

# Relationship between the optimal solutions of least squares regularized with $\ell_0$ -norm and constrained by k-sparsity

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**Abstract.** Given an  $M \times N$  real-valued matrix  $A$  with  $M < N$  and a data-vector  $d$ , consider that  $d$  should be expressed as a linear combination of a small number of basis vectors taken from  $A$ . Two popular options to find the sought-after sparse solution are problem  $(\mathcal{R}_\beta)$  – to minimize the least-squares *regularized* with the counting function  $\|\cdot\|_0$  (called usually the  $\ell_0$ -norm) via a trade-off parameter  $\beta > 0$ , and problem  $(\mathcal{C}_k)$  – to solve the least-squares *constrained* by k-sparsity, i.e.  $\|\cdot\|_0 \leq k$ .

Our focus is on the *sets of globally optimal solutions* (called “optimal solution sets”) of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . We analyse in depth the relationship between the optimal solution sets of these two nonconvex (combinatorial) problems. These problems are shown to be *quasi-equivalent* in the following sense. There is a set  $J \subseteq \{0, \dots, M\}$  and a strictly decreasing sequence  $\{\beta_k\}_{k \in J}$  of parameters that partitions  $\mathbb{R}_+$  into  $\text{size}(J)$  proper intervals. *For every  $\beta$  inside the  $k$ 'th interval, problem  $(\mathcal{R}_\beta)$  and problem  $(\mathcal{C}_k)$  have the same optimal solution set.* This holds for any  $k \in J$ . Problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are *fully* quasi-equivalent if  $J = \{0, \dots, M\}$ . All  $\beta_k$ 's and  $J$  are obtained from the optimal values of problems  $(\mathcal{C}_k)$ . The optimal values of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are analyzed. The optimal solution sets of both problems are shown to be composed of a finite number of isolated points. Under mild assumptions, these sets are singletons if  $k < M$ .

Examples and small-size exact numerical tests illustrate the obtained theoretical results.

**Keywords:**  $\ell_0$ -regularization; k-sparsity constraint; globally optimal solutions; optimal solution analysis; parameter selection; quasi-equivalence between nonconvex problems; sparse signal recovery; under-determined linear systems.

## 1 Introduction

The recovery of sparse objects (e.g., signals, images) or representations  $u \in \mathbb{R}^N$  using a few basis vectors from incomplete and possibly inaccurate data  $d \in \mathbb{R}^N$  is a tremendously growing topic, especially with the recent progress in compressed sensing [14, 9, 16, 8, 38, 18]. The most natural way to measure sparsity is the counting function  $\|\cdot\|_0$ , called usually the  $\ell_0$ -norm

$$\|u\|_0 := \#\{i \in \{0, 1, \dots, N\} : u[i] \neq 0\}, \quad (1)$$

where  $\#S$  is the number of elements in the set  $S$  and  $u[i]$  stands for the  $i$ th components of  $u$ . We consider a frame (a dictionary)  $A \in \mathbb{R}^{M \times N}$  with  $M < N$  for fixed  $M$  and  $N$ .

Two popular options to find a sparse solution  $\hat{u}$  are defined by the following optimization problems:

- the  $k$ -sparsity constrained minimization problem

$$(\mathcal{C}_k) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|_2^2, \quad \text{subject to } \|u\|_0 \leq k; \quad (2)$$

- the  $\|\cdot\|_0$ -regularised minimization given by  $\mathcal{F}_\beta : \mathbb{R}^N \rightarrow \mathbb{R}$  below

$$(\mathcal{R}_\beta) \quad \mathcal{F}_\beta(u) := \|Au - d\|_2^2 + \beta \|u\|_0, \quad \beta > 0, \quad (3)$$

where  $\beta > 0$  is a regularization parameter.

Even though overlooked for several decades, problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  were essentially considered from a numerical standpoint. The relationship between these problems has never been studied in a systematic way. A footnote in [5, p. 631] mentions the relation between  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  in that there is a  $\beta$ , depending on  $d$  and  $k$ , so that these problems have the same solution.

*The goal of this work is to analyze in depth the connections between the globally optimal solution sets of  $(\mathcal{C}_k)$  and of  $(\mathcal{R}_\beta)$ .* As usual, we say “optimal set” or “optimal solutions” (respectively, “optimal values”) for *globally optimal* solutions (respectively, globally optimal values) [36, 2]. The optimal sets and the optimal values for the problems we consider are given below. Let  $d \in \mathbb{R}^M$ .

- In problem  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_N^0$  the constraint set of reads as  $\{u \in \mathbb{R}^N \mid \|u\|_0 \leq k\}$ , so we have

$$\text{optimal value} \quad \theta_k := \inf \left\{ \|Au - d\|^2 \mid u \in \mathbb{R}^N \text{ and } \|u\|_0 \leq k \right\}, \quad (4)$$

$$\text{set of optimal solutions} \quad \hat{\mathcal{C}}_k := \left\{ u \in \mathbb{R}^N \text{ and } \|u\|_0 \leq k \mid \|Au - d\|^2 = \theta_k \right\}. \quad (5)$$

- In problem  $(\mathcal{R}_\beta)$  for  $\beta > 0$  one has

$$\text{optimal value} \quad f_\beta := \inf \left\{ \mathcal{F}_\beta(u) \mid u \in \mathbb{R}^N \right\}, \quad (6)$$

$$\text{set of optimal solutions} \quad \hat{\mathcal{R}}_\beta := \left\{ u \in \mathbb{R}^N \mid \mathcal{F}_\beta(u) = f_\beta \right\}. \quad (7)$$

We anticipate that for any  $d \in \mathbb{R}^M$ , one has  $\hat{\mathcal{C}}_k \neq \emptyset, \forall k \geq 0$  and  $\hat{\mathcal{R}}_\beta \neq \emptyset, \forall \beta > 0$  (Lemma 1 and Theorem 2(b), respectively). Thus, our goal is to explore the relationship between the *optimal sets*  $\hat{\mathcal{C}}_k$  and  $\hat{\mathcal{R}}_\beta$ . To this end, we adopt a common assumption on  $A$ :

**H 1.** *The matrix  $A \in \mathbb{R}^{M \times N}$  satisfies  $\text{rank}(A) = M < N$ .*

It is systematically assumed that  $d \neq 0$ .

The quite standard Definition 1 shall be used to evaluate the extent of some properties.

DEFINITION 1. A property is generic on  $\mathbb{R}^M$  if it holds on a subset of  $\mathbb{R}^M \setminus S$  where  $S$  is closed in  $\mathbb{R}^M$  and its Lebesgue measure in  $\mathbb{R}^M$  is null.

A generic property is clearly stronger than a property that holds only with probability one because  $\mathbb{R}^M \setminus S$  contains a dense open subset of  $\mathbb{R}^M$ .

Finding an optimal solution of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  is NP-hard in general [13, 37]. Let us mention that recent advances in stochastic optimization have made these problems tractable [35].

## 1.1 Related work

The amount of papers dealing with problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  is huge. We present a brief summary that helps to position the goals of our work.

**On algorithms.** The solutions of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are usually approximated by greedy pursuit, relaxation of the  $\|\cdot\|_0$  penalty often combined with nonconvex minimization [19, 27, 20, 11, 23], as well as direct optimization [30, 34, 1]. Tropp and Wright [38] gave a comprehensive overview, mainly focused on greedy pursuits and convex relaxation. These algorithms require strong assumptions, e.g., RIP, bounds on  $\text{spark}(A)$  and on the sparsity of the solution. Iterative thresholding algorithms has become quite popular after the local convergence results of Blumensath and Davies [5], further expanded by the authors in [6, 7]. Recent results allow to alleviate the RIP condition [28, 4, 10].

Problem  $(\mathcal{R}_\beta)$  is a particular case of an objective whose global minimizer is computed in finite time with high probability by the stochastic continuation algorithm conceived by Robini, Lachal and Magnin [33] and refined by Robini and Magnin in [34]. Recently, Robini and Reissman [35] extended the methodology to general combinatorial objectives and gave results on the probability for global convergence versus the running time. So [35] can be adapted to solve optimally problem  $(\mathcal{C}_k)$  as well.

**Some applications.** Problem  $(\mathcal{C}_k)$  involves a natural sparse coding constraint; it is well known as the k-best term approximation model [14, 12]. It has been used for low-rank matrix decomposition [3], sparse inverse problems [7], dictionary learning [17], among many others. Problem  $(\mathcal{R}_\beta)$  has been widely considered for subset selection [30, 5], model selection [27], variable selection [24], feature selection [31, 19], signal and image reconstruction [22, 20, 11, 39, 15].

**Comparison between  $\|\cdot\|_0$ -related problems.** Here are few references. The analysis of Lorenz [25, sec. 5.2.] holds for *infinite dimensional*  $\ell_p$ -spaces. The author shows that the regularized  $\|\cdot\|_0$  problem can have an empty optimal set and that the minimization of  $\|u\|_0$  subject to  $\|Au - d\| \leq \tau$  can yield an infinite number of solutions. In *finite dimensional* real spaces, Fung and Mangasarian [21] consider the minimization of  $\|u\|_p$  subject to  $Au = d$ ,  $Bu \geq b$  and  $\|u\|_\infty \leq 1$  for  $p \in [0, 1)$ , where  $B$  and  $b$  are a matrix and a vector, respectively. They prove that the  $\|u\|_0$ -problem is equivalent to the  $\|u\|_p$ -problem for a sufficiently small  $p > 0$ .

## 1.2 Main contributions and Outline

This work provides an in-depth analysis of the relationship between the sets of globally optimal solutions (called optimal sets) of the two nonconvex (combinatorial) problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ , formulated in (2) and (3), respectively. Their optimal sets, specified in (5) and (7), are always nonempty (Lemma 1 and Theorem 2). Our main results are summarized below.

- (a) Given  $A$  and  $d$ , let  $L \leq M$  be the least integer so that the optimal value of  $\widehat{\mathcal{C}}_L$  is  $\theta_L = 0$ . For any  $k \leq L$ , any optimal solution  $\widehat{u}$  of  $(\mathcal{C}_k)$  obeys  $\|\widehat{u}\|_0 = k$  and the  $M \times k$  submatrix formed from the columns of  $A$  with indices in  $\text{supp}(\widehat{u})$  has full column rank (Theorem 1). For any  $\beta > 0$ , any global minimizer  $\widehat{u}$  of  $\mathcal{F}_\beta$  is an optimal solution of  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_L^0$  (Theorem 4); hence  $\|\widehat{u}\|_0 \leq L$ .

We emphasize that in a generic sense,  $L = M$  (Proposition 6 and Remark 9).

- (b) Necessary and sufficient conditions for optimality of  $(\mathcal{R}_\beta)$  in terms of the optimal sets of  $(\mathcal{C}_k)$  are derived (Theorem 5). This is the key to finding the links between the optimal sets of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ .

The results in (a) and (b) are developed in section 2.

- (c) A sequence of critical parameter values  $\{\beta_k, \beta_k^U\}_{k=0}^L$  is proposed using the optimal values  $\{\theta_k\}_{k=0}^L$  of  $(\mathcal{C}_k)$  (Definition 2). It holds that the optimal set of  $(\mathcal{C}_k)$  is included in (respectively, equal to) the optimal set of  $(\mathcal{R}_\beta)$  if and only if  $\beta_k \leq \beta \leq \beta_k^U$  (respectively,  $\beta_k < \beta < \beta_k^U$ ) (Theorem 6).
- (d) Problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are *quasi-equivalent* in the following sense. Let  $J := \{k \in \mathbb{I}_L^0 \mid \beta_k < \beta_k^U\}$  and  $J^E := \{m \in \mathbb{I}_L^0 \mid \beta_m = \beta_m^U\}$ . One has  $\{\beta_m\}_{m \in J^E} \subset \{\beta_k\}_{k \in J}$ ,  $\{0, L\} \in J$  and by setting  $J = \{k_n\}_{n \geq 0}$ ,  $\beta_{k_n}^U = \beta_{k_{n-1}}$  and  $\{\beta_{k_n}\}_{n \geq 0}$  is strictly decreasing (Proposition 2). For any  $\beta \in (\beta_{k_n}, \beta_{k_{n-1}})$  problem  $(\mathcal{R}_\beta)$  and problem  $(\mathcal{C}_{k_n})$  have the same optimal set. For  $\beta = \beta_{k_n}$ , the optimal set of  $(\mathcal{R}_\beta)$  contains the optimal sets of  $(\mathcal{C}_{k_n})$  and  $(\mathcal{C}_{k_{n+1}})$ , and those of  $(\mathcal{C}_m)$  for  $m \in \{J^E \mid \beta_m = \beta_{k_n}\}$  (Theorem 7, our main result). This quasi-equivalence, described in (8), holds if and only if  $k \in J$ :

$$\begin{array}{ccccccccccc} \widehat{\mathcal{R}}_\beta = & & \widehat{\mathcal{C}}_L & & & \widehat{\mathcal{C}}_{k_{n+1}} & & \underbrace{\widehat{\mathcal{C}}_{k_n} \cup \widehat{\mathcal{C}}_{k_{n+1}} \cup \widehat{\mathcal{C}}_{k_n}^0}_{\uparrow} & & \widehat{\mathcal{C}}_{k_n} & & \widehat{\mathcal{C}}_0 & & (8) \\ 0 = & \beta_L & < & \dots < & \beta_{k_{n+1}} & < & \beta_{k_n} & < & \beta_{k_{n-1}} < & \dots < & \beta_{k_0} < & +\infty, \end{array}$$

where  $\widehat{\mathcal{C}}_{k_n}^0 := \bigcup_{\{m \in J^E \mid \beta_m = \beta_{k_n}\}} \widehat{\mathcal{C}}_m$ . Further,  $\{J^E \mid \beta_m = \beta_{k_n}\} = \{m \in J^E \mid k_n < m < k_{n+1}\}$  (Lemma 7).

In a generic sense,  $J^E$  is empty (Proposition 4) in which case all  $\widehat{\mathcal{C}}_{k_n}^0$ 's in (8) are empty.

The sequence  $\{\beta_k\}_{k \in J}$  is characterized in Proposition 3: it is the largest strictly decreasing subsequence of  $\{\beta_k\}$  in Definition 2 containing  $\beta_0$ .

- (e) One has  $J = \mathbb{I}_L^0$ , i.e., problem  $(\mathcal{C}_k)$  and problem  $(\mathcal{R}_\beta)$  for any  $\beta \in (\beta_k, \beta_{k-1})$  have the same optimal set if and only if  $\{\beta_k\}_{k=0}^L$  in Definition 2 is strictly decreasing (Theorem 8). In this case,  $\beta_k = \theta_k - \theta_{k+1}$  for all  $k$ , and we say that  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are fully quasi-equivalent.

The results on  $\{\beta_k\}$  and  $(J, J^E)$  are established in section 3 and quasi-equivalence in section 4.

- (f) The optimal values  $\theta_k$  of problem  $(\mathcal{C}_k)$  generically form a strictly decreasing sequence (Proposition 6). Necessary and sufficient conditions for optimality of  $\beta \mapsto f_\beta$  are deduced. See section 5.
- (g) The optimal sets of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are composed out of a finite number of isolated points (Theorem 9, section 6). For  $k \leq \min\{L, M - 1\}$  and for  $\beta > \beta_{k^*}$ , where  $\beta_{k^*}$  is the next to last entry of  $\{\beta_k\}_{k \in J}$ , the optimal solutions of  $(\mathcal{C}_k)$  and of  $(\mathcal{R}_\beta)$  are generically unique (subsection 6.1).

Exact numerical tests for  $(M, N) = (5, 10)$  illustrate all theoretical findings in section 7. We used Monte Carlo tests on the parameters  $\beta_k$  in Definition 2. We always had  $L = M$  and  $\{\beta_k\}_{k=0}^M$  was strictly decreasing in more than 93% of the cases. We emphasize that our tests are on small-size problems.

### 1.3 Notation

For ease of presentation, the  $\ell_2$ -norm is *systematically* denoted by

$$\|\cdot\| := \|\cdot\|_2 .$$

Let  $n$  be a positive integer. We denote by  $\mathbb{I}_n$  and  $\mathbb{I}_n^0$  the *totally and strictly ordered* index sets

$$\mathbb{I}_n := (\{1, \dots, n\}, <) \quad \text{and} \quad \mathbb{I}_n^0 := (\{0, 1, \dots, n\}, <) , \quad (9)$$

where the symbol  $<$  stands for the natural order of integers. Thus any subset  $\omega \subseteq \mathbb{I}_n$  is also totally and strictly ordered. Without this precision, some formulas will be ambiguous; e.g., (12) below.

For any  $u \in \mathbb{R}^N$ , the *support* of  $u$  is defined by

$$\text{supp}(u) := \left\{ i \in \mathbb{I}_N \mid u[i] \neq 0 \right\} \subseteq \mathbb{I}_N . \quad (10)$$

As usual, a vector  $u$  is said to be  $k$ -sparse if  $\|u\|_0 = \#(\text{supp}(u)) \leq k$ .

REMARK 1. For  $(\mathcal{C}_k)$  we consider also two trivial cases:  $k = 0$  because  $\mathcal{F}_\beta$  always has a strict (local) minimum at  $\hat{u} = 0$  and  $k = M$  since  $\mathcal{F}_\beta$  can have strict (local) minimizers  $\tilde{u}$  with  $\|\tilde{u}\|_0 = M$  [32]. According to the value of  $\beta$ ,  $\hat{u}$  or  $\tilde{u}$  can be global minimizers of  $\mathcal{F}_\beta$ .  $\diamond$

#### General notation

- The  $i$ th column in matrix  $A \in \mathbb{R}^{M \times N}$  is denoted by  $a_i$ , i.e.,  $A = (a_1, \dots, a_N)$ .
- For  $A \in \mathbb{R}^{M \times N}$  and  $u \in \mathbb{R}^N$ , with any  $\omega \subseteq \mathbb{I}_N$ , the associated *submatrix*  $A_\omega$  and *subvector*  $u_\omega$  read as

$$A_\omega := (a_{\omega[1]}, \dots, a_{\omega[\#\omega]}) \in \mathbb{R}^{M \times \#\omega} \quad \text{and} \quad u_\omega := \left( u[\omega[1]], \dots, u[\omega[\#\omega]] \right)^T \in \mathbb{R}^{\#\omega} , \quad (11)$$

respectively, where the superscript  $^T$  means transposed. By (10) and (11), for any  $u \in \mathbb{R}^N$  one has

$$\omega \in \mathbb{I}_N \quad \text{and} \quad \omega \supseteq \text{supp}(u) \quad \implies \quad Au = A_\omega u_\omega . \quad (12)$$

- We also set  $A_\omega^T := (A_\omega)^T$  and  $A_\omega^{-1} := (A_\omega)^{-1}$ .
- In view of Remark 1, to unify the presentation we set  $A_\emptyset := [ ] \in \mathbb{R}^{M \times 0}$  and  $\text{rank}(A_\emptyset) := 0$ .
- $I$  stands for the identity operator (usually on  $\mathbb{R}^M$ ).
- $0$  denotes a vector of zeros in a real space of arbitrary dimension  $\geq 1$ .
- $\Omega_k := \{\omega \subset \mathbb{I}_N \mid \#\omega = k = \text{rank}(A_\omega)\}$  lists the  $k$ -length subsets  $\omega \in \mathbb{I}_N^0$  obeying  $\text{rank}(A_\omega) = k$ .

### Notation for problems $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

- $L := \min \{k \in \mathbb{I}_N \mid \theta_k = 0\}$  where  $\theta_k$  is the optimal value of  $(\mathcal{C}_k)$ , see (4).
- $\widehat{\mathcal{C}} := \bigcup_{k=0}^L \widehat{\mathcal{C}}_k$  where  $\widehat{\mathcal{C}}_k$  is the set of optimal solutions of  $(\mathcal{C}_k)$ , see (5).
- $\widehat{\mathcal{R}} := \bigcup_{\beta > 0} \widehat{\mathcal{R}}_\beta$  where  $\widehat{\mathcal{R}}_\beta$  is the optimal solution set of  $(\mathcal{R}_\beta)$  (the global minimizers of  $\mathcal{F}_\beta$ ), see (7).
- $\beta_k, \beta_k^U$  for  $k \in \mathbb{I}_L^0$  – critical parameter values, Definition 2 (section 3).
- $J \subseteq \mathbb{I}_L^0$  – the integers  $k \in \mathbb{I}_L^0$  such that  $\beta_k < \beta_k^U$  and  $\widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta$  for  $\beta \in (\beta_k, \beta_k^U)$  by Theorem 6.
- $J^E \subsetneq \mathbb{I}_L^0$  – the integers  $m \in \mathbb{I}_L^0$  leading to  $\beta_m = \beta_m^U$ . The superscript  $E$  in  $J^E$  evokes equality.

## 2 Common optimality conditions for $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

Here we present results aimed to establish relations between the optimality conditions for problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . The proofs of nearly all statements in this section are outlined in Appendix 9.1.

Given  $d \in \mathbb{R}^M$  and  $\omega \subseteq \mathbb{I}_N$ , consider the following constrained quadratic optimization problem:

$$(\mathcal{P}_\omega) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|^2, \quad \text{subject to } u[i] = 0 \quad \forall i \in \mathbb{I}_N^0 \setminus \omega. \quad (13)$$

Obviously, the convex problem  $(\mathcal{P}_\omega)$  is related to problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . Since problem  $(\mathcal{P}_\omega)$  has solutions for any  $d \in \mathbb{R}^M$  and for any  $\omega \subset \mathbb{I}_N$ , it is a good tool for analyzing the nonconvex combinatorial problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . We shall use it mainly in this section and in section 6.

We begin with a study of the optimal sets of problem  $(\mathcal{C}_k)$ . For problem  $(\mathcal{R}_\beta)$  we will use known results.

### 2.1 On the optimal solutions of problem $(\mathcal{C}_k)$

The set of all  $\omega \in \mathbb{I}_N$  with at most  $k$  different components reads as

$$\Sigma_k := \bigcup_{n=0}^k \left\{ \omega \subset \mathbb{I}_N \mid \#\omega = n \right\}. \quad (14)$$

Clearly,  $\Sigma_k$  lists the supports of all  $k$ -sparse vectors in  $\mathbb{R}^N$ . The constraint set of  $(\mathcal{C}_k)$  in (2) also reads as

$$\left\{ u \in \mathbb{R}^N \mid \text{supp}(u) \in \Sigma_k \right\} \quad (15)$$

and the corresponding optimal set  $\widehat{C}_k$  in (5) is

$$\widehat{C}_k = \left\{ u \in \mathbb{R}^N, \text{supp}(u) \in \Sigma_k \mid \|Au - d\|^2 = \theta_k \right\}. \quad (16)$$

A central question is to know whether problem  $(C_k)$  admits an optimal solution.

LEMMA 1. *Let  $d \in \mathbb{R}^M$ . For any  $k \in \mathbb{I}_N^0$  one has  $\widehat{C}_k \neq \emptyset$ , i.e.  $(C_k)$  always has an optimal solution.*

Using  $(P_\omega)$  in (13) and (15), it follows the optimal value  $\theta_k$  of  $(C_k)$  equals

$$\theta_k = \min \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (P_\omega) \mid \omega \in \Sigma_k \right\}. \quad (17)$$

At this point, we need the following simple lemma.

LEMMA 2. *Let  $d \in \mathbb{R}^M$ . Then  $\theta_0 = \|d\|^2$  and  $\{\theta_k\}_{k \geq 0}$  is decreasing. If H1 holds,  $\theta_k = 0 \quad \forall k \geq M$ .*

The next lemma has a pivotal role in this work.

LEMMA 3. *Let H1 hold. For  $k \in \mathbb{I}_M$ , assume that  $(C_k)$  has an optimal solution  $\widehat{u}$  obeying*

$$\|\widehat{u}\|_0 = k - n \quad \text{for } n \geq 1. \quad (18)$$

Then  $A\widehat{u} = d$ . Furthermore,

$$\theta_m = 0 \quad \text{and} \quad \widehat{u} \in \widehat{C}_m \quad \forall m \geq k - n.$$

Based on Lemma 3 and assuming that H1 holds, we introduce the constant

$$\boxed{L := \min \{k \in \mathbb{I}_N \mid \theta_k = 0\}}. \quad (19)$$

$L$  is uniquely defined since  $\{\theta_k\}$  is decreasing with  $L \leq M$  because  $\theta_M = 0$  (Lemma 2). We emphasize that  $L$  relies on  $A$  and on  $d$  but that  $L = M$  generically (Proposition 6(b) and Remark 9).

EXAMPLE 1. One has  $L \leq M - 1$  if  $d = Au$  for  $\|u\|_0 \leq M - 1$ . Then  $d$  belongs to a subspace of  $\mathbb{R}^M$  of dimension  $\|u\|_0$  which has null Lebesgue measure in  $\mathbb{R}^M$ . Usual data range on the whole  $\mathbb{R}^M$  and  $L = M$ .

THEOREM 1. *Let H1 be satisfied and  $L$  read according to (19). Then*

$$\widehat{u} \in \widehat{C}_k \text{ for } k \in \mathbb{I}_L^0 \implies \|\widehat{u}\|_0 = k = \text{rank}(A_{\widehat{\sigma}}) \text{ for } \widehat{\sigma} := \text{supp}(\widehat{u}). \quad (20)$$

Furthermore,  $\widehat{C}_L \subset \widehat{C}_k \quad \forall k \geq L$ .

Theorem 1 gives a necessary condition for an optimal solution of  $(C_k)$  for  $k \in \mathbb{I}_L^0$ .

The algorithm aimed at solving  $(C_k)$  proposed in [5] was shown in [5, Lemma 6] to produce, under certain conditions, solutions that fulfill this necessary condition.

The optimal sets  $\widehat{C}_k$  for  $k > L$  are of limited interest: they have  $\ell_0$ -norms in  $\{L, \dots, k\}$ , can be nonstrict if they solve  $(P_\omega)$  for an  $\omega$  so that  $A_\omega$  does not have full column rank; they always contain  $\widehat{C}_L$ .

REMARK 2. [On Assumption H1] This usual assumption ensures that  $A$  is rich enough to represent any  $d \in \mathbb{R}^M$ . It is also needed to prove the pivotal Lemma 3 and Theorem 1.  $\diamond$

A direct and useful consequence of Theorem 1 is stated below.

COROLLARY 1. *Let H1 hold. Then  $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_n = \emptyset$  for all  $(k, n) \in (\mathbb{I}_L^0)^2$  such that  $k \neq n$ .*

If  $\widehat{u}$  solves optimally  $(\mathcal{C}_k)$  for  $k \leq L$ , then  $\widehat{u}$  is not an optimal solution of  $(\mathcal{C}_n)$  for  $n \leq L$ ,  $n \neq k$ .

By Theorem 1, many subsets in  $\{\Sigma_k\}_{k=0}^N$  are not the supports of optimal solutions sets of  $(\mathcal{C}_k)$ . Accordingly, we focus only on the subsets  $\omega \subset \mathbb{I}_N^0$  with exactly  $k$  entries such that  $\text{rank}(A_\omega) = k$ :

$$\Omega_k := \left\{ \omega \subset \mathbb{I}_N \mid \#\omega = k = \text{rank}(A_\omega) \right\}. \quad (21)$$

REMARK 3. Let H1 hold. By Theorem 1, for any  $k \in \mathbb{I}_L^0$ , the optimal value of problem  $(\mathcal{C}_k)$  obeys

$$\theta_k = \min \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \mid \omega \in \Omega_k \right\} \quad (22)$$

and the corresponding set of optimal solutions satisfies

$$\widehat{\mathcal{C}}_k = \left\{ u \in \mathbb{R}^N, \text{supp}(u) \in \Omega_k \mid \|Au - d\|^2 = \theta_k \right\}. \quad (23)$$

The fact that  $\#\Omega_k \ll \#\Sigma_k$  may be interesting.  $\diamond$

## 2.2 Necessary and sufficient conditions for optimality of $(\mathcal{R}_\beta)$ in terms of $(\mathcal{C}_k)$

First we cite several results on problem  $(\mathcal{R}_\beta)$ , used in what follows. It worths noting that  $\widehat{u} \in \mathbb{R}^N$  is a (local) minimizer of  $\mathcal{F}_\beta$  if and only if  $\widehat{u}$  solves  $(\mathcal{P}_\omega)$  for some  $\omega \subset \mathbb{I}_N^0$  [32]. For reminder,  $\widehat{u}$  is a *strict* (local) minimizer of  $\mathcal{F}_\beta$  if there is a neighborhood  $\mathcal{O} \ni \widehat{u}$  so that  $\mathcal{F}_\beta(\widehat{u}) < \mathcal{F}_\beta(u)$  for any  $u \in \mathcal{O} \setminus \{\widehat{u}\}$ .

THEOREM 2 ([32] Theorem 4.4). *Let  $d \in \mathbb{R}^M$  and let  $\beta > 0$ .*

- (a)  $\widehat{\mathcal{R}}_\beta \neq \emptyset$ , i.e.  $\mathcal{F}_\beta$  always has a global minimizer.
- (b) If  $\widehat{u}$  is a global minimizer of  $\mathcal{F}_\beta$ , then  $\widehat{u}$  is a strict minimizer.

The global minimizers of  $\mathcal{F}_\beta$  being strict, we want to select those strict minimizers of  $\mathcal{F}_\beta$  that can also be global minimizers. With the notation in (21), the strict minimizers of  $\mathcal{F}_\beta$  are characterised next.

THEOREM 3 ([32], Theorem 3.2). *Let  $d \in \mathbb{R}^M$  and  $\beta > 0$ . A point  $\widehat{u} \in \mathbb{R}^N$  is a strict (local) minimizer of  $\mathcal{F}_\beta$  if and only if its support  $\omega := \text{supp}(\widehat{u})$  satisfies  $\#\omega = \text{rank}(A_\omega)$ .*

COROLLARY 2 ([32] Corollary 3.3). *Let  $d \in \mathbb{R}^M$  and let  $\widehat{u}$  solve  $(\mathcal{P}_\omega)$  for  $\omega \in \Omega_k$  where  $k \in \mathbb{I}_M^0$ . Then  $\widehat{u}$  is a strict (local) minimizer of  $\mathcal{F}_\beta$  for any  $\beta > 0$ .*



These facts, together with the results on  $(\mathcal{C}_k)$  obtained in § 2.1, will enable us to derive common optimality conditions for  $(\mathcal{R}_\beta)$  and for  $(\mathcal{C}_k)$  in Theorem 5. Proposition 1 relates the optimal sets and the optimal values of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . It helps to develop this section.

PROPOSITION 1. *Let H1 hold and let  $\beta > 0$ .*

(a) *For any  $k \in \mathbb{I}_L^0$  it holds that*

$$\mathcal{F}_\beta(\hat{u}) = \theta_k + \beta k \quad \forall \hat{u} \in \hat{\mathcal{C}}_k .$$

(b)  $\hat{u} \in \hat{\mathcal{R}}_\beta \implies \hat{u} \in \hat{\mathcal{C}}_k$  *where  $k := \|\hat{u}\|_0 \in \mathbb{I}_L^0$ .*

(c)  $\hat{u} \in \hat{\mathcal{R}}_\beta \implies \hat{\mathcal{C}}_k \subseteq \hat{\mathcal{R}}_\beta$  *for  $k := \|\hat{u}\|_0 \in \mathbb{I}_L^0$ .*

By (c), the global minimizers of  $\mathcal{F}_\beta$  are composed of some optimal sets  $\hat{\mathcal{C}}_k$  *only* for  $k \leq L$ .

The claim in Lemma 4 is usually false for ordinary subsets  $C$  and  $R$ .

LEMMA 4. *Let H1 hold. For any  $\beta > 0$  and for any  $k \in \mathbb{I}_N^0$  one has*

$$\hat{\mathcal{C}}_k \not\subseteq \hat{\mathcal{R}}_\beta \iff \hat{\mathcal{R}}_\beta \cap \hat{\mathcal{C}}_k = \emptyset .$$

We denote by  $\hat{\mathcal{C}}$  the collection of all optimal solutions of problems  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_L^0$  and likewise, by  $\hat{\mathcal{R}}$  – the set of all global minimizers of  $\mathcal{F}_\beta$  for all  $\beta > 0$ :

$$\boxed{\hat{\mathcal{C}} := \bigcup_{k=0}^L \hat{\mathcal{C}}_k \quad \text{and} \quad \hat{\mathcal{R}} := \bigcup_{\beta>0} \hat{\mathcal{R}}_\beta .} \quad (24)$$

With this notation, the Theorem 4 is an important consequence of Proposition 1(b).

THEOREM 4. *Let H1 hold. Then  $\hat{\mathcal{R}} \subset \hat{\mathcal{C}}$ .*

Theorem 4 shows that when  $\beta$  ranges on  $(0, +\infty)$ ,  $\mathcal{F}_\beta$  can have *at most*  $L + 1$  different sets of global minimizers which are optimal solutions of  $(\mathcal{C}_k)$  for  $k \in \{0, \dots, L\}$ . Even though  $\hat{\mathcal{C}}_k \neq \emptyset$  for any  $k \geq 0$ , if  $k \geq L + 1$ , then  $\forall \beta > 0$ , the entries of  $\hat{\mathcal{C}}_k$  are not global minimizers of  $\mathcal{F}_\beta$ ; i.e.,

$$\hat{\mathcal{C}}_k \cap \hat{\mathcal{R}} = \emptyset \quad \forall k \geq L + 1 .$$

REMARK 4. Let  $\beta > 0$  and  $k \in \mathbb{I}_N^0$ . Since  $\hat{\mathcal{R}}_\beta$  is the set of the global minimizers of  $\mathcal{F}_\beta$ , it is clear that

$$\begin{aligned} \hat{\mathcal{C}}_k \subseteq \hat{\mathcal{R}}_\beta &\iff \mathcal{F}_\beta(u) \geq \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall u \in \mathbb{R}^N ; \\ \hat{\mathcal{C}}_k = \hat{\mathcal{R}}_\beta &\iff \mathcal{F}_\beta(u) > \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \forall u \in \mathbb{R}^N \setminus \hat{\mathcal{C}}_k . \end{aligned}$$

The next theorem provides the basic tool to compare the optimal sets of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ .

THEOREM 5. *Let H1 hold and let  $\beta > 0$ . For any  $k \in \mathbb{I}_L^0$  the following holds:*

(a)  $\widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta$  if and only if

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}; \quad (25)$$

(b)  $\widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta$  if and only if

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) > 0 \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}} \setminus \widehat{\mathcal{C}}_k. \quad (26)$$

*Proof.* From Remark 4 it is straightforward that

$$\widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta \text{ (resp., } \widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta) \implies \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \forall \bar{u} \in \widehat{\mathcal{C}} \text{ (resp., } > 0, \forall \bar{u} \in \widehat{\mathcal{C}} \setminus \widehat{\mathcal{C}}_k) \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k.$$

The rest of the proof is by contraposition.

(a) Assume that  $\widehat{\mathcal{C}}_k \not\subseteq \widehat{\mathcal{R}}_\beta$ . Then  $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{R}}_\beta = \emptyset$  by Lemma 4. Since  $\widehat{\mathcal{R}}_\beta \neq \emptyset$  (Theorem 2(a)), Proposition 1(c) entails that there exists  $n \in \mathbb{I}_L^0 \setminus \{k\}$  such that  $\widehat{\mathcal{C}}_n \subseteq \widehat{\mathcal{R}}_\beta$ . It follows that

$$\mathcal{F}_\beta(\hat{u}) > \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}_n. \quad (27)$$

The obtained inequality contradicts (25).

(b) Let  $\widehat{\mathcal{C}}_k \neq \widehat{\mathcal{R}}_\beta$ . This, together with  $\widehat{\mathcal{R}}_\beta \neq \emptyset$ , implies that there is  $\bar{u} \in \widehat{\mathcal{R}}_\beta$  such that  $\bar{u} \notin \widehat{\mathcal{C}}_k$ . Using Proposition 1(c) shows that  $\widehat{\mathcal{C}}_n \subseteq \widehat{\mathcal{R}}_\beta$  for  $n := \|\bar{u}\|_0 \in \mathbb{I}_L^0$  where  $n \neq k$ . Therefore

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}_n,$$

where  $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_n = \emptyset$  (see Corollary 1). This result contradicts (26).  $\square$

Theorem 5 is the key to finding the links between the optimal sets of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ : it provides necessary and sufficient conditions for optimality of  $(\mathcal{R}_\beta)$  only in terms of the optimal sets of  $\{(\mathcal{C}_k)\}_{k=0}^L$ .

A simple useful claim is stated next.

LEMMA 5. *Let H1 hold. Let  $(k, k+p) \in (\mathbb{I}_L^0)^2$  for  $p \geq 1$ . The following implications holds:*

$$\widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta \quad \left( \text{respectively } \widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta \right) \quad \text{and} \quad \widehat{\mathcal{C}}_{k+p} = \widehat{\mathcal{R}}_{\beta'} \quad \left( \text{respectively } \widehat{\mathcal{C}}_{k+p} \subseteq \widehat{\mathcal{R}}_{\beta'} \right) \implies \beta' < \beta.$$

This lemma confirms the intuition that when  $\beta$  increases on  $(0, +\infty)$ , the optimal sets  $\widehat{\mathcal{R}}_\beta$  are given by a subsequence of  $\{\widehat{\mathcal{C}}_k\}$  with decreasing indexes.

### 3 Exact connections between the optimal sets of $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

#### 3.1 Optimal sets versus parameter values

DEFINITION 2. (Critical parameter values) Let  $\mathbb{L}$  be as in (19). We define  $(\beta_k, \beta_k^U)$  by

$$\beta_k := \max \left\{ \frac{\theta_k - \theta_{k+n}}{n} \mid n \in \{1, \dots, \mathbb{L} - k\} \right\} \quad \forall k \in \mathbb{I}_{\mathbb{L}-1}^0 \quad \text{and} \quad \beta_{\mathbb{L}} = 0, \quad (28)$$

$$\beta_k^U := \min \left\{ \frac{\theta_{k-n} - \theta_k}{n} \mid n \in \{1, \dots, k\} \right\} \quad \forall k \in \mathbb{I}_{\mathbb{L}} \quad \text{and} \quad \beta_0^U \equiv \beta_{-1} := +\infty. \quad (29)$$

The superscript  $^U$  in (29) suggests that  $\beta_k^U$  can be an upper bound.

REMARK 5. Since  $d \neq 0$ , one has  $\beta_0 < \beta_0^U$  and  $\beta_{\mathbb{L}} < \beta_{\mathbb{L}}^U$ . Indeed, Definition 2 shows that  $\beta_{\mathbb{L}}^U = \min_{n=1}^{\mathbb{L}} \frac{\theta_{\mathbb{L}-n} - \theta_{\mathbb{L}}}{n} > 0 = \beta_{\mathbb{L}}$ . Further,  $\beta_0 < \beta_0^U$  because  $\beta_0$  is finite.  $\diamond$

The theorem below is an important consequence of Theorem 5 and Definition 2.

THEOREM 6. Let H1 hold. Then  $\forall k \in \mathbb{I}_{\mathbb{L}}^0$  it holds that

$$(a) \quad \widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta \quad \text{if and only if} \quad \begin{cases} \beta_0 \leq \beta < \beta_0^U & \text{for } k = 0; \\ \beta_k \leq \beta \leq \beta_k^U & \text{for } k \in \{1, \dots, \mathbb{L} - 1\}; \\ \beta_{\mathbb{L}} < \beta \leq \beta_{\mathbb{L}}^U & \text{for } k = \mathbb{L}. \end{cases}$$

$$(b) \quad \widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta \quad \text{if and only if} \quad \beta_k < \beta < \beta_k^U.$$

*Proof.* We use the equalities below coming from Proposition 1(a):

– If  $\widehat{u} \in \widehat{\mathcal{C}}_k$  for  $k \in \mathbb{I}_{\mathbb{L}-1}^0$ , then

$$\forall n \in \mathbb{I}_{\mathbb{L}-k} \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k+n} \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = \theta_{k+n} - \theta_k + n\beta = n \left( \beta - \frac{\theta_k - \theta_{k+n}}{n} \right). \quad (30)$$

– If  $\widehat{u} \in \widehat{\mathcal{C}}_k$  for  $k \in \mathbb{I}_{\mathbb{L}}$ , then

$$\forall n \in \mathbb{I}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}}_{k-n} \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = \theta_{k-n} - \theta_k - n\beta = n \left( \frac{\theta_{k-n} - \theta_k}{n} - \beta \right). \quad (31)$$

Theorem 5 equivalently reads as given in (33) and (32) below:

$$\forall k \in \mathbb{I}_{\mathbb{L}}^0 \quad \left[ \widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{R}}_\beta \iff \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) \geq 0 \quad \forall \widehat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}} \right]. \quad (32)$$

$$\forall k \in \mathbb{I}_{\mathbb{L}}^0 \quad \left[ \widehat{\mathcal{C}}_k = \widehat{\mathcal{R}}_\beta \iff \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) > 0 \quad \forall \widehat{u} \in \widehat{\mathcal{C}}_k \quad \forall \bar{u} \in \widehat{\mathcal{C}} \setminus \widehat{\mathcal{C}}_k \right]. \quad (33)$$

(a) Consider the expression below:

$$\begin{cases} \forall k \in \mathbb{I}_{\mathbb{L}-1}^0 & \beta \geq \frac{\theta_k - \theta_{k+n}}{n} \quad \forall n \in \mathbb{I}_{\mathbb{L}-k} & \iff & \beta \geq \beta_k; \\ \forall k \in \mathbb{I}_{\mathbb{L}} & \beta \leq \frac{\theta_{k-n} - \theta_k}{n} \quad \forall n \in \mathbb{I}_k & \iff & \beta \leq \beta_k^U. \end{cases} \quad (34)$$

From (30) the first system in the middle column in (34) is equivalent to  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) \geq 0$ ,  $\forall \widehat{u} \in \widehat{\mathcal{C}}_k$ ,  $\forall \bar{u} \in \widehat{\mathcal{C}}_{k+n}$ ,  $\forall n \in \mathbb{I}_{\mathbb{L}-k}$ . Using (31), the second system in the middle column in (34) is equivalent

to  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0$ ,  $\forall \hat{u} \in \widehat{\mathcal{C}}_k$ ,  $\forall \bar{u} \in \widehat{\mathcal{C}}_{k-n}$ ,  $\forall n \in \mathbb{I}_k$ . The equivalence between the middle column and the third column in (34) follows from Definition 2. We have thus obtained for any  $k \in \mathbb{I}_L^0$  that

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0, \quad \forall \hat{u} \in \widehat{\mathcal{C}}_k, \quad \forall \bar{u} \in \widehat{\mathcal{C}} \quad \text{if and only if} \quad \begin{cases} \beta_k \leq \beta \leq \beta_k^U & \text{for } k \in \mathbb{I}_{L-1} \\ \beta_0 \leq \beta & \text{for } k = 0 \\ \beta \leq \beta_L^U & \text{for } k = L \end{cases}$$

For  $k = 0$ , the necessary and sufficient condition is  $\beta \geq \beta_0$  which is equivalent to  $\beta_0 \leq \beta < +\infty =: \beta_0^U$ . For  $k = L$ , the obtained necessary and sufficient condition is  $\beta \leq \beta_L^U$  which condition is the same as  $\beta_L := 0 < \beta \leq \beta_L^U$ . This proves (a).

(b) In the same way one proves that (33) holds if and only if all inequalities in (34) are replaced by strict inequalities. This establishes (b).  $\square$

### 3.2 Critical parameters for the global minimizers of $(\mathcal{R}_\beta)$

Since the global minimizers of  $\mathcal{F}_\beta$  are always in  $\widehat{\mathcal{C}}$  (Theorem 4), we are interested in the indexes  $k$  for which there exist values of  $\beta$  such that  $\mathcal{F}_\beta$  has global minimizers containing  $\widehat{\mathcal{C}}_k$ . Their set is obtained from Theorem 6. This set is split into  $J$  and  $J^E$  as it follows:

$$\boxed{J := \{k \in \mathbb{I}_L^0 \mid \beta_k < \beta_k^U\} \quad \text{and} \quad J^E := \{k \in \mathbb{I}_L^0 \mid \beta_k = \beta_k^U\}} \quad (35)$$

Set

$$p := \#J - 1 \quad \text{and} \quad J = \{k_0, k_1, \dots, k_p\} \quad \text{with} \quad k_{n-1} < k_n \quad \forall (k_{n-1}, k_n) \in J^2. \quad (36)$$

By Definition 2 and Remark 5 we have

$$\{k_0 = 0, k_p = L\} \in J^2 \quad \text{where} \quad \beta_{k_0}^U \equiv \beta_0^U = +\infty \quad \text{and} \quad \beta_{k_p} \equiv \beta_L = 0. \quad (37)$$

The next claim is a cautionary consequence of Theorems 4 and 6.

LEMMA 6. *Let H1 hold. One has  $\widehat{\mathcal{R}} \cap \widehat{\mathcal{C}}_k = \emptyset$  if and only if  $k \in \mathbb{I}_N^0 \setminus \{J \cup J^E\}$ .*

Clearly,  $\widehat{\mathcal{R}} = \bigcup_{k \in \{J \cup J^E\}} \widehat{\mathcal{C}}_k$ . A simplification of the parameters  $\{\beta_k, \beta_k^U\}_{k \in J \cup J^E}$  is derived next.

PROPOSITION 2. *Let H1 hold and  $\{\beta_k, \beta_k^U\}$  be as in Definition 2. Set  $\beta_{k_{-1}} := \beta_{k_0}^U = +\infty$ . Then:*

- (a)  $\{\beta_m \mid m \in J^E\} \subset \{\beta_{k_n} \mid k_n \in J \setminus \{k_p\}\}$  ;
- (b)  $\beta_{k_n} < \beta_{k_n}^U = \beta_{k_{n-1}}$  if and only if  $(k_{n-1}, k_n) \in J^2$  ;
- (c)  $\left( \bigcup_{n=0}^{p-1} [\beta_{k_n}, \beta_{k_{n-1}}] \right) \cup [\beta_{k_0}, \beta_{k_{-1}}] = \mathbb{R}_+$  .

*Proof.* From Theorem 6(b),  $\widehat{\mathcal{R}}_\beta = \widehat{\mathcal{C}}_{k_n}$  if and only if  $\beta \in (\beta_{k_n}, \beta_{k_n}^U)$ ; and from the definition of  $J$  in (35),  $(\beta_{k_n}, \beta_{k_n}^U) \neq \emptyset$  if and only if  $k_n \in J$ . This, together with Lemma 5 shows that

$$\beta_{k_n} < \beta_{k_n}^U \leq \beta_{k_{n-1}} < \beta_{k_{n-1}}^U \quad \text{if and only if} \quad (k_{n-1}, k_n) \in J^2. \quad (38)$$

Now we need to locate  $\beta_m, \forall m \in J^E$ . By (37),  $\beta_m \notin \{\beta_{k_0}^U, \beta_{k_p}\}$ . From Theorem 6(b),  $\beta_m \notin (\beta_{k_n}, \beta_{k_n}^U)$  for any  $k_n \in J$ . These facts, together with (37), show that

$$\forall m \in J^E \quad \exists k_n \in J \setminus \{k_p\} \quad \text{such that} \quad \beta_m \in [\beta_{k_n}^U, \beta_{k_{n-1}}] . \quad (39)$$

Since  $\widehat{R}_\beta \neq \emptyset$ , if  $\beta_{k_n}^U < \beta_{k_{n-1}}$  in (38), Proposition 1(c) implies that for  $\beta \in (\beta_{k_n}^U, \beta_{k_{n-1}}) \setminus \{\beta_m\}$  there is  $n \notin \{J \cup J^E\}$  obeying  $\widehat{C}_n \subset \widehat{R}_\beta$ , in contradiction to Lemma 6. Then (38) and (39) become

$$\beta_{k_n} < \beta_{k_n}^U = \beta_{k_{n-1}} < \beta_{k_{n-1}}^U \quad \text{if and only if} \quad (k_{n-1}, k_n) \in J^2 , \quad (40)$$

$$\forall m \in J^E \quad \exists k_n \in J \setminus \{k_p\} \quad \text{such that} \quad \beta_m = \beta_{k_n} .$$

These results prove (a) and (b). Together (a) and (b) lead to (c).  $\square$

It is worth emphasizing that  $\{\beta_k\}_{k \in J}$  is strictly decreasing and that its first entry is  $\beta_0$ .

In Example 2 we designed a ‘‘pathological’’ sequence  $\{\theta_k\}_{k=0}^L$  that illustrates all singular cases that could occur (in particular, the case  $\beta_k = \beta_k^U$  in Definition 2 which is unlikely by Proposition 4).

EXAMPLE 2. Let  $\{\theta_k\}_{k=0}^L$  for  $L = 6$  reads as

$$\theta_0 = 39 \quad \theta_1 = 33 \quad \theta_2 = 23 \quad \theta_3 = 15 \quad \theta_4 = 11 \quad \theta_5 = 3 \quad \theta_6 = 0 . \quad (41)$$

According to Definition 2 the sequences  $\{\beta_k, \beta_k^U\}_{k=0}^6$  are given by

$$\begin{aligned} \beta_0 &= \mathbf{8} & \beta_1 &= 10 & \beta_2 &= 8 & \beta_3 &= \mathbf{6} & \beta_4 &= 8 & \beta_5 &= \mathbf{3} & \beta_6 &= \mathbf{0} \\ \beta_0^U &= +\infty & \beta_1^U &= 6 & \beta_2^U &= 8 & \beta_3^U &= 8 & \beta_4^U &= 4 & \beta_5^U &= 6 & \beta_6^U &= 3 \end{aligned} \quad (42)$$

Then the sets  $J$  and  $J^E$  in (35) are given by  $J = \{0, 3, 5, 6\}$  and  $J^E = \{2\}$ . Obviously, Proposition 2 holds. Further, it is easily seen that  $\{\beta_k\}_{k \in J} = \{8, 6, 3, 0\}$  is the largest strictly decreasing subsequence of  $\{\beta_k\}_{k=0}^6$  with  $\beta_0$  as its first entry.  $\diamond$

The proofs of the other statements in this subsection are delegated to Appendix 9.2.

We want to know how  $\{\beta_k\}_{k \in \mathbb{I}_L^0}$  is related to  $\{\beta_k\}_{k \in J}$ . It worths to recall that  $0 \in J$  (Remark 82).

PROPOSITION 3. *Let H1 hold and  $\{\beta_k\}_{k=0}^L$  reads as in Definition 2. Then  $\{\beta_k\}_{k \in J}$  is the largest strictly decreasing subsequence of  $\{\beta_k\}_{k=0}^L$  whose first entry is  $\beta_0$ .*

In order to better clarify the repartition of  $\{\beta_m \mid m \in J^E\}$ , we examine the (possibly empty) subsets

$$J_{k_n}^E := \{m \in J^E \mid \beta_m = \beta_{k_n}\} \quad \forall k_n \in J . \quad (43)$$

LEMMA 7. *Let H1 hold. The sets  $J_{k_n}^E$  in (43) fulfill  $J_{k_n}^E = \emptyset$ , and for any  $k_n \in J \setminus \{L\}$*

$$J_{k_n}^E = \{m \in J^E \mid k_n < m < k_{n+1}\} . \quad (44)$$

In view of Definition 2, the intuition suggests that the set  $J^E$  should be “very small”.

PROPOSITION 4. *Let H1 hold and  $\{\beta_k, \beta_k^U\}_{k=0}^L$  be as in Definition 2. There exists a finite union of vector subspaces of dimension  $\leq M - 1$ , denoted by  $S$ , such that*

$$d \in \mathbb{R}^M \setminus S \implies \beta_k \neq \beta_k^U \quad \forall k \in \mathbb{I}_L^0.$$

Data generically live in  $\mathbb{R}^M \setminus S$ . So  $\beta_k \neq \beta_k^U \quad \forall k \in \mathbb{I}_L^0$  in Definition 2 and  $J^E = \emptyset$  hold generically.

## 4 Quasi-equivalences between the optimal sets of $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

### 4.1 The general case: necessary and sufficient conditions

From Definition 2 and Remark 5, the set  $J$  in (35)-(36) cannot be empty. Our main result follows.

THEOREM 7. *Let H1 hold,  $\{\beta_k\}_{k=0}^L$  be as in Definition 2 and  $J$  as in (35)-(36). Set  $\beta_{k_{-1}} := \beta_{k_0}^U = +\infty$ .*

$$\widehat{C}_{k_n} = \left\{ \widehat{R}_\beta \mid \beta \in (\beta_{k_n}, \beta_{k_{n-1}}) \right\} \quad (45)$$

$$\begin{aligned} \text{and for } k_n \neq L \quad \widehat{R}_{\beta_{k_n}} &= \widehat{C}_{k_n} \cup \widehat{C}_{k_{n+1}} \cup \left( \bigcup_{m \in J_{k_n}^E} \widehat{C}_m \right) \text{ where } J_{k_n}^E = \{m \in J^E \mid \beta_m = \beta_{k_n}\} \\ \text{and } \widehat{C}_k \cap \widehat{C}_n &= \emptyset \quad \forall (k, n) \in \left( \{k_n, k_{n+1}\} \cup J_{k_n}^E \right)^2, \quad k \neq n, \end{aligned} \quad (46)$$

if and only if  $\{k_n\}_{n \geq 0} = J$ .

*Proof.* By Proposition 2, for any  $k_n \in J$  one has  $\beta_{k_n} = \beta_m$ ,  $\forall m \in J_{k_n}^E$  where  $J_{k_n}^E$  is given in (43). Applying Theorem 6(a) yields  $\widehat{C}_{k_n} \cup \widehat{C}_{k_{n+1}} \cup \left( \bigcup_{m \in J_{k_n}^E} \widehat{C}_m \right) \subseteq \widehat{R}_{\beta_{k_n}}$ . Further,  $\beta_{k_n} \neq \beta_n \quad \forall n \in \{J \cup J^E\} \setminus \{k_n \cup J_{k_n}^E\}$  and  $\widehat{C}_n \cap \widehat{R}_{\beta_{k_n}} = \emptyset \quad \forall n \notin \{J \cup J^E\}$  by Lemma 6. Hence the equality for  $\widehat{R}_{\beta_{k_n}}$  in (46). The last result in (46) comes from Corollary 1.

Proposition 2(b) shows that  $(\beta_{k_n}, \beta_{k_n}^U) = (\beta_{k_n}, \beta_{k_{n-1}}) \neq \emptyset$  if and only if  $\{k_n\}_{n \geq 0} = J$ . Then Theorem 6(b) implies that (45) holds if and only if  $\{k_n\}_{n \geq 0} = J$ .  $\square$

Theorem 7 shows that a quasi-equivalence between problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  always exists – as illustrated in (8). For the  $\sharp J - 1$  isolated values  $\{\beta_k\}_{k \in J \setminus \{L\}}$  problem  $(\mathcal{R}_\beta)$  has typically two (and up to  $1 + k_{n+1} - k_n$  if  $J_{k_n}^E \neq \emptyset$ ) optimal solutions.

REMARK 6. In view of Proposition 3, it will be interesting to have some statistical knowledge on  $\{\theta_k\}$  that could help to estimate the largest strictly decreasing subsequence of  $\{\beta_k\}_{k=0}^L$ .  $\diamond$

EXAMPLE 3. [Continuation of Example 2] Let  $\{\beta_k\}_{k=0}^6$ ,  $J$  and  $J^E$  be as in Example 2. We recall that  $J = \{0, 3, 5, 6\}$  and that  $J^E = \{2\}$ , so  $J_0^E = \{2\}$  and  $J_k^E = \emptyset$  otherwise. By Theorem 7 one has

$$\widehat{C}_0 = \left\{ \widehat{R}_\beta \mid \beta \in (8, +\infty) \right\} \quad \widehat{C}_3 = \left\{ \widehat{R}_\beta \mid \beta \in (6, 8) \right\} \quad \widehat{C}_5 = \left\{ \widehat{R}_\beta \mid \beta \in (3, 6) \right\} \quad \widehat{C}_6 = \left\{ \widehat{R}_\beta \mid \beta \in (0, 3) \right\}$$

$$\text{and } \widehat{R}_{\beta=8} = \widehat{C}_0 \cup \widehat{C}_2 \cup \widehat{C}_3 \quad \widehat{R}_{\beta=6} = \widehat{C}_3 \cup \widehat{C}_5 \quad \widehat{R}_{\beta=3} = \widehat{C}_5 \cup \widehat{C}_6 .$$

The next Corollary 3 is easy to verify using  $\{\theta_k\}_{k=0}^6$  and the relevant set  $J$  in Example 2.  $\diamond$

The statement below contains important precisions on the parameter values in Definition 2.

**COROLLARY 3.** *Let H1 hold and  $J$  be as in (35)-(36). The parameters  $(\beta_{k_n}, \beta_{k_n}^U)$  in Definition 2 obey*

$$\beta_{k_n} = \frac{\theta_{k_n} - \theta_{k_{n+1}}}{k_{n+1} - k_n} \quad \text{if } k_n < L \quad \text{and} \quad \beta_L = 0 , \quad (47)$$

$$\text{and} \quad \beta_{k_n} < \beta_{k_n}^U = \beta_{k_{n-1}} \quad \text{with} \quad \beta_{k_{-1}} := \beta_0^U = +\infty , \quad (48)$$

if and only if  $\{k_n\}_{n \geq 0} = J$ .

*Proof.* Let  $\widehat{u} \in \widehat{C}_{k_n}$  and  $\widehat{u} \in \widehat{C}_{k_{n+1}}$  for  $k_n \in J \setminus \{L\}$ . Theorem 7 shows that  $\mathcal{F}_{\beta_{k_n}}(\widehat{u}) = \mathcal{F}_{\beta_{k_n}}(\widehat{u})$ , i.e.,

$$\theta_{k_n} + \beta_{k_n} k_n = \theta_{k_{n+1}} + \beta_{k_n} k_{n+1} .$$

Hence the value of  $\beta_{k_n}$  in (47). The second result in (47) follows from Proposition 2(b).

For any  $k \in \mathbb{I}_L^0 \setminus J$ , (47)-(48) cannot hold by the definitions of  $J$  and of  $J^E$ .  $\square$

For any  $m \in J_{k_n}^E$ , see (43), we also have  $\mathcal{F}_{\beta_m}(\bar{u}) = \mathcal{F}_{\beta_{k_n}}(\widehat{u})$  for any  $\bar{u} \in \widehat{C}_m$  since  $\beta_m = \beta_{k_n}$ .

## 4.2 Full quasi-equivalence: necessary and sufficient conditions

Here we explore the case of full quasi-equivalence between  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ , i.e.  $J = \mathbb{I}_L^0$ .

**PROPOSITION 5.** *Let H1 hold. Let  $J$  be defined according to (35).*

(a) *If the sequence  $\{\beta_k\}_{k=0}^L$  in Definition 2 is strictly decreasing, then its entries read as*

$$\boxed{\beta_k = \theta_k - \theta_{k+1} \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \text{and} \quad \beta_L = 0} \quad (49)$$

(b)  $J = \mathbb{I}_L^0$  *if and only if the sequence  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing.*

*Proof.* (a) Since  $\{\beta_k\}_{k=0}^L$  is strictly decreasing, Proposition 3 shows that the set  $J$  in (35) is given by  $J = \mathbb{I}_L^0$ . Applying (47) in Corollary 3 with  $k_n = k$  and  $k_{n+1} = k + 1$  delivers the formula in (49).

(b) Assume that  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing. Then  $\beta_k$  in (49) satisfy (47) for any  $k \in J = \mathbb{I}_L^0$ . By setting  $\beta_k^U := \beta_{k-1}$ , (48) holds for any  $(k, k-1) \in J^2$ . Hence,  $J = \mathbb{I}_L^0$  by Corollary 3.

Conversely, let  $J = \mathbb{I}_L^0$ . Then  $J = \{k \in \mathbb{I}_L^0 \mid \beta_k < \beta_k^U\} = \mathbb{I}_L^0$  where  $\{\beta_k, \beta_k^U\}_{k=0}^L$  are as in Definition 2. Applying Proposition 2(b) to  $\{\beta_k\}_{k \in J}$  shows that

$$\beta_k < \beta_k^U = \beta_{k-1} \quad \forall (k-1, k) \in J^2 = (\mathbb{I}_L^0)^2 ;$$

i.e.,  $\{\beta_k\}_{k=0}^L$  in Definition 2 is strictly decreasing. Hence  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing by (a).  $\square$

Proposition 5(a) can also be proven using induction.

THEOREM 8. *Let H1 hold. Let  $\{\beta_k\}_{k=0}^L$  read as in (49) and set  $\beta_{-1} = +\infty$ . Then*

$$\forall k \in \mathbb{I}_L^0 \quad \widehat{C}_k = \left\{ \widehat{R}_\beta \mid \beta \in (\beta_k, \beta_{k-1}) \right\} \quad (50)$$

$$\text{and} \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \widehat{R}_{\beta_k} = \widehat{C}_k \cup \widehat{C}_{k+1} \quad \text{where} \quad \widehat{C}_k \cap \widehat{C}_{k+1} = \emptyset, \quad (51)$$

*if and only if the sequence  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing.*

*Proof.* Let  $\{\beta_k\}_{k=0}^L$  in (49) be strictly decreasing. Then  $J = \mathbb{I}_L^0$  by Lemma 5(b). So,  $J^E = \emptyset$  in (35) and  $J_k^E = \emptyset$  in Lemma 7. The results in (50) and in (51) follow directly from Theorem 7.

Conversely, (50)-(51) means  $J = \mathbb{I}_L^0$ . Then  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing by Proposition 5(b).  $\square$

REMARK 7. [Continuation of Remark 6] The condition that  $\{\beta_k\}_{k=0}^L$  in (49) is strictly decreasing reads as

$$\beta_{k-1} > \beta_k \quad \forall k \in \mathbb{I}_{L-1} \quad \iff \quad \theta_k < \frac{1}{2} (\theta_{k-1} + \theta_{k+1}) \quad \forall k \in \mathbb{I}_{L-1}.$$

Its realization depends on  $A$  and on  $d$ . We refer to section 7 for some numerical tests.

## 5 On the optimal values of $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

The proofs of the statements in this section are outlined in Appendix 9.3.

REMARK 8. [Continuation of Remark 3.] For  $k \in \mathbb{I}_L^0$  let  $\widehat{u} \in \widehat{C}_k$ ; set  $\widehat{\sigma} := \text{supp}(\widehat{u})$ . Using Theorem 1 and the notation in (21),  $\widehat{\sigma} \in \Omega_k$ . Denoting by  $\Pi_{\widehat{\sigma}}$  the orthogonal projector onto  $\text{range}(A_{\widehat{\sigma}})$ ,

$$\theta_k = d^T (I - \Pi_{\widehat{\sigma}}) d \quad \text{where} \quad \Pi_{\widehat{\sigma}} = A_{\widehat{\sigma}} (A_{\widehat{\sigma}}^T A_{\widehat{\sigma}})^{-1} A_{\widehat{\sigma}}^T. \quad (52)$$

The optimal value  $\theta_k$  in (22), Remark 3, equivalently reads as  $\theta_k = \min \{ d^T (I - \Pi_\omega) d \mid \omega \in \Omega_k \}$ .  $\diamond$

Using the definition of  $\Omega_k$  in (21), we introduce the subsets of  $\mathbb{R}^M$  given below:

$$E_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega)^\perp \quad \text{and} \quad G_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega). \quad (53)$$

Clearly,  $E_0 = G_M = \mathbb{R}^M$  and  $E_M = G_0 = \emptyset$  by H1.

The next Proposition 6 gives interesting results on  $\{\theta_k\}_{k=0}^M$  in connection with  $d \in \mathbb{R}^M$ .

PROPOSITION 6. *Let H1 hold. Let  $L' \leq M$  be arbitrarily fixed. Then*

- (a)  $\theta_k > 0 \quad \forall k \leq L' - 1 \quad \iff \quad d \in \mathbb{R}^M \setminus G_{L'-1};$
- (b)  $d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \quad \implies \quad \theta_{k-1} > \theta_k \quad \forall k \in \mathbb{I}_{L'}.$

Proposition 6(a) shows that the constant  $L$  in (19) corresponds to  $d \in G_L \setminus G_{L-1}$ .

REMARK 9. The subsets  $E_2$  and  $G_{M-1}$  are finite unions of vector subspaces of dimensions  $M - 2$  and  $M - 1$ , respectively. Hence,  $d \in \mathbb{R}^M \setminus (E_2 \cup G_{M-1})$  is a generic property.



Therefore,  $\{\theta_k\}_{k=0}^M$  is strictly decreasing and  $L = M$  generically.  $\diamond$

By Proposition 1(a) and Theorem 4, for any  $\beta > 0$  the optimal value of problem  $(\mathcal{R}_\beta)$  in (6) reads as

$$f_\beta = \min \{ \theta_k + \beta k \mid k \in \mathbb{I}_L^0 \} . \quad (54)$$

The claim below is a direct consequence of this observation and of Theorem 7.

**COROLLARY 4.** *Let H1 hold and J read as in (35). The application  $\beta \mapsto f_\beta : (0, +\infty) \rightarrow \mathbb{R}$  fulfills*

$$(a) \quad \begin{cases} f_\beta = \theta_{k_n} + \beta k_n \\ = \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \widehat{\mathcal{C}}_{k_n} \end{cases} \quad \text{if and only if} \quad \beta \in \begin{cases} [\beta_{k_0}, +\infty) & \text{for } k_0 = 0 ; \\ [\beta_{k_n}, \beta_{k_{n-1}}] & \text{for } k_n \in J \setminus \{0, L\} ; \\ (0, \beta_{k_{p-1}}] & \text{for } k_p = L . \end{cases}$$

(b)  $\beta \mapsto f_\beta$  is continuous and concave.

(c)  $f_{\beta_{k_{n-1}}} > f_{\beta_{k_n}} \quad \forall (k_{n-1}, k_n) \in (J)^2$ ,  $f_{\beta_{k_0}} = \theta_{k_0} = f_\beta \quad \forall \beta \geq \beta_{k_0}$  and  $f_{\beta_{k_0}} > f_\beta \quad \forall \beta < \beta_{k_0}$ .

$f_\beta$  is affine increasing on each interval  $(\beta_{k_n}, \beta_{k_{n-1}})$  with upward kinks at  $k_n \in J \setminus \{L\}$  and bounded by  $\theta_0$ .

## 6 Geometry of the optimal sets of $(\mathcal{R}_\beta)$ and of $(\mathcal{C}_k)$

For the convex surrogates of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  where  $\|u\|_0$  is replaced by  $\|u\|_1$ , it is well known that the optimal sets are convex closed with possibly a a continuum of solutions.

**THEOREM 9.** *Let H1 hold. For any  $\beta > 0$  and for any  $k \in \mathbb{I}_L^0$ , each optimal set  $\widehat{\mathcal{R}}_\beta$  and  $\widehat{\mathcal{C}}_k$  is finite.*

*Proof.* From Remark 3,  $\hat{u} \in \widehat{\mathcal{C}}_k$  solves problem  $(\mathcal{P}_\omega)$  for  $\omega := \text{supp}(\hat{u})$  where  $\omega \in \Omega_k$  (Theorem 1 and the notation in (21)). Since  $(\mathcal{P}_\omega)$  has a unique solution for each  $\omega \in \Omega_k$  (given by  $(A_\omega^T A_\omega)^{-1} A_\omega^T d$ ), it follows that this  $\hat{u}$  is unique. Therefore, if  $\widehat{\mathcal{C}}_k$  is not a singleton, then each optimal solution belonging to  $\widehat{\mathcal{C}}_k$  is the unique solution of  $(\mathcal{P}_\omega)$  for a different  $\omega \in \Omega_k$ . Hence  $\#\widehat{\mathcal{C}}_k$  is finite (with  $\#\widehat{\mathcal{C}}_k \leq \#\Omega_k$ ). This, together with Theorem 7, shows that  $\#\widehat{\mathcal{R}}_\beta$  is finite for every  $\beta > 0$ .  $\square$

For any  $\beta > 0$  and  $k \in \mathbb{I}_L^0$  the optimal sets of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are composed out of a certain finite number of isolated points.

### 6.1 Uniqueness of the optimal solutions of $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

If  $L = M$ , Remark 3 shows that  $\#\widehat{\mathcal{C}}_M = \#\Omega_M$  and by Remark 5,  $M \in J$  in which case Theorem 7 yields  $\#\{\widehat{\mathcal{R}}_\beta \mid \beta \in (0, \beta_{k^*}]\} > \#\widehat{\mathcal{C}}_M$  where  $\beta_{k^*}$  is the next to last entry of  $\{\beta_k\}_{k \in J}$ ; see (36).

Let  $k \leq \min\{L, M - 1\}$  and  $(\hat{u}, \tilde{u}) \in (\widehat{\mathcal{C}}_k)^2$  for  $\hat{u} \neq \tilde{u}$ . Set  $\hat{\sigma} := \text{supp}(\hat{u})$  and  $\tilde{\sigma} := \text{supp}(\tilde{u})$ . By Theorem 1,  $(\hat{\sigma}, \tilde{\sigma}) \in (\Omega_k)^2$ . Then

$$\theta_k = \|A_{\hat{\sigma}} \hat{u}_{\hat{\sigma}} - d\|^2 = \|A_{\tilde{\sigma}} \tilde{u}_{\tilde{\sigma}} - d\|^2 \quad \text{where } \hat{\sigma} \neq \tilde{\sigma} .$$

The expression for  $\theta_k$  in (52) shows that

$$\|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 - \|A_{\tilde{\sigma}}\tilde{u}_{\tilde{\sigma}} - d\|^2 = d^T (\Pi_{\tilde{\sigma}} - \Pi_{\hat{\sigma}}) d = 0. \quad (55)$$

The last equality in (55) suggests that  $\hat{C}_k$  can be a singleton under the assumption  $H^*$  below.

**$H^*$ .** For  $K \leq M - 1$  fixed,  $A \in \mathbb{R}^{M \times N}$  obeys  $\Pi_{\omega} \neq \Pi_{\bar{\omega}} \quad \forall (\omega, \bar{\omega}) \in (\Omega_k)^2, \quad \omega \neq \bar{\omega} \quad \forall k \in \mathbb{I}_K.$

$H^*$  is a *generic property* (Definition 1) of all matrices in  $\mathbb{R}^{M \times N}$  [32, Theorem 5.3]. If  $H^*$  holds, the set

$$\Delta_K := \bigcup_{k=1}^K \bigcup_{(\omega, \bar{\omega}) \in (\Omega_k)^2} \{g \in \mathbb{R}^M \mid \omega \neq \bar{\omega} \text{ and } g \in \ker(\Pi_{\bar{\omega}} - \Pi_{\omega})\}$$

is a finite union of vector subspaces of dimension  $\leq M - 1$ , so *data generically live in*  $\mathbb{R}^M \setminus \Delta_K$ .

Under the *generic assumptions*  $H^*$  and  $d \in \mathbb{R}^M \setminus \Delta_K$ , (55) shows that problem  $(\mathcal{C}_k) \forall k \in \mathbb{I}_K$  has a unique optimal solution. Using that  $\{\beta_k\}_{k \in J}$  is strictly decreasing, we set  $K' := \max\{k \in J \mid k \leq K\}$ . Under the assumptions given above, Theorem 7 implies that problem  $(\mathcal{R}_\beta)$  has a unique optimal solution for any  $\beta \in (\beta_{K'}, +\infty) \setminus \{\beta_k\}_{k \in J}$  and hence *generically for any*  $\beta > \beta_{K'}$  since  $\{\beta_k\}_{k \in J}$  is finite.

For  $k \leq M - 1$  and for  $\beta > \beta_{k^*}$ , where  $\beta_{k^*}$  is the next to last entry of  $\{\beta_k\}_{k \in J}$ , problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are generically singletons.

For the sake of generality, we did not consider the assumptions evoked in this subsection.

## 7 Numerical tests

Here we present two kind of experiments using matrices  $A \in \mathbb{R}^{M \times N}$  for  $(M, N) = (5, 10)$ , original vectors  $u^\circ \in \mathbb{R}^N$  and data samples  $d = Au^\circ(+\text{noise})$  with *two different goals*:

- to get a rough idea on behaviour of the parameters  $\beta_k$  in Definition 2;
- to verify and illustrate our theoretical findings.

All results were calculated using an exhaustive combinatorial search.

### 7.1 Monte Carlo experiments on $\{\beta_k\}$ with $10^5$ tests for $(M, N) = (5, 10)$

We realized two experiments, each one composed of  $10^5$  trials with  $(M, N) = (5, 10)$ . In each trial, the “original”  $u^\circ \in \mathbb{R}^N$  had a random support on  $\{1, \dots, N\}$  satisfying  $\|u^\circ\|_0 \leq M - 1 = 4$  with mean 3.79. The coefficients of each  $A$  and the non-zero entries of each  $u^\circ$  were independent and identically distributed (i.i.d.). Data were obtained as  $d = Au^\circ +$  i.i.d. centered Gaussian noise. In each trial we computed the exact optimal values  $\{\theta_k\}$  and then computed  $(\beta_k, \beta_k^U)$  according to Definition 2. We considered two different distributions for  $A$  and for the non-zero entries of  $u^\circ$ .

- **Experiment  $\mathcal{N}(0, 10)$ .** All coefficients of each  $A$  and all non-zero entries of  $u^\circ$  had a normal distribution with mean 0 and variance 10. The SNR in dB was in  $[10, 61]$  with mean value 33.75 dB.
- **Experiment Uni  $[0, 10]$ .** The coefficients of  $A$  and of  $u_{\text{supp}(u^\circ)}^\circ$  were uniform on  $[0, 10]$ . We had SNR in  $[20, 55]$  with a mean of 28.95 dB.

**Observations** In these experiments, the following facts were observed:

- We had  $L = M$  in each trial which confirms Proposition 6(b) and Remark 9;
- $\{\theta_k\}_{k=0}^M$  was always strictly decreasing – as expected from Proposition 6;
- We never found  $\beta_m = \beta_m^U$ , so the set  $J^E$  in (35) was always empty; see Proposition 4.

The other results in percentage are shown in Tab. 1 where  $N_k$  reads as

$$N_k := \# \{k \in \mathbb{I}_M^0 \mid \beta_k > \beta_{k-1}\} . \quad (56)$$

In both experiments, the sequence  $\{\beta_k\}_{k=0}^M$  in Definition 2 was strictly decreasing in a huge amount

Table 1: Results on the behaviour of  $\{\beta_k\}$  in Definition 2 for two experiments, each one composed of  $10^5$  random trials. For  $k \geq 3$  we had  $N_k = 0$ .

	$\beta_k < \beta_{k-1}, \forall k \in \mathbb{I}_M^0$	$N_k = 1$	$N_k = 2$	mean(SNR)
$\mathcal{N}(0, 10)$	<b>93.681 %</b>	6.254 %	0.065 %	33.75
Uni $[0, 10]$	<b>98.783 %</b>	1.216 %	0.001 %	28.95

of cases; by Proposition 5(a) in all these cases  $\{\beta_k\}_{k=0}^M$  equals the sequence in (49) and the full quasi-equivalence in Theorem 8 holds. One should suppose that these percentages are high because of the small size of the matrices. Anyway, these percentages clearly depend on the distribution of the coefficients.

## 7.2 Tests on the quasi-equivalence with a selected matrix and selected data

Next we present in detail three experiments for  $(M, N) = (5, 10)$  where

$$A = \begin{pmatrix} 13.94 & 16.36 & 4.88 & -3.09 & -15.42 & 1.31 & -3.18 & -12.13 & -4.26 & -10.09 \\ 7.06 & -6.48 & -9.07 & -8.37 & -2.72 & -17.42 & -5.83 & -3.81 & 3.87 & -1.80 \\ 11.63 & 6.73 & -4.75 & -6.28 & 3.42 & 6.68 & -1.64 & 13.23 & 9.03 & -20.27 \\ -7.54 & 12.74 & -6.66 & 5.01 & 4.84 & 8.98 & -9.35 & 3.85 & 7.18 & 4.09 \\ 3.22 & -10.40 & -5.02 & 16.70 & 9.53 & -5.49 & 11.88 & -3.62 & 17.36 & 7.34 \end{pmatrix}$$

$$u^o = (0 \ 4 \ 0 \ 0 \ 0 \ 9 \ 0 \ 0 \ 3 \ 0)^T . \quad (57)$$

The entries of  $A$  follow a nearly normal distribution. The coefficients of  $A$ ,  $u^o$ , and  $d$  in (58), (60) and (62) are *exact*. H1 holds since  $\text{rank}(A) = M = 5$ . Problem  $(\mathcal{C}_M)$  has  $\#\Omega_M = 252$  optimal solutions; none of them is shown. We have  $\beta_0 < \beta_0^U = +\infty$  (Remark 5), so  $\widehat{\mathcal{C}}_0 = \{\widehat{\mathcal{R}}_\beta \mid \beta > \beta_0\}$  in all cases (Theorem 6). In the tests presented below the optimal set of  $(\mathcal{C}_k)$  for  $k \leq M - 1$  is a singleton (see § 6.1).

In order to illustrate various cases of quasi-equivalence, we selected a “pathological” couple  $(A, u^o)$  in (57) that behaves very badly compared to Tab. 1. Results for  $10^5$  random trials are shown in Tab. 2.

Table 2: The behaviour of  $\{\beta_k\}$  in Definition 2 for an experiment with  $10^5$  trials where  $A$  and  $u^\circ$  are given by (57)  $d = Au^\circ +$  i.i.d. centered Gaussian noise. We had  $N_k = 0, \forall k \geq 3$ .

	$\ \beta_k < \beta_{k-1}, \forall k \in \mathbb{I}_M^0\ $	$N_k = 1$	$N_k = 2$	mean(SNR)
<b>A, <math>u^\circ</math> in (57)</b>	<b>29.41 %</b>	70.59 %	0 %	36.25

**Noise-free data** According to (57), data read as

$$d = Au^\circ = (64.45 \quad -171.09 \quad 114.13 \quad 153.32 \quad -38.93)^\top. \quad (58)$$

Since data are noise-free and  $\|u^\circ\|_0 = 3$ , clearly  $\hat{u} = u_0$  is an optimal solution to problems  $(\mathcal{C}_k)$  with  $\theta_k = 0$  for  $k \in \{3, 4, 5\}$  and  $L = 3$ . The other optimal values  $\theta_k$  are seen in Tab. 3. By Theorem 4, any  $\hat{u} \in \hat{\mathbf{R}}$  obeys  $\|\hat{u}\|_0 \leq 3$ . The critical parameters  $\{\beta_k\}$  by Definition 2 are

$$\beta_3 = 0 < \beta_3^U = \beta_1 = 3872.46 < \beta_1^U = \beta_0 = 63729 \quad \text{and} \quad \beta_2 = 3968 > \beta_2^U = 3776.82. \quad (59)$$

$\beta_k > \beta_k^U$  only for  $k = 2$ , so  $J = \{0, 1, 3\}$  in (35). By Lemma 6,  $\hat{\mathbf{R}} \cap \hat{\mathbf{C}}_k = \emptyset$  for  $k \in \{2, 4, 5\}$ . By Theorem 7,  $\hat{\mathbf{C}}_3 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_3, \beta_1)\}$  and  $\hat{\mathbf{C}}_1 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_1, \beta_0)\}$ . The numerical results are seen in Tab. 3.

Table 3: The optimal values  $\theta_k$  and the optimal sets of  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_3^0$  where  $d$  is as in (58). The values of  $\beta_k$  are given in (59). We recall that  $\hat{\mathbf{R}}_\beta$  is the optimal set of problem  $(\mathcal{R}_\beta)$ .

k	$\theta_k$	$\hat{\mathbf{C}}_k =$ the optimal solution of $(\mathcal{C}_k)$ , singleton	$\hat{\mathbf{C}}_k = \hat{\mathbf{R}}_\beta$
3	0	0 <b>4</b> 0 0 0 <b>9</b> 0 0 <b>3</b> 0	$\beta \in (\beta_3, \beta_1)$
2	3968	0 <b>3.25</b> 0 0 0 <b>9.29</b> 0 0 0 0	<b>no</b>
1	7745	0 0 0 0 0 <b>11.76</b> 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	71474	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

**Noisy data 1.** Data are corrupted with nearly normal, centered, i.i.d. noise and SNR= 32.32 dB:

$$d = (69.13 \quad -171.95 \quad 113.74 \quad 150.27 \quad -36.09)^\top. \quad (60)$$

The optimal values  $\theta_k$  of problems  $(\mathcal{C}_k)$  in Tab. 4 with  $\theta_5 = 0$  yield  $L = M = 5$ . From Definition 2,

$$\beta_5 = 0 < \beta_5^U = \beta_4 = 0.068 < \beta_4^U = \beta_3 = 36.25 < \beta_3^U = \beta_1 = 3987.68 < \beta_1^U = \beta_0 = 63154, \quad (61)$$

while  $\beta_2 = 4002.83 > \beta_2^U = 3972.54$ . Hence,  $J = \mathbb{I}_5^0 \setminus \{2\}$  in (35) and  $\{\beta_k\}_{k \in J}$  confirms Propositions 2 and 3. By Lemma 6,  $\hat{\mathbf{R}} \cap \hat{\mathbf{C}}_2 = \emptyset$  and by Theorem 7,  $\hat{\mathbf{C}}_5 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (0, \beta_4)\}$ ,  $\hat{\mathbf{C}}_4 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_4, \beta_3)\}$ ,  $\hat{\mathbf{C}}_3 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_3, \beta_1)\}$  and  $\hat{\mathbf{C}}_1 = \{\hat{\mathbf{R}}_\beta \mid \beta \in (\beta_1, \beta_0)\}$ . The numerical tests are shown in Tab. 4.

Table 4: The optimal values  $\theta_k$  and the optimal solutions of  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_4^0$  where  $d$  is given in (60). The values of  $\beta_k$  are given in (61). We recall that  $\widehat{\mathbf{R}}_\beta$  is the set of the global minimizers of  $\mathcal{F}_\beta$ .

k	$\theta_k$	$\widehat{\mathbf{C}}_k =$ the optimal solution of $(\mathcal{C}_k)$ , singleton	$\widehat{\mathbf{C}}_k = \widehat{\mathbf{R}}_\beta$
4	0.068	0 <b>4.40</b> 0 0 0 <b>8.71</b> <b>0.54</b> 0 <b>2.95</b> 0	$\beta \in (\beta_4, \beta_3)$
3	36.31	0 <b>4.09</b> 0 0 0 <b>8.88</b> 0 0 <b>3.01</b> 0	$\beta \in (\beta_3, \beta_1)$
2	4039	0 <b>3.33</b> 0 0 0 <b>9.17</b> 0 0 0 0	<b>no</b>
1	8012	0 0 0 0 0 <b>11.71</b> 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	71166	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

**Noisy data 2.** The noise is nearly normal, centered, i.i.d., SNR= 25.74 dB:

$$d = ( 66.67 \quad -169.08 \quad 101.56 \quad 149.38 \quad -39.50 )^T . \quad (62)$$

The optimal values  $\{\theta_k\}$  in Tab. 5 show that  $\mathbf{L} = \mathbf{M}$ . The sequence  $\{\beta_k\}$  by Definition 2 reads as

$$\beta_0 = 60287 \quad \beta_1 = 3825 \quad \beta_2 = 3037.1 \quad \beta_3 = 72.734 \quad \beta_4 = 0.0259 \quad \beta_5 = 0 . \quad (63)$$

This  $\{\beta_k\}$  is strictly decreasing and clearly it equals  $\{\beta_k\}$  in (49), as claimed in Proposition 5(a). From Theorem 8, problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are fully quasi-equivalent. This is confirmed by the tests in Tab. 5.

Table 5: The optimal values and solutions of  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_4$  where  $d$  is given in (62). Here  $\{\beta_k\}$  is strictly decreasing, see (63), so  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  are fully quasi-equivalent.

k	$\theta_k$	$\widehat{\mathbf{C}}_k =$ the optimal solution of $(\mathcal{C}_k)$ , singleton	$\widehat{\mathbf{C}}_k = \widehat{\mathbf{R}}_\beta$
4	0.0259	0 <b>8.54</b> 0 0 <b>4.59</b> <b>4.90</b> <b>2.73</b> 0 0 0	$\beta \in (\beta_4, \beta_3)$
3	72.76	0 <b>3.93</b> 0 0 0 <b>8.70</b> 0 0 <b>2.63</b> 0	$\beta \in (\beta_3, \beta_2)$
2	3109.86	0 <b>3.27</b> 0 0 0 <b>8.95</b> 0 0 0 0	$\beta \in (\beta_2, \beta_1)$
1	6934.85	0 0 0 0 0 <b>11.44</b> 0 0 0 0	$\beta \in (\beta_1, \beta_0)$
0	67222	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$

## 8 Conclusions and open questions

We have analyzed the relationship between the optimal solutions of least-squares constrained by k-sparsity (problem  $(\mathcal{C}_k)$  in (2)) and regularized by  $\|\cdot\|_0$  via a parameter  $\beta > 0$  (problem  $(\mathcal{R}_\beta)$  in (3)). These problems were shown to be quasi-equivalent in the sense explained in (d) and (8) in § 1.2. This issue was quite surprising for us. Other interesting results were listed in the same § 1.2 (Main contributions). Our theoretical findings pose intriguing questions, of both a theoretical and practical flavor.

- The obtained results can clarify a proper choice between models  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  in applications. If one needs optimal solutions with a fixed number of nonzero entries,  $(\mathcal{C}_k)$  is obviously the best choice. If only information on the perturbations is available,  $(\mathcal{R}_\beta)$  is a more flexible option.

- Many algorithms are built on good knowledge on the optimal solutions. One can expect our detailed results to give rise to innovative and efficient algorithms enabling one to compute relevant solutions.
- It will be interesting to obtain some statistical knowledge on the optimal values of problem  $(\mathcal{C}_k)$  that can help to clarify the largest strictly decreasing subsequence of critical parameter values. The Monte-Carlo tests in § 7.1 have shown that its length depends on the statistics of the matrix  $A$  and the data  $d$ .
- Extensions to penalties of the form  $\|Du\|_0$  for  $D$  a linear operator, or to low rank matrix recovery, seem important. The preliminary step should involve an extension of the knowledge on the optimal solutions of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  to these more complex penalties.

Hopefully, these problems will be addressed in future work.

## 9 Appendix

### 9.1 Proofs relevant to section 2

**Proof of Proposition 1.** Using  $(\mathcal{P}_\omega)$  in (13) and (15), the optimal value of  $(\mathcal{C}_k)$  for any  $k$  is given by

$$\theta_k = \inf \left\{ \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \mid \omega \in \Sigma_k \right\}.$$

For  $k \in \mathbb{I}_N^0$  and  $\omega \in \Sigma_k$ , define  $\theta^\omega \geq 0$  by

$$\theta^\omega := \|A\tilde{u} - d\|^2 \text{ where } \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Sigma_k. \quad (64)$$

The set of numbers  $\{\theta^\omega \mid \omega \in \Sigma_k\}$  is nonempty and finite. Then  $\theta_k = \min\{\theta^\omega \mid \omega \in \Sigma_k\}$  is well defined. By (64) there exists  $\hat{u} \in \mathbb{R}^N$  such that  $\|A\hat{u} - d\|^2 = \theta_k$ . Hence  $\hat{u} \in \hat{\mathcal{C}}_k$  and thus  $\hat{\mathcal{C}}_k \neq \emptyset$ .

**Proof of Lemma 2** Clearly,  $\theta_0 = \|d\|^2$ . Since  $\Sigma_{k-n} \subset \Sigma_k$ ,  $\forall n \in \mathbb{I}_k^0$  it follows from (13) and (17) that

$$\theta_k \leq \|Au - d\|^2 \quad \forall u \in \mathbb{R}^N \text{ such that } \text{supp}(u) \in \Sigma_{k-n}, \quad \forall n \in \mathbb{I}_k^0. \quad (65)$$

So,  $\theta_k \leq \theta_{k-n}$ ,  $\forall n \in \mathbb{I}_k$ . If H1 holds,  $\exists \omega \in \Sigma_M$  meeting  $\text{rank}(A_\omega) = M = \#\omega$ . Then  $\|A\hat{u} - d\|^2 = \theta_M = 0$  for  $\hat{u}$  given by  $\hat{u}_\omega = (A_\omega)^{-1}d$  and  $\hat{u}_{\mathbb{I}_N \setminus \omega} = 0$ . Hence  $\theta_k = 0$ ,  $\forall k \geq M$ .

**Proof of Lemma 3** Set  $\hat{\sigma} := \text{supp}(\hat{u})$ ; by (18),  $\#\hat{\sigma} = \|\hat{u}\|_0 = k - n$ . Define  $z \in \mathbb{R}^M$  by

$$z := A\hat{u} - d = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d, \quad (66)$$

where we recall that  $A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A\hat{u}$ . The proof is by contraposition. So, assume that  $A\hat{u} \neq d$ ; i.e.,

$$z \neq 0. \quad (67)$$

Since  $\hat{u}$  solves  $(\mathcal{P}_{\hat{\sigma}})$ , one has  $A_{\hat{\sigma}}^T(A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d) = 0$  which, together with (66) leads to  $A_{\hat{\sigma}}^T z = 0$ . Select a  $\tilde{\sigma} \subseteq \hat{\sigma}$  yielding  $\text{rank}(A_{\tilde{\sigma}}) = r := \text{rank}(A_{\hat{\sigma}})$ ; then  $r \leq k - n < M$ . By H1, there is  $\omega \subset \mathbb{I}_N$  such that  $\omega \supseteq \tilde{\sigma}$  and

$\text{rank}(A_\omega) = \mathbf{M} = \#\omega$ . Since  $\langle a_n, z \rangle = 0$  for any  $n \in \tilde{\sigma}$ ,  $\hat{\sigma} \setminus \tilde{\sigma} \not\subseteq \omega$  and  $A_\omega^T z \neq 0$  by (67), there is  $m \in \omega \setminus \hat{\sigma}$  obeying  $\langle a_m, z \rangle \neq 0$ . Clearly,  $a_m$  is linearly independent of  $\text{range}(A_{\hat{\sigma}})$ . One has

$$\hat{u}_m := \frac{\langle a_m, z \rangle}{\|a_m\|^2} \implies \hat{u}_m \langle a_m, z \rangle > 0.$$

Using the definition of  $z$  in (66), we obtain

$$\begin{aligned} \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} + a_m\hat{u}_m - d\|^2 - \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 &= \|z + a_m\hat{u}_m\|^2 - \|z\|^2 = \|a_m\|^2\hat{u}_m^2 - 2\hat{u}_m\langle z, a_m \rangle \\ &= \frac{\hat{u}_m}{\|a_m\|^2} (\langle a_m, z \rangle - 2\langle a_m, z \rangle) = -\frac{\hat{u}_m \langle a_m, z \rangle}{\|a_m\|^2} < 0, \end{aligned}$$

a contradiction to (18). Hence  $\|A\hat{u} - d\|^2 = 0$ . This combined with  $\text{supp}(\hat{u}) \in \Sigma_{k-n}$  yields  $\theta_{k-n} = 0$  and  $\hat{u} \in \widehat{\mathbf{C}}_{k-n}$ . For any  $m \geq n - k$  one has  $\theta_m = 0$  by Lemma 2 and  $\hat{u} \in \widehat{\mathbf{C}}_m$  because  $\Sigma_m \supset \Sigma_{k-n}$ .

### 9.1.1 Proofs of Theorem 1 and Corollary 1

We begin with two auxiliary claims.

**COROLLARY 5.** *Let H1 hold. Then  $\left[ k \in \mathbb{I}_L^0 \text{ and } \hat{u} \in \widehat{\mathbf{C}}_k \implies \|\hat{u}\|_0 = k \right]$ .*

*Proof.* The case  $k = 0$  being trivial we focus on  $k \in \mathbb{I}_L$ . Assume that  $\|\hat{u}\|_0 = k - n$  for  $n \geq 1$ . Then  $\theta_{k-n} = 0$  by Lemma 3 which contradicts the definition of  $L$  in (19) because  $k \leq L$ . Hence  $n = 0$ .  $\square$

**LEMMA 8.** *Let H1 hold. Consider that  $\hat{u} \in \widehat{\mathbf{C}}_k$  for  $k \in \mathbb{I}_L^0$ . Set  $\hat{\sigma} := \text{supp}(\hat{u})$ . Then*

$$\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma} \equiv \|\hat{u}\|_0. \quad (68)$$

*Proof.* One has  $\|\hat{u}\|_0 = k$  by Corollary 5. For  $k = 0$  (68) is obvious. Suppose that (68) fails for  $k \geq 1$ :

$$\text{rank}(A_{\hat{\sigma}}) \leq \#\hat{\sigma} - 1. \quad (69)$$

The rank-nullity theorem [29] entails that  $\dim \ker(A_{\hat{\sigma}}) = \#\hat{\sigma} - \text{rank}(A_{\hat{\sigma}}) \geq 1$ . We can take an arbitrary  $v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \setminus \{0\}$ , set  $v_{\mathbb{I}_N \setminus \hat{\sigma}} := 0$  and select an  $i \in \hat{\sigma}$  in order to define  $\tilde{u}$  by

$$\tilde{u} := \hat{u} - \hat{u}[i] \frac{v}{v[i]}.$$

Clearly,  $\tilde{u}[i] = 0$  and  $\hat{u}[i] \neq 0$ , so  $\tilde{\sigma} := \text{supp}(\tilde{u}) \subsetneq \hat{\sigma}$ , which leads to

$$\|\tilde{u}\|_0 = k - n \quad \text{for } n := \|\hat{u}\|_0 - \|\tilde{u}\|_0 \geq 1. \quad (70)$$

From  $v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \in \ker(A_{\hat{\sigma}})$  one has  $A\hat{u} = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}} \left( \hat{u}_{\hat{\sigma}} - v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \right) = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A_{\tilde{\sigma}}\tilde{u}_{\tilde{\sigma}} = A\tilde{u}$ . Then

$$\theta_k = \|A\hat{u} - d\|^2 = \|A\tilde{u} - d\|^2. \quad (71)$$

This, together with the fact that  $\text{supp}(\tilde{u}) \in \Sigma_k$  shows that  $\tilde{u} \in \widehat{\mathbf{C}}_k$ . Thus  $\tilde{u} \in \widehat{\mathbf{C}}_k$  and  $\|\tilde{u}\|_0 \leq k - 1$  by (70), in contradiction to Corollary 5. So the assumption in (69) fails.  $\square$

**Proof of Theorem 1.** Let  $\hat{u} \in \widehat{\mathbf{C}}_k$  for  $k \in \mathbb{I}_L^0$ . By Corollary 5 and Lemma 8,  $\text{rank}(A_{\hat{\sigma}}) = k = \|\hat{u}\|_0$  where  $\hat{\sigma} := \text{supp}(\hat{u})$ ; hence (20). The last claim follows from Lemma 3 and Corollary 5.

**Proof of Corollary 1** Let  $\hat{u} \in \widehat{\mathbf{C}}_k$  and  $\bar{u} \in \widehat{\mathbf{C}}_n$  for  $(k, n) \in (\mathbb{I}_L^0)^2$ ,  $k \neq n$ . By Theorem 1,  $\|\hat{u}\|_0 = k$  and  $\|\bar{u}\|_0 = n$ , hence the result.

### 9.1.2 Proofs relevant to subsection 2.2

Lemmas 9, 10 and 11 below help to prove Proposition 1. With the notation in (21), it is suitable to set

$$\Omega := \bigcup_{k=0}^M \Omega_k .$$

LEMMA 9. *Let  $d \in \mathbb{R}^M$  and  $\beta > 0$ . Then*

$$\hat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_\beta \iff \hat{u} \in U := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega\}. \quad (72)$$

*Proof.* Let  $\hat{u}$  is a strict (local) minimizers of  $\mathcal{F}_\beta$ . Then  $\hat{u}$  solves  $(\mathcal{P}_\omega)$  for  $\omega := \text{supp}(\hat{u})$ . By Theorem 3  $\omega \in \Omega$  and thus  $\hat{u} \in U$ . Conversely, any  $\hat{u} \in U$  is a strict (local) minimizer of  $\mathcal{F}_\beta$  by Corollary 2.  $\square$

Now we partition  $U$  in (72) as follows:

$$U = \bigcup_{k=0}^M U_k \quad \text{where} \quad U_k := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega \text{ and } \|\tilde{u}\|_0 = k\} . \quad (73)$$

LEMMA 10. *Let H1 be satisfied and  $L$  be as in (19). Then  $\hat{C}_k \subset U_k \quad \forall k \in \mathbb{I}_L^0$ .*

*Proof.* Let  $\hat{u} \in \hat{C}_k$  for  $k \in \mathbb{I}_L^0$ . Set  $\omega := \text{supp}(\hat{u})$ . The expression for  $\hat{C}_k$  in (23) and Theorem 1 show that  $\hat{u}$  solves  $(\mathcal{P}_\omega)$  for  $\omega \in \Omega_k \subset \Omega$  and that  $\|\hat{u}\|_0 = k$ . Hence  $\hat{u} \in U_k$ .  $\square$

LEMMA 11. *Let H1 hold,  $L$  be as in (19) and let  $\beta > 0$ .*

(a) *Let  $k \in \mathbb{I}_L^0$ . Then*

$$\mathcal{F}_\beta(\hat{u}) = \theta_k + \beta k \quad \forall \hat{u} \in \hat{C}_k ; \quad (74)$$

$$\mathcal{F}_\beta(\tilde{u}) > \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_k \setminus \hat{C}_k . \quad (75)$$

(b) *Let  $\hat{u} \in \hat{C}_L$ . If  $L \leq M - 1$ , then*

$$\mathcal{F}_\beta(\tilde{u}) > \theta_L + \beta L = \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_n \text{ for } \forall n \in \{L + 1, \dots, M\} ; \quad (76)$$

*and thus any  $\tilde{u} \in U_n$  for  $\forall n \in \{L + 1, \dots, M\}$  obeys  $\tilde{u} \notin \hat{R}_\beta$  for any  $\beta > 0$ .*

*Proof.* From the definition of  $U_k$  in (73), if  $U_k \neq \emptyset$ , then  $\|\tilde{u}\|_0 = k$  for any  $\tilde{u} \in U_k$ .

(a) Since  $k \in \mathbb{I}_L^0$ ,  $\hat{C}_k \subset U_k$  by Lemma 10. Any  $\hat{u} \in \hat{C}_k$  yields  $\|A\hat{u} - d\|^2 = \theta_k$ , hence (74). Any  $\tilde{u} \in U_k \setminus \hat{C}_k$  is not an optimal solution of  $(\mathcal{C}_k)$ , so  $\|A\tilde{u} - d\|^2 > \theta_k$ . Then  $\mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta k > \theta_k + \beta k = \mathcal{F}_\beta(\hat{u})$ .

(b) By the definition of  $L$ ,  $\theta_n = \theta_L = 0$ ,  $\forall n \geq L$ . It follows that for any  $\tilde{u} \in U_n$ ,  $\forall n \geq L + 1$  one has  $\mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta n > \theta_L + \beta L = \beta L$ . Clearly, such a  $\tilde{u}$  cannot be global minimizer of  $\mathcal{F}_\beta$ .  $\square$

**Proof of Proposition 1.** (a) The statement follows from Lemma 11(a).

(b-c) Let  $\hat{u} \in \hat{R}_\beta$ . Then  $\hat{u}$  is a strict minimizer of  $\mathcal{F}_\beta$  (Theorem 2(b)) and  $\hat{u} \in U$  by Lemma 9. Set



$k := \|\widehat{u}\|_0$ ; then  $\widehat{u} \in U_k$  according to (73). In addition,  $k \leq L$  because otherwise  $\widehat{u} \notin \widehat{R}_\beta$  by Lemma 11(b). Also,  $\widehat{u} \in \widehat{R}_\beta$  means that  $f_\beta = \mathcal{F}_\beta(\widehat{u})$  is the optimal value of problem  $(\mathcal{R}_\beta)$ . Then  $\widehat{u} \in \widehat{C}_k$  by Lemma 11(a). Further, by item (a) in this proposition,  $f_\beta = \mathcal{F}_\beta(\widehat{u})$  for any  $\widehat{u} \in \widehat{C}_k$ . Thus,  $\widehat{C}_k \subseteq \widehat{R}_\beta$ .

**Proof of Lemma 4.** The backward implication is obvious. We focus on the forward one. For  $k > L$ , one has  $\widehat{C}_k \cap \widehat{R}_\beta = \emptyset, \forall \beta > 0$  from Lemma 11(b). For  $k \in \mathbb{I}_L^0$ , we proceed by contraposition. Assume that  $\widehat{C}_k \cap \widehat{R}_\beta \neq \emptyset$ , i.e., there exists  $\widehat{u} \in \widehat{C}_k \cap \widehat{R}_\beta$ . Then  $\widehat{C}_k \subseteq \widehat{R}_\beta$  by Proposition 1(c).

**Proof of Theorem 4.** By Proposition 1(b), any  $\widehat{u} \in \widehat{R}_\beta$  satisfies  $\widehat{u} \in \widehat{C}_k$  for  $k \leq L$  and thus  $\widehat{u} \in \widehat{C}$ . Therefore,  $\widehat{R}_\beta \subset \widehat{C}$ . The same holds for any  $\beta > 0$  which proves the theorem.

**Proof of Lemma 5** By Remark 4,  $\widehat{C}_k \subseteq \widehat{R}_\beta$  implies  $\mathcal{F}_\beta(\widehat{u}) \leq \mathcal{F}_\beta(\bar{u}) \quad \forall \widehat{u} \in \widehat{C}_k \quad \forall \bar{u} \in \widehat{C}_{k+p}$ . Using Proposition 1(a) this inequality reads as  $\theta_k + \beta k \leq \theta_{k+p} + \beta(k+p)$ , which leads to

$$\beta \geq \frac{\theta_k - \theta_{k+p}}{p}. \quad (77)$$

On the other hand,  $\widehat{C}_{k+p} = \widehat{R}_{\beta'}$  entails  $\mathcal{F}_{\beta'}(\bar{u}) < \mathcal{F}_{\beta'}(\widehat{u}) \quad \forall \bar{u} \in \widehat{C}_{k+p} \quad \forall \widehat{u} \in \widehat{C}_k$ . Therefore

$$\theta_{k+p} + \beta'(k+p) < \theta_k + \beta'k \quad \Rightarrow \quad \beta' < \frac{\theta_k - \theta_{k+p}}{p}. \quad (78)$$

Comparing (78) and (77) proves the first part of the lemma. The proof of second one is similar.

## 9.2 Proofs relevant to subsection 3.2

**Proof of Lemma 6** The definition of  $J$  and  $J^E$  shows that

$$k \in \mathbb{I}_L^0 \quad \text{and} \quad \beta_k > \beta_k^U \quad \iff \quad k \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}.$$

By Theorem 6(a),  $\widehat{C}_k \not\subseteq \widehat{R}_\beta, \forall \beta > 0$ , if and only if  $k \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$ . By Lemma 4,  $\widehat{C}_k \not\subseteq \widehat{R}_\beta$  means  $\widehat{C}_k \cap \widehat{R}_\beta = \emptyset, \forall \beta > 0$ . This, together with  $\widehat{C}_k \cap \widehat{R} = \emptyset, \forall k \geq L+1$  by Theorem 4, proves the claim.

**Proof of Proposition 3** Let  $J^* \subset \mathbb{I}_L^0$  be such that  $\{\beta_k\}_{k \in J^*}$  is the largest strictly decreasing subsequence of  $\{\beta_k\}_{k=0}^L$  in Definition 2 containing  $\beta_0$ . From Proposition 2(b),  $J \subseteq J^*$ . To get a contradiction, assume that  $J \subsetneq J^*$ . For  $m \notin J$ , we consider that  $\{J \cup \{m\}\}$  is ordered in an increasing way.

- Let  $m \in J^E$  for  $J^E$  as given in (35). By Proposition 2(a) there is  $k_n \in J$  such that  $\beta_{k_n} = \beta_m$ . Then  $\{\beta_k \mid k \in J \cup \{m\}\}$  is not strictly decreasing, hence  $\{J \cup \{m\}\} \not\subseteq J^*$ .
- Otherwise, let  $J^* \subseteq \{J \cup \{m\}\}$  for  $m \in \mathbb{I}_L^0 \setminus \{J \cup J^E\}$ . Since  $\{\beta_k\}_{k \in J^*}$  is strictly decreasing and  $0 \in J$ , there are  $(k_{n-1}, k_n) \in (J)^2$  such that  $k_{n-1} < m < k_n$  and  $\beta_{k_n} < \beta_m < \beta_{k_{n-1}}$ . From Theorem 6(b) and Proposition 2(b), one has  $\widehat{R}_\beta = \widehat{C}_{k_n}$  if and only if  $\beta \in (\beta_{k_n}, \beta_{k_{n-1}})$ . It follows that (see Remark 4)

$$\mathcal{F}_{\beta_m}(\bar{u}) > \mathcal{F}_{\beta_m}(\widehat{u}) \quad \forall \bar{u} \in \widehat{C}_m \quad \forall \widehat{u} \in \widehat{C}_{k_n},$$

which by Proposition 1(a) reads as  $\theta_m + \beta_m m > \theta_{k_n} + \beta_m k_n$ . Consequently,

$$\beta_m < \frac{\theta_m - \theta_{k_n}}{k_n - m}.$$

Since  $k_n > m$ , Definition 2 shows that  $\beta_m \geq \frac{\theta_m - \theta_{k_n}}{k_n - m}$ . Hence the desired contradiction.

**Proof of Lemma 7** Let  $m \in J^E$ . Theorem 6(a) shows that  $\widehat{C}_m \subset \widehat{R}_{\beta_m}$ . By the definition of  $J$  and Theorem 6(b),  $\widehat{C}_{k_n} = \widehat{R}_\beta$  if and only if  $\beta_{k_n} < \beta < \beta_{k_n}^U$ . Assume that  $m < k_n$ . Then Lemma 5 shows that  $\beta_m > \beta$ . It follows that  $\beta_m > \beta_{k_n}$  and hence  $\beta_m \neq \beta_{k_n}$ . This, together with (43), yields

$$\{m \in J^E \mid m < k_n\} \cap J_{k_n}^E = \emptyset. \quad (79)$$

By Theorem 6(b) and Proposition 2(b),  $\widehat{C}_{k_{n+1}} = \widehat{R}_\beta$  if and only if  $\beta \in (\beta_{k_{n+1}}, \beta_{k_{n+1}}^U) = (\beta_{k_{n+1}}, \beta_{k_n})$ . Consider that  $m > k_{n+1}$ . Then Lemma 5 shows that  $\beta_m < \beta$  and thus  $\beta_m < \beta_{k_n}$ . Consequently,

$$\{m \in J^E \mid m > k_{n+1}\} \cap J_{k_n}^E = \emptyset. \quad (80)$$

Jointly (79), (80) and  $J \cap J^E = \emptyset$  prove (44). Finally,  $\beta_{k_p} \equiv \beta_L = 0$  in (37) shows that  $J_L^E = \emptyset$ .

**Proof of Proposition 4** For  $k \in \{0, L\}$ , see Remark 5. Consider  $S$  given by

$$S := \bigcup_{k=1}^{L-1} S_k \quad \text{where} \quad S_k := \bigcup_{n=1}^{L-k} \bigcup_{m=1}^k \bigcup_{\omega \in \Omega_k} \bigcup_{\bar{\omega} \in \Omega_{k+n}} \bigcup_{\widehat{\omega} \in \Omega_{k-m}} \ker(m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (n+m)\Pi_\omega). \quad (81)$$

Let  $k \in \mathbb{I}_{L-1}$ . Since  $\text{rank}(\Pi_\omega) = k$ ,  $\text{rank}(\Pi_{\bar{\omega}}) = k + n$  and  $\text{rank}(\Pi_{\widehat{\omega}}) = k - m$ , we are guaranteed that  $m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (n+m)\Pi_\omega \neq 0$  and thus  $\dim(\ker(m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (n+m)\Pi_\omega)) \leq M - 1$ . Hence,  $S$  is a finite union of vector subspaces of dimension  $\leq M - 1$ .

There exists  $n \in \{1, \dots, L - k\}$  such that  $\beta_k = \frac{\theta_k - \theta_{k+n}}{n}$ . By Theorem 1, there are  $\omega \in \Omega_k$  and  $\bar{\omega} \in \Omega_{k+n}$  obeying  $\theta_k = d^T(I - \Pi_\omega)d$  and  $\theta_{k+n} = d^T(I - \Pi_{\bar{\omega}})d$ . Similarly, there is  $m \in \{1, \dots, k\}$  satisfying  $\beta_k^U = \frac{\theta_{k-m} - \theta_k}{m}$  and  $\widehat{\omega} \in \Omega_{k-m}$  such that  $\theta_{k-m} = d^T(I - \Pi_{\widehat{\omega}})d$ . Assume that  $\beta_k = \beta_k^U$ , i.e.,

$$0 = \beta_k - \beta_k^U = \frac{d^T(\Pi_{\bar{\omega}} - \Pi_\omega)d}{n} - \frac{d^T(\Pi_\omega - \Pi_{\widehat{\omega}})d}{m} = \frac{d^T(m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (m+n)\Pi_\omega)d}{nm}.$$

Since the term between the big parentheses is non-null, it follows that

$$d \in \ker(m\Pi_{\bar{\omega}} + n\Pi_{\widehat{\omega}} - (m+n)\Pi_\omega) \subset S_k;$$

a contradiction to the assumption that  $d \notin S$ .

### 9.3 Proofs of the statements in section 5

#### 9.3.1 Proof of Proposition 6

We shall use the two lemmas stated below.

LEMMA 12. *Let H1 hold and let  $k \in \{1, \dots, M-1\}$ . Then*

$$d \in \mathbb{R}^M \setminus G_k \iff \theta_k > 0.$$

*Proof.* Let  $d \in \mathbb{R}^M \setminus G_k$  and let  $\hat{u} \in \hat{C}_k$ . Set  $\hat{\sigma} := \text{supp}(\hat{u})$ . By Theorem 1 and the notation  $\Omega_k$  in (21),  $\hat{\sigma} \in \Omega_k$ . Since  $(I - \Pi_{\hat{\sigma}})$  is the orthogonal projector onto  $(\text{range}(A_{\hat{\sigma}}))^\perp$ , one has

$$d \in \mathbb{R}^M \setminus G_k \implies d \notin \text{range}(A_{\hat{\sigma}}) \text{ and } d \neq 0 \implies \theta_k = \|A\hat{u} - d\|^2 = d^T(I - \Pi_{\hat{\sigma}})d > 0.$$

Conversely, let  $\theta_k > 0$ . If  $d \in G_k$ , there is  $\omega \in \Omega_k$  meeting  $d \in \text{range}(A_\omega)$ . For  $u_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d$  one has  $\|A_\omega u_\omega - d\|^2 = d^T(I - \Pi_\omega)d = 0$ , a contradiction to  $\theta_k > 0$ .  $\square$

LEMMA 13. *Let H1 hold and let  $E_k$  be given by (53). For any  $k \geq 2$  such that  $\theta_{k-1} > 0$  one has*

$$\theta_{k-1} > \theta_k \quad \forall d \in \mathbb{R}^M \setminus E_k.$$

*Proof.* Let  $\hat{u} \in \hat{C}_{k-1}$  and  $\theta_{k-1} > 0$ . Set  $\hat{\sigma} := \text{supp}(\hat{u})$  and denote by  $B_{\hat{\sigma}}$  a matrix whose columns form an orthonormal basis for  $A_{\hat{\sigma}}$ . From Theorem 1,  $\hat{\sigma} \in \Omega_{k-1}$ . By H1, there is  $n \in \mathbb{I}_N \setminus \hat{\sigma}$  such that  $\omega := \hat{\sigma} \cup \{n\} \in \Omega_k$ . Then there is  $b_k \in \text{range}(A_\omega)$  such that  $B_\omega = (B_{\hat{\sigma}}, b_k)$  forms an orthonormal basis for  $A_\omega$  (see, e.g., [26]). The orthogonal projectors onto  $\text{range}(A_{\hat{\sigma}})$  and  $\text{range}(A_\omega)$  are  $\Pi_{\hat{\sigma}} = B_{\hat{\sigma}} B_{\hat{\sigma}}^T$  and  $\Pi_\omega = B_\omega B_\omega^T = B_{\hat{\sigma}} B_{\hat{\sigma}}^T + b_k b_k^T$ , respectively. Clearly,  $\theta_k \leq T_k := d^T(I - \Pi_\omega)d$ . Applying (52) yields

$$\theta_{k-1} - \theta_k \geq \theta_{k-1} - T_k = d^T(\Pi_\omega - \Pi_{\hat{\sigma}})d = \langle b_k, d \rangle^2.$$

Since  $d \in \mathbb{R}^M \setminus E_k$ , one has  $d \neq 0$  and  $d \notin (\text{range}((B_{\hat{\sigma}}, b_k)))^\perp$ . Hence  $\langle b_k, d \rangle^2 > 0$ .  $\square$

REMARK 10. Let H1 hold. If  $d \neq 0$ , there is  $n \in \mathbb{I}_N$  meeting  $\langle a_n, d \rangle \neq 0$ . Set  $\hat{u}_n = \arg \min_{v \in \mathbb{R}} \|a_n v - d\|^2$ . Then

$$\hat{u}_n = \frac{\langle a_n, d \rangle}{\|a_n\|^2} \neq 0 \quad \text{and} \quad \theta_1 \leq \|a_n \hat{u}_n - d\|^2 = \|d\|^2 - \frac{\langle a_n, d \rangle^2}{\|a_n\|^2} < \theta_0 = \|d\|^2.$$

**Proof of Proposition 6** (a) Let  $d \in \mathbb{R}^M \setminus G_{L'-1}$ . By Lemma 12,  $\theta_{L'-1} > 0$ . Since  $\{\theta_k\}$  is decreasing (Lemma 2), one has  $\theta_k > 0 \forall k \leq L' - 1$ . Conversely, if  $d \notin \mathbb{R}^M \setminus G_{L'-1}$ , then  $\theta_{L'-1} = 0$  by Lemma 12.

(b) Using (a)

$$d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies \theta_k > 0 \quad \forall k \in \mathbb{I}_{L'-1}. \quad (82)$$

Since  $d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1})$  one has  $d \neq 0$ . Using Remark 10,

$$d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies \theta_0 > \theta_1. \quad (83)$$

Let  $k \in \mathbb{I}_{L'}$ . For any  $\omega \in \Omega_k$  and  $\bar{\omega} \subsetneq \omega$  with  $\sharp \bar{\omega} = k - 1$  one has  $\bar{\omega} \in \Omega_{k-1}$  and  $\text{range}(A_{\bar{\omega}}) \subsetneq \text{range}(A_{\omega})$ . Since  $(\text{range}(A_{\bar{\omega}}))^{\perp} \supsetneq (\text{range}(A_{\omega}))^{\perp}$  we obtain  $(\mathbb{R}^M \setminus E_k) \supsetneq (\mathbb{R}^M \setminus E_{k-1})$  for any  $k \in \{2, \dots, L'\}$ . Using Lemma 13 together with (83) and (82) shows that

$$d \in \mathbb{R}^M \setminus (E_2 \cup G_{L'-1}) \implies \theta_{k-1} > \theta_k > 0 \quad \forall k \in \mathbb{I}_{L'-1} \quad \text{and} \quad \theta_{L'-1} > \theta_{L'}.$$

### 9.3.2 Proof of Corollary 4.

(a) follows from Theorem 7 and Proposition 1(a).

(b) By (54),  $f_{\beta}$  is the lower envelope of  $L + 1$  affine increasing functions. Hence (b).

(c) One has  $\beta_{k_n}(k_{n+1} - k_n) = \theta_{k_n} - \theta_{k_{n+1}}$  and  $\{\beta_{k_n}\}_{n=0}^p$  strictly decreasing by Corollary 3. Then

$$\begin{aligned} f_{\beta_{k_n}} - f_{\beta_{k_{n+1}}} &= \theta_{k_n} - \theta_{k_{n+1}} + \beta_{k_n} k_n - (\beta_{k_n} - (\beta_{k_n} - \beta_{k_{n+1}})) k_{n+1} \\ &= \theta_{k_n} - \theta_{k_{n+1}} + \beta_{k_n}(k_n - k_{n+1}) + (\beta_{k_n} - \beta_{k_{n+1}}) k_{n+1} = (\beta_{k_n} - \beta_{k_{n+1}}) k_{n+1} > 0. \end{aligned} \tag{84}$$

Since  $k_0 = 0$ , see (37),  $f_{\beta_{k_0}} = \theta_{k_0} = f_{\beta}$ ,  $\forall \beta \geq \beta_{k_0}$ . Using (a) and (84) yields  $f_{\beta_{k_0}} > f_{\beta}$ ,  $\forall \beta < \beta_{k_0}$ .

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## References

- [1] H. ATTOUCH, J. BOLTE, AND B. F. SVAITER, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized gaussseidel methods*, Math. Program., 137 (2013).
- [2] A. AUSLENDER AND M. TEBoulLE, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York, 2003.
- [3] E. VAN DEN BERG AND M. P. FRIEDLANDER, *Sparse optimization with least-squares constraints*, SIAM J. Optim., 21 (2011), pp. 1201–1229.
- [4] D. BLANCHARD, C. CARTIS, J. TANNER, AND A. THOMPSON, *Phase transitions for greedy sparse approximation algorithms*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 188–203.
- [5] T. BLUMENSATH AND M. DAVIES, *Iterative thresholding for sparse approximations*, J. Fourier Anal. Appl., 14 (2008), pp. 629–654.
- [6] T. BLUMENSATH AND M. DAVIES, *Iterative hard thresholding for compressed sensing*, Appl. Comput. Harmon. Anal., 27 (2009), pp. 265–274.
- [7] T. BLUMENSATH AND M.E. DAVIES, *Normalized iterative hard thresholding: Guaranteed stability and performance*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 298–309.

- [8] A. M. BRUCKSTEIN, D. L. DONOHO, AND M. ELAD, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev., 51 (2009), pp. 34–81.
- [9] E. CANDÈS, J. ROMBERG, AND T. TAO, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information*, IEEE Trans. Inform. Theory, 52 (2006), pp. 489–509.
- [10] C. CARTIS AND A. THOMPSON, *A new and improved quantitative recovery analysis for iterative hard thresholding algorithms in compressed sensing*, arXiv:1309.5406, (2013).
- [11] E. CHOUZENOUX, A. JEZIEWSKA, J.-C. PESQUET, AND H. TALBOT, *A majorize-minimize subspace approach for  $\ell_2 - \ell_0$  image regularization*, SIAM J. Imaging Sci., 6 (2013), pp. 563–591.
- [12] A. COHEN, W. DAHMEN, AND R. A. DEVORE, *Compressed sensing and best  $k$ -term approximation*, J. Amer. Math. Soc., 22 (2009), pp. 211–231.
- [13] G. DAVIS, S. MALLAT, AND M. AVELLANEDA, *Adaptive greedy approximations*, Constructive approximation, 13 (1997), pp. 57–98.
- [14] R. A. DEVORE, *Nonlinear approximation*, Acta Numerica, 7 (1998).
- [15] B. DONG AND Y. ZHANG, *An efficient algorithm for  $\ell_0$  minimization in wavelet frame based image restoration*, J. Sci. Comput., 54 (2013).
- [16] D.L. DONOHO, *Compressed sensing*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.
- [17] M. ELAD AND M. AHARON, *Image denoising via learned dictionaries and sparse representation*, in Computer Vision and Pattern Recognition, vol. 1, IEEE, 2006, pp. 895–900.
- [18] Y. C. ELDAR AND G. KUTYNIOK, *Compressed Sensing: Theory and Applications*, Cambridge Univ. Press, 2012.
- [19] J. FAN AND R. LI, *Statistical challenges with high dimensionality: feature selection in knowledge discovery*, in Proc. Intern. Congr. Math., vol. 3, Eur. Math. Soc., Zürich, 2006, pp. 595–622.
- [20] M. FORNASIER AND R. WARD, *Iterative thresholding meets free-discontinuity problems*, Found. Comput. Math., 10 (2010), pp. 527–567.
- [21] G. M. FUNG AND O. L. MANGASARIAN, *Equivalence of minimal  $\ell_0$ - and  $\ell_p$ -norm solutions of linear equalities, inequalities and linear programs for sufficiently small  $p$* , J. Optim. Theory Appl., 151 (2011), pp. 1–10.
- [22] J. HAUPT AND R. NOWAK, *Signal reconstruction from noisy random projections*, IEEE Trans. Inform. Theory, 52 (2006), pp. 4036–4048.
- [23] Y. JIAO, B. JIN, AND X. LU, *A primal dual active set algorithm for a class of nonconvex sparsity optimization*, arXiv:1310.1147v1, (2013).
- [24] Y. LIU AND Y. WU, *Variable selection via a combination of the  $\ell_0$  and  $\ell_1$  penalties*, J. Comp. Graph. Stat., 16 (2007), pp. 782–798.
- [25] D. LORENZ, *Convergence rates and source conditions for Tikhonov regularization with sparsity constraints*, Journal of Inverse and Ill-posed Problems, 16 (2008), pp. 463–478.

- [26] D.G. LUENBERGER, *Optimization by Vector Space Methods*, Wiley, J., New York, 1 ed., 1969.
- [27] J. LV AND Y. FAN, *A unified approach to model selection and sparse recovery using regularized least squares*, The Annals of Statistics, 37 (2009).
- [28] A. MALEKI AND D. L. DONOHO, *Optimally tuned iterative reconstruction algorithms for compressed sensing*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 330–341.
- [29] C. D. MEYER, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.
- [30] A. J. MILLER, *Subset Selection in Regression*, Chapman and Hall, London, U.K., 2 ed., 2002.
- [31] J. NEUMANN, C. SCHÖRR, AND G. STEIDL, *Combined SVM-based Feature Selection and classification*, Machine Learning 61, 2005, pp. 129–150.
- [32] M. NIKOLOVA, *Description of the minimizers of least squares regularized with  $\ell_0$ -norm. uniqueness of the global minimizer*, SIAM J. Imaging Sci., 6 (2013), pp. 904–937.
- [33] M.C. ROBINI, A. LACHAL, AND I.E. MAGNIN, *A stochastic continuation approach to piecewise constant reconstruction*, IEEE Trans. Image Process., 16 (2007), pp. 2576–2589.
- [34] M. C. ROBINI AND I. E. MAGNIN, *Optimization by stochastic continuation*, SIAM J. Imaging Sci., 3 (2010), pp. 1096–1121.
- [35] M. C. ROBINI AND P.-J. REISSMAN, *From simulated annealing to stochastic continuation: a new trend in combinatorial optimization*, J. Global Optim., 56 (2013), pp. 185–215.
- [36] R. T. ROCKAFELLAR AND J. B WETS, *Variational analysis*, Springer-Verlag, New York, 1997
- [37] J.A. TROPP, *Just relax: convex programming methods for identifying sparse signals in noise*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1030–1051.
- [38] J. TROPP AND S. J. WRIGHT, *Computational methods for sparse solution of linear inverse problems*, Proc. IEEE, 98 (2010), pp. 948–958.
- [39] Y. ZHANG, B. DONG, AND Z. LU,  *$\ell_0$  minimization of wavelet frame based image restoration*, Mathematics of Computation, 82 (2013), pp. 995–1015.