{d} \equiv (\mathbf{d}_1, \dots, \mathbf{d}_m)$ , and  $\mathbf{C}^{\text{eff}}$  is called an effective elasticity tensor of multi-parameter 2D-theory for laminated shells with interlaminar imperfections. (14) will be referred to as the effective shell constitutive equation.

Define  $p_A \equiv \langle T^{33} \rangle_A$  as a mean pressure (averaged normal stress) acting on the interface  $\Phi_A$ . The simple physical reasoning leads to *Mean Pressure Assumption (MPA)*. The following implications are assumed to hold

$$p_A > 0 \Rightarrow \tilde{j}_3^A > 0 \quad \text{and} \quad p_A \leq 0 \Rightarrow \tilde{j}_3^A = 0 \quad \text{in } \Phi_A, \quad A = 1, \dots, M. \quad (15)$$

*Conclusion:* because the condition  $p_A > 0$  implies  $\tilde{t}_A^3 = 0$  then the solution of the variational inequality (13) is

$$\tilde{j}_3^A = -(\text{sg } p_A)^+ [(C_A^{3333})^{-1} C_A^{33\mu\nu} (w_{(\mu|\nu)} + \sum \alpha_a d_{(\mu|\nu)}^a) + \sum \alpha_{a,3} d_3^a + \nu_{A-1} q_3^{A-1} - q_3^A]. \quad (16)$$

Taking into account conclusion (16) and using (13)<sub>1</sub>, (12), (9), (11) we can eliminate correctors  $q_A^i, j_A^i$  from the governing equations of the averaged 2D-theory obtained in Sect. 5. The calculations are rather lengthy and complicated. We restrict ourselves to the final results. To this end we introduce the denotations

$$(\text{sg } p)^+ = \begin{cases} 1 & \text{if } p > 0, \\ 0 & \text{if } p \leq 0 \end{cases}$$

where  $p \equiv \langle T^{33} \rangle(\theta, \zeta)$  is a mean pressure on the surface  $\zeta = \text{const}$  and we describe the averaged distribution of imperfections by means of the function

$$\tilde{\chi}(\zeta) \equiv \sum_{A=1}^M i_{A,3}(\zeta) \tilde{\chi}_A.$$

In order to write down the final formulae we also introduce what will be called *heterogeneity correction tensors*

$$\begin{aligned} \{H^{\alpha\beta\mu\nu}, H_a^{\alpha\beta\mu\nu}, H_{a3}^{\alpha\beta 33}\} &\equiv C^{\alpha\beta 33} \{H^{\mu\nu}, H_a^{\mu\nu}, H_{a3}\}, \\ \{H^{33\mu\nu}, H_a^{33\mu\nu}, H_{a3}^{33 33}\} &\equiv C^{33 33} \{H^{\mu\nu}, H_a^{\mu\nu}, H_{a3}\}, \\ \{H^{\alpha 3\mu 3}, H_a^{\alpha 3\mu 3}, H_{a3}^{\alpha 3\mu 3}\} &\equiv C^{\alpha 3\mu 3} \{\eta, \eta_a, \eta_{a3}\}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} H^{\mu\nu} &\equiv \sum_{A,B=1}^M h_{A,3} [C^{33\mu\nu}]_B Q_{33}^{AB}, \\ H_a^{\mu\nu} &\equiv \sum h_{A,3} [C^{33\mu\nu}]_B \alpha_a^B Q_{33}^{AB}, \\ H_{a3} &\equiv \sum h_{A,3} [C^{33 33}]_B \alpha_{a,3}^B Q_{33}^{AB}, \\ \eta &\equiv \sum h_{A,3} [C^{3\mu 3\nu}]_B Q_{\mu\nu}^{AB}, \\ \eta_a &\equiv \sum h_{A,3} [C^{3\mu 3\nu}]_B \alpha_a^B Q_{\mu\nu}^{AB}, \\ \eta_{a3} &\equiv \sum h_{A,3} [C^{3\mu 3\nu}]_B \alpha_{a,3}^B Q_{\mu\nu}^{AB}, \end{aligned} \quad (18)$$

and where  $Q_{ij}^{AB}$  are obtained as solutions to the linear algebraic equations of the form

$$-v_{A-1} C_A^{i3j3} Q_{ij}^{A-1,B} + (C_A^{i3j3} + v_A C_{A+1}^{i3j3}) Q_{ij}^{AB} - v_A C_{A+1}^{i3j3} Q_{ij}^{A+1,B} = \delta^{AB}. \quad (19)$$

It can be proved that the shell constitutive equations with initial interlaminar imperfections can be written down in the final form (summation convention for  $a = 1, \dots, m$  holds)

$$\begin{aligned} T^{\alpha\beta} = & \left\{ (1 - \tilde{\chi}(\text{sg } p)^+) (C^{\alpha\beta\mu\nu} - H^{\alpha\beta\mu\nu}) + \tilde{\chi}(\text{sg } p)^+ \bar{C}^{\alpha\beta\mu\nu} \right\} W_{(\mu|\nu)} \\ & + \left\{ (1 - \tilde{\chi}(\text{sg } p)^+) (C^{\alpha\beta\mu\nu} \alpha_a - H_a^{\alpha\beta\mu\nu}) + \tilde{\chi}(\text{sg } p)^+ \bar{C}^{\alpha\beta\mu\nu} \alpha_a \right\} d_{(\mu|\nu)}^a \\ & + (1 - \tilde{\chi}(\text{sg } p)^+) (C^{\alpha\beta 33} \alpha_{a,3} - H_{a3}^{\alpha\beta 33}) d_3^a, \end{aligned}$$

where  $\bar{C}^{\alpha\beta\mu\nu} \equiv C^{\alpha\beta\mu\nu} - C^{\alpha\beta 33} C^{33\mu\nu} / C^{3333}$ ;

$$\begin{aligned} T^{33} = & (1 - \tilde{\chi}(\text{sg } p)^+) (C^{33\mu\nu} - H^{33\mu\nu}) w_{(\mu|\nu)} + (1 - \tilde{\chi}(\text{sg } p)^+) (C^{33\mu\nu} \alpha_a - H_a^{33\mu\nu}) d_{(\mu|\nu)}^a \\ & + (1 - \tilde{\chi}(\text{sg } p)^+) (C^{3333} \alpha_{a,3} - H_{a3}^{3333}) d_3^a, \end{aligned} \quad (20)$$

$$\begin{aligned} T^{\alpha 3} = & 2 \cdot (1 - \tilde{\chi}) (C^{\alpha 3\mu 3} - H_a^{\alpha 3\mu 3}) w_{(3|\mu)} + 2 \cdot (1 - \tilde{\chi}) (C^{\alpha 3\mu 3} \alpha_a - H_a^{\alpha 3\mu 3}) d_{(3|\mu)}^a \\ & + (1 - \tilde{\chi}) (C^{\alpha 3\mu 3} \alpha_{a,3} - H_{a3}^{\alpha 3\mu 3}) d_\mu^a. \end{aligned}$$

Heterogeneity correction tensors  $\mathbf{H}$ ,  $\mathbf{H}_a$ ,  $\mathbf{H}_{a3}$  in (20) are determined by formulae (17)–(19).

It has to be remembered that (20) hold at any point  $(\boldsymbol{\theta}, \phi)$  belonging to an arbitrary laminae. That physical components of tensor  $\mathbf{C}$  in every laminae are constant.

(20) together with (17)–(19) represent the final result of the analysis and are called *effective constitutive relations for laminated linear elastic shells with initial interlaminar imperfections*. At the same time (20) yield the explicite form of the effective elasticity tensor  $\mathbf{C}^{\text{eff}}$  (compare (14)).

Combining field equations (6) with (7), where stress components  $T^{ij}$  are given by means of (20), we obtain the governing equations of 2D-theory for laminated shells with initial interlaminar imperfections. It has to be emphasized that these equations do not involve unknown corrector fields  $q_i^A(\cdot), j_i^A(\cdot)$ . Hence, from the formal point of view the obtained governing equations have a form similar to that of the  $3 + 3m$ -parameter shell theory with  $w_i(\cdot), d_i^a(\cdot), a = 1, \dots, m$  as the basic unknowns.

It can be shown that if  $\tilde{\chi} \in [0, 1]$  then formulae (20) lead to the quadratic positive definite form of the strain energy; the proof of this statement is rather lengthy and will be not presented here.

For  $\tilde{\chi} \equiv 0$  we arrive at the theory of laminated shells without imperfections; it can be seen that the main advantage of approach proposed in this paper is that the number  $3 + 3m$  of the basic unknowns is independent of the number of laminae. Such situation does not hold if the theory of a laminated shell is based on the *broken line kinematic hypothesis* where the number of basic unknowns is very large. Let us also observe that if  $\tilde{\chi} \rightarrow 1$  and  $p > 0$  then the state of stress tends to the plain stress state.

## 7 Discussion

Combining (14) and (20) it can be seen that  $\mathbf{C}^{\text{eff}}$  has the form

$$\mathbf{C}^{\text{eff}} = \mathbf{C} - \mathbf{H} - \mathbf{C}^{\text{imp}}(\tilde{\chi}, \text{sg } p^+) + \mathbf{H}^{\text{imp}}(\tilde{\chi}, \text{sg } p^+) \quad (21)$$

where  $\mathbf{C}$  is a standard elasticity tensor and  $\mathbf{H}$  is the heterogeneity correction tensor derived from (17)–(19). Tensor  $\mathbf{C}^{\text{imp}}(\tilde{\chi}, \text{sg } p^+)$  characterizes the effect of imperfections on the shell behavior independently of the shell laminated structure while tensor  $\mathbf{H}^{\text{imp}}(\tilde{\chi}, \text{sg } p^+)$  describes the effect of coupling between imperfections and heterogeneity on the state of stress.

It can be shown that

$$\text{sg } p = \text{sg} \left( \left\{ C^{33\mu\nu} - H^{33\mu\nu}, C^{33\mu\nu} \alpha_a - H_a^{33\mu\nu}, C^{3333} \alpha_{a,3} - H_{a3}^{3333} \right\} \begin{Bmatrix} W_{(\mu|\nu)} \\ d_{(\mu|\nu)}^a \\ d_3^a \end{Bmatrix} \right)$$

and hence, the effective constitutive equations (20) are nonlinear. It follows that the initial interlaminar imperfections leads to a nonlinear shell behavior. However, in the first approximation



we can introduce what will be called: *Mean Pressure Approximation*. The mean pressure  $p = \langle T^{33} \rangle$  acting on an arbitrary surface  $\zeta = \text{const}$  can be approximated by

$$p(\boldsymbol{\theta}, \zeta) \cong \frac{p_+{}^3(\boldsymbol{\theta}) - p_-{}^3(\boldsymbol{\theta})}{\delta} \zeta + \frac{p_+{}^3(\boldsymbol{\theta}) + p_-{}^3(\boldsymbol{\theta})}{2}, \quad \zeta \in [-\delta/2, \delta/2].$$

where  $p_+{}^3(\boldsymbol{\theta})$ ,  $p_-{}^3(\boldsymbol{\theta})$  are the known normal loadings acting on the shell boundary surfaces  $\zeta = -\delta/2$ ,  $\zeta = \delta/2$ , respectively.

Hence, we see that the effective form of constitutive equations (20) (under Mean Pressure Approximation) depends on the character of normal loadings acting on shell boundary surfaces; if  $p(\boldsymbol{\theta}, \zeta)$  is a compression then in (20)  $(\text{sg } p)^+ = 0$ , if otherwise then  $(\text{sg } p)^+ = 1$ .

The effective constitutive equations (20) for laminated shells with initial interlaminar imperfections have an explicit form and can be used in engineering problems provided that the imperfection densities are known. (20) describe the effect of imperfections on the shell behavior in the framework of  $3 + 3m$  parameter shell theory. The analysis of special shell problems as well as the possible generalizations of the proposed approach will be studied separately.

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