



HAL
open science

Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity

Mila Nikolova

► **To cite this version:**

Mila Nikolova. Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity. Applied and Computational Harmonic Analysis, 2015, 10.1016/j.acha.2015.10.010 . hal-00944006v3

HAL Id: hal-00944006

<https://hal.science/hal-00944006v3>

Submitted on 6 Nov 2015 (v3), last revised 27 Aug 2017 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Relationship between the optimal solutions of least squares regularized with ℓ_0 -norm and constrained by k-sparsity

Mila NIKOLOVA

CMLA, CNRS, ENS Cachan, Université Paris-Saclay, 94230 Cachan, France
Email: nikolova@cmla.ens-cachan.fr

Abstract. The recovery of a sparse vector from a noisy underdetermined linear system arises in various fields. Two desirable models to find a sparse solution are the constrained problem where the quadratic error is minimized subject to a given level of sparsity and the regularized problem where both the quadratic error and the sparsity are minimized using a regularization parameter.

The existence of connections between these two problems is intuitive. However, the mechanism of this relation has remained unclear so far. This work provides an exhaustive description of the relationship between the globally optimal solutions of these two problems. A partial equivalence between them always exists. We exhibit formulae for a sequence of critical parameters that can enable equality between the global minimizers of these problems. This sequence always admits a strictly decreasing subsequence that partitions the positive axis into a certain number of intervals. For every value of the regularization parameter inside an interval, there is a sparsity level k such that the global minimizers of the regularized problem and those of the k -constrained problem coincide. At the values of the subsequence of critical parameters, the optimal set of the regularized problem contains two optimal sets of the constrained problem. In some cases the whole sequence of critical parameters is strictly decreasing. Then the number of the intervals on the positive axis equals the number of all sparsity levels and both problems are quasi-completely equivalent. The critical parameters are obtained from the optimal values of the constrained problem. Examples and small-size exact numerical tests are given to illustrate our theoretical results. Our contributions yield various open questions and can help the design of innovative numerical schemes.

Keywords: ℓ_0 -regularization; k-sparsity constraint; globally optimal solutions; optimal solution analysis; parameter selection; quasi-equivalence between nonconvex problems; sparse recovery; under-determined linear systems.

1 Introduction

The recovery of sparse objects (e.g., signals, images) or representations $u \in \mathbb{R}^N$ using a few basis vectors from few and possibly inaccurate data $d \in \mathbb{R}^M$ is an extremely lively area of research in linear inverse problems and in compressed sensing [10, 16, 9, 41, 17]. The most natural measure of sparsity is the counting function $\| \cdot \|_0$, called usually the ℓ_0 -norm

$$\|u\|_0 := \# \{i \in \{0, 1, \dots, N\} : u_i \neq 0\}, \quad (1)$$

where $\#S$ is the number of elements in the set S and u_i is the i th components of u . We consider a matrix (e.g., a dictionary, a measurement system) $A \in \mathbb{R}^{M \times N}$ with $M < N$ for fixed M and N .

Two desirable models to find a sparse solution are given by the following optimization problems:

- the k -sparsity constrained minimization problem where one looks for the minimum squared error at a given level of sparsity k

$$(\mathcal{C}_k) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|_2^2, \quad \text{subject to } \|u\|_0 \leq k, \quad (2)$$

- the $\|\cdot\|_0$ -regularised problem where a positive parameter β is used to balance the minimization of both squared error and sparsity in the objective function $\mathcal{F}_\beta : \mathbb{R}^N \rightarrow \mathbb{R}$

$$(\mathcal{R}_\beta) \quad \mathcal{F}_\beta(u) := \|Au - d\|_2^2 + \beta \|u\|_0, \quad \beta > 0. \quad (3)$$

Let us evoke a few fields where problems (\mathcal{C}_k) and (\mathcal{R}_β) arise. Problem (\mathcal{C}_k) involves a natural sparse coding constraint; it is a particular case of the well known best k -term approximation [14, 12]. It has been used for low-rank matrix decomposition [4], sparse inverse problems [8]. Problem (\mathcal{R}_β) has been widely considered for subset selection [30, 6], model selection [25], variable selection [23], feature selection [33, 18], signal and image reconstruction [22, 19, 42, 15].

Even though overlooked for several decades, problems (\mathcal{C}_k) and (\mathcal{R}_β) were essentially considered from a numerical standpoint. The existence of some connections between these problems seems intuitive. However, the relationship between these problems has never been studied in a systematic way.

The goal of this work is to analyze in depth the connections between the sets of global minimizers of (\mathcal{C}_k) and of (\mathcal{R}_β) . Our theoretical results raise salient questions about the existing algorithms and can help the design of innovative numerical schemes.

As usual, we say “optimal set” or “optimal solutions” (resp., “optimal values”) for *globally optimal* solutions, i.e., *global minimizers* (resp., globally optimal values) [38, 3]. For clarity we recall that:

- In problem (\mathcal{C}_k) for $k \leq N$ the constraint set of u reads as $\{u \in \mathbb{R}^N \mid \|u\|_0 \leq k\}$, so we have

$$\text{optimal value} \quad c_k := \inf \left\{ \|Au - d\|_2^2 \mid u \in \mathbb{R}^N \text{ and } \|u\|_0 \leq k \right\}, \quad (4)$$

$$\text{optimal solutions} \quad \widehat{\mathcal{C}}_k := \left\{ u \in \mathbb{R}^N \text{ and } \|u\|_0 \leq k \mid \|Au - d\|_2^2 = c_k \right\}. \quad (5)$$

- In problem (\mathcal{R}_β) for $\beta > 0$ one has

$$\text{optimal value} \quad r_\beta := \inf \left\{ \mathcal{F}_\beta(u) \mid u \in \mathbb{R}^N \right\}, \quad (6)$$

$$\text{optimal solutions} \quad \widehat{\mathcal{R}}_\beta := \left\{ u \in \mathbb{R}^N \mid \mathcal{F}_\beta(u) = r_\beta \right\}. \quad (7)$$

We anticipate that for any $d \in \mathbb{R}^N$, it holds that $\widehat{\mathcal{C}}_k \neq \emptyset, \forall k \geq 0$ and that $\widehat{\mathcal{R}}_\beta \neq \emptyset, \forall \beta > 0$ (Lemma 1 and Theorem 2(b), respectively). In view of these definitions, we are aimed at clarifying the relationship between the sets of global minimizers $\widehat{\mathcal{C}}_k$ and $\widehat{\mathcal{R}}_\beta$. To this end, we adopt a blanket assumption:

H 1. *The matrix $A \in \mathbb{R}^{M \times N}$ satisfies $\text{rank}(A) = M < N$. It is also assumed that $d \neq 0$.*

The quite standard Definition 1 shall be used to evaluate the extent of some properties.

Definition 1. *A property is generic on \mathbb{R}^M if it holds on a subset of $\mathbb{R}^M \setminus S$ where S is closed in \mathbb{R}^M and its Lebesgue measure in \mathbb{R}^M is null.*

A generic property is clearly stronger than a property that holds only with probability one because $\mathbb{R}^M \setminus S$ contains a dense open subset of \mathbb{R}^M . Equivalently, we say that a property holds generically.

1.1 Related work

The amount of papers dealing with problems (\mathcal{C}_k) and (\mathcal{R}_β) is huge. We present a brief summary.

Solving problems (\mathcal{C}_k) and (\mathcal{R}_β) by exhaustive search is combinatorial and NP-hard in general [13, 40]. A major difficulty raised by these problems is the design of practical numerical schemes.

The solutions of problems (\mathcal{C}_k) and (\mathcal{R}_β) are usually approximated by greedy pursuit [31], relaxation of the $\|\cdot\|_0$ penalty [40] often combined with nonconvex minimization [18, 25, 19], as well as direct optimization [30, 1]. Tropp and Wright [41] gave a comprehensive overview, mainly focused on greedy pursuits and convex relaxation. Iterative hard thresholding algorithms has become a major technique after the local convergence results of Blumensath and Davies [6], further expanded by the authors to solve (\mathcal{C}_k) in [7, 8]. In the compressed sensing context, one typically seeks for guarantees for exact recovery if there is no noise and for unique and stable recovery when data are noisy. Such guarantees can be given under very strong assumptions, e.g., restricted isometry property (RIP) [10], bounds on $\text{spark}(A)$ and on the sparsity of the solution. Verifying whether a given matrix A satisfies such conditions is NP-hard in general [9]. Instead, certain random measurement matrices A , known to satisfy such conditions, are employed. Recent results in compressed sensing allow to alleviate the burden of the RIP condition [26, 5, 11]. In the context of ill-posed inverse problems from limited data, the matrix A is fixed by the physics underlying the problem. Convergence of descent methods – proximal, operator splitting, and regularized Gauss-Seidel – for a wide class of problems including (\mathcal{R}_β) was recently established in [1].

Problem (\mathcal{R}_β) is a particular case of a class of objectives where the counting function $\|\cdot\|_0$ is used in the context of Markov random field models. In the inaugural work [21] Geman and Geman (1984) designed a stochastic relaxation method for labeled images that achieves global minimization asymptotically. Various stochastic algorithms have been proposed to improve the convergence speed. Robini, Lachal and Magnin [35, 36] introduced the stochastic continuation approach and proved high probability for convergence to a global minimizer in finite time. They applied the method to reconstruct 3D tomographic images. Recently, Robini and Reissman [37] extended the methodology to general combinatorial objectives and gave results on the probability for global convergence with respect to the running time.

Comparison between $\|\cdot\|_0$ -related problems In finite dimensional real spaces, Fung and Mangasarian [20] considered the minimization of $\|u\|_p$ subject to $Au = d$, $Bu \geq b$ and $\|u\|_\infty \leq 1$ for $p \in [0, 1)$, where B and b are a matrix and a vector, respectively. They proved that the $\|u\|_0$ -problem is equivalent to the $\|u\|_p$ -problem for a sufficiently small $p > 0$. A promising continuous tight continuous relaxation of model (\mathcal{R}_β) was recently proposed in [39].

1.2 Main contributions

This work provides a detailed description of the relationship between the sets of global minimizers (called also optimal sets) of the two nonconvex (combinatorial) problems (\mathcal{C}_k) and (\mathcal{R}_β) , given in (2) and (3), respectively. These sets, see (5) and (7), are always nonempty (Lemma 1 and Theorem 2(b)). Our main results are summarized below. The couple (A, d) satisfies H1.

- (a) We define L as the least number so that the optimal value of (\mathcal{C}_L) is null; note that $L = M$ generically (Proposition 5). For any $k \leq L$, any optimal solution \hat{u} of (\mathcal{C}_k) is *strict* and obeys $\|\hat{u}\|_0 = k$ (Theorem 1). Problem (\mathcal{R}_β) for all $\beta \in (0, +\infty)$ has *at most* $L + 1$ different sets of global minimizers which are global minimizers of (\mathcal{C}_k) for $k \in \{0, \dots, L\}$ (Theorem 4).
- (b) Optimality of (\mathcal{R}_β) can be reduced to a search over the optimal sets of (\mathcal{C}_k) (Theorem 5).
- (c) A sequence of parameter values $\{\beta_k, \beta_k^U\}_{k=0}^L$ is proposed (Definition 3) using the optimal values of problem (\mathcal{C}_k) . The global minimizers of (\mathcal{C}_k) and of (\mathcal{R}_β) coincide if and only if $\beta_k < \beta < \beta_k^U$ (Theorem 6). However, according to the data, it can occur that $(\beta_k, \beta_k^U) = \emptyset$. So we focus on $J := \{k \in \{0, \dots, L\} \mid \beta_k < \beta_k^U\}$ which subset is always nonempty and yields $\beta_{J_k}^U = \beta_{J_{k-1}}$ (Proposition 3).
- (d) *Problem (\mathcal{R}_β) for any $\beta \in (\beta_{J_k}, \beta_{J_{k-1}})$ and problem (\mathcal{C}_{J_k}) have the same global minimizers* (Theorem 7). This agreement is referred to as *partial equivalence*. For the isolated values $\beta = \beta_{J_k}$, the optimal set of problem (\mathcal{R}_β) contains the global minimizers of (\mathcal{C}_{J_k}) and $(\mathcal{C}_{J_{k+1}})$ (Theorem 8).
- (e) $\{\beta_k\}_{k \in J}$ is the largest strictly decreasing subsequence in Definition 3 containing β_0 (Proposition 4).
- (f) When the whole sequence $\{\beta_k\}_{k=0}^L$ in Definition 3 is strictly decreasing, problem (\mathcal{C}_k) and problem (\mathcal{R}_β) for all $\beta \in (\beta_k, \beta_{k-1})$ have the same optimal set (Theorem 9). This case is referred to as *quasi-complete equivalence*.
- (g) The optimal solutions of (\mathcal{C}_k) and of (\mathcal{R}_β) are generically uniques (subsection 6.1).

1.3 Paper overview

Section 2 establishes necessary and sufficient conditions for global minimizers of problem (\mathcal{R}_β) only in terms of the global minimizers of problem (\mathcal{C}_k) . It begins with a study of the optimal sets of problem (\mathcal{C}_k) . Facts on the optimal sets of problem (\mathcal{R}_β) are taken from [34]. Section 3 is devoted to parameter values and conditions for agreement between the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) . The main results on the relationship between the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) – partial equivalence and possible quasi-complete equivalence – are established in section 4. Some facts on the optimal values of these problems are given in section 5. Uniqueness of the global minimizers of problems (\mathcal{C}_k) and (\mathcal{R}_β) under additional mild conditions is discussed in section 6. The theoretical findings are illustrated using exact numerical tests for $(M, N) = (5, 10)$ in section 7. Conclusions are presented in section 8.

1.4 Notation

For ease of presentation, we give here all important notations used throughout the paper.

The ℓ_2 -norm is denoted by $\|\cdot\| := \|\cdot\|_2$. Let n be a positive integer. We denote by \mathbb{I}_n and \mathbb{I}_n^0 the totally and strictly ordered index sets

$$\mathbb{I}_n := (\{1, \dots, n\}, <) \quad \text{and} \quad \mathbb{I}_n^0 := (\{0, 1, \dots, n\}, <), \quad (8)$$

where the symbol $<$ stands for the natural order of integers (the superscript 0 recalls that zero is included). Thus any subset $\omega \subseteq \mathbb{I}_n$ is also totally and strictly ordered. The support of $u \in \mathbb{R}^n$ is $\text{supp}(u) := \{i \in \mathbb{I}_n^0 \mid u[i] \neq 0\}$. A vector u is said to be k -sparse if $\|u\|_0 = \#\text{supp}(u) \leq k$.

Remark 1. For (\mathcal{C}_k) we consider also the trivial case $k = 0$ because \mathcal{F}_β always has a strict (local) minimum at $\hat{u} = 0$ [34]. According to the value of β , \hat{u} can be global minimizer of \mathcal{F}_β . \diamond

For any $\omega \subseteq \mathbb{I}_N$, we denote by A_ω the $M \times \#\omega$ submatrix of A formed from the columns of A with indexes in ω and similarly u_ω is the $\#\omega$ -length restriction of $u \in \mathbb{R}^N$ whose indexes are in ω :

$$A_\omega := (A_{\omega_1}, \dots, A_{\omega_{\#\omega}}) \quad \text{and} \quad u_\omega := (u_{\omega_1}, \dots, u_{\omega_{\#\omega}})^T,$$

where the superscript T means transposed. We also set $A_\omega^T := (A_\omega)^T$. In view of Remark 1, we define $A_\emptyset := [\] \in \mathbb{R}^{M \times 0}$ and $\text{rank}(A_\emptyset) := 0$ in order to handle the case $\hat{u} = 0$. The identity operator on \mathbb{R}^n is denoted by I_n ; the index n is omitted when clear from the context. A vector or a matrix of zeros of arbitrary dimension is denoted by 0 .

The notation specific to problem (\mathcal{C}_k) (resp., (\mathcal{R}_β)) is based on the letter “c” (resp., the letter “r”). Lowercase (resp., uppercase) letters stand for optimal values (resp., optimal sets). Furthermore:

- $L := \min \{k \in \mathbb{I}_N \mid c_k = 0\}$ where c_k is the optimal value of problem (\mathcal{C}_k) defined in (4).
- $\Omega_k := \{\omega \subset \mathbb{I}_N \mid \#\omega = k = \text{rank}(A_\omega)\}$ – introduced in (17) (section 2).
- $\hat{C} := \bigcup_{k=0}^L \hat{C}_k$ where \hat{C}_k are the global minimizers of (\mathcal{C}_k) defined in (5).
- $\hat{R} := \bigcup_{\beta > 0} \hat{R}_\beta$ where \hat{R}_β is the optimal set of (\mathcal{R}_β) (the global minimizers of \mathcal{F}_β) given in (7).
- β_k, β_k^U for $k \in \mathbb{I}_L^0$ – critical parameter values, Definition 3 (section 3).
- J (resp. J^E) – all $k \in \mathbb{I}_L^0$ such that $\beta_k < \beta_k^U$ (resp., $\beta_k = \beta_k^U$), Definition 4 (section 3).

2 Joint optimality conditions for (\mathcal{C}_k) and (\mathcal{R}_β)

In this section we shall derive necessary and sufficient conditions for the global minimizers of problem (\mathcal{R}_β) only in terms of the global minimizers of problem (\mathcal{C}_k) . The obtained results enable us to compare the optimal sets of these problems.

2.1 Preliminaries

We shall refer to the constrained quadratic optimization problem stated next. Given $d \in \mathbb{R}^M$ and $\omega \subseteq \mathbb{I}_N$, problem (\mathcal{P}_ω) reads as:

$$(\mathcal{P}_\omega) \quad \begin{aligned} & \min_{u \in \mathbb{R}^N} \|Au - d\|^2, \quad \text{subject to} \quad u[i] = 0 \quad \forall i \in \mathbb{I}_N^0 \setminus \omega \\ & \iff \hat{u}_\omega = \arg \min_{v \in \mathbb{R}^{\#\omega}} \|A_\omega v - d\|^2 \quad \text{and} \quad \hat{u}_{\mathbb{I}_N \setminus \omega} = 0 \end{aligned} \quad (9)$$

The convex problem (\mathcal{P}_ω) is related to problems (\mathcal{C}_k) and (\mathcal{R}_β) . So problem (\mathcal{P}_ω) is a good tool for analyzing the combinatorial problems (\mathcal{C}_k) and (\mathcal{R}_β) . This is used in sections 2 and 6. We remind that for any $\omega \subset \mathbb{I}_N^0$, the solution of problem (\mathcal{P}_ω) is a (local) minimizer of the nonconvex objective \mathcal{F}_β in (3), see [34]. This fact is independent of the value of β .

For clarity, we recall the following definitions:

Definition 2. For a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ and a set $S \subseteq \mathbb{R}^N$, \hat{u} is a strict (local) minimizer of the problem $\min \{f(u) \mid u \in S\}$ if there is a neighborhood $\mathcal{O} \subset S$ containing \hat{u} so that $f(u) > f(\hat{u})$ for any $u \in \mathcal{O} \setminus \{\hat{u}\}$. Further, \hat{u} is an isolated (local) minimizer if \hat{u} is the only minimizer in an open subset $\mathcal{O}' \subset \mathcal{O}$; see, e.g., [32].

An isolated minimizer is always a strict minimizer.

Remark 2. For any $\omega \subset \mathbb{I}_N$ such that $\text{rank}(A_\omega) = \#\omega$, it is readily seen from (9) that the solution \hat{u} of (\mathcal{P}_ω) is an isolated minimizer. Since any solution of problems (\mathcal{C}_k) and (\mathcal{R}_β) is a solution problem (\mathcal{P}_ω) for a particular ω , all strict minimizers discussed in this work are also isolated minimizers.

2.2 On the global minimizers of problem (\mathcal{C}_k)

We begin with a brief study of the optimal sets of problem (\mathcal{C}_k) that are essential to develop the paper. The proofs of all results in this subsection are given in Appendix A.1.

The list of the supports $\omega \in \mathbb{I}_N$ of all k -sparse vectors in \mathbb{R}^N is given bellow:

$$\Sigma_k := \bigcup_{n=0}^k \left\{ \omega \subset \mathbb{I}_N \mid \#\omega = n \right\}. \quad (10)$$

Using this notation problem (\mathcal{C}_k) in (2) also reads as

$$\min_{u \in \mathbb{R}^N} \|Au - d\|^2, \quad \text{subject to } \text{supp}(u) \in \Sigma_k. \quad (11)$$

The corresponding optimal set $\hat{\mathcal{C}}_k$ given in (5) is

$$\hat{\mathcal{C}}_k = \left\{ u \in \mathbb{R}^N, \text{supp}(u) \in \Sigma_k \mid \|Au - d\|^2 = c_k \right\}. \quad (12)$$

It is straightforward that if $\hat{u} \in \hat{\mathcal{C}}_k$ then \hat{u} solves (\mathcal{P}_ω) for $\omega := \text{supp}(\hat{u})$.

A central question is to know whether problem (\mathcal{C}_k) admits an optimal solution.

Lemma 1. For any $k \in \mathbb{I}_N^0$ problem (\mathcal{C}_k) has a global minimizer, i.e., $\hat{\mathcal{C}}_k \neq \emptyset$.

Using (\mathcal{P}_ω) in (9) and (11), the optimal value c_k of (\mathcal{C}_k) is

$$c_k = \min \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \mid \omega \in \Sigma_k \right\}. \quad (13)$$

At this point, we need the following simple lemma.

Lemma 2. Let H1 hold. Then $c_0 = \|d\|^2 > 0$ and $\{c_k\}_{k \geq 0}$ is decreasing with $c_k = 0 \quad \forall k \geq M$.

The next simple lemma has a pivotal role in this work.

Lemma 3. *Let H1 hold. For $k \in \mathbb{I}_M$, assume that (\mathcal{C}_k) has an optimal solution \hat{u} obeying*

$$\|\hat{u}\|_0 = k - n \quad \text{for } n \geq 1. \quad (14)$$

Then $A\hat{u} = d$. Furthermore, $c_m = 0$ and $\hat{u} \in \hat{\mathcal{C}}_m \quad \forall m \geq k - n$.

Based on Lemma 3 and assuming that H1 holds, we introduce the constant

$$\boxed{L := \min \{k \in \mathbb{I}_N \mid c_k = 0\}}. \quad (15)$$

L is uniquely defined since $\{c_k\}$ is decreasing with $L \leq M$ (Lemma 2). We emphasize that L depends only on d (Lemma 5(a)) and that $L = M$ generically (Proposition 5(b) and Remark 9).

Example 1. One has $L \leq M - 1$ if $d = Au$ for $\|u\|_0 \leq M - 1$. Then d belongs to a subspace of \mathbb{R}^M of dimension $\|u\|_0$ which has null Lebesgue measure in \mathbb{R}^M . Usual data range on the whole \mathbb{R}^M and $L = M$.

Theorem 1. *Let H1 hold and L be as in (15). One has:*

(a) *If $k \in \mathbb{I}_L^0$, then*

$$\hat{u} \in \hat{\mathcal{C}}_k \implies \|\hat{u}\|_0 = k = \text{rank}(A_{\hat{\sigma}}) \quad \text{for } \hat{\sigma} := \text{supp}(\hat{u}) \quad (16)$$

and \hat{u} is a strict global minimizer of problem (\mathcal{C}_k) .

(b) *If $k \geq L + 1$ then $\hat{\mathcal{C}}_L \subset \hat{\mathcal{C}}_k$.*

Observe that \hat{u} in (16) is a strict solution of problem (\mathcal{C}_k) since \hat{u} solves problem $(\mathcal{P}_{\hat{\sigma}})$ in (9) for $\hat{\sigma} := \text{supp}(\hat{u})$ where $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ (Lemma 10). Further, the graph $\{(k, c_k) \mid k \leq L\}$ is Pareto optimal [27, Definition 1]: for any $k \leq L$, there is no $u \notin \hat{\mathcal{C}}_k$ that can decrease both $\|Au - d\|^2$ and $\|u\|_0$.

Theorem 1(a) gives necessary conditions for an optimal solution of (\mathcal{C}_k) for $k \leq L$.

The optimal solutions of (\mathcal{C}_k) for $k \leq L$ are strict minimizers.

The algorithm aimed at solving (\mathcal{C}_k) proposed in [6] was shown in [6, Lemma 6] to produce, under certain conditions, solutions that fulfill this necessary condition.

Example 2. Let I_2 be the 2×2 identity matrix and set $\mathbb{1}_2 := (1, 1)^T$.

- Consider that $A = (I_2, \mathbb{1}_2)$ and $d = \mathbb{1}_2$. There is an optimal solution $\hat{u} = (0, 0, 1)^T$ with $c_1 = 0$, hence $L = 1$. Further, $\hat{u} = (1, 1, 0)^T$ is a strict minimizer of (\mathcal{C}_2) because $\text{rank}(A_{\text{supp}(\hat{u})}) = 2$. One has $\|\hat{u}\|_0 = 3$ for a continuum of optimal solutions of the form $\hat{u} = (x, x, 1 - x)^T$ where $x \in \mathbb{R} \setminus \{0, 1\}$.
- Let now $A = (I_2, I_2)$ and $d = (1, 0)^T$. The optimal solutions of (\mathcal{C}_1) are $(1, 0, 0, 0)^T$ and $(0, 0, 1, 0)^T$. One has $c_1 = 0$ and $L = 1$. For $k \geq 2$ all *other* optimal solutions are nonstrict and have the form $\hat{u} = (x, y, 1 - x, -y)^T$, $x \in \mathbb{R} \setminus \{0, 1\}$. If $y = 0$ then $\|\hat{u}\|_0 = 2$ and otherwise $\|\hat{u}\|_0 = 4$. \diamond

The optimal solutions of (\mathcal{C}_k) for $k > L$ have ℓ_0 -norms in $\{L, \dots, k\}$ and they can be nonstrict.

Remark 3. [On Assumption H1] This usual assumption ensures that A is rich enough to represent any $d \in \mathbb{R}^M$. It is also needed to prove the pivotal Lemma 3 and to define L in (15). \diamond

A direct and useful consequence of Theorem 1(a) is stated below.

Corollary 1. *Let H1 hold. Then $\widehat{C}_k \cap \widehat{C}_n = \emptyset$ for all $(k, n) \in (\mathbb{I}_L^0)^2$ such that $k \neq n$.*

If \widehat{u} solves optimally (C_k) for $k \leq L$, then \widehat{u} is not an optimal solution of (C_n) for $n \leq L$, $n \neq k$.

By Theorem 1(a), many subsets in $\{\Sigma_k\}_{k=0}^N$ are not the supports of optimal solutions of (C_k) . Accordingly, we focus only on the subsets $\omega \subset \mathbb{I}_N^0$ with exactly k entries such that $\text{rank}(A_\omega) = k$:

$$\Omega_k := \left\{ \omega \subset \mathbb{I}_N \mid \#\omega = k = \text{rank}(A_\omega) \right\}. \quad (17)$$

Remark 4. From H1 and Theorem 1(a), the optimal value of problem (C_k) for any $k \in \mathbb{I}_L^0$ obeys

$$c_k = \min \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \mid \omega \in \Omega_k \right\} \quad (18)$$

and the corresponding set of optimal solutions satisfies

$$\widehat{C}_k = \left\{ u \in \mathbb{R}^N, \text{supp}(u) \in \Omega_k \mid \|Au - d\|^2 = c_k \right\}. \quad (19)$$

Let $\widehat{u} \in \widehat{C}_k$; for $\widehat{\sigma} := \text{supp}(\widehat{u})$ one has $\widehat{\sigma} \in \Omega_k$ and \widehat{u} solves $(\mathcal{P}_{\widehat{\sigma}})$, hence \widehat{u} is an isolated minimizer (Remark 2). Denoting by $\Pi_{\widehat{\sigma}}$ the orthogonal projector onto $\text{range}(A_{\widehat{\sigma}})$, see e.g. [29], one has

$$c_k = d^T (I - \Pi_{\widehat{\sigma}}) d \quad \text{where} \quad \Pi_{\widehat{\sigma}} = A_{\widehat{\sigma}} (A_{\widehat{\sigma}}^T A_{\widehat{\sigma}})^{-1} A_{\widehat{\sigma}}^T. \quad (20)$$

From (18) and (20), the optimal value c_k satisfies also $c_k = \|d\|^2 - \max \{ d^T \Pi_\omega d \mid \omega \in \Omega_k \}$. \diamond

The fact that $\#\Omega_k \ll \#\Sigma_k$ might be useful.

2.3 Necessary and sufficient conditions

Problem (\mathcal{R}_β) in (3) is equivalently given by the global minimization of

$$\mathcal{F}_\beta(u) = \|Au - d\|^2 + \beta \#\text{supp}(u).$$

We give two known results on its optimal sets (Theorems 2 and 3) needed in what follows. They hold for any $A \in \mathbb{R}^{M \times N}$ with $M < N$.

Theorem 2 ([34] Theorem 4.4). *Let \mathcal{F}_β read as in (3). The following statements hold:*

- (a) \mathcal{F}_β always has a global minimizer, i.e., $\widehat{R}_\beta \neq \emptyset$ for any $\beta > 0$ and any $d \in \mathbb{R}^M$.
- (b) If \widehat{u} is a global minimizer of \mathcal{F}_β , then \widehat{u} is a strict minimizer.

The global minimizers of \mathcal{F}_β are strict. The strict (local) minimizers of \mathcal{F}_β are characterised next.

Theorem 3 ([34], Theorem 3.2). *A point $\widehat{u} \in \mathbb{R}^N$ is a strict (local) minimizer of \mathcal{F}_β if and only if $\text{rank}(A_{\widehat{\sigma}}) = \#\widehat{\sigma}$ where $\widehat{\sigma} := \text{supp}(\widehat{u})$.*

The global minimizers \widehat{u} of (C_k) for $k \leq L$ and those of \mathcal{F}_β are strict and moreover isolated: they solve problem $(\mathcal{P}_{\widehat{\sigma}})$ in (9) for $\widehat{\sigma} := \text{supp}(\widehat{u})$ such that $\text{rank}(A_{\widehat{\sigma}}) = \#\widehat{\sigma}$ (Remark 2).

The proofs of the statements below (except for Theorem 5) are given in Appendix A.2.

Proposition 1 relates the optimal sets and the optimal values of problems (C_k) and (\mathcal{R}_β) .

Proposition 1. *Let H1 hold. The following statements hold:*

(a) *For any $k \in \mathbb{I}_L^0$ it holds that*

$$\mathcal{F}_\beta(\hat{u}) = c_k + \beta k \quad \forall \hat{u} \in \hat{\mathcal{C}}_k .$$

(b) $\hat{u} \in \hat{\mathcal{R}}_\beta \implies \hat{u} \in \hat{\mathcal{C}}_k$ *where $k := \|\hat{u}\|_0 \in \mathbb{I}_L^0$.*

(c) $\hat{u} \in \hat{\mathcal{R}}_\beta \implies \hat{\mathcal{C}}_k \subseteq \hat{\mathcal{R}}_\beta$ *for $k := \|\hat{u}\|_0 \in \mathbb{I}_L^0$.*

By (c), the global minimizers of \mathcal{F}_β are composed of some optimal sets $\hat{\mathcal{C}}_k$ *only* for $k \leq L$.

The statements that follow are given in terms of the optimal sets $\hat{\mathcal{C}}_k$ and $\hat{\mathcal{R}}_\beta$, as defined in (5) and (7), respectively. The claim in the next Lemma 4 is usually false for ordinary subsets C and R .

Lemma 4. *Let H1 hold. For any $\beta > 0$ and for any $k \in \mathbb{I}_L^0$ one has*

$$\hat{\mathcal{C}}_k \not\subseteq \hat{\mathcal{R}}_\beta \iff \hat{\mathcal{C}}_k \cap \hat{\mathcal{R}}_\beta = \emptyset .$$

We denote by $\hat{\mathcal{C}}$ the collection of all optimal solutions of problems (\mathcal{C}_k) for $k \in \mathbb{I}_L^0$ and likewise, by $\hat{\mathcal{R}}$ – the set of all global minimizers of \mathcal{F}_β for all $\beta > 0$:

$$\boxed{\hat{\mathcal{C}} := \bigcup_{k=0}^L \hat{\mathcal{C}}_k \quad \text{and} \quad \hat{\mathcal{R}} := \bigcup_{\beta>0} \hat{\mathcal{R}}_\beta .} \quad (21)$$

With this notation, Theorem 4 is a direct consequence of Proposition 1(b).

Theorem 4. *Let H1 hold. Then $\hat{\mathcal{R}} \subset \hat{\mathcal{C}}$.*

Theorem 4 shows that when β ranges on $(0, +\infty)$, \mathcal{F}_β can have *at most* $L + 1$ different sets of global minimizers which are optimal solutions of (\mathcal{C}_k) for $k \in \{0, \dots, L\}$.

In view of Proposition 1(c) and Theorem 4, each global minimizer of (\mathcal{R}_β) can be composed out of the optimal sets of several problems of the form of (\mathcal{C}_k) or it can be equal to the optimal set of exactly one problem (\mathcal{C}_k) . This is made explicit in the following remark:

Remark 5. Let $\beta > 0$ and $k \in \mathbb{I}_L^0$. Since $\hat{\mathcal{R}}_\beta$ is the set of the global minimizers of \mathcal{F}_β , one has

$$\begin{aligned} \hat{\mathcal{C}}_k \subseteq \hat{\mathcal{R}}_\beta &\iff \mathcal{F}_\beta(u) \geq \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall u \in \mathbb{R}^N ; \\ \hat{\mathcal{C}}_k = \hat{\mathcal{R}}_\beta &\iff \mathcal{F}_\beta(u) > \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall u \in \mathbb{R}^N \setminus \hat{\mathcal{C}}_k . \end{aligned}$$

The next theorem provides the basic tool to compare the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) .

Theorem 5. *Let H1 hold and let $\beta > 0$. For any $k \in \mathbb{I}_L^0$ the following holds:*

(a) $\hat{\mathcal{C}}_k \subseteq \hat{\mathcal{R}}_\beta$ *if and only if*

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall \bar{u} \in \hat{\mathcal{C}} ; \quad (22)$$

(b) $\hat{\mathcal{C}}_k = \hat{\mathcal{R}}_\beta$ *if and only if*

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) > 0 \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall \bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k . \quad (23)$$

Proof. From Remark 5 it is straightforward that

$$\widehat{C}_k \subseteq \widehat{R}_\beta \quad (\text{resp., } \widehat{C}_k = \widehat{R}_\beta) \implies \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \forall \bar{u} \in \widehat{C} \quad (\text{resp., } > 0, \quad \forall \bar{u} \in \widehat{C} \setminus \widehat{C}_k) \quad \forall \hat{u} \in \widehat{C}_k.$$

The rest of the proof is by contraposition.

(a) Assume that $\widehat{C}_k \not\subseteq \widehat{R}_\beta$. Then $\widehat{C}_k \cap \widehat{R}_\beta = \emptyset$ by Lemma 4. Since $\widehat{R}_\beta \neq \emptyset$ (Theorem 2(a)), Proposition 1(c) entails that there exists $n \in \mathbb{I}_L^0 \setminus \{k\}$ such that $\widehat{C}_n \subseteq \widehat{R}_\beta$. It follows that

$$\mathcal{F}_\beta(\hat{u}) > \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \widehat{C}_k \quad \forall \bar{u} \in \widehat{C}_n, \quad (24)$$

a contradiction to (22).

(b) Let $\widehat{C}_k \neq \widehat{R}_\beta$. This, together with $\widehat{R}_\beta \neq \emptyset$, implies that there is $\bar{u} \in \widehat{R}_\beta$ such that $\bar{u} \notin \widehat{C}_k$. Proposition 1(c) shows that $\widehat{C}_n \subseteq \widehat{R}_\beta$ for $n := \|\bar{u}\|_0 \in \mathbb{I}_L^0$ where $n \neq k$ (Theorem 1(a)). Therefore

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \widehat{C}_k \quad \forall \bar{u} \in \widehat{C}_n,$$

which contradicts (23). □

Theorem 5 is the key to finding the links between the optimal sets of (\mathcal{C}_k) and (\mathcal{R}_β) : it provides necessary and sufficient conditions for optimality of (\mathcal{R}_β) only in terms of the optimal sets of $\{(\mathcal{C}_k)\}_{k=0}^L$.

A simple useful result is stated next.

Lemma 5. *Let H1 hold. Let $(k, k+p) \in (\mathbb{I}_L^0)^2$ for $p \geq 1$. The following implications holds:*

$$\widehat{C}_k \subseteq \widehat{R}_\beta \quad (\text{resp., } \widehat{C}_k = \widehat{R}_\beta) \quad \text{and} \quad \widehat{C}_{k+p} = \widehat{R}_{\beta'} \quad (\text{resp., } \widehat{C}_{k+p} \subseteq \widehat{R}_{\beta'}) \implies \beta' < \beta.$$

This lemma confirms the intuition that when β increases on $(0, +\infty)$, the optimal sets \widehat{R}_β are given by a subsequence of $\{\widehat{C}_k\}$ with decreasing indexes.

3 Parameter values for equality between optimal sets

The links between the optimal solutions of problems (\mathcal{C}_k) and (\mathcal{R}_β) are driven by the values of k and of β . This section is devoted to parameter selection.

3.1 The entire list of parameter values

Based on the optimal values $\{c_k\}_{k=0}^L$ of problems (\mathcal{C}_k) 's, see (20), for each $k \in \mathbb{I}_L^0$ we give explicit formulae for the lower and the upper bounds of β that can enable an agreement between the optimal sets \widehat{C}_k and \widehat{R}_β .

Definition 3. (Critical parameter values) *Let L be as in (15). The parameters (β_k, β_k^U) are defined by*

$$\beta_k := \max \left\{ \frac{c_k - c_{k+n}}{n} \mid n \in \{1, \dots, L-k\} \right\} \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \text{and} \quad \beta_L := 0, \quad (25)$$

$$\beta_k^U := \min \left\{ \frac{c_{k-n} - c_k}{n} \mid n \in \{1, \dots, k\} \right\} \quad \forall k \in \mathbb{I}_L \quad \text{and} \quad \beta_0^U := +\infty. \quad (26)$$

The superscript u in (26) suggests that β_k^u can be the upper bound.

Remark 6. Since $d \neq 0$, this definition indicates that $\beta_L = 0 < \beta_L^u = \min_{n=1}^L \frac{c_{L-n}}{n}$ and that $\beta_0 < \beta_0^u$ because β_0 is finite. The cases where $\beta_k < \beta_k^u$ will be of particular interest, as seen in section 4 (Theorems 7 and 9). However, this inequality can fail (see Discussion on Theorem 9). A simplification of these parameters is derived in Proposition 3. Some insight can be gained from the numerical tests in section 7. \diamond

In view of Definition 3, the intuition suggests that the set $\{k \mid \beta_k = \beta_k^u\}$ should be “small”.

Proposition 2. *Let H1 hold and $\{\beta_k, \beta_k^u\}_{k=0}^L$ be as in Definition 3. There exists a finite union of vector subspaces of dimension $\leq M - 1$, denoted by S , such that*

$$d \in \mathbb{R}^M \setminus S \implies \beta_k \neq \beta_k^u \quad \forall k \in \mathbb{I}_L^0 .$$

Thus $\mathbb{R}^M \setminus S$ contains a dense open subset of \mathbb{R}^M . The proof is given in Appendix B.1.

Data generically live in $\mathbb{R}^M \setminus S$. So $\beta_k \neq \beta_k^u \quad \forall k \in \mathbb{I}_L^0$ in Definition 3 holds generically.

3.2 Conditions for agreement between the optimal sets of (\mathcal{C}_k) and (\mathcal{R}_β)

Theorem 6 relates Theorem 5 and Definition 3. It provides a general mechanism for comparing the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) .

Theorem 6. *Let H1 hold. Then $\forall k \in \mathbb{I}_L^0$ it holds that*

$$(a) \quad \widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta \quad \text{if and only if} \quad \begin{cases} \beta_0 \leq \beta < \beta_0^u & \text{for } k = 0 ; \\ \beta_k \leq \beta \leq \beta_k^u & \text{for } k \in \{1, \dots, L-1\} ; \\ \beta_L < \beta \leq \beta_L^u & \text{for } k = L . \end{cases}$$

$$(b) \quad \widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta \quad \text{if and only if} \quad \beta_k < \beta < \beta_k^u .$$

Proof. The following equalities come from Proposition 1(a):

– If $\widehat{u} \in \widehat{\mathcal{C}}_k$ for $k \in \mathbb{I}_{L-1}^0$, then

$$\forall n \in \mathbb{I}_{L-k} \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k+n} \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = c_{k+n} - c_k + n\beta = n \left(\beta - \frac{c_k - c_{k+n}}{n} \right) . \quad (27)$$

– If $\widehat{u} \in \widehat{\mathcal{C}}_k$ for $k \in \mathbb{I}_L$, then

$$\forall n \in \mathbb{I}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k-n} \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = c_{k-n} - c_k - n\beta = n \left(\frac{c_{k-n} - c_k}{n} - \beta \right) . \quad (28)$$

(a) Using (27) together with (25) in Definition 3, for any $k \in \mathbb{I}_{L-1}^0$ one has

$$\mathcal{F}_\beta(\bar{u}) \geq \mathcal{F}_\beta(\widehat{u}) \quad \forall \widehat{u} \in \widehat{\mathcal{C}}_k \quad \forall n \in \mathbb{I}_{L-k} \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k+n} \iff \beta \geq \frac{c_k - c_{k+n}}{n} \quad \forall n \in \mathbb{I}_{L-k} \iff \beta \geq \beta_k .$$

Using (28) together with (26) in Definition 3, for any $k \in \mathbb{I}_L$ one has

$$\mathcal{F}_\beta(\bar{u}) \geq \mathcal{F}_\beta(\widehat{u}) \quad \forall \widehat{u} \in \widehat{\mathcal{C}}_k \quad \forall n \in \mathbb{I}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k-n} \iff \beta \leq \frac{c_{k-n} - c_k}{n} \quad \forall n \in \mathbb{I}_k \iff \beta \leq \beta_k^u .$$

For $k = 0$ (resp., for $k = L$), $\beta \geq \beta_0$ (resp., $\beta \leq \beta_L^U$) is equivalent to $\beta_0 \leq \beta < +\infty =: \beta_0^U$ (resp., $\beta_L := 0 < \beta \leq \beta_L^U$). Combining the obtained results, one has

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0, \quad \forall \hat{u} \in \hat{\mathcal{C}}_k, \quad \forall \bar{u} \in \hat{\mathcal{C}}$$

if and only if $\beta_k \leq \beta \leq \beta_k^U$ for $k \in \mathbb{I}_{L-1}$, $\beta_0 \leq \beta \leq \beta_0^U$ (resp., $\beta_L < \beta \leq \beta_L^U$) for $k = 0$ (resp., for $k = L$). Applying Theorem 5(a) proves statement (a).

(b) The proof of (b) follows the same recipe as above (nonstrict inequalities are replaced by strict inequalities) and the conclusion is obtained using Theorem 5(b). \square

The proof of Theorem 6 reveals how the critical parameters in Definition 3 were defined.

3.3 The effective parameters values

Since the global minimizers of \mathcal{F}_β are always in $\hat{\mathcal{C}}$ (Theorem 4), we are interested in the indexes k for which there exist values of β such that \mathcal{F}_β has global minimizers containing components of $\hat{\mathcal{C}}_k$. Their set, referred to as effective parameter set, is obtained from Theorem 6.

Definition 4. Let $\{\beta_k, \beta_k^U\}_{k=0}^L$ be as in Definition 3. The effective index set $J \cup J^E$ is defined by

$$\boxed{J := \{k \in \mathbb{I}_L^0 \mid \beta_k < \beta_k^U\} \quad \text{and} \quad J^E := \{m \in \mathbb{I}_L^0 \mid \beta_m = \beta_m^U\} .} \quad (29)$$

The entries J_k of J are ordered as it follows:

$$J = \{J_0, J_1, \dots, J_p\} \quad \text{where} \quad p := \#J - 1 \quad \text{and} \quad J_{k-1} < J_k \quad \forall k . \quad (30)$$

Using Definition 3 and Remark 6,

$$(J_0 = 0, J_p = L) \in J^2 \quad \text{with} \quad \beta_{J_{-1}} := \beta_{J_0}^U \equiv \beta_0^U = +\infty \quad \text{and} \quad \beta_{J_p} \equiv \beta_L = 0 . \quad (31)$$

It worths emphasizing that the set J is always nonempty (Remark 6). The superscript E in J^E evokes equality.

The proofs of several statements in this subsection are delegated to Appendix B.2. The next claim is a cautionary consequence of Theorems 4 and 6.

Lemma 6. Let H1 hold. One has $\hat{\mathcal{R}} \cap \hat{\mathcal{C}}_k = \emptyset$ if and only if $k \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$.

In words: for any $\beta > 0$, the optimal set $\hat{\mathcal{R}}$ of problem (\mathcal{R}_β) does not contain optimal solutions of (\mathcal{C}_k) , $k \leq L$, unless k belongs to $J \cup J^E$. Thus $\hat{\mathcal{R}} = \bigcup_{k \in \{J \cup J^E\}} \hat{\mathcal{C}}_k$.

A simplification of the parameters $\{\beta_k, \beta_k^U\}_{k \in J \cup J^E}$ is derived in Proposition 3.

Proposition 3. Let H1 hold, $\{\beta_k, \beta_k^U\}$ and J be as in Definition 3 and Definition 4, respectively. Then

- (a) $\beta_{J_k} < \beta_{J_k}^U = \beta_{J_{k-1}} \quad \forall J_k \in J \setminus \{J_0\} \quad \text{and} \quad \beta_{J_0}^U \equiv \beta_{J_{-1}} = +\infty .$
- (b) $\beta_{J_k} = \frac{c_{J_k} - c_{J_{k+1}}}{J_{k+1} - J_k} \quad \forall J_k \in J \setminus \{J_p\} \quad \text{and} \quad \beta_{J_p} \equiv \beta_L = 0 .$
- (c) $\{\beta_m \mid m \in J^E\} \subset \{\beta_{J_k} \mid J_k \in J \setminus \{J_p\}\} .$

Proof. (a)-(b) Let $(J_{k-1}, J_k) \in J^2$. Applying Definition 3 for $\beta_{J_{k-1}}$ and for $\beta_{J_k}^U$ yields

$$\beta_{J_k}^U \leq \frac{c_{J_{k-1}} - c_{J_k}}{J_k - J_{k-1}} \leq \beta_{J_{k-1}} . \quad (32)$$

Assume that $\beta_{J_k}^U < \beta_{J_{k-1}}$ and that $(m_1, \dots, m_q) \in (J^E)^q$ satisfy $\beta_{m_i} \in (\beta_{J_k}^U, \beta_{J_{k-1}})$, $\forall i \in \mathbb{I}_q$. Since $\widehat{R}_\beta \neq \emptyset \quad \forall \beta > 0$, Proposition 1(c) implies that for $\beta \in (\beta_{J_k}^U, \beta_{J_{k-1}}) \setminus \{\beta_{m_i}\}_{i=1}^q$ there is $n \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$ obeying $\widehat{C}_n \subset \widehat{R}_\beta$, in contradiction to Lemma 6. Therefore, $\beta_{J_k}^U = \beta_{J_{k-1}}$. This, together with (32) and the formula of J in (29) gives that

$$\beta_{J_k} < \beta_{J_k}^U = \frac{c_{J_{k-1}} - c_{J_k}}{J_k - J_{k-1}} = \beta_{J_{k-1}} .$$

(c) Let $m \in J^E$. By Definition 4, $\beta_m > \beta_{J_p} = 0$. From Theorem 6(b) and statement (a), $\beta_m \notin (\beta_{J_k}, \beta_{J_{k-1}})$ for any $J_k \in J$. Consequently, there exists $J_k \in J \setminus \{J_p\}$ such that $\beta_m = \beta_{J_k}$. \square

We emphasize that $\{\beta_k\}_{k \in J}$ is strictly decreasing and that its first entry is β_0 .

In Example 3 we designed a decreasing sequence $\{c_k\}_{k=0}^L$ that illustrates several special cases.

Example 3. Let $\{c_k\}_{k=0}^L$ for $L = 7$ reads as

$$c_0 = 48 \quad c_1 = 40 \quad c_2 = 30 \quad c_3 = 22 \quad c_4 = 14 \quad c_5 = 10 \quad c_6 = 4 \quad c_7 = 0 . \quad (33)$$

According to Definition 3 the sequences $\{\beta_k, \beta_k^U\}_{k=0}^7$ are given by

$$\begin{aligned} \beta_0 &= \mathbf{9} & \beta_1 &= 10 & \beta_2 &= \mathbf{8} & \beta_3 &= 8 & \beta_4 &= \mathbf{5} & \beta_5 &= 6 & \beta_6 &= \mathbf{4} & \beta_7 &= \mathbf{0} \\ \beta_0^U &= +\infty & \beta_1^U &= 8 & \beta_2^U &= \mathbf{9} & \beta_3^U &= 8 & \beta_4^U &= \mathbf{8} & \beta_5^U &= 4 & \beta_6^U &= \mathbf{5} & \beta_7^U &= 4 \end{aligned} \quad (34)$$

From Definition 4, $p = 4$, $J = \{J_0 = \mathbf{0}, J_1 = \mathbf{2}, J_2 = \mathbf{4}, J_3 = \mathbf{6}, J_4 = \mathbf{7}\}$ and $J^E = \{\mathbf{3}\}$.

- One has $\beta_{J_k} = \beta_{J_{k+1}}^U$ for any $J_k \in J$ as asserted in Proposition 3(a).
- The formula in Proposition 3(b) is easy to verify.
- $\{\beta_3 \mid 3 \in J^E\}$ yields $\beta_3 = \beta_{J_1} = 8$ and thus $\{\beta_3 \mid 3 \in J^E\} \subset \{\beta_{J_k} \mid J_k \in J \setminus \{J_4\}\}$ (Proposition 3(c)).
- One has $J_{\beta_{J_1}}^E := \{m \in J^E \mid \beta_m = \beta_{J_1}\} = \{3 \in J^E \mid J_1 < 3 < J_2\}$, as seen in Lemma 7.
- Observe that J has the smallest indexes so that $\{\beta_k\}_{k \in J} = \{9, 8, 5, 4, 0\}$ is the longest strictly decreasing subsequence of $\{\beta_k\}_{k=0}^7$ containing β_0 – see Proposition 4. Another set yielding the same $\{\beta_k\}_{k \in J}$ is $J' := \{0, 3, 4, 6, 7\}$; however its indexes are larger than those of J : $J'_2 > J_2$. \diamond

The location of $\{\beta_m \mid m \in J^E\}$ is given by the (probably empty) subsets

$$J_{\beta_{J_k}}^E := \{m \in J^E \mid \beta_m = \beta_{J_k}\} . \quad (35)$$

Lemma 7. *Let H1 hold. The sets $J_{\beta_{J_k}}^E$ in (35) fulfill $J_{\beta_{J_k}}^E = \emptyset$ for $k = p$, and for any $k \leq p - 1$*

$$J_{\beta_{J_k}}^E = \{m \in J^E \mid J_k < m < J_{k+1}\} . \quad (36)$$

We want to know how J and $\{\beta_k\}_{k \in J}$ are related to $\{\beta_k\}_{k \in \mathbb{I}_L^0}$ in (25), Definition 3. Both J and $\{\beta_k\}_{k \in J}$ are characterized in the following proposition:

Proposition 4. *Let H1 hold, $\{\beta_k\}_{k=0}^L$ read as in Definition 3 and J as in Definition 4. Then $0 \in J$ and J contains the smallest indexes such that $\{\beta_k\}_{k \in J}$ is the longest strictly decreasing subsequence of $\{\beta_k\}_{k=0}^L$ containing β_0 .*

In order to find the effective indexes J and values $\{\beta_k\}_{k \in J}$ we need only $\{\beta_k\}_{k=0}^L$ in (25), Definition 3.

The effective parameter values according to Definition 4 can be derived using Proposition 4.

4 Equivalence relations between the optimal sets of (\mathcal{C}_k) and (\mathcal{R}_β)

In this section we derive the main results of this paper. The most general relationship between the global minimizers of (\mathcal{C}_k) and (\mathcal{R}_β) is a *partial equivalence* formulated in Theorems 7 and 8 (subsection 4.1). According to the content of the data d , *quasi-complete equivalence* can hold in the sense that the global minimizers of both problems differ only at $\beta = \beta_k$ for $k = 0, \dots, L - 1$. The result is presented in Theorem 9 (subsection 4.2).

4.1 Partial equivalence

The main result of this work can be stated in the following theorems. We recall that the set J cannot be empty (Remark 6).

Theorem 7. *Let H1 hold, $\{\beta_k\}$ be as in Definition 3 and J as in Definition 4. Then the following hold:*

$$\left\{ \widehat{\mathcal{R}}_\beta \mid \beta \in (\beta_{J_k}, \beta_{J_{k-1}}) \right\} = \widehat{\mathcal{C}}_{J_k} \quad \forall J_k \in J, \quad (37)$$

$$\left(\bigcup_{n=1}^p [\beta_{J_n}, \beta_{J_{n-1}}] \right) \cup [\beta_{J_0}, \beta_{J_{-1}}] = [0, +\infty). \quad (38)$$

Proof. From Proposition 3(a), $(\beta_{J_k}, \beta_{J_k}^U) = (\beta_{J_k}, \beta_{J_{k-1}}) \neq \emptyset$ for any $J_k \in J$. Then (37) is an immediate consequence of Theorem 6(b). Definition 4 and Proposition 3(a) directly lead to (38). \square

Discussion on Theorem 7 The theorem states that for any $\beta \notin \{\beta_k \mid k \in J\}$ there is an optimal set $\widehat{\mathcal{C}}_k$ of problem (\mathcal{C}_k) that coincides with the optimal set $\widehat{\mathcal{R}}_\beta$ of problem (\mathcal{R}_β) for a whole range of parameter values β . More precisely, the effective parameter values $\{\beta_{J_0}, \dots, \beta_{J_{p-1}}\}$ in Definition 4 partition the positive axis $(0, +\infty)$ into $\sharp J$ proper intervals. For any $\beta \in (\beta_{J_k}, \beta_{J_{k-1}})$ the optimal set of problem (\mathcal{R}_β) equals the optimal set of problem (\mathcal{C}_n) for $n = J_k$. The agreement described above can be referred to as partial equivalence because when $\mathbb{I}_L^0 \setminus J$ is nonempty, the optimal sets (\mathcal{C}_k) for $k \in \mathbb{I}_L^0 \setminus J$ cannot be optimal solutions of (\mathcal{R}_β) for any $\beta > 0$.

Remark 7. Since $\{\beta_k \mid k \in J \setminus \{L\}\}$ is a finite set of isolated values in $(0, \infty)$, the selection of a β belonging to this set can be considered as extremely exceptional ($\beta \neq \beta_k$ for any k is a generic property).

In spite of this remark, we will describe the optimal sets of problem (\mathcal{R}_β) for β_k , $k \in J$.

Theorem 8. *Let H1 hold. Let $\{\beta_k\}$ be as in Definition 3 and J as in Definition 4. Then*

$$\widehat{\mathcal{R}}_{\beta_{J_k}} = \widehat{\mathcal{C}}_{J_k} \cup \widehat{\mathcal{C}}_{J_{k+1}} \cup \left(\bigcup_{m \in J_{\beta_{J_k}}^E} \widehat{\mathcal{C}}_m \right) \quad \forall J_k \in J \setminus \{J_p\}, \quad (39)$$

where $J_{\beta_{J_k}}^E$ obeys (36) and $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_n = \emptyset$ for any $(k, n) \in (J \cup J^E)^2$, $k \neq n$.

Proof. For any $k \leq p-1$, Proposition 3(a) and Theorem 6(a) show that $\widehat{\mathcal{C}}_{J_k} \subseteq \widehat{\mathcal{R}}_{\beta} \forall \beta \in [\beta_{J_k}, \beta_{J_{k-1}}]$ and that $\widehat{\mathcal{C}}_{J_{k+1}} \subseteq \widehat{\mathcal{R}}_{\beta} \forall \beta \in [\beta_{J_{k+1}}, \beta_{J_k}]$. Therefore, $\widehat{\mathcal{C}}_{J_k} \cup \widehat{\mathcal{C}}_{J_{k+1}} \subseteq \widehat{\mathcal{R}}_{\beta_{J_k}}$. In addition, $\beta_m = \beta_{J_k}$ for any $m \in J_{\beta_{J_k}}^E$ (Proposition 3(c) and (35)) which yields $\bigcup_{m \in J_{\beta_{J_k}}^E} \widehat{\mathcal{C}}_m \subseteq \widehat{\mathcal{R}}_{\beta_{J_k}}$. The conditions in Theorem 6 for $\beta = \beta_{J_k}$ can be satisfied only for $n \in \{J_k \cup J_{\beta_{J_k}}^E\}$ because $\{\beta_{J_k}\}$ is strictly decreasing (Proposition 3(a)). Hence the equality in (39). The set $J_{\beta_{J_k}}^E$ satisfies (36) by Lemma 7. The result on the intersection of the sets $\widehat{\mathcal{C}}_n$ comes from Corollary 1. \square

Example 4. [Example 3, continued] Let $\{\beta_k\}_{k=0}^7$, J and J^E be as in Example 3. We recall that $J = \{0, 2, 4, 6, 7\}$ and that $J^E = \{3\}$, so $J_2^E = \{3\}$ and $J_k^E = \emptyset$ otherwise. By Theorem 7 one has

$$\begin{aligned} \{\widehat{\mathcal{R}}_{\beta} \mid \beta > 9\} &= \widehat{\mathcal{C}}_0 & \{\widehat{\mathcal{R}}_{\beta} \mid \beta \in (8, 9)\} &= \widehat{\mathcal{C}}_2 & \{\widehat{\mathcal{R}}_{\beta} \mid \beta \in (5, 8)\} &= \widehat{\mathcal{C}}_4 & \{\widehat{\mathcal{R}}_{\beta} \mid \beta \in (4, 5)\} &= \widehat{\mathcal{C}}_6 & \{\widehat{\mathcal{R}}_{\beta} \mid \beta \in (0, 4)\} &= \widehat{\mathcal{C}}_7 \\ \text{and } \widehat{\mathcal{R}}_{\beta=9} &= \widehat{\mathcal{C}}_0 \cup \widehat{\mathcal{C}}_2 & \widehat{\mathcal{R}}_{\beta=8} &= \widehat{\mathcal{C}}_2 \cup \widehat{\mathcal{C}}_3 \cup \widehat{\mathcal{C}}_4 & \widehat{\mathcal{R}}_{\beta=5} &= \widehat{\mathcal{C}}_4 \cup \widehat{\mathcal{C}}_6 & \widehat{\mathcal{R}}_{\beta=4} &= \widehat{\mathcal{C}}_6 \cup \widehat{\mathcal{C}}_7. \end{aligned}$$

Discussion on Theorem 8 Proposition 2 have proved that the sets $J_{\beta_{J_k}}^E$ are empty with an overwhelming probability. So (39) in Theorem 7 show that for $\beta \in \{\beta_k \mid k \in J \setminus \{L\}\}$, the optimal set of problem (\mathcal{R}_{β}) is normally composed out of two optimal sets of problem (\mathcal{C}_k) , namely $\widehat{\mathcal{R}}_{\beta_{J_k}} = \widehat{\mathcal{C}}_{J_k} \cup \widehat{\mathcal{C}}_{J_{k+1}}$. Further, if an $J_{\beta_{J_k}}^E$ is nonempty, we know from Lemma 7 that the optimal set of (\mathcal{R}_{β}) for $\beta = \beta_{J_k}$ can involve at most $J_{k+1} - J_k - 1$ additional optimal sets $\widehat{\mathcal{C}}_m$ where $m \in (J_k, J_{k+1})$.

A partial equivalence between problems (\mathcal{C}_k) and (\mathcal{R}_{β}) always exists.

For the $\#J - 1$ isolated values $\{\beta_k \mid k \in J \setminus \{L\}\}$ problem (\mathcal{R}_{β}) has normally two optimal sets.

4.2 Quasi-complete equivalence

Here we explore the conditions enabling the equivalence results in Theorem 7 to hold for any $k \leq L$.

Lemma 8. *Let H1 hold. Let J be as in Definition 4. Then the following hold:*

(a) *If the sequence $\{\beta_k\}_{k=0}^L$ in Definition 3 is strictly decreasing, then its entries read as*

$$\boxed{\beta_k = c_k - c_{k+1} \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \text{and} \quad \beta_L = 0, \quad \beta_{-1} := \beta_0^U = +\infty.} \quad (40)$$

(b) *If the sequence $\{\beta_k\}_{k=0}^L$ in (40) is strictly decreasing then $J = \mathbb{I}_L^0$.*

Proof. (a) Since $\{\beta_k\}_{k=0}^L$ is strictly decreasing, Proposition 4 implies that the set J in (29) obeys $J = \mathbb{I}_L^0$. Applying Proposition 3(b) with $J_k = k$ and $J_{k+1} = k+1$ delivers the formula in (40).

(b) Let $\{\beta_k\}_{k=0}^L$ in (40) be strictly decreasing. Then β_k in (40) satisfies Proposition 3(b) for any $k \in J = \mathbb{I}_L^0$. By setting $\beta_k^U := \beta_{k-1}$, one has $J = \mathbb{I}_L^0$ according to Definition 4. \square

Theorem 9. *Let H1 hold. Let $\{\beta_k\}_{k=0}^L$ in (40) be strictly decreasing. Then*

$$\left\{ \widehat{\mathbf{R}}_\beta \mid \beta \in (\beta_k, \beta_{k-1}) \right\} = \widehat{\mathbf{C}}_k \quad \forall k \in \mathbb{I}_L^0 \quad (41)$$

$$\widehat{\mathbf{R}}_{\beta_k} = \widehat{\mathbf{C}}_k \cup \widehat{\mathbf{C}}_{k+1} \quad \text{with} \quad \widehat{\mathbf{C}}_k \cap \widehat{\mathbf{C}}_{k+1} = \emptyset \quad \forall k \in \mathbb{I}_{L-1}^0 . \quad (42)$$

Proof. Since $\{\beta_k\}_{k=0}^L$ in (40) is strictly decreasing, one has $\mathbf{J} = \mathbb{I}_L^0$ by Lemma 8(b). So, $\mathbf{J}^E = \emptyset$ in Definition 4 and $\mathbf{J}_k^E = \emptyset$ in (35). Then the statement follows from Theorems 7 and 8. \square

Discussion on Theorem 9 The condition that $\{\beta_k\}_{k=0}^L$ in (40) is strictly decreasing reads as

$$\beta_{k-1} > \beta_k \quad \forall k \in \mathbb{I}_{L-1} \quad \iff \quad c_{k-1} - c_k > c_k - c_{k+1} \quad \forall k \in \mathbb{I}_{L-1} . \quad (43)$$

For generic data, $\{c_k\}$ is strictly decreasing (Proposition 5(b)). For such generic data and $k \leq L-1$, let \bar{u} , \hat{u} and \tilde{u} be optimal solutions of problems (\mathcal{C}_{k-1}) , (\mathcal{C}_k) and (\mathcal{C}_{k+1}) , respectively. Denote $\bar{\sigma} := \text{supp}(\bar{u})$, $\hat{\sigma} := \text{supp}(\hat{u})$ and $\tilde{\sigma} := \text{supp}(\tilde{u})$. The condition on the right hand side in (43) reads as

$$d^T (\Pi_{\hat{\sigma}} - \Pi_{\bar{\sigma}}) d > d^T (\Pi_{\tilde{\sigma}} - \Pi_{\bar{\sigma}}) d \quad (44)$$

where both sides in the inequality above are positive. It is reasonable to expect that there are data d such that the above condition holds for any $k \leq L$. Note such data would belong to an open subset of \mathbb{R}^M where (44) is satisfied. Given a series of numerical tests on 5×10 matrices (see section 7) the realization of (43) essentially depends on d .

Theorem 9 shows that it is *possible* to have equivalence between problems (\mathcal{C}_k) and (\mathcal{R}_β) , except for the isolated parameter values $\{\beta_k\}_{k=0}^L$. If the matrix A is specified and if there are assumptions on the data, one could infer knowledge whether the context of Theorem 9 is a regular regime or not. If the answer turns to be positive, this can be used in the development of numerical schemes.

A mid-way scenario appears as an immediate consequence of Proposition 3(b), (35) and Theorem 7.

Remark 8. Let H1 hold, $\{\beta_k, \beta_k^U\}$ and \mathbf{J} be as in Definition 3 and Definition 4, respectively. Suppose that for $m \geq 0$ and $n \geq 1$ the entries of \mathbf{J} satisfy

$$\mathbf{J}' := \{\mathbf{J}_m, \dots, \mathbf{J}_{m+n}\} = \{\mathbf{J}_m, \mathbf{J}_{m+1}, \mathbf{J}_{m+2}, \dots, \mathbf{J}_{m+n}\} .$$

Then

$$\beta_{\mathbf{J}_{m+k}} = c_{\mathbf{J}_{m+k}} - c_{\mathbf{J}_{m+k+1}} \quad \left\{ \widehat{\mathbf{R}}_\beta \mid \beta \in (\beta_{\mathbf{J}_{m+k}}, \beta_{\mathbf{J}_{m+k-1}}) \right\} = \widehat{\mathbf{C}}_{\mathbf{J}_{m+k}} \quad \text{and} \quad \mathbf{J}_{\beta_{\mathbf{J}_k}}^E = \emptyset \quad 0 \leq k \leq n-1 .$$

If the subset \mathbf{J}' contains all sparsity levels of interest in a given situation, we are again in the setup of Theorem 9 and the comments given after it.

5 On the optimal values of (\mathcal{C}_k) and (\mathcal{R}_β)

The proofs of the statements in this section are outlined in Appendix C.

Using the definition of Ω_k in (17), we introduce the subsets of \mathbb{R}^M given below:

$$\mathbf{E}_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega)^\perp \quad \text{and} \quad \mathbf{G}_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega) . \quad (45)$$

Clearly, $\mathbf{E}_0 = \mathbf{G}_M = \mathbb{R}^M$ and $\mathbf{E}_M = \mathbf{G}_0 = \{0\}$ by H1.

The next Proposition 5 gives results on $\{c_k\}_{k=0}^M$ in connection with $d \in \mathbb{R}^M$.

Proposition 5. *Let H1 hold. Let $L' \leq M$ be arbitrarily fixed. Then*

- (a) $c_k > 0 \quad \forall k \leq L' - 1 \iff d \in \mathbb{R}^M \setminus G_{L'-1} ;$
- (b) $d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies c_{k-1} > c_k \quad \forall k \in \mathbb{I}_{L'} .$

Proposition 5(a) shows that the constant L in (15) corresponds to $d \in G_L \setminus G_{L-1}$.

Remark 9. The subsets E_2 and G_{M-1} are finite unions of vector subspaces of dimensions $M - 2$ and $M - 1$, respectively. Hence, $d \in \mathbb{R}^M \setminus (E_2 \cup G_{M-1})$ is a generic property (Definition 1).

Therefore, $\{c_k\}_{k=0}^M$ is strictly decreasing and $L = M$ generically. \diamond

By Proposition 1(a) and Theorem 4, for any $\beta > 0$ the optimal value of problem (\mathcal{R}_β) in (6), $r_\beta = \inf \{ \mathcal{F}_\beta(u) \mid u \in \mathbb{R}^N \}$, equivalently reads as

$$r_\beta = \min \{ c_k + \beta k \mid k \in \mathbb{I}_L^0 \} . \quad (46)$$

From this observation and Theorems 7 and 8 one infers the following:

Corollary 2. *Let H1 hold and J be as in Definition 4. The application $\beta \mapsto r_\beta : (0, +\infty) \rightarrow \mathbb{R}$ fulfills*

$$(a) \quad \begin{cases} r_\beta = c_{J_k} + \beta J_k \\ = \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{C}_{J_k} \end{cases} \quad \text{if and only if} \quad \beta \in \begin{cases} [\beta_{J_0}, +\infty) & \text{for } J_0 = 0 ; \\ [\beta_{J_k}, \beta_{J_{k-1}}] & \text{for } J_k \in J \setminus \{0, L\} ; \\ (0, \beta_{J_p}] & \text{for } J_p = L . \end{cases}$$

(b) $\beta \mapsto r_\beta$ is continuous and concave.

(c) $r_{\beta_{J_{k-1}}} > r_{\beta_{J_k}} \quad \forall J_k \in J, \quad r_{\beta_{J_0}} = c_{J_0} = r_\beta \quad \forall \beta \geq \beta_{J_0} \quad \text{and} \quad r_{\beta_{J_0}} > r_\beta \quad \forall \beta < \beta_{J_0}.$

r_β is affine increasing on each interval $(\beta_{k_n}, \beta_{k_{n-1}})$ with upward kinks at $k_n \in J \setminus \{L\}$ and bounded by c_0 .

Example 5. [Continuation of Example 3] Let us consider again $\{c_k\}_{k=0}^7$ in (33) along with $\{\beta_k\}_{k=0}^7$ in (4) and J . From Corollary 2, the mapping $\beta \mapsto r_\beta$ is given by

$$\begin{array}{lll} \beta \in (0, 4] & r_\beta = c_7 + 7\beta = 7\beta & r_{\beta=4} = 28 \\ \beta \in [4, 5] & r_\beta = c_6 + 6\beta = 4 + 6\beta & r_{\beta=5} = 34 \\ \beta \in [5, 8] & r_\beta = c_4 + 4\beta = 14 + 4\beta & r_{\beta=8} = 46 \\ \beta \in [8, 9] & r_\beta = c_2 + 2\beta = 30 + 2\beta & r_{\beta=9} = 48 \\ \beta \in [9, +\infty) & r_\beta = c_0 + 0\beta = 48 & \end{array}$$

6 Cardinality of the optimal sets of (\mathcal{R}_β) and of (\mathcal{C}_k)

For the convex surrogates of problems (\mathcal{C}_k) and (\mathcal{R}_β) where $\|u\|_0$ is replaced by $\|u\|_1$, it is well known that the optimal sets are convex closed with possibly a continuum of solutions.

Proposition 6. *Let H1 hold. For any $\beta > 0$ and for any $k \leq L$, the optimal sets \hat{R}_β and \hat{C}_k are finite.*

Proof. From Remark 4, each $\hat{u} \in \hat{C}_k$ is the unique minimizer of problem (\mathcal{P}_ω) for $\omega := \text{supp}(\hat{u})$ where $\omega \in \Omega_k$ (Theorem 1(a) and the notation in (17)). Therefore, $\#\hat{C}_k$ is finite (with $\#\hat{C}_k \leq \#\Omega_k$). This, together with Theorem 7 and Theorem 8, shows that $\#\hat{R}_\beta$ is finite for every $\beta > 0$. \square

For any $\beta > 0$ and $k \in \mathbb{I}_L^0$ the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) are composed out of a certain finite number of isolated (hence strict) minimizers.

6.1 Uniqueness of the global minimizers of (\mathcal{C}_k) and (\mathcal{R}_β)

If $L = M$, Remark 4 shows that $\#\widehat{\mathcal{C}}_M = \#\Omega_M$ and by Definition 4, $M \in J$ in which case Theorem 7 and Theorem 8 yield $\#\left\{\widehat{\mathcal{R}}_\beta \mid \beta \in (0, \beta_{k_{p-1}}]\right\} > \#\widehat{\mathcal{C}}_M$.

Let $k \leq \min\{L, M - 1\}$ and $(\widehat{u}, \widetilde{u}) \in (\widehat{\mathcal{C}}_k)^2$ for $\widehat{u} \neq \widetilde{u}$. Set $\widehat{\sigma} := \text{supp}(\widehat{u})$ and $\widetilde{\sigma} := \text{supp}(\widetilde{u})$. By Theorem 1(a), $(\widehat{\sigma}, \widetilde{\sigma}) \in (\Omega_k)^2$. Then

$$c_k = \|A_{\widehat{\sigma}}\widehat{u}_{\widehat{\sigma}} - d\|^2 = \|A_{\widetilde{\sigma}}\widetilde{u}_{\widetilde{\sigma}} - d\|^2 \quad \text{where } \widehat{\sigma} \neq \widetilde{\sigma}.$$

The expression for c_k in (20) shows that

$$\|A_{\widehat{\sigma}}\widehat{u}_{\widehat{\sigma}} - d\|^2 - \|A_{\widetilde{\sigma}}\widetilde{u}_{\widetilde{\sigma}} - d\|^2 = d^T (\Pi_{\widetilde{\sigma}} - \Pi_{\widehat{\sigma}}) d = 0. \quad (47)$$

The last equality in (47) suggests that $\widehat{\mathcal{C}}_k$ could be a singleton under the assumption H^* below.

H^* . For $K \leq \min\{M - 1, L\}$ fixed, $A \in \mathbb{R}^{M \times N}$ obeys $\Pi_\omega \neq \Pi_{\bar{\omega}} \quad \forall (\omega, \bar{\omega}) \in (\Omega_k)^2, \omega \neq \bar{\omega} \quad \forall k \in \mathbb{I}_K$.

H^* is a generic property of all matrices in $\mathbb{R}^{M \times N}$ [34, Theorem 5.3]. Under H^* , the set Δ_K below

$$\Delta_K := \bigcup_{k=1}^K \bigcup_{(\omega, \bar{\omega}) \in (\Omega_k)^2} \{g \in \mathbb{R}^M \mid \omega \neq \bar{\omega} \text{ and } g \in \ker(\Pi_{\bar{\omega}} - \Pi_\omega)\}$$

is a finite union of vector subspaces of dimension $\leq M - 1$, so *data generically live in* $\mathbb{R}^M \setminus \Delta_K$.

Remark 10. Let the *two generic assumptions*, A satisfies H^* and $d \in \mathbb{R}^M \setminus \Delta_K$, hold. From (47), problem $(\mathcal{C}_k) \forall k \in \mathbb{I}_K$ has a unique optimal solution. Using that $\{\beta_k\}_{k \in J}$ is strictly decreasing, we set $K' := \max\{k \in J \mid k \leq K\}$. Then Theorems 7 and 8 show that problem (\mathcal{R}_β) has a unique optimal solution for any $\beta \in (\beta_{K'}, +\infty) \setminus \{\beta_k\}_{k \in J}$ and hence generically for any $\beta > \beta_{K'}$ by Remark 7. \diamond

For any $k \leq \min\{M - 1, L\}$ and for any $\beta > \beta_{k_{p-1}}$ problems (\mathcal{C}_k) and (\mathcal{R}_β) generically have a unique global minimizer.

For the sake of generality, we did not consider the assumptions evoked in this subsection.

7 Numerical tests

Here we present two kind of experiments using matrices $A \in \mathbb{R}^{M \times N}$ for $(M, N) = (5, 10)$, original vectors $u^\circ \in \mathbb{R}^N$ and data samples $d = Au^\circ (+\text{noise})$ with *two different goals*:

- to get a rough idea on behaviour of the parameters β_k in Definitions 3 and 4;
- to verify and illustrate our theoretical findings.

All results were calculated using an exhaustive combinatorial search.

7.1 Monte Carlo experiments for $(M, N) = (5, 10)$

We realized two experiments, each one composed of 10^5 trials with $(M, N) = (5, 10)$. In each trial, the “original” $u^\circ \in \mathbb{R}^N$ had a random support on $\{1, \dots, N\}$ satisfying $\|u^\circ\|_0 \leq M - 1 = 4$ with mean 3.79. The coefficients of each A and the non-zero entries of each u° were independent and identically distributed (i.i.d.). Data were obtained as $d = Au^\circ +$ i.i.d. centered Gaussian noise. In each trial we computed the exact optimal values $\{c_k\}$ and then computed (β_k, β_k^U) according to Definition 3. We considered two different distributions for A and for the non-zero entries of u° .

- **Experiment $\mathcal{N}(0, 10)$.** All coefficients of each A and all non-zero entries of u° had a normal distribution with mean 0 and variance 10. The SNR in dB was in $[10, 61]$ with mean value 33.75 dB.
- **Experiment Uni $[0, 10]$.** The coefficients of A and of $u^\circ_{\text{supp}(u^\circ)}$ were uniform on $[0, 10]$. We had SNR in $[20, 55]$ with a mean of 28.95 dB.

In these experiments, the following facts were observed:

- We had $L = M$ in each trial which confirms Proposition 5(b) and Remark 9;
- $\{c_k\}_{k=0}^M$ was always strictly decreasing – as expected from Proposition 5;
- We never found $\beta_m = \beta_m^U$, so the set J^E in (29) was always empty; see Proposition 2.
- For every A there were data d so that the sequence $\{c_k - c_{k-1}\}_{k=0}^L$, see (40), was strictly decreasing.

The other results in percentage are shown in Table 1 where N_k reads as

$$N_k := \# \{k \in \mathbb{I}_M^0 \mid \beta_k > \beta_{k-1}\} . \quad (48)$$

In both experiments, the sequence $\{\beta_k\}_{k=0}^M$ in Definition 3 was strictly decreasing in a huge amount

Table 1: Results on the behaviour of $\{\beta_k\}$ in Definition 3 for two experiments, each one composed of 10^5 random trials. For $k \geq 3$ we had $N_k = 0$.

	$\beta_k < \beta_{k-1}, \forall k \in \mathbb{I}_M^0$	$N_k = 1$	$N_k = 2$	mean(SNR)
$\mathcal{N}(0, 10)$	93.681 %	6.254 %	0.065 %	33.75
Uni $[0, 10]$	98.783 %	1.216 %	0.001 %	28.95

of cases; by Lemma 8(a) in all these cases $\{\beta_k\}_{k=0}^M$ equals the sequence in (40) and the quasi-complete equivalence in Theorem 9 holds. One should suppose that these percentages are high because of the small size of the matrices. Anyway, these percentages clearly depend on the distribution of the coefficients of (A, d) .

7.2 Tests on (partial) equivalence with a selected matrix and selected data

Next we present in detail three experiments for $(M, N) = (5, 10)$ where

$$A = \begin{pmatrix} 13.94 & 16.36 & 4.88 & -3.09 & -15.42 & 1.31 & -3.18 & -12.13 & -4.26 & -10.09 \\ 7.06 & -6.48 & -9.07 & -8.37 & -2.72 & -17.42 & -5.83 & -3.81 & 3.87 & -1.80 \\ 11.63 & 6.73 & -4.75 & -6.28 & 3.42 & 6.68 & -1.64 & 13.23 & 9.03 & -20.27 \\ -7.54 & 12.74 & -6.66 & 5.01 & 4.84 & 8.98 & -9.35 & 3.85 & 7.18 & 4.09 \\ 3.22 & -10.40 & -5.02 & 16.70 & 9.53 & -5.49 & 11.88 & -3.62 & 17.36 & 7.34 \end{pmatrix}$$

$$u^\circ = (0 \quad 4 \quad 0 \quad 0 \quad 0 \quad 9 \quad 0 \quad 0 \quad 3 \quad 0)^\top . \quad (49)$$

The entries of A follow a nearly normal distribution. The coefficients of A , u° , and d in (50), (52) and (54) are *exact*. H1 holds since $\text{rank}(A) = M = 5$. Problem (\mathcal{C}_M) has $\#\Omega_M = 252$ optimal solutions; none of them is shown. We have $\beta_0 < \beta_0^U = +\infty$ (Remark 6), so $\widehat{\mathcal{C}}_0 = \{\widehat{R}_\beta \mid \beta > \beta_0\}$ in all cases (Theorem 6). In the tests presented below the optimal set of (\mathcal{C}_k) for $k \leq M - 1$ is a singleton (see subsection 6.1).

In order to illustrate various cases of partial or quasi-complete equivalence, we selected a couple (A, u°) in (49) that behaves differently compared to Table 1: it does not favor quasi-complete equivalence as seen from the 10^5 random trials summarized in Table 2.

Table 2: The behaviour of $\{\beta_k\}$ in Definition 3 for an experiment with 10^5 trials where A and u° are given by (49) $d = Au^\circ +$ i.i.d. centered Gaussian noise. We had $N_k = 0, \forall k \geq 3$.

	$\beta_k < \beta_{k-1}, \forall k \in \mathbb{I}_M^0$	$N_k = 1$	$N_k = 2$	mean(SNR)
A, u° in (49)	29.41 %	70.59 %	0 %	36.25

Noise-free data According to (49), data read as

$$d = Au^\circ = (64.45 \quad -171.09 \quad 114.13 \quad 153.32 \quad -38.93)^T . \quad (50)$$

Since data are noise-free and $\|u^\circ\|_0 = 3$, clearly $\hat{u} = u^\circ$ is an optimal solution to problems (\mathcal{C}_k) with $c_k = 0$ for $k \in \{3, 4, 5\}$ and $L = 3$. The other optimal values c_k are seen in Table 3. By Theorem 4, any $\hat{u} \in \hat{\mathbf{R}}$ obeys $\|\hat{u}\|_0 \leq 3$. The critical parameters $\{\beta_k\}$ by Definition 3 are

$$\beta_3 = 0 < \beta_3^U = \beta_1 = 3872.46 < \beta_1^U = \beta_0 = 63729 \quad \text{and} \quad \beta_2 = 3968 > \beta_2^U = 3776.82 . \quad (51)$$

$\beta_k > \beta_k^U$ only for $k = 2$, so $J = \{0, 1, 3\}$ in (29). By Lemma 6, $\hat{\mathbf{R}} \cap \hat{\mathbf{C}}_k = \emptyset$ for $k = 2$. By Theorem 7, $\hat{\mathbf{C}}_3 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_3, \beta_1)\}$ and $\hat{\mathbf{C}}_1 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_1, \beta_0)\}$. The numerical results are seen in Table 3.

Table 3: The optimal values c_k and the optimal sets of (\mathcal{C}_k) for $k \in \mathbb{I}_3^0$ where d is as in (50). The values of β_k are given in (51). We recall that $\hat{\mathbf{R}}_\beta$ is the optimal set of problem (\mathcal{R}_β) .

k	c_k	$\hat{\mathbf{C}}_k =$ the optimal solution of (\mathcal{C}_k) , singleton	$\hat{\mathbf{C}}_k = \hat{\mathbf{R}}_\beta$
3	0	0 4 0 0 0 9 0 0 3 0	$\beta \in (\beta_3, \beta_1)$
2	3968	0 3.25 0 0 0 9.29 0 0 0 0	no
1	7745	0 0 0 0 0 11.76 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	71474	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

Noisy data 1. Data are corrupted with nearly normal, centered, i.i.d. noise and SNR= 32.32 dB:

$$d = (69.13 \quad -171.95 \quad 113.74 \quad 150.27 \quad -36.09)^T . \quad (52)$$

The optimal values c_k of problems (\mathcal{C}_k) in Table 4 with $c_5 = 0$ yield $L = M = 5$. From Definition 3,

$$\beta_5 = 0 < \beta_5^U = \beta_4 = 0.068 < \beta_4^U = \beta_3 = 36.25 < \beta_3^U = \beta_1 = 3987.68 < \beta_1^U = \beta_0 = 63154 , \quad (53)$$

while $\beta_2 = 4002.83 > \beta_2^U = 3972.54$. Hence, $J = \mathbb{I}_5^0 \setminus \{2\}$ in (29) and $\{\beta_k\}_{k \in J}$ confirms Propositions 3 and 4. By Lemma 6, $\hat{\mathbf{R}} \cap \hat{\mathbf{C}}_2 = \emptyset$ and by Theorem 7, $\hat{\mathbf{C}}_5 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (0, \beta_4)\}$, $\hat{\mathbf{C}}_4 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_4, \beta_3)\}$, $\hat{\mathbf{C}}_3 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_3, \beta_1)\}$ and $\hat{\mathbf{C}}_1 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_1, \beta_0)\}$. The numerical tests are shown in Table 4.

Table 4: The optimal values c_k and the optimal solutions of (\mathcal{C}_k) for $k \in \mathbb{I}_4^0$ where d is given in (52). The values of β_k are given in (53). We recall that $\widehat{\mathbf{R}}_\beta$ is the set of the global minimizers of \mathcal{F}_β .

k	c_k	$\widehat{\mathbf{C}}_k =$ the optimal solution of (\mathcal{C}_k) , singleton	$\widehat{\mathbf{C}}_k = \widehat{\mathbf{R}}_\beta$
4	0.068	0 4.40 0 0 0 8.71 0.54 0 2.95 0	$\beta \in (\beta_4, \beta_3)$
3	36.3141	0 4.09 0 0 0 8.88 0 0 3.01 0	$\beta \in (\beta_3, \beta_1)$
2	4039	0 3.33 0 0 0 9.17 0 0 0 0	no
1	8011.68	0 0 0 0 0 11.71 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	71166	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

Noisy data 2. The noise is nearly normal, centered, i.i.d., SNR= 25.74 dB:

$$d = (66.67 \quad -169.08 \quad 101.56 \quad 149.38 \quad -39.50)^T . \quad (54)$$

The optimal values $\{c_k\}$ in Table 5 show that $L = M$. The sequence $\{\beta_k\}$ by Definition 3 reads as

$$\beta_0 = 60287 \quad \beta_1 = 3825 \quad \beta_2 = 3037.1 \quad \beta_3 = 72.734 \quad \beta_4 = 0.0259 \quad \beta_5 = 0 . \quad (55)$$

This $\{\beta_k\}$ is strictly decreasing and equals $\{\beta_k\}$ in (40), as claimed in Lemma 8(a). From Theorem 9, problems (\mathcal{C}_k) and (\mathcal{R}_β) are quasi-completely equivalent. This is confirmed by the tests reported in Table 5.

Table 5: The optimal values and solutions of (\mathcal{C}_k) for $k \in \mathbb{I}_4$ where d is given in (54). Here $\{\beta_k\}$ is strictly decreasing, see (55), so (\mathcal{C}_k) and (\mathcal{R}_β) are quasi-completely equivalent.

k	c_k	$\widehat{\mathbf{C}}_k =$ the optimal solution of (\mathcal{C}_k) , singleton	$\widehat{\mathbf{C}}_k = \widehat{\mathbf{R}}_\beta$
4	0.0259	0 8.54 0 0 4.59 4.90 2.73 0 0 0	$\beta \in (\beta_4, \beta_3)$
3	72.7601	0 3.93 0 0 0 8.70 0 0 2.63 0	$\beta \in (\beta_3, \beta_2)$
2	3109.86	0 3.27 0 0 0 8.95 0 0 0 0	$\beta \in (\beta_2, \beta_1)$
1	6934.85	0 0 0 0 0 11.44 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	67222	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

8 Conclusions and open questions

We have derived the precise mechanism of the relationship between the optimal solutions of least-squares constrained by k -sparsity (problem (\mathcal{C}_k) in (2)) and regularized by $\|\cdot\|_0$ via a parameter $\beta > 0$ (problem (\mathcal{R}_β) in (3)). Subsection 1.2 (Main contributions) made seven claims regarding the obtained new results. These problems were shown to be partially equivalent in general and under conditions, quasi-completely equivalent. Our theoretical findings pose intriguing questions, of both a theoretical and practical flavor.

- Our work provided useful hints how to analyze $\|\cdot\|_0$ -related problems.
- The obtained results can clarify a proper choice between models (\mathcal{C}_k) and (\mathcal{R}_β) in applications. If one needs optimal solutions with a fixed number of nonzero entries, (\mathcal{C}_k) is obviously the best choice. If only information on the perturbations is available, (\mathcal{R}_β) is a more flexible model.

- A proper extension of our results to matrices A and data d with complex entries should not present inherent difficulties but it is important in many applications (e.g., phase retrieval).
- Many algorithms have been built on good knowledge on the optimal solutions. One can expect our detailed results to give rise to innovative and efficient algorithms enabling one to compute relevant solutions.
- Extensions to penalties of the form $\|Du\|_0$ for D a linear operator. The preliminary step should involve an extension of the knowledge on the optimal solutions of (\mathcal{C}_k) and (\mathcal{R}_β) to these more complex penalties.
- The Monte-Carlo tests (subsection 7.1) have shown that the degree of partial equivalence, i.e., the size and the distribution of effective critical sequence $\{\beta_k\}$, depends on the statistics of the matrix A and the data d .
- By specifying a class of matrices A and assumptions on data d , one might want to infer statistical knowledge on the optimal values c_k 's of problems (\mathcal{C}_k) and thus on the critical parameters $\{\beta_k\}$. In the partial equivalent context, it would be intriguing to see if problem (\mathcal{R}_β) is able to eliminate some uninteresting optimal solutions of problem (\mathcal{C}_k) .

Hopefully, these problems will be addressed in future work.

A Proofs for joint optimality conditions for (\mathcal{C}_k) and (\mathcal{R}_β) , sec. 2

A.1 On the optimal solutions of problem (\mathcal{C}_k) , subsection 2.2

Proof of Lemma 1. Using (\mathcal{P}_ω) in (9) and (11), the optimal value of (\mathcal{C}_k) for any k is given by

$$c_k = \inf \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \mid \omega \in \Sigma_k \right\}.$$

For $k \in \mathbb{I}_N^0$ and $\omega \in \Sigma_k$, define $c^\omega \geq 0$ by

$$c^\omega := \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Sigma_k. \quad (56)$$

The set of numbers $\{c^\omega \mid \omega \in \Sigma_k\}$ is nonempty and finite. Then $c_k = \min\{c^\omega \mid \omega \in \Sigma_k\}$ is well defined. By (56) there exists $\hat{u} \in \mathbb{R}^N$ such that $\|A\hat{u} - d\|^2 = c_k$. Hence $\hat{u} \in \hat{\mathcal{C}}_k$ and thus $\hat{\mathcal{C}}_k \neq \emptyset$.

Proof of Lemma 2 Since $\Sigma_{k-n} \subset \Sigma_k$, $\forall n \in \mathbb{I}_k^0$ it follows from (9) and (13) that

$$c_k \leq \|Au - d\|^2 \quad \forall u \in \mathbb{R}^N \text{ such that } \text{supp}(u) \in \Sigma_{k-n}, \quad \forall n \in \mathbb{I}_k^0. \quad (57)$$

Hence $c_k \leq c_{k-n}$, $\forall n \in \mathbb{I}_k$. By H1, there is $\omega \in \Sigma_M$ so that $\text{rank}(A_\omega) = M = \#\omega$. Then $\|A\hat{u} - d\|^2 = c_M = 0$ for \hat{u} given by $\hat{u}_\omega = (A_\omega)^{-1}d$ and $\hat{u}_{\mathbb{I}_N \setminus \omega} = 0$.

Proof of Lemma 3 Since $n \geq 1$, \hat{u} solves the problem $\min_u \|Au - d\|^2$ subject to $\|\hat{u}\|_0 < k$. This is an unconstrained problem, therefore the gradient of $u \mapsto \|Au - d\|^2$ must be null at \hat{u} :

$$A^T(A\hat{u} - d) = 0.$$

By H1 we immediately get that $A\hat{u} - d = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d = 0$ and $\|A\hat{u} - d\|^2 = 0$. This combined with $\text{supp}(\hat{u}) \in \Sigma_{k-n}$ yields $c_{k-n} = 0$ and $\hat{u} \in \hat{C}_{k-n}$. For any $m \geq n - k$ one has $c_m = 0$ by Lemma 2 and $\hat{u} \in \hat{C}_m$ because $\Sigma_m \supset \Sigma_{k-n}$.

In order to prove Theorem 1(a) we need some auxiliary results.

Corollary 3. *Let H1 hold. Then $\left[k \in \mathbb{I}_{\mathbb{L}}^0 \quad \text{and} \quad \hat{u} \in \hat{C}_k \implies \|\hat{u}\|_0 = k \right]$.*

Proof. The case $k = 0$ being trivial we focus on $k \in \mathbb{I}_{\mathbb{L}}$. Assume that $\|\hat{u}\|_0 = k - n$ for $n \geq 1$. Then $c_{k-n} = 0$ by Lemma 3 which contradicts the definition of \mathbb{L} in (15) Hence $n = 0$. \square

Lemma 9. *Let H1 hold. Consider that $\hat{u} \in \hat{C}_k$ for $k \in \mathbb{I}_{\mathbb{L}}^0$. Set $\hat{\sigma} := \text{supp}(\hat{u})$. Then*

$$\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma} \equiv \|\hat{u}\|_0. \quad (58)$$

Proof. One has $\|\hat{u}\|_0 = k$ by Corollary 3. For $k = 0$ (58) is obvious. Suppose that (58) fails for $k \geq 1$:

$$\text{rank}(A_{\hat{\sigma}}) \leq \#\hat{\sigma} - 1. \quad (59)$$

The rank-nullity theorem [29] entails that $\dim \ker(A_{\hat{\sigma}}) = \#\hat{\sigma} - \text{rank}(A_{\hat{\sigma}}) \geq 1$. Take an arbitrary $v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \setminus \{0\}$, set $v_{\mathbb{I}_{\mathbb{N}} \setminus \hat{\sigma}} := 0$ and select an $i \in \hat{\sigma}$ in order to define \tilde{u} by

$$\tilde{u} := \hat{u} - \hat{u}_i \frac{v}{v_i}.$$

Then $\tilde{u}_i = 0$ and $\hat{u}_i \neq 0$, so $\tilde{\sigma} := \text{supp}(\tilde{u}) \subsetneq \hat{\sigma}$, which leads to

$$\|\tilde{u}\|_0 = k - n \quad \text{for} \quad n := \|\hat{u}\|_0 - \|\tilde{u}\|_0 \geq 1. \quad (60)$$

From $v_{\hat{\sigma}} \frac{\hat{u}_i}{v_i} \in \ker(A_{\hat{\sigma}})$ one has $A\hat{u} = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}}\left(\hat{u}_{\hat{\sigma}} - v_{\hat{\sigma}} \frac{\hat{u}_i}{v_i}\right) = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A\tilde{u}$. Then

$$c_k = \|A\hat{u} - d\|^2 = \|A\tilde{u} - d\|^2. \quad (61)$$

This, together with the fact $\text{supp}(\tilde{u}) \in \Sigma_k$ shows that $\tilde{u} \in \hat{C}_k$. Thus $\tilde{u} \in \hat{C}_k$ and $\|\tilde{u}\|_0 \leq k - 1$ by (60), hence $c_{k-1} = 0$ by Lemma 3, in contradiction to the definition of \mathbb{L} in (15). So (59) fails. \square

Lemma 10. *Let H1 hold. Let \hat{u} be a solution of problem (C_k) such that $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ for $\hat{\sigma} := \text{supp}(\hat{u})$. Then \hat{u} is a strict minimizer of problem (C_k) .*

Proof. Observe that \hat{u} solves the problem $\min \{\|Au - d\|^2 \mid u \in S\}$ where

$$S = \left\{ v \in \mathbb{R}^{\mathbb{N}} : \text{supp}(v) \subset \hat{\sigma} \right\}. \quad (62)$$

Since \hat{u} solves problem $(\mathcal{P}_{\hat{\sigma}})$ where $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ the mapping $\hat{u}_{\hat{\sigma}} \mapsto \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2$ is strictly convex and its unique solution $\hat{u}_{\hat{\sigma}}$ satisfies

$$v \in S \setminus \{0\} \implies \|A_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}} + v_{\hat{\sigma}}) - d\|^2 > \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2.$$

This, combined with (62) shows that $\|A(\hat{u} + v) - d\|^2 > \|A\hat{u} - d\|^2$ for any $v \in S \setminus \{0\}$. Hence \hat{u} is a strict minimizer of problem (C_k) (Definition 2). \square

Proof of Theorem 1. (a) Let $\hat{u} \in \hat{C}_k$ for $k \in \mathbb{I}_L^0$. By Corollary 3 and Lemma 9, $\text{rank}(A_{\hat{\sigma}}) = k = \|\hat{u}\|_0$ where $\hat{\sigma} := \text{supp}(\hat{u})$, which proves (16). Since \hat{u} solves problem $(\mathcal{P}_{\hat{\sigma}})$ in (9) where $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ it follows by Lemma 10 that \hat{u} is a strict minimizer.

(b) From Lemma 3 and the definition of L in (15) one finds $\hat{C}_L \subset \hat{C}_k$ for any $k \geq L + 1$.

Proof of Corollary 1 Let $\hat{u} \in \hat{C}_k$ and $\bar{u} \in \hat{C}_n$ for $(k, n) \in (\mathbb{I}_L^0)^2$, $k \neq n$. By Theorem 1(a), $\|\hat{u}\|_0 = k$ and $\|\bar{u}\|_0 = n$, hence the result.

A.2 Proofs for necessary and sufficient conditions, subsection 2.3

From Theorems 2 and 3, if H1 holds, all global minimizers \hat{u} of \mathcal{F}_β satisfy $\|\hat{u}\|_0 \leq M$.

Corollary 4, Lemmas 11, 12 and 13 below help to prove Proposition 1.

Corollary 4 ([34] Corollary 3.3). *Let \hat{u} solve (\mathcal{P}_ω) for $\omega \in \Omega_k$ where $k \in \mathbb{I}_M^0$. Then \hat{u} is a strict (local) minimizer of \mathcal{F}_β for any $\beta > 0$.*

With the notation in (17), it is suitable to set

$$\Omega := \bigcup_{k=0}^M \Omega_k .$$

Lemma 11. *Let $d \in \mathbb{R}^M$ and $\beta > 0$. Then*

$$\hat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_\beta \iff \hat{u} \in U := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega\} . \quad (63)$$

Proof. Let \hat{u} be a strict (local) minimizers of \mathcal{F}_β . Then \hat{u} solves (\mathcal{P}_ω) for $\omega := \text{supp}(\hat{u})$. By Theorem 3 $\omega \in \Omega$ and thus $\hat{u} \in U$. Conversely, any $\hat{u} \in U$ is a strict (local) minimizer of \mathcal{F}_β by Corollary 4. \square

Now we partition U in (63) as follows:

$$U = \bigcup_{k=0}^M U_k \quad \text{where} \quad U_k := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega \text{ and } \|\tilde{u}\|_0 = k\} . \quad (64)$$

Lemma 12. *Let H1 be satisfied and L be as in (15). Then $\hat{C}_k \subset U_k \quad \forall k \in \mathbb{I}_L^0$.*

Proof. Let $\hat{u} \in \hat{C}_k$ for $k \in \mathbb{I}_L^0$. Set $\omega := \text{supp}(\hat{u})$. The expression for \hat{C}_k in (19) and Theorem 1(a) show that \hat{u} solves (\mathcal{P}_ω) for $\omega \in \Omega_k \subset \Omega$ and that $\|\hat{u}\|_0 = k$. Hence $\hat{u} \in U_k$. \square

Lemma 13. *Let H1 hold, L be as in (15) and let $\beta > 0$.*

(a) *Let $k \in \mathbb{I}_L^0$. Then*

$$\mathcal{F}_\beta(\hat{u}) = c_k + \beta k \quad \forall \hat{u} \in \hat{C}_k ; \quad (65)$$

$$\mathcal{F}_\beta(\tilde{u}) > \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_k \setminus \hat{C}_k . \quad (66)$$

(b) *Let $\hat{u} \in \hat{C}_L$. If $L \leq M - 1$, then*

$$\mathcal{F}_\beta(\tilde{u}) > c_L + \beta L = \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_n \text{ for } \forall n \in \{L + 1, \dots, M\} ; \quad (67)$$

and thus any $\tilde{u} \in U_n$ for $\forall n \in \{L + 1, \dots, M\}$ obeys $\tilde{u} \notin \hat{R}_\beta$ for any $\beta > 0$.

Proof. From the definition of U_k in (64), if $U_k \neq \emptyset$, then $\|\tilde{u}\|_0 = k$ for any $\tilde{u} \in U_k$.

(a) Since $k \in \mathbb{I}_L^0$, $\widehat{C}_k \subset U_k$ by Lemma 12. Any $\hat{u} \in \widehat{C}_k$ yields $\|A\hat{u} - d\|^2 = c_k$, hence (65). Any $\tilde{u} \in U_k \setminus \widehat{C}_k$ is not an optimal solution of (C_k) , so $\|A\tilde{u} - d\|^2 > c_k$. Then $\mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta k > c_k + \beta k = \mathcal{F}_\beta(\hat{u})$.

(b) By the definition of L , $c_n = c_L = 0$, $\forall n \geq L$. It follows that for any $\tilde{u} \in U_n$, $\forall n \geq L + 1$ one has $\mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta n > c_L + \beta L = \beta L$. Such a \tilde{u} is not a global minimizer of \mathcal{F}_β . \square

Proof of Proposition 1. (a) follows from Lemma 13(a).

(b)-(c) Let $\hat{u} \in \widehat{R}_\beta$. Then \hat{u} is a strict minimizer of \mathcal{F}_β (Theorem 2(b)) and $\hat{u} \in U$ by Lemma 11. Set $k := \|\hat{u}\|_0$; then $\hat{u} \in U_k$ according to (64). In addition, $k \leq L$ because otherwise $\hat{u} \notin \widehat{R}_\beta$ by Lemma 13(b). Also, $\hat{u} \in \widehat{R}_\beta$ means that $\mathcal{F}_\beta(\hat{u})$ is the optimal value of problem (R_β) . Then $\hat{u} \in \widehat{C}_k$ by Lemma 13(a). Further, $\hat{u} \mapsto \mathcal{F}_\beta(\hat{u})$ is constant for any $\hat{u} \in \widehat{C}_k$; see (65). Therefore, $\widehat{C}_k \subseteq \widehat{R}_\beta$.

Proof of Lemma 4. The backward implication is obvious. We focus on the forward one. Let $k \in \mathbb{I}_L^0$; we proceed by contraposition. Assume that $\widehat{C}_k \cap \widehat{R}_\beta \neq \emptyset$, i.e., there exists $\hat{u} \in \widehat{C}_k \cap \widehat{R}_\beta$. Then $\widehat{C}_k \subseteq \widehat{R}_\beta$ by Proposition 1(c), a contradiction to the fact that $\widehat{C}_k \not\subseteq \widehat{R}_\beta$.

Proof of Theorem 4. By Proposition 1(b), any $\hat{u} \in \widehat{R}_\beta$ satisfies $\hat{u} \in \widehat{C}_k$ for $k \leq L$ and thus $\hat{u} \in \widehat{C}$. Therefore, $\widehat{R}_\beta \subset \widehat{C}$. The same holds for any $\beta > 0$ which proves the theorem.

Proof of Lemma 5 By Remark 5, $\widehat{C}_k \subseteq \widehat{R}_\beta$ implies $\mathcal{F}_\beta(\hat{u}) \leq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \widehat{C}_k \quad \forall \bar{u} \in \widehat{C}_{k+p}$. Using Proposition 1(a) this inequality reads as $c_k + \beta k \leq c_{k+p} + \beta(k+p)$, which leads to

$$\beta \geq \frac{c_k - c_{k+p}}{p}. \quad (68)$$

On the other hand, $\widehat{C}_{k+p} = \widehat{R}_{\beta'}$ entails $\mathcal{F}_{\beta'}(\bar{u}) < \mathcal{F}_{\beta'}(\hat{u}) \quad \forall \bar{u} \in \widehat{C}_{k+p} \quad \forall \hat{u} \in \widehat{C}_k$. Therefore

$$c_{k+p} + \beta'(k+p) < c_k + \beta'k \quad \Rightarrow \quad \beta' < \frac{c_k - c_{k+p}}{p}. \quad (69)$$

Comparing (69) and (68) proves the first part of the lemma. The proof of second one is similar.

B Proofs for parameter values for equality between optimal sets, section 3

B.1 Proofs for the entire list parameters values, subsection 3.1

Proof of Proposition 2 For $k \in \{0, L\}$ one has $\beta_k < \beta_k^U$ by Remark 6. We recall the notation Ω_k introduced in (17) and that Π_ω is the orthogonal projector onto $\text{range}(A_\omega)$ given in Remark 4.

There exists $n \in \mathbb{I}_{L-k}$ such that $\beta_k = \frac{c_k - c_{k+n}}{n}$. By Theorem 1, there are $\omega \in \Omega_k$ and $\bar{\omega} \in \Omega_{k+n}$ obeying $c_k = d^T(I - \Pi_\omega)d$ and $c_{k+n} = d^T(I - \Pi_{\bar{\omega}})d$. Similarly, there is $m \in \mathbb{I}_k$ satisfying $\beta_k^U = \frac{c_{k-m} - c_k}{m}$ and $\widehat{\omega} \in \Omega_{k-m}$ such that $c_{k-m} = d^T(I - \Pi_{\widehat{\omega}})d$. Then $\beta_k - \beta_k^U$ reads as

$$\beta_k - \beta_k^U = \frac{d^T(\Pi_{\bar{\omega}} - \Pi_\omega)d}{n} - \frac{d^T(\Pi_\omega - \Pi_{\widehat{\omega}})d}{m} \quad (70)$$

$$= \frac{d^T(m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (m+n)\Pi_\omega)d}{nm}. \quad (71)$$

It follows that all $d \in \mathbb{R}^M$ able to yield $\beta_k - \beta_k^U = 0$ for some $k \in \mathbb{I}_{L-1}$ belong to the set S given below:

$$S := \bigcup_{k=1}^{L-1} S_k \quad \text{where} \quad S_k := \bigcup_{n=1}^{L-k} \bigcup_{m=1}^k \bigcup_{\omega \in \Omega_k} \bigcup_{\bar{\omega} \in \Omega_{k+n}} \bigcup_{\hat{\omega} \in \Omega_{k-m}} \ker(m\Pi_{\bar{\omega}} + n\Pi_{\hat{\omega}} - (n+m)\Pi_{\omega}). \quad (72)$$

Let $k \in \mathbb{I}_{L-1}$. Since $\text{rank}(\Pi_{\bar{\omega}}) = k+n$, the SVD of $\Pi_{\bar{\omega}}$ yields an orthonormal matrix $Q \in \mathbb{R}^{M \times M}$ so that

$$Q^T \Pi_{\bar{\omega}} Q = \begin{bmatrix} I_{k+n} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}.$$

Observe that $Q^T \Pi_{\bar{\omega}} Q$ is symmetric semi-positive definite. By setting

$$P := \frac{n}{m} Q^T \Pi_{\bar{\omega}} Q = \begin{bmatrix} P_0 & \vdots & P_1 \\ \cdots & \cdots & \cdots \\ P_1^T & \vdots & P_2 \end{bmatrix}$$

where P_0 is the first $(k+n) \times (k+n)$ principal minor of P , we have

$$\begin{aligned} \text{rank}(m\Pi_{\bar{\omega}} + n\Pi_{\hat{\omega}}) &= \text{rank}\left(\Pi_{\bar{\omega}} + \frac{n}{m}\Pi_{\hat{\omega}}\right) = \text{rank}\left(Q^T \left(\Pi_{\bar{\omega}} + \frac{n}{m}\Pi_{\hat{\omega}}\right) Q\right) \\ &= \text{rank} \begin{bmatrix} I_{k+n} + P_0 & \vdots & P_1 \\ \cdots & \cdots & \cdots \\ P_1^T & \vdots & P_2 \end{bmatrix} \geq k+n. \end{aligned} \quad (73)$$

The rank inequality above comes from the fact that P_0 is positive semidefinite and thus $I_{k+n} + P_0$ is positive definite; see [28]. This, together with the facts that $\text{rank}((n+m)\Pi_{\omega}) = k$ and that $n \geq 1$ in (73) gives that $m\Pi_{\bar{\omega}} + n\Pi_{\hat{\omega}} - (n+m)\Pi_{\omega} \neq 0$; hence $\dim(\ker(m\Pi_{\bar{\omega}} + n\Pi_{\hat{\omega}} - (n+m)\Pi_{\omega})) \leq M-1$. Therefore, S is a finite union of vector subspaces of dimension $\leq M-1$.

Finally, if $d \in \mathbb{R}^M \setminus S$, then $\beta_k - \beta_k^U \neq 0$ in (70) because the term in (71) is non-null.

B.2 Proofs for effective parameters values, subsection 3.3

Proof of Lemma 6 The definition of J and J^E in (29) shows that

$$k \in \mathbb{I}_L^0 \quad \text{and} \quad \beta_k > \beta_k^U \quad \iff \quad k \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}.$$

By Theorem 6(a), one has $\hat{C}_k \not\subseteq \hat{R}_\beta, \forall \beta > 0$, if and only if $k \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$. By Lemma 4, $\hat{C}_k \not\subseteq \hat{R}_\beta$ means $\hat{C}_k \cap \hat{R}_\beta = \emptyset, \forall \beta > 0$.

Proof of Lemma 7 Let $m \in J^E$. Theorem 6(a) shows that $\hat{C}_m \subset \hat{R}_{\beta_m}$. By Theorem 6(b) and Proposition 3(a), $\hat{C}_{J_k} = \hat{R}_\beta$ if and only if $\beta_{J_k} < \beta < \beta_{J_{k-1}}$. Assume that $m < k_n$. Then Lemma 5 shows that $\beta_m > \beta$ for any $\beta \in (\beta_{J_k}, \beta_{J_{k-1}})$. It follows that $\beta_m \neq \beta_{J_k}$. This, together with (35), yields

$$\{m \in J^E \mid m < J_k\} \cap J_{\beta_{J_k}}^E = \emptyset. \quad (74)$$

Using the same statements, $\hat{C}_{J_{k+1}} = \hat{R}_\beta$ if and only if $\beta \in (\beta_{J_{k+1}}, \beta_{J_k})$. Consider that $m > J_{k+1}$. From Lemma 5, $\beta_m < \beta$ for any $\beta \in (\beta_{J_{k+1}}, \beta_{J_k})$ and thus $\beta_m \neq \beta_{J_k}$. Hence

$$\{m \in J^E \mid m > J_{k+1}\} \cap J_{\beta_{J_k}}^E = \emptyset. \quad (75)$$

Jointly (74), (75) and $J \cap J^E = \emptyset$ prove (36). Finally, $\beta_{k_p} \equiv \beta_L = 0$ in (31) shows that $J_{\beta_L}^E = \emptyset$.

Proof of Proposition 4 From Definition 4 one has $0 \in J$ and from Proposition 3(a) $\{\beta_k\}_{k \in J}$ is strictly decreasing. By Lemma 7 it holds that $J_k < m$ for any m satisfying $\beta_{J_k} = \beta_m$. Hence J contains the smallest indexes.

If $J = \mathbb{I}_L^0$, the result is proved. Consider that $J \subsetneq \mathbb{I}_L^0$. Suppose that there is $m \in \mathbb{I}_L \setminus J$ such that $\{\beta_k\}_{k \in J^*}$ is strictly decreasing where J^* denotes the increasingly ordered $\{J \cup \{m\}\}$. By Proposition 3(c), $m \notin J^E$. Then $m \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$. Since $\beta_{J_{k-1}} < \beta_{J_k}, \forall J_k \in J$, there are (J_{k-1}, J_k) such that $J_{k-1} < m < J_k$ and $\beta_{J_k} < \beta_m < \beta_{J_{k-1}}$. From Theorem 6(b) and Proposition 3(a), $\widehat{R}_\beta = \widehat{C}_{J_k}$ if and only if $\beta \in (\beta_{J_k}, \beta_{J_{k-1}})$. Since $\beta_m \in (\beta_{J_k}, \beta_{J_{k-1}})$ one has

$$\mathcal{F}_{\beta_m}(\bar{u}) > \mathcal{F}_{\beta_m}(\hat{u}) \quad \forall \bar{u} \in \widehat{C}_m \quad \forall \hat{u} \in \widehat{C}_{J_k},$$

which yields $c_m + \beta_m m > c_{J_k} + \beta_m J_k$ (see Proposition 1(a)). Consequently,

$$\beta_m < \frac{c_m - c_{J_k}}{J_k - m}.$$

However, using that $J_k > m$, Definition 3 shows that $\beta_m \geq \frac{c_m - c_{J_k}}{J_k - m}$, a contradiction.

C Proofs for optimal values of (\mathcal{C}_k) and (\mathcal{R}_β) , section 5

In order to prove Proposition 5 we shall use Lemmas 14 and 15 given below.

Lemma 14. *Let H1 hold and let $k \in \{1, \dots, M-1\}$. Then*

$$d \in \mathbb{R}^M \setminus G_k \iff c_k > 0.$$

Proof. Let $d \in \mathbb{R}^M \setminus G_k$ and let $\hat{u} \in \widehat{C}_k$. Set $\hat{\sigma} := \text{supp}(\hat{u})$. By Theorem 1(a) and the notation Ω_k in (17), $\hat{\sigma} \in \Omega_k$. Since $(I - \Pi_{\hat{\sigma}})$ is the orthogonal projector onto $(\text{range}(A_{\hat{\sigma}}))^\perp$, one has

$$d \in \mathbb{R}^M \setminus G_k \implies d \notin \text{range}(A_{\hat{\sigma}}) \text{ and } d \neq 0 \implies c_k = \|A\hat{u} - d\|^2 = d^T(I - \Pi_{\hat{\sigma}})d > 0.$$

Conversely, let $c_k > 0$. If $d \in G_k$, there is $\omega \in \Omega_k$ meeting $d \in \text{range}(A_\omega)$. For $u_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d$ one has $\|A_\omega u_\omega - d\|^2 = d^T(I - \Pi_\omega)d = 0$, a contradiction to $c_k > 0$. \square

Lemma 15. *Let H1 hold and let E_k be given by (45). For any $k \geq 1$ such that $c_{k-1} > 0$ one has*

$$c_{k-1} > c_k \quad \forall d \in \mathbb{R}^M \setminus E_k.$$

Proof. From H1, there is $n \in \mathbb{I}_N$ meeting $\langle A_n, d \rangle \neq 0$. Set $\hat{u}_n = \arg \min_{v \in \mathbb{R}} \|A_n v - d\|^2$. Then

$$\hat{u}_n = \frac{\langle A_n, d \rangle}{\|A_n\|^2} \neq 0 \quad \text{and} \quad c_1 \leq \|A_n \hat{u}_n - d\|^2 = \|d\|^2 - \frac{\langle A_n, d \rangle^2}{\|A_n\|^2} < c_0 = \|d\|^2.$$

Consider that $k \geq 2$. Let $\hat{u} \in \widehat{C}_{k-1}$ and $c_{k-1} > 0$. Set $\hat{\sigma} := \text{supp}(\hat{u})$ and denote by $B_{\hat{\sigma}}$ a matrix whose columns form an orthonormal basis for $A_{\hat{\sigma}}$. From Theorem 1(a), $\hat{\sigma} \in \Omega_{k-1}$. By H1, there is $n \in \mathbb{I}_N \setminus \hat{\sigma}$ such that $\omega := \hat{\sigma} \cup \{n\} \in \Omega_k$. Then there is $b_k \in \text{range}(A_\omega)$ such that $B_\omega = (B_{\hat{\sigma}}, b_k)$ forms an orthonormal basis for A_ω (Gram-Schmidt theorem, see, e.g., [24]). The orthogonal projectors

onto $\text{range}(A_{\hat{\sigma}})$ and $\text{range}(A_{\omega})$ are $\Pi_{\hat{\sigma}} = B_{\hat{\sigma}}B_{\hat{\sigma}}^T$ and $\Pi_{\omega} = B_{\omega}B_{\omega}^T = B_{\hat{\sigma}}B_{\hat{\sigma}}^T + b_k b_k^T$, respectively. Then $c_k \leq \zeta_k := d^T(I - \Pi_{\omega})d$. Applying (20) yields

$$c_{k-1} - c_k \geq c_{k-1} - \zeta_k = d^T(\Pi_{\omega} - \Pi_{\hat{\sigma}})d = \langle b_k, d \rangle^2.$$

Since $d \in \mathbb{R}^M \setminus E_k$, one has $d \notin (\text{range}((B_{\hat{\sigma}}, b_k)))^{\perp}$. Hence $\langle b_k, d \rangle^2 > 0$. \square

Proof of Proposition 5 (a) Let $d \in \mathbb{R}^M \setminus G_{L'-1}$. By Lemma 14, $c_{L'-1} > 0$. Since $\{c_k\}$ is decreasing (Lemma 2), one has $c_k > 0 \forall k \leq L' - 1$. Conversely, if $d \notin \mathbb{R}^M \setminus G_{L'-1}$, then $c_{L'-1} = 0$ by Lemma 14.

(b) Using (a)

$$d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies c_k > 0 \quad \forall k \in \mathbb{I}_{L'-1}^0. \quad (76)$$

Let $k \in \mathbb{I}_{L'}$. For any $\omega \in \Omega_k$ and $\bar{\omega} \subsetneq \omega$ with $\#\bar{\omega} = k - 1$ one has $\bar{\omega} \in \Omega_{k-1}$ and $\text{range}(A_{\bar{\omega}}) \subsetneq \text{range}(A_{\omega})$. Since $(\text{range}(A_{\bar{\omega}}))^{\perp} \supsetneq (\text{range}(A_{\omega}))^{\perp}$ we obtain $(\mathbb{R}^M \setminus E_k) \supsetneq (\mathbb{R}^M \setminus E_{k-1})$ for any $k \in \{2, \dots, L'\}$. Using Lemma 15 together with (76) shows that

$$d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies c_{k-1} > c_k > 0 \quad \forall k \in \mathbb{I}_{L'-1} \quad \text{and} \quad c_{L'-1} > c_{L'}.$$

Proof of Corollary 2. (a) follows from Theorem 7 and Proposition 1(a).

(b) By (46), r_{β} is the lower envelope of $L + 1$ affine increasing functions. Hence (b).

(c) One has $\beta_{J_k}(J_{k+1} - J_k) = c_{J_k} - c_{J_{k+1}}$ and $\{\beta_{J_k}\}_{k=0}^P$ strictly decreasing by Proposition 3. Then

$$\begin{aligned} r_{\beta_{J_k}} - r_{\beta_{J_{k+1}}} &= c_{J_k} - c_{J_{k+1}} + \beta_{J_k} J_k - (\beta_{J_k} - (\beta_{J_k} - \beta_{J_{k+1}})) J_{k+1} \\ &= c_{J_k} - c_{J_{k+1}} + \beta_{J_k}(J_k - J_{k+1}) + (\beta_{J_k} - \beta_{J_{k+1}}) J_{k+1} = (\beta_{J_k} - \beta_{J_{k+1}}) J_{k+1} > 0. \end{aligned} \quad (77)$$

Since $J_0 = 0$, see (31), $r_{\beta_{J_0}} = c_{J_0} = r_{\beta}$, $\forall \beta \geq \beta_{J_0}$. Using (a) and (77) yields $r_{\beta_{J_0}} > r_{\beta}$, $\forall \beta < \beta_{J_0}$.

Acknowledgments

Our work was supported by the ‘‘FMJH Program Gaspard Monge in optimization and operation research’’, and by the support to this program from EDF.

References

- [1] H. Attouch, J. Bolte, and B. F. Svaiter, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized gausseidel methods*, Math. Program., 137 (2013).
- [2] H. Bauschke, D. R. Luke, H. M. Phang, and X. Wang, *Restricted Normal Cones and Sparsity Optimization with Affine Constraints*, Found. Comput. Math., 14 (2014), pp. 63–83.
- [3] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York, 2003.

- [4] E. Van Den Berg and M. P. Friedlander, *Sparse optimization with least-squares constraints*, SIAM J. Optim., 21 (2011), pp. 1201–1229.
- [5] D. Blanchard, C. Cartis, J. Tanner, and A. Thompson, *Phase transitions for greedy sparse approximation algorithms*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 188–203.
- [6] T. Blumensath and M. Davies, *Iterative thresholding for sparse approximations*, J. Fourier Anal. Appl., 14 (2008), pp. 629–654.
- [7] ———, *Iterative hard thresholding for compressed sensing*, Appl. Comput. Harmon. Anal., 27 (2009), pp. 265–274.
- [8] ———, *Normalized iterative hard thresholding: Guaranteed stability and performance*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 298–309.
- [9] A. M. Bruckstein, D. L. Donoho, and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev., 51 (2009), pp. 34–81.
- [10] E. J. Candès, J. Romberg, and T. Tao, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information*, IEEE Trans. Inform. Theory, 52 (2006), pp. 489 – 509.
- [11] C. Cartis and A. Thompson, *A new and improved quantitative recovery analysis for iterative hard thresholding algorithms in compressed sensing*, IEEE Trans. Inform. Theory, 61 (2015), pp. 2019–2042.
- [12] A. Cohen, W. Dahmien, and R. A. DeVore, *Compressed sensing and best k -term approximation*, J. Amer. Math. Soc., 22 (2009), pp. 211–231.
- [13] G. Davis, S. Mallat, and M. Avellaneda, *Adaptive greedy approximations*, Constructive approximation, 13 (1997), pp. 57–98.
- [14] R. A. DeVore, *Nonlinear approximation*, Acta Numerica, 7 (1998).
- [15] B. Dong and Y. Zhang, *An efficient algorithm for ℓ_0 minimization in wavelet frame based image restoration*, J. Sci. Comput., 54 (2013).
- [16] D.L. Donoho, *Compressed sensing*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.
- [17] Y. C. Eldar and G. Kutyniok, *Compressed Sensing: Theory and Applications*, Cambridge Univ. Press, 2012.
- [18] J. Fan and R. Li, *Statistical challenges with high dimensionality: feature selection in knowledge discovery*, in Proc. Intern. Congr. Math., vol. 3, Eur. Math. Soc., Zürich, 2006, pp. 595–622.
- [19] M. Fornasier and R. Ward, *Iterative thresholding meets free-discontinuity problems*, Found. Comput. Math., 10 (2010), pp. 527–567.

- [20] G. M. Fung and O. L. Mangasarian, *Equivalence of minimal ℓ_0 - and ℓ_p -norm solutions of linear equalities, inequalities and linear programs for sufficiently small p* , J. Optim. Theory Appl., 151 (2011), pp. 1–10.
- [21] S. Geman and D. Geman, *Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images*, IEEE Trans. Pattern Anal. Mach. Intell., PAMI-6 (1984), pp. 721–741.
- [22] J. Haupt and R. Nowak, *Signal reconstruction from noisy random projections*, IEEE Trans. Inform. Theory, 52 (2006), pp. 4036–4048.
- [23] Y. Liu and Y. Wu, *Variable selection via a combination of the ℓ_0 and ℓ_1 penalties*, J. Comp. Graph. Stat., 16 (2007), pp. 782–798.
- [24] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, J., New York, 1 ed., 1969.
- [25] J. Lv and Y. Fan, *A unified approach to model selection and sparse recovery using regularized least squares*, The Annals of Statistics, 37 (2009).
- [26] A. Maleki and D. L. Donoho, *Optimally tuned iterative reconstruction algorithms for compressed sensing*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 330–341.
- [27] R. T. Marler and J. S. Arora, *Survey of multi-objective optimization methods for engineering*, Struct; Multidisc; Optim., 26 (2004), pp. 369–395
- [28] C. D. Jr. Meyer, *Generalized inverses and ranks of block matrices*, SIAM J. Appl. Math., 25 (1973), pp. 597–602.
- [29] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.
- [30] A. J. Miller, *Subset Selection in Regression*, Chapman and Hall, London, U.K., 2 ed., 2002.
- [31] D. Needell, and J. Tropp, *CoSaMP: Iterative signal recovery from incomplete and inaccurate samples*, Appl. Comput. Harmon. Anal., 26 (3) (2009), pp. 301–321.
- [32] J. Nocedal and S. Wright, *Numerical Optimization*, Springer, NY, 2 ed., 2006.
- [33] J. Neumann, C. Schörr, and G. Steidl, *Combined SVM-based Feature Selection and classification*, Machine Learning 61, 2005, pp. 129–150.
- [34] M. Nikolova, *Description of the minimizers of least squares regularized with ℓ_0 -norm. uniqueness of the global minimizer*, SIAM J. Imaging Sci., 6 (2013), pp. 904–937.
- [35] M. C. Robini, A. Lachal, and I.E. Magnin, *A stochastic continuation approach to piecewise constant reconstruction*, IEEE Trans. Image Process., 16 (2007), pp. 2576–2589.
- [36] M. C. Robini and I. E. Magnin, *Optimization by stochastic continuation*, SIAM J. Imaging Sci., 3 (2010), pp. 1096–1121.
- [37] M. C. Robini and P.-J. Reissman, *From simulated annealing to stochastic continuation: a new trend in combinatorial optimization*, J. Global Optim., 56 (2013), pp. 185–215.

- [38] R. T. Rockafellar and J. B. Wets, *Variational analysis*, Springer-Verlag, New York, 1998.
- [39] E. Soubies, L. Blanc-Féraud and G. Aubert, *A Continuous Exact ℓ_0 penalty (CEL0) for least squares regularized problem*, SIAM J. Imaging Sci., 8 (2015), pp. 1607–1639.
- [40] J.A. Tropp, *Just relax: convex programming methods for identifying sparse signals in noise*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1030–1051.
- [41] J. Tropp and S. J. Wright, *Computational methods for sparse solution of linear inverse problems*, Proc. IEEE, 98 (2010), pp. 948–958.
- [42] Y. Zhang, B. Dong, and Z. Lu, *ℓ_0 minimization of wavelet frame based image restoration*, Mathematics of Computation, 82 (2013), pp. 995–1015.