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Relationship between the optimal solutions of least squares regularized with ℓ_0 -norm and constrained by k-sparsity

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Abstract. Given an $M \times N$ real-valued matrix A with $M < N$ and a data-vector d , consider that d must be expressed as a linear combination of a small number of basis vectors taken from A . Two popular options to find the sought-after sparse solution are (1) to minimize the least-squares *regularized* with the counting function $\|\cdot\|_0$ (called usually the ℓ_0 -norm) via a trade-off parameter $\beta > 0$, and (2) to solve the least-squares *constrained* by k-sparsity, i.e. $\|\cdot\|_0 \leq k$. This work provides an exhaustive description of the relationship between the optimal solutions of these two nonconvex (combinatorial) optimization problems. Small-size exact numerical tests give a flavour of the meaning of the obtained theoretical results.

Keywords: ℓ_0 -regularization; k-sparsity constraint; quasi-equivalence between nonconvex problems; optimal solution analysis; parameter selection; sparse signal recovery; under-determined linear systems.

1 Introduction

The recovery of sparse objects (e.g., signals, images) or representations $u \in \mathbb{R}^N$ using a few basis vectors from incomplete and possibly inaccurate data $d \in \mathbb{R}^N$ is a tremendously growing topic, especially with the recent progress in compressed sensing [13, 8, 15, 7, 36, 17]. The most natural way to measure sparsity is the counting function $\|\cdot\|_0$, called usually the ℓ_0 -norm

$$\|u\|_0 := \sharp \{i \in \{0, 1, \dots, N\} : u[i] \neq 0\} , \quad (1.1)$$

where \sharp denotes cardinality and $u[i]$ stands for the i th components of u . We consider a frame (a dictionary) $A \in \mathbb{R}^{M \times N}$ with $M < N$ where M and N are fixed.

Two popular options to find a sparse solution \hat{u} are defined by the following optimization problems:

- the $\|\cdot\|_0$ -regularised problem where one looks for minimizers of $\mathcal{F}_\beta : \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$(\mathcal{R}_\beta) \quad \mathcal{F}_\beta(u) := \|Au - d\|_2^2 + \beta \|u\|_0 , \quad \beta > 0 , \quad (1.2)$$

where $\beta > 0$ is a regularization parameter;

- the k-sparsity constrained minimization problem defined by

$$(\mathcal{C}_k) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|_2^2 , \quad \text{subject to } \|u\|_0 \leq k , \quad (1.3)$$

where k is a fixed integer. As usual, a vector u is said to be k-sparse if $\|u\|_0 \leq k$.

Finding an optimal solution of these problems is NP-hard in general [12, 35]. We note that recent advances in stochastic optimization can or could make these tasks feasible [34].

In the literature these problems are often considered as somehow “equivalent”; see, e.g. [4, p. 631]. The goal of this work is to clarify the connection between *the optimal solutions of (\mathcal{R}_β) and of (\mathcal{C}_k)* . More precisely, we look for guarantees that *for a k and some values $\beta > 0$, (\mathcal{R}_β) and (\mathcal{C}_k) have exactly the same optimal set*. Since (\mathcal{R}_β) and (\mathcal{C}_k) are nonconvex problems, our goal is highly non trivial.

1.1 Related work

The amount of papers dealing with problems (\mathcal{R}_β) and (\mathcal{C}_k) is huge. We present a brief summary that helps to position the goals of our work.

On algorithms. The solutions of (\mathcal{R}_β) and (\mathcal{C}_k) are usually approximated by greedy pursuit, relaxation of the $\|\cdot\|_0$ penalty often combined with nonconvex minimization [18, 25, 19, 10, 22], as well as direct optimization [29, 33, 1]. Tropp and Wright [36] gave a comprehensive overview, mainly focused on greedy pursuits and convex relaxation. These algorithms require strong assumptions on A , e.g., RIP or bounds on $\text{spark}(A)$, and on the sparsity of the solution. Iterative thresholding algorithms has become quite popular after the local convergence results of Blumensath and Davies [4], further expanded by the authors in [5, 6]. Recent results has enabled the RIP assumption to be lighten [26, 3, 9].

Problem (\mathcal{R}_β) is a particular case of an objective whose global minimizer is computed in finite time with high probability by the stochastic continuation algorithm conceived by Robini, Lachal and Magnin [32] and refined by Robini and Magnin in [33]. Recently, Robini and Reissman [34] extended the methodology to general (combinatorial) objectives and gave results on the probability for global convergence versus the running time. So [34] can be adapted to solve optimally problems (\mathcal{C}_k) as well.

Some applications. Problem (\mathcal{R}_β) has been widely considered for subset selection [29, 4], model selection [25], variable selection [23], feature selection [30, 18], signal and image reconstruction [21, 19, 10, 37, 14]. Problem (\mathcal{C}_k) involves a natural sparse coding constraint; it is well known as the k -best term approximation model [13, 11]. It has been used for low-rank matrix decomposition [2], sparse inverse problems [6], dictionary learning [16], among others.

Comparison between $\|\cdot\|_0$ -related problems. Here are few references. Fung and Mangasarian [20] consider the problem: minimize $\|u\|_p$ subject to $Au = d$, $Bu \geq b$ and $\|u\|_\infty \leq 1$, where $p \in [0, 1)$, and B and b are a matrix and a vector, respectively. They prove that the $\|u\|_0$ -problem is equivalent to the $\|u\|_p$ -problem for a sufficiently small $p > 0$. Malgouyres and Nikolova [27] focus on the approximation performances of the problem: minimize $\|u\|_0$ subject to $\|Au - d\| \leq \tau$. They give a geometrical description of the data sets $d \in \mathbb{R}^M$ yielding an optimal solution of (\mathcal{C}_k) .

1.2 Our goals and contributions

In this work we provide an exhaustive description of the connection between *the optimal solutions of the two nonconvex, combinatorial problems* (\mathcal{R}_β) and (\mathcal{C}_k) , stated in (1.2) and (1.3), respectively. This goal is achieved using simple mathematical tools. Our main results are summarized below.

- There is an integer $L \leq M$ (dependent on A and on d) for which it holds that: (a) For any $k \leq L$, any optimal solution \hat{u} of (\mathcal{C}_k) obeys $\|\hat{u}\|_0 = k$ and the columns of A indexed by the support of \hat{u} are linearly independent; (b) For any $\beta > 0$, any global minimizer \hat{u} of \mathcal{F}_β verifies $\|\hat{u}\|_0 \leq L$.
- We exhibit critical values β_k , $k \in \{0, \dots, L\}$, and we prove that for any k , $(\mathcal{R}_\beta) \forall \beta \in (\beta_k, \beta_{k-1})$ and (\mathcal{C}_k) have the same optimal set *if and only if* $\{\beta_k\}$ is strictly decreasing. But for any $k \leq L - 1$ and $\beta = \beta_k$, the optimal set of (\mathcal{R}_β) is the union of the (disjoint) optimal sets of (\mathcal{C}_k) and (\mathcal{C}_{k+1}) . Thus when $\{\beta_k\}$ is strictly decreasing, we can say that (\mathcal{R}_β) and (\mathcal{C}_k) are *quasi-equivalent*.
- It is shown that $\{\beta_k\}$ *can be oscillating*, relying on A or d . Then there is a subset $\{k_n\}$ so that for any n and for any $\beta > 0$, any global minimizer \hat{u} of \mathcal{F}_β is not an optimal solution of (\mathcal{C}_{k_n}) and $\|\hat{u}\|_0 \neq k_n$. In such a case, only *partial quasi-equivalence* can hold: for $k \notin \{k_n\}$, there are $\{\beta'_k, \beta'_{k-1}\}$ and $\{\bar{b}, \underline{b}\}$ so that (\mathcal{R}_β) for any $\beta \in (\beta'_k, \beta'_{k-1})$ and (\mathcal{C}_k) share the same optimal set if $\bar{b} \geq \beta'_{k-1} > \beta'_k \geq \underline{b}$.
- The union of all global minimizers of \mathcal{F}_β for all $\beta > 0$ is *included* in the union of all optimal solutions of (\mathcal{C}_k) for $k \in \{0, \dots, L\}$.

1.3 Outline of the paper

Results relating the optimality conditions for (\mathcal{R}_β) and (\mathcal{C}_k) are derived in section 2. In particular, useful properties on the optimal sets of (\mathcal{C}_k) are exhibited in subsection 2.2. Necessary and sufficient conditions for quasi-equivalence between problems (\mathcal{R}_β) and (\mathcal{C}_k) are provided in section 3. Otherwise, partial equivalence occurs which is also examined. Some facts on the critical parameter values are given in section 4. In section 5, small-size exact numerical tests illustrate the main results of the paper.

1.4 Main notation and definitions

To simplify the notation, the ℓ_2 -norm is *systematically* denoted by

$$\|\cdot\| := \|\cdot\|_2.$$

Let n be any positive integer. The identity operator on \mathbb{R}^n is denoted by I_n . We denote by \mathbb{I}_n and \mathbb{I}_n^0 the *totally and strictly ordered* index sets

$$\mathbb{I}_n := (\{1, \dots, n\}, <) \quad \text{and} \quad \mathbb{I}_n^0 := (\{0, 1, \dots, n\}, <) , \quad (1.4)$$

where the symbol $<$ stands for the natural order of integers. Thus any subset $\omega \subseteq \mathbb{I}_n$ is also totally and strictly ordered. Without this precision, the expressions in (1.6) and (1.7) below are ambiguous.

For any $u \in \mathbb{R}^N$, the *support* $\sigma(u)$ of u is defined by

$$\sigma(u) := \left\{ i \in \mathbb{I}_N : u[i] \neq 0 \right\} \subseteq \mathbb{I}_N . \quad (1.5)$$

The i th column in a matrix $A \in \mathbb{R}^{M \times N}$ is denoted by a_i . For a matrix $A \in \mathbb{R}^{M \times N}$ and a vector $u \in \mathbb{R}^N$, with any $\omega \subseteq \mathbb{I}_N$, we associate the *submatrix* A_ω and the *subvector* u_ω given by

$$A_\omega := (a_{\omega[1]}, \dots, a_{\omega[\#\omega]}) \in \mathbb{R}^{M \times \#\omega} , \quad (1.6)$$

$$u_\omega := (u[\omega[1]], \dots, u[\omega[\#\omega]]) \in \mathbb{R}^{\#\omega} , \quad (1.7)$$

respectively, as well as the zero padding operator $Z_\omega : \mathbb{R}^{\#\omega} \rightarrow \mathbb{R}^N$ given by

$$u = Z_\omega(u_\omega) \quad u[i] = \begin{cases} 0 & \text{if } i \notin \omega , \\ u_\omega[k] & \text{for the unique } k \text{ such that } \omega[k] = i. \end{cases} \quad (1.8)$$

Using (1.5) and the notation in (1.6)-(1.7), for any $u \in \mathbb{R}^N$ we have

$$\omega \in \mathbb{I}_N \text{ and } \omega \supseteq \sigma(u) \quad \Rightarrow \quad Au = A_\omega u_\omega \quad \text{and} \quad u = Z_\omega(u_\omega) . \quad (1.9)$$

The superscript “ T ” denotes transposed. For definiteness, we set $A_\omega^T := (A_\omega)^T$ and $A_\omega^{-1} := (A_\omega)^{-1}$.

Example 1 Let $A = (a_1, a_2, a_3, a_4)$ and $u = (1, 0, 1, 0)^T$. Then $\sigma := \sigma(u) = \{1, 3\}$. Let $\omega = \{1, 2, 3\} \supset \sigma$. Clearly, $Au = A_\sigma u_\sigma = A_\omega u_\omega = a_1 + a_3$.

Remark 1 For (\mathcal{C}_k) we have to consider also two trivial cases: $k = 0$ because \mathcal{F}_β always has a strict (local) minimum at $\hat{u} = 0$ [31, Lemma 2.2] and $k = M$ since \mathcal{F}_β can have strict (local) minimizers \tilde{u} with $\|\tilde{u}\|_0 = M$ [31, Proposition 3.9]. According to the value of β , \hat{u} or \tilde{u} can be global minimizers of \mathcal{F}_β .

In order to unify the presentation, we adopt the definitions: $A_\emptyset := [\] \in \mathbb{R}^{M \times 0}$ and $\text{rank}(A_\emptyset) := 0$. Then all claims hold when $k = 0$ for (\mathcal{C}_k) and when the optimal set of (\mathcal{R}_β) is null.

We shall consider the standard hypothesis on A stated below.

H 1 *The matrix $A \in \mathbb{R}^{M \times N}$ satisfies $\text{rank}(A) = M < N$.*

It is systematically assumed that $d \neq 0$ to prevents from considering other trivial cases.

2 Preliminary results

In this section we derive simple tests relating the optimality conditions for (\mathcal{R}_β) and for (\mathcal{C}_k) . This task needs a few developments. Some results on (\mathcal{R}_β) obtained in [31] are used. We shall often refer to the constrained quadratic optimization problem stated in subsection 2.1.

2.1 A constrained quadratic optimization problem

Given $d \in \mathbb{R}^M$ and $\omega \subseteq \mathbb{I}_N$, problem (\mathcal{P}_ω) reads as:

$$(\mathcal{P}_\omega) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|^2, \quad \text{subject to} \quad u[i] = 0 \quad \forall i \in \mathbb{I}_N \setminus \omega. \quad (2.1)$$

Using the notation in (1.6), (1.7) and (1.8), the following equivalence is obvious and useful:

$$\left[\hat{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \right] \Leftrightarrow \left[\hat{u} = Z_\omega(\hat{u}_\omega) \text{ and } \hat{u}_\omega \text{ solves } \min_{v \in \mathbb{R}^{\sharp\omega}} \|A_\omega v - d\|^2 \right]. \quad (2.2)$$

The optimality conditions for the quadratic problem on the right side of (2.2) amount to solving the normal equation relevant to A_ω , i.e. to finding a $v \in \mathbb{R}^{\sharp\omega}$ satisfying $A_\omega^T A_\omega v = A_\omega^T d$. Such a v always exists; see e.g., [28]. Therefore *for any $d \in \mathbb{R}^M$ and for any $\omega \subset \mathbb{I}_N$ problem (\mathcal{P}_ω) has solutions.*

Note that any \hat{u} solving (\mathcal{P}_ω) for some $\omega \in \mathbb{I}_N$ is a (local) minimizer of \mathcal{F}_β [31, Proposition 2.3].

2.2 On the optimal solutions of problem (\mathcal{C}_k)

The set of the supports of all k -sparse vectors reads as

$$\Theta_k := \bigcup_{n=0}^k \left\{ \omega \subset \mathbb{I}_N : \sharp\omega = n \right\}. \quad (2.3)$$

Using (\mathcal{P}_ω) and this notation, for any k , problem (\mathcal{C}_k) in (1.3) equivalently reads as

$$\begin{aligned} (\mathcal{C}_k) \quad \|A\hat{u} - d\|^2 &= \min_{\omega \in \Theta_k} \left\{ \|A\tilde{u} - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k \right\} \\ &= \min_{\omega \in \Theta_k} \left\{ \|A_\omega \tilde{u}_\omega - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k \right\}. \end{aligned} \quad (2.4)$$

We first verify the existence of optimal solution of (\mathcal{C}_k) .

Lemma 1 *For any $k \in \mathbb{I}_N^0$, the optimal set of problem (\mathcal{C}_k) is nonempty.*

Proof. Consider the equivalent formulation of (\mathcal{C}_k) in (2.4). For $k \in \mathbb{I}_N^0$, define $\theta^\omega \in \mathbb{R}_+$ as

$$\theta^\omega := \|A\tilde{u} - d\|^2, \quad \text{where } \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k.$$

The facts that $\sharp\Theta_k$ is finite and that (\mathcal{P}_ω) has solutions for any $\omega \in \Theta_k \subset \mathbb{I}_N$ ensure that the set $\{\theta^\omega : \omega \in \Theta_k\}$ is nonempty and finite. Then the number $\hat{\theta} := \min\{\theta^\omega : \omega \in \Theta_k\}$ is uniquely defined and there exists \hat{u} such that $\|A\hat{u} - d\|^2 = \hat{\theta}$; so \hat{u} is an optimal solution of (\mathcal{C}_k) . \square

By Lemma 1, for any $k \in \mathbb{I}_N^0$ the optimal value $\theta_d(k)$ of problem (\mathcal{C}_k) is well defined:

$$\boxed{\theta_d(k) := \|A\hat{u} - d\|^2 \quad \text{where } \hat{u} \text{ is an optimal solution of } (\mathcal{C}_k).} \quad (2.5)$$

The optimal set of any problem (\mathcal{C}_k) is denoted by

$$\boxed{\hat{\mathcal{C}}_k := \left\{ \hat{u} \in \mathbb{R}^N : \hat{u} \text{ is an optimal solution of } (\mathcal{C}_k) \right\}.} \quad (2.6)$$

If (\mathcal{C}_k) has a unique solution \hat{u} , then clearly $\hat{\mathcal{C}}_k = \{\hat{u}\}$.

Below we prove that the sequence $\{\theta_d(k)\}_k$ is indeed decreasing.

Lemma 2 Let $d \in \mathbb{R}^M$. Then $\theta_d(0) = \|d\|^2$ and

$$\theta_d(k-1) \geq \theta_d(k) \quad \forall k \geq 1. \quad (2.7)$$

If H1 holds, then $\theta_d(M) = 0$ and $\theta_d(k) = 0$ for any $k > M$.

Proof. It is obvious that $\theta_d(0) = \|d\|^2$. Let $\hat{u} \in \hat{\mathcal{C}}_k$ for $k \in \mathbb{I}_N$. Set $\hat{\sigma} := \sigma(\hat{u})$. From (2.4) one has

$$\theta_d(k) = \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 \leq \|A_{\omega}u_{\omega} - d\|^2 \quad \forall \omega \in \Theta_n \quad \forall u_{\omega} \in \mathbb{R}^{\sharp\omega}, \quad \forall n \leq k. \quad (2.8)$$

Consequently, $\theta_d(k) \leq \theta_d(k-n)$ for any $n \in \mathbb{I}_k$, which proves (2.7). From H1, there exists $\omega \subset \mathbb{I}_N$ such that $\text{rank}(A_{\omega}) = M = \sharp\omega$. Then $\hat{u}_{\omega} = (A_{\omega})^{-1}d$ leads to $\|A_{\omega}\hat{u}_{\omega} - d\|^2 = 0 = \theta_d(M)$. This, combined with (2.7), shows that $\theta_d(k) = 0$ for any $k > M$. \square

We verify that if $\hat{u} \in \hat{\mathcal{C}}_k$ is an unconstrained solution, then \hat{u} solves exactly $Au = d$.

Lemma 3 Let H1 hold. For $k \in \mathbb{I}_M$, assume that (\mathcal{C}_k) has an optimal solution \hat{u} obeying

$$\|\hat{u}\|_0 = k - n \quad \text{for } n \geq 1. \quad (2.9)$$

Then

$$A\hat{u} = d. \quad (2.10)$$

Further, $\hat{u} \in \hat{\mathcal{C}}_m$ and $\theta_d(m) = 0$ for any $m \geq k - n$.

Proof. Set $\hat{\sigma} := \sigma(\hat{u})$ then $\sharp\hat{\sigma} = k - n$. Recalling that $A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A\hat{u}$, define $z \in \mathbb{R}^M$ by

$$z := A\hat{u} - d = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d. \quad (2.11)$$

Assume that (2.10) fails, i.e.

$$z \neq 0. \quad (2.12)$$

Since by (2.4) \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$, one has $A_{\hat{\sigma}}^T(A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d) = 0$ which combined with (2.11) leads to $A_{\hat{\sigma}}^T z = 0$. Select a $\tilde{\sigma} \subseteq \hat{\sigma}$ yielding $\text{rank}(A_{\tilde{\sigma}}) = r := \text{rank}(A_{\hat{\sigma}})$; recall that $r \leq k - n < M$. By H1, there is $\omega \in \mathbb{I}_N$ such that $\omega \supsetneq \tilde{\sigma}$ and $\text{rank}(A_{\omega}) = M = \sharp\omega$. Then $A_{\omega}^T z \neq 0$ by (2.12). So there is $m \in \omega \setminus \hat{\sigma}$ obeying

$$\langle a_m, z \rangle \neq 0.$$

Clearly, a_m is linearly independent of $\text{range}(A_{\hat{\sigma}})$. One has

$$\hat{u}_m := \frac{\langle a_m, z \rangle}{\|a_m\|^2} \Rightarrow \hat{u}_m \langle a_m, z \rangle > 0. \quad (2.13)$$

Using the definition of z in (2.11), one derives

$$\begin{aligned} \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} + a_m\hat{u}_m - d\|^2 - \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 &= \|z + a_m\hat{u}_m\|^2 - \|z\|^2 = \|a_m\|^2\hat{u}_m^2 - 2\hat{u}_m\langle z, a_m \rangle \\ [\text{by (2.13)}] &= \frac{\hat{u}_m}{\|a_m\|^2}(\langle a_m, z \rangle - 2\langle a_m, z \rangle) = -\frac{\hat{u}_m\langle a_m, z \rangle}{\|a_m\|^2} < 0. \end{aligned}$$

Hence any optimal solution \hat{u} of (\mathcal{C}_k) satisfies $\|\hat{u}\|_0 > \sharp\hat{\sigma} = k - n$. But this contradicts (2.9); hence the assumption in (2.12) fails. This proves (2.10). Then $\hat{u} \in \hat{\mathcal{C}}_{k-n}$ since $\theta_d(k-n) = 0$. By Lemma 2, for any $m \geq n - k$ one has $\theta_d(m) = 0$ and $\hat{u} \in \hat{\mathcal{C}}_m$. \square

Corollary 1 *Let H1 hold. For some $k \in \mathbb{I}_M^0$, suppose that $\hat{u} \in \hat{C}_k$. Then*

$$\theta_d(k) > 0 \quad \Rightarrow \quad \|\hat{u}\|_0 = k .$$

Proof. The case $k = 0$ is trivial. Focus on $k \geq 1$. If $\|\hat{u}\|_0 = k - n$ for $n \in \mathbb{I}_k$, Lemma 3 shows that $\theta_d(k) = 0$. This contradicts the fact that $\theta_d(k) > 0$. It follows that $\|\hat{u}\|_0 = k$. \square

Given A satisfying H1 and $d \neq 0$, we introduce the constant

$$\boxed{L := \max \{k \in \mathbb{I}_M : \theta_d(k) > 0\} + 1 .} \quad (2.14)$$

Since $\theta_d(\cdot)$ is decreasing (Lemma 2), the constant L is uniquely defined. From the same lemma, $\theta_d(M) = 0$, hence $L \leq M$. Note that the constant L relies on A or d .

Remark 2 One has $L \leq M - 1$ if $d = Au$ for $\|u\|_0 \leq M - 1$. Then d belongs to a subspace of \mathbb{R}^M of dimension $\|u\|_0$ which is a negligible subset of \mathbb{R}^M . Usual noisy data range on the whole \mathbb{R}^M and $L = M$.

Lemma 4 *Let H1 be satisfied and L read as in (2.14). If \hat{u} is an optimal solution of (C_L) , then*

$$\theta_d(L) = 0 \quad \text{and} \quad \|\hat{u}\|_0 = L .$$

Proof. From (2.14), $\theta_d(L) = \|A\hat{u} - d\|^2 = 0$. If $\|\hat{u}\|_0 = L - n$ for $n \geq 1$, then \hat{u} also satisfies $\hat{u} \in \hat{C}_{L-n}$ and $\theta_d(L - n) = 0$ by Lemma 3. But this contradicts the definition of L . Hence $\|\hat{u}\|_0 = L$. \square

The result on the optimal solutions of (C_k) below is important.

Lemma 5 *Let H1 hold. For $k \in \mathbb{I}_L^0$, suppose that $\hat{u} \in \hat{C}_k$. Set $\hat{\sigma} := \sigma(\hat{u})$. Then*

$$\text{rank}(A_{\hat{\sigma}}) = \sharp \hat{\sigma} . \quad (2.15)$$

Proof. If $k = 0$, (2.15) holds. The proof is by contradiction. Suppose that (2.15) fails for $k \geq 1$, i.e.,

$$\text{rank}(A_{\hat{\sigma}}) \leq \sharp \hat{\sigma} - 1 . \quad (2.16)$$

The rank-nullity theorem [28] entails that $\dim \ker(A_{\hat{\sigma}}) = \sharp \hat{\sigma} - \text{rank}(A_{\hat{\sigma}}) \geq 1$. We can take an arbitrary $v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \setminus \{0\}$, set $v := Z_{\hat{\sigma}}(v_{\hat{\sigma}})$ and select an $i \in \hat{\sigma}$ obeying $v[i] \neq 0$. Define \tilde{u} by

$$\tilde{u} := \hat{u} - \hat{u}[i] \frac{v}{v[i]} .$$

Clearly, $\tilde{u}[i] = 0$ and $\hat{u}[i] \neq 0$, so $\tilde{\sigma} := \sigma(\tilde{u}) \subsetneq \hat{\sigma}$, which leads to

$$\|\tilde{u}\|_0 \leq \|\hat{u}\|_0 - 1 . \quad (2.17)$$

From $v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \in \ker(A_{\hat{\sigma}})$ one has $A\hat{u} = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}} \left(\hat{u}_{\hat{\sigma}} - v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \right) = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A_{\tilde{\sigma}}\tilde{u}_{\tilde{\sigma}} = A\tilde{u}$. Then

$$\theta_d(k) = \|A\hat{u} - d\|^2 = \|A\tilde{u} - d\|^2 . \quad (2.18)$$

Consequently, \tilde{u} is also an optimal solution of (\mathcal{C}_k) which by (2.17) satisfies

$$\|\tilde{u}\|_0 = k - n \quad \text{for} \quad n := \|\hat{u}\|_0 - \|\tilde{u}\|_0 \geq 1 .$$

From Lemma 3, $\tilde{u} \in \hat{\mathcal{C}}_{k-n}$ and $\theta_d(k-n) = 0$. This result contradicts the fact that by (2.14), one has $\theta_d(L-n) > 0$ for any $n \geq 1$. Hence the assumption in (2.16) fails which proves the lemma. \square

Remark 3 The algorithm aimed at solving (\mathcal{C}_k) proposed in [4] was shown in [4, Lemma 6] to produce, under certain conditions, solutions that fulfill the necessary optimality condition in (2.15).

Based on Lemma 5, we focus on all M -row full column rank submatrices of A . All supports $\omega \subset \mathbb{I}_N$ corresponding to such submatrices are described below:

$$\Omega_k := \left\{ \omega \subset \mathbb{I}_N : \# \omega = k = \text{rank}(A_\omega) \right\} \quad \forall k \in \mathbb{I}_M^0 \quad \text{and} \quad \Omega := \bigcup_{k=0}^M \Omega_k . \quad (2.19)$$

Theorem 1 *Let H1 be satisfied and L read according to (2.14). Then*

$$\boxed{\hat{u} \text{ is an optimal solution of } (\mathcal{C}_k) \text{ for } k \in \mathbb{I}_L^0 \Rightarrow \|\hat{u}\|_0 = k \text{ and } \sigma(\hat{u}) \in \Omega_k .} \quad (2.20)$$

Further, if $\hat{u} \in \hat{\mathcal{C}}_L$, then $\hat{u} \in \bigcap_{k=L}^N \hat{\mathcal{C}}_k$.

Proof. For any $k \in \mathbb{I}_L^0$, Lemma 5 shows that $\sigma(\hat{u}) \in \Omega_k$. If $k = L$, $\|\hat{u}\|_0 = L$ from Lemma 4. Let $k \leq L - 1$. From the definition of L in (2.14), $\theta_d(k) > 0$, which by Corollary 1 shows that $\|\hat{u}\|_0 = k$. The last claim follows directly from Lemmas 2 and Lemma 4. \square

We note that problem (\mathcal{C}_M) is unconstrained and has $\#\Omega_M$ optimal solutions.

Remark 4 Let H1 hold. By Theorem 1, for any $k \in \mathbb{I}_M^0$, problem (\mathcal{C}_k) equivalently reads as

$$(\mathcal{C}_k) \quad \min_{\omega \in \Omega_k} \left\{ \|A\tilde{u} - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega_k \right\} . \quad (2.21)$$

For a given k , the formulation of (\mathcal{C}_k) in (2.4) suggests that getting an optimal solution of (\mathcal{C}_k) needs to compare the values of $\#\Theta_k = \sum_{n=0}^k \#\Theta_n$ different solutions of (\mathcal{P}_ω) . The new formulation in (2.21) shows that in fact, the values of only $\#\Omega_k \leq \#\Theta_k$ solutions of (\mathcal{P}_ω) , for $\omega \in \Omega_k$, must be compared. Correspondingly, the optimal value of (\mathcal{C}_k) , introduced in (2.5), satisfies

$$\theta_d(k) = \min \left\{ \|A_\omega u_\omega - d\|^2 : \omega \in \Omega_k \right\} \quad \forall k \in \mathbb{I}_M^0 .$$

Another direct and useful consequence of Theorem 1 is stated below.

Corollary 2 *Let H1 hold. Then $\hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_n = \emptyset$ for all $(k, n) \in \mathbb{I}_L^0$ such that $k \neq n$.*

In words, if \hat{u} solves optimally (\mathcal{C}_k) for $k \leq L$, then \hat{u} cannot be an optimal solution of (\mathcal{C}_n) for any $n \neq k$ obeying $n \leq L$.

2.3 Problem (\mathcal{P}_ω) – a bridge between problems (\mathcal{R}_β) and (\mathcal{C}_k)

We recall that a (local) minimizer \hat{u} of \mathcal{F}_β is strict if there is a neighborhood $\mathcal{O} \subset \mathbb{R}^N$, containing \hat{u} , such that $\mathcal{F}_\beta(\hat{u}) < \mathcal{F}_\beta(u)$ for any $u \in \mathcal{O}$. Obviously, such a minimizer \hat{u} is isolated.

Using the definition of Ω in (2.19), we introduce the set $U \subset \mathbb{R}^N$ given by

$$U = \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega\} . \quad (2.22)$$

Lemma 6 *Let $d \in \mathbb{R}^M$ and $\beta > 0$. Then*

$$\hat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_\beta \Leftrightarrow \hat{u} \in U . \quad (2.23)$$

Proof. From [31, Theorem 3.2], \hat{u} is a strict (local) minimizers of \mathcal{F}_β if and only if \hat{u} solves $\mathcal{P}_{\hat{\sigma}}$ for $\hat{\sigma} := \sigma(\hat{u}) \in \Omega$; hence $\hat{u} \in U$. And [31, Corollary 3.3] tells us that if \hat{u} solves (\mathcal{P}_ω) for $\omega \in \Omega$ (i.e., if $\hat{u} \in U$) then \hat{u} is a strict (local) minimizer of \mathcal{F}_β . The proof is complete. \square

Now we partition U in (2.22) according to

$$U = \bigcup_{k=0}^M U_k \quad \text{where} \quad U_k := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega \text{ and } \|\tilde{u}\|_0 = k\} \quad \forall k \in \mathbb{I}_M^0 . \quad (2.24)$$

Lemma 7 *Let H1 be satisfied. Then $\hat{C}_k \subset U_k$, $\forall k \in \mathbb{I}_L^0$.*

Proof. Let $\hat{u} \in \hat{C}_k$ for $k \in \mathbb{I}_L^0$. Set $\hat{\sigma} := \sigma(\hat{u})$. The formulation of (\mathcal{C}_k) in (2.21), combined with Theorem 1 shows that \hat{u} solves (\mathcal{P}_ω) for $\omega := \hat{\sigma} \in \Omega_k \subset \Omega$ and that $\|\hat{u}\|_0 = k$. Hence $\hat{u} \in U_k$. \square

Lemma 8 *Let H1 be satisfied and let $\beta > 0$.*

(i) *Let $\hat{u} \in \hat{C}_k$ for $k \in \mathbb{I}_L^0$. Then*

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k \quad \forall \hat{u} \in \hat{C}_k ; \quad (2.25)$$

$$\mathcal{F}_\beta(\tilde{u}) > \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_k \setminus \hat{C}_k . \quad (2.26)$$

(ii) *Let $\hat{u} \in \hat{C}_L$. If $L \leq M - 1$, then*

$$n \in \{L + 1, \dots, M\} \text{ and } \tilde{u} \in U_n \Rightarrow \mathcal{F}_\beta(\tilde{u}) > \theta_d(L) + \beta L = \mathcal{F}_\beta(\hat{u}) . \quad (2.27)$$

Proof. (i). Using that $k \in \mathbb{I}_L^0$, Theorem 1 shows that $\|\hat{u}\|_0 = k$. This leads to (2.25). If $\tilde{u} \in U_k \setminus \hat{C}_k$, then $\|\tilde{u}\|_0 = k$ from the definition of U_k and \tilde{u} is not an optimal solution of (\mathcal{C}_k) , hence

$$\tilde{u} \in U_k \setminus \hat{C}_k \Rightarrow \|A\tilde{u} - d\|^2 > \theta_d(k) .$$

(ii). The definition of L in (2.14) entails that $\theta_d(n) = \theta_d(L) = 0$, $\forall n \geq L + 1$. Therefore

$$n \geq L + 1 \text{ and } \tilde{u} \in U_n \Rightarrow \mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta n > \theta_d(L) + \beta L . \quad \square$$

2.4 Joint optimality conditions for (\mathcal{C}_k) and (\mathcal{R}_β)

We shall use the following result on \mathcal{F}_β :

Theorem 2 (Nikolova, [31, Theorem 4.4]) *Let $d \in \mathbb{R}^M$ and $\beta > 0$. Then*

- (i) \mathcal{F}_β has a global minimizer: there exists $\hat{u} \in \mathbb{R}^N$ such that $\min_{u \in \mathbb{R}^N} \mathcal{F}_\beta(u) = \mathcal{F}_\beta(\hat{u}) \leq \mathcal{F}_\beta(u)$, $\forall u \in \mathbb{R}^N$;
- (ii) If \hat{u} is a global minimizer of \mathcal{F}_β , then \hat{u} is a strict minimizer of \mathcal{F}_β , i.e. $\sigma(\hat{u}) \in \Omega$.

The optimal set of an objective \mathcal{F}_β with regularization parameter $\beta > 0$ is denoted by

$$\hat{\mathcal{F}}_\beta := \left\{ \hat{u} \in \mathbb{R}^N : \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for a given } \beta > 0 \right\}. \quad (2.28)$$

Our first result relating the optimal sets of (\mathcal{R}_β) and (\mathcal{C}_k) is given below.

Proposition 1 *Let H1 hold and let $\beta > 0$. Then*

$$\hat{u} \in \hat{\mathcal{F}}_\beta \implies \hat{u} \in \hat{\mathcal{C}}_k \text{ for } k := \|\hat{u}\|_0 \in \mathbb{I}_L^0. \quad (2.29)$$

Proof. From Theorem 2(ii), \hat{u} is a strict minimizer of \mathcal{F}_β . Then $\hat{u} \in U$ by Lemma 6 and $\sigma(\hat{u}) \in \Omega$. Set $k := \|\hat{u}\|_0$. By the definition of U_k in (2.24), $\hat{u} \in U_k$. Further, $k \leq L$ because otherwise $\hat{u} \notin \hat{\mathcal{F}}_\beta$ by Lemma 8(ii). And Lemma 8(i) shows that if $\hat{u} \in \hat{\mathcal{F}}_\beta$, then $\hat{u} \notin U_k \setminus \hat{\mathcal{C}}_k$. Therefore $\hat{u} \in \hat{\mathcal{C}}_k$. \square

We denote by $\hat{\mathcal{C}}$ the collection of all optimal solutions of problems (\mathcal{C}_k) for $k \in \mathbb{I}_L^0$:

$$\hat{\mathcal{C}} := \bigcup_{k=0}^L \hat{\mathcal{C}}_k, \quad (2.30)$$

where all $\hat{\mathcal{C}}_k$'s read as in (2.6). Likewise, $\hat{\mathcal{F}}$ is the set of all global minimizers of \mathcal{F}_β for all $\beta > 0$:

$$\hat{\mathcal{F}} := \bigcup_{\beta > 0} \hat{\mathcal{F}}_\beta. \quad (2.31)$$

With these notation, the next claim is an important direct consequence of Proposition 1.

Theorem 3 *Let H1 hold. Then $\hat{\mathcal{F}} \subset \hat{\mathcal{C}}$.*

Proof. Let $\hat{u} \in \bigcup_{\beta > 0} \hat{\mathcal{F}}_\beta$ for some $\beta > 0$, hence $\hat{u} \in \hat{\mathcal{F}}$. By Proposition 1 and (2.30), $\hat{u} \in \hat{\mathcal{C}}$. \square

Remark 5 By [31, Corollary 3.3], each solution of problem (\mathcal{P}_ω) , for any $\omega \in \Omega$, is a strict (local) minimizer of \mathcal{F}_β . So \mathcal{F}_β typically has numerous strict (local) minimizers. Among them, only those that solve optimally problems (\mathcal{C}_k) for $k \in \mathbb{I}_L^0$ could be global minimizers of \mathcal{F}_β for some $\beta > 0$.

Next we give the main tool to compare the optimal sets of problems (\mathcal{C}_k) and (\mathcal{R}_β) .

Theorem 4 *Let H1 hold, and let $k \in \mathbb{I}_L^0$ and $\beta > 0$. Suppose that $\hat{u} \in \hat{\mathcal{C}}_k$ and that*

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) > 0 \quad \forall \bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k. \quad (2.32)$$

Then

$$u \in \mathbb{R}^N \setminus \hat{\mathcal{C}}_k \implies \mathcal{F}_\beta(u) > \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k; \quad (2.33)$$

in other words, any $\hat{u} \in \hat{\mathcal{C}}_k$ is a global minimizer of \mathcal{F}_β .

Proof. By (2.25) in Lemma 8(i) one has

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k \quad \forall \hat{u} \in \hat{\mathcal{C}}_k. \quad (2.34)$$

The proof is conducted by contradiction. So suppose that (2.33) fails, that is

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\tilde{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some} \quad \tilde{u} \in \mathbb{R}^N \setminus \hat{\mathcal{C}}_k. \quad (2.35)$$

From Theorem 2(i), \mathcal{F}_β has a global minimizer, say $\bar{u} \in \hat{\mathcal{F}}_\beta$, which obeys $\bar{u} \in \hat{\mathcal{C}}$ by Theorem 3. Hence

$$u \in \mathbb{R}^N \Rightarrow \mathcal{F}_\beta(u) \geq \mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\tilde{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some} \quad \bar{u} \in \hat{\mathcal{C}}. \quad (2.36)$$

- Let $\bar{u} \in \hat{\mathcal{C}}_k$. From (2.34) and (2.36) one has $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u}) = \mathcal{F}_\beta(\bar{u})$, hence $\tilde{u} \in \hat{\mathcal{F}}_\beta$. By (2.35) and Theorem 3 it follows that $\tilde{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k$. Hence

$$\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some} \quad \tilde{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k.$$

This result contradicts (2.32). Hence the assumption in (2.35) fails.

- Otherwise $\bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k$. This, combined with (2.36), leads to

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some} \quad \bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k. \quad (2.37)$$

The last result contradicts the condition in (2.32), hence the assumption in (2.35) fails. \square

3 The same optimal set for (\mathcal{C}_k) and (\mathcal{R}_β)

In this section we develop our findings on the connection between the optimal solutions of problems (\mathcal{C}_k) and (\mathcal{R}_β) . We shall use the previously obtained Corollary 2, Lemma 8, and Theorems 3 and 4.

3.1 Critical parameter values

Remark 6 Let $\hat{u} \in \hat{\mathcal{C}}_k$ and let $\bar{u} \in \hat{\mathcal{C}}_{k+1}$. The equality $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\bar{u})$ equivalently reads as

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k = \theta_d(k+1) + \beta(k+1) = \mathcal{F}_\beta(\bar{u}),$$

by Lemma 8(i). Then for $\beta := \beta_k := \theta_d(k) - \theta_d(k+1)$ one finds $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\bar{u})$.

For $d \in \mathbb{R}^M$ given, this remark suggests the following set of $M+1$ constants to be introduced:

$$\boxed{\beta_k := \theta_d(k) - \theta_d(k+1), \quad \forall k \in \mathbb{I}_M^0.} \quad (3.1)$$

A few facts on $\{\beta_k\}$ are given next.

Remark 7 Let H1 hold. Using Lemma 2, $\beta_k \geq 0$ for $\forall k \in \mathbb{I}_M^0$. Recall that $d \neq 0$.

(a) From H1, there is $n \in \mathbb{I}_N$ such that $\langle a_n, d \rangle \neq 0$. Set $\hat{u}_n = \arg \min_{v \in \mathbb{R}} \|a_n v - d\|^2 \in \mathbb{R}$. Clearly,

$$\hat{u}_n = \frac{\langle a_n, d \rangle}{\|a_n\|^2} \neq 0 \quad \text{and} \quad \theta_d(1) \leq \|a_n \hat{u}_n - d\|^2 = \|d\|^2 - \frac{\langle a_n, d \rangle^2}{\|a_n\|^2} < \theta_d(0) = \|d\|^2.$$

Then (3.1) yields

$$\beta_0 := \theta_d(0) - \theta_d(1) > 0. \quad (3.2)$$

To unify the notation, we set $\beta_{-1} := +\infty$.

(b) From the definition of L in 2.14, $\theta_d(L-1) > 0$ and $\theta_d(k) = 0$ if $k \geq L$. So (3.1) leads to

$$\beta_{L-1} > 0 \quad \text{and} \quad \beta_L = 0. \quad (3.3)$$

And if $L < M$, one has $\beta_k = 0$ for any $k \in \{L+1, \dots, M\}$.

The next claim gives a flavor on the role of $\{\beta_k\}$ as introduced in (3.1).

Lemma 9 *Let H1 hold. Assume that \mathcal{F}_β has a global minimum at $\hat{u} \in \hat{C}_k$. Then $\beta \in [\beta_k, \beta_{k-1}]$.*

Proof. Let $\bar{u} \in \hat{C}_{k+1}$ and $\tilde{u} \in \hat{C}_{k-1}$. Since $\hat{u} \in \hat{F}_\beta$ for some $\beta > 0$, the following conditions must hold:

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \text{and} \quad \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0. \quad (3.4)$$

Combining this with (2.25) in Lemma 8(i) leads to

$$\begin{aligned} \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k+1) - \theta_d(k) + \beta = -\beta_k + \beta \geq 0, \\ \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k-1) - \theta_d(k) - \beta = \beta_{k-1} - \beta \geq 0. \end{aligned} \quad (3.5)$$

Hence β must satisfy $\beta_k \leq \beta \leq \beta_{k-1}$. □

The relation below is quite practical:

$$\beta_k + \beta_{k+1} + \dots + \beta_{k+n} = \theta_d(k) - \theta_d(k+n+1), \quad \forall k \in \mathbb{I}_{M-1}^0, \quad \forall n \in \mathbb{I}_{M-k-1}^0. \quad (3.6)$$

Indeed, $\sum_{m=k}^{k+n} \beta_m = \theta_d(k) - \theta_d(k+1) + \theta_d(k+1) - \theta_d(k) + \dots + \theta_d(k+n-1) - \theta_d(k+n) + \theta_d(k+n) - \theta_d(k+n+1)$. For $n = 0$, (3.6) amounts to the definition of β_k in (3.1).

3.2 Particular cases

We exhibit a set of values for β so that (\mathcal{C}_k) and (\mathcal{R}_β) have the same optimal set.

Theorem 5 *Let H1 hold and L be defined according to (2.14). For $k \in \mathbb{I}_L^0$ assume that*

$$\bar{b}_{k-1} := \min_{n=1}^k \beta_{k-n} \geq \beta_{k-1} > \beta_k \geq \max_{n=0}^{L-k} \beta_{k+n} =: \underline{b}_k, \quad (3.7)$$

where all β_k read as in (3.1). Statements (i) and (ii) below are equivalent:

(i) \hat{u} is an optimal solution of (\mathcal{C}_k) ;

(ii) \hat{u} is a global minimizer of \mathcal{F}_β for $\beta \in (\beta_k, \beta_{k-1})$.

Proof. We break the proof into two parts.

[(i) \Rightarrow (ii)]. To prove the claim, we test the sufficient condition for global minimizer of \mathcal{F}_β in Theorem 4. Recall that by Lemma 8(i), if $\tilde{u} \in \hat{C}_n$ for $n \in \mathbb{I}_L^0$, then $\mathcal{F}_\beta(\tilde{u}) = \theta_d(n) + \beta n$. Let $\hat{u} \in \hat{C}_k$.

- Let $k \in \mathbb{I}_L$ and let $\bar{u} \in \hat{C}_{k-n}$ for $n \in \mathbb{I}_k$. Then

$$\begin{aligned} \forall n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k-n) - \theta_d(k) - n\beta \\ &\quad [\text{by (3.6)}] = \beta_{k-n} + \cdots + \beta_{k-1} - n\beta \\ &= \sum_{m=1}^n (\beta_{k-m} - \beta) \end{aligned} \quad (3.8)$$

$$[\beta < \beta_{k-1} \leq \bar{b}_{k-1} \text{ by (3.7)}] > 0. \quad (3.9)$$

- Consider that $k \in \mathbb{I}_{L-1}^0$ and let $\bar{u} \in \hat{C}_{k+n}$ for $n \in \mathbb{I}_{L-k}$. Then

$$\begin{aligned} \forall n \in \mathbb{I}_{L-k} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k+n) - \theta_d(k) + n\beta \\ &\quad [\text{by (3.6)}] = -\beta_{k+n-1} - \cdots - \beta_k + n\beta \\ &= \sum_{m=0}^{n-1} (\beta - \beta_{k+m}) \end{aligned} \quad (3.10)$$

$$[\beta > \beta_k \geq \underline{b}_k \text{ by (3.7)}] > 0. \quad (3.11)$$

Inserting (3.11) and (3.9) in (2.32) in Theorem 4 shows that $\hat{u} \in \hat{F}_\beta$.

[(ii) \Rightarrow (i)]. Since $\hat{u} \in \hat{F}$, Theorem 3 shows that $\hat{u} \in \hat{C}$, i.e. that $\hat{u} \in \hat{C}_n$ for some $n \in \mathbb{I}_L^0$ which depends on β . We will show that any $\bar{u} \in \hat{C}_n$, $n \neq k$, cannot be global minimizer of \mathcal{F}_β for $\beta \in (\beta_k, \beta_{k-1})$.

Let $\hat{u} \in \hat{C}_k$. A necessary condition for \mathcal{F}_β to have a global minimum at $\bar{u} \in \hat{C}_n$, $n \neq k$ is that

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \leq 0 \quad \text{for some } \bar{u} \in \left(\bigcup_{n=1}^k \hat{C}_{k-n} \right) \cup \left(\bigcup_{n=1}^{L-k} \hat{C}_{k+n} \right). \quad (3.12)$$

- Using (3.8), the necessary condition (3.12) for $\bar{u} \in \bigcup_{n=1}^k \hat{C}_{k-n}$ reads as

$$\bar{u} \in \hat{C}_{k-n} \quad n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = \sum_{m=1}^n (\beta_{k-m} - \beta) \leq 0.$$

According to this inequality, combined with (3.7), β must fulfill

$$\beta \geq \frac{1}{n} \sum_{m=1}^n \beta_{k-m} \geq \beta_{k-1}. \quad (3.13)$$

- From (3.10), the necessary condition (3.12) for $\bar{u} \in \bigcup_{n=1}^{L-k} \hat{C}_{k+n}$ is equivalent to

$$\bar{u} \in \hat{C}_{k+n} \quad n \in \mathbb{I}_{L-k} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = \sum_{m=0}^{n-1} (\beta - \beta_{k+m}) \leq 0.$$

This inequality, combined with (3.7), show that β must satisfy

$$\beta \leq \frac{1}{n} \sum_{m=0}^{n-1} \beta_{k+m} \leq \beta_k . \quad (3.14)$$

Combining (3.14) and (3.13) yields $\beta_{k-1} \leq \beta \leq \beta_k$ which contradicts (3.7). \square

Let us slightly modify the condition in (3.7):

Lemma 10 *Let H1 hold and let $k \in \mathbb{I}_{L-1}$. For $p \in \{0, \dots, k-1\}$ assume that*

$$\bar{b}_{k-p-1} := \min_{n=1}^{k-p} \beta_{k-p-n} \geq \beta_{k-p-1} > \beta_{k-p} = \dots = \beta_k > \beta_{k+1} \geq \max_{n=1}^{L-k} \beta_{k+n} =: \underline{b}_{k+1} . \quad (3.15)$$

(i) *If $\beta = \beta_k$ then $\widehat{F}_\beta = \bigcup_{n=k-p}^{k+1} \widehat{C}_n$ where $\widehat{C}_n \cap \widehat{C}_m = \emptyset$ for any $m \neq n$ obeying $(m, n) \in \{k-p, \dots, k+1\}$.*

(ii) *If $p \geq 1$ then \mathcal{F}_β can have a global minimum at $\widehat{u} \in \widehat{C}_k$ only if $\beta = \beta_k$.*

Proof. Let $\bar{u} \in \widehat{C}_{k-n}$ for $n \in \mathbb{I}_k$. By (3.8) and (3.15) we have $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = 0$ for any $n \in \{1, \dots, p\}$, while $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) > 0$ for any $n \in \{p+1, \dots, k-p\}$. Let now $\bar{u} \in \widehat{C}_{k+n}$ for $n \in \mathbb{I}_{L-k}$. From (3.10) and (3.15), $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = 0$ for $n = 1$ whereas $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) > 0$ for any $n \in \{2, \dots, L-k\}$. Hence \mathcal{F}_β has a global minimizer for any $\widehat{u} \in \widehat{C}_{k-p} \cup \dots \cup \widehat{C}_{k+1}$. The rest of claim (i) follows from Theorem 3. By Lemma 9, $\widehat{u} \in \widehat{C}_k$ can satisfy $\widehat{u} \in \widehat{F}_\beta$ only if $\beta_k \leq \beta \leq \beta_{k-1}$. Since $p \geq 1$, we find $\beta = \beta_k = \beta_{k-1}$. \square

According to the facts on β_k delivered in section 4, the case $\beta_k = \beta_{k-1}$ should be quite exceptional. Another possibility is that \mathcal{F}_β can never have global minimizers with a desired ℓ_0 -norm.

Lemma 11 *Let H1 be satisfied. Suppose that there are $k \in \mathbb{I}_{L-1}^0$ and $m \in \mathbb{I}_{L-k-1}^0$ such that*

$$n \in \{0, \dots, m\} \quad \Rightarrow \quad f_k(n) := \frac{1}{n+1} \sum_{p=0}^n \beta_{k-1+p} < \beta_{k+n} . \quad (3.16)$$

Then any $\widehat{u} \in \widehat{F}$ satisfies $\widehat{u} \in \widehat{C} \setminus \bigcup_{n=0}^m \widehat{C}_{k+n}$ and $\|\widehat{u}\|_0 \notin \{k, \dots, k+m\}$.

Proof. Observe that by (3.6),

$$f_k(n) = \frac{\theta_d(k-1) - \theta_d(k+n)}{n+1} .$$

The proof is by contradiction. For $n \leq m$, assume that \mathcal{F}_β has a global minimizer \widehat{u} satisfying $\widehat{u} \in \widehat{C}_{k+n}$. Let $\bar{u} \in \widehat{C}_{k+n+1}$ and let $\tilde{u} \in \widehat{C}_{k-1}$. Then the necessary conditions below must be satisfied:

$$\begin{aligned} \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) &= \theta_d(k+n+1) - \theta_d(k+n) + \beta = -\beta_{k+n} + \beta \geq 0 , \\ \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\widehat{u}) &= \theta_d(k-1) - \theta_d(k+n) - (n+1)\beta = (n+1)f_k(n) - (n+1)\beta \geq 0 . \end{aligned}$$

One obtains $\beta_{k+n} \leq \beta \leq f_k(n)$, which contradicts (3.16). This proves the lemma. \square

Since $\beta_0 > 0$ and $\beta_L = 0$ (Remark 7), if (3.16) occurs, the sequence $\{\beta_k\}$ defined in (3.1) oscillates.

3.3 Necessary and sufficient conditions for quasi-equivalence

Theorem 6 *Let H1 hold and \mathbb{L} be defined according to (2.14). Then*

$$k \in \mathbb{I}_{\mathbb{L}}^0 \quad \Rightarrow \quad \left[\hat{u} \in \hat{\mathcal{C}}_k \quad \Longleftrightarrow \quad \hat{u} \in \hat{\mathcal{F}}_{\beta} \text{ for } \beta \in (\beta_k, \beta_{k-1}) \right] \quad (3.17)$$

if and only if

$$\beta_0 > \beta_1 > \cdots > \beta_{\mathbb{L}-1} > \beta_{\mathbb{L}} . \quad (3.18)$$

Proof. Let (3.18) hold. Then the condition on β_k given in (3.7), Theorem 5 is satisfied for any k . Applying Theorem 5 for any $k \in \mathbb{I}_{\mathbb{L}}^0$ proves (3.17).

From Remark 7, see (3.3), $\beta_{\mathbb{L}-1} > \beta_{\mathbb{L}} = 0$ in all cases. To see that (3.18) is necessary for all other values of k , we argue by contradiction. Suppose there is $k \in \mathbb{I}_{\mathbb{L}-1}$ so that $\beta_{k-1} \leq \beta_k$.

- The case $\beta_{k-1} < \beta_k$ corresponds to $m = 0$ in (3.16), Lemma 11. So (3.17) fails by this lemma.
- Let $\beta_{k-1} = \beta_k$. Using Lemma 10 for $p = 1$ yields $\hat{\mathcal{F}}_{\beta} = \hat{\mathcal{C}}_{k-1} \cup \hat{\mathcal{C}}_k \cup \hat{\mathcal{C}}_{k+1}$ with $\hat{\mathcal{C}}_{k-1} \cap \hat{\mathcal{C}}_k = \hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_{k+1} = \emptyset$. So (3.17) fails. \square

Naturally, the boundary cases $\beta = \beta_k$ for $k \in \mathbb{I}_{\mathbb{L}-1}$ arise the curiosity.

Proposition 2 *Let H1 hold. Assume that (3.18) is satisfied. Let $k \in \mathbb{I}_{\mathbb{L}-1}$. Then*

$$\beta = \beta_k \quad \Rightarrow \quad \hat{\mathcal{F}}_{\beta} = \hat{\mathcal{C}}_k \cup \hat{\mathcal{C}}_{k+1} \quad \text{where} \quad \hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_{k+1} = \emptyset .$$

Proof. Applying Lemma 10 for $p = 0$ proves the proposition. \square

Under the condition in (3.18), we can say that problems (\mathcal{C}_k) and (\mathcal{R}_{β}) are *quasi-equivalent*: we have a partition $\mathbb{R}_+ = \bigcup_{k=0}^{\mathbb{L}} (\beta_{k-1}, \beta_k]$ so that

- For every $k \in \mathbb{I}_{\mathbb{L}}^0$, problems (\mathcal{C}_k) and (\mathcal{R}_{β}) for $\beta \in (\beta_{k-1}, \beta_k)$ have exactly the same optimal set;
- For each $\beta \in \{\beta_k\}_{k=0}^{\mathbb{L}-1}$ there is no equivalence since Proposition 2 holds.

3.4 Partial quasi-equivalence

Remark 8 The condition in (3.18) reads as

$$\beta_{k-1} > \beta_k \quad \forall k \in \mathbb{I}_{\mathbb{L}-1} \quad \Leftrightarrow \quad \theta_d(k) < \frac{1}{2} (\theta_d(k-1) + \theta_d(k+1)) \quad \forall k \in \mathbb{I}_{\mathbb{L}-1} .$$

Its realization seems unstable, relying on A or d ; see Lemmas 10 and 11, and the tests in section 5.

Theorem 7 *Let H1 hold. For $k \geq 1$ and $m \in \{k+2, \dots, \mathbb{L}\}$, assume that the following holds:*

- (a) *If $\bar{u} \in \bigcup_{n=k+1}^{m-1} \hat{\mathcal{C}}_n$ then for any $\beta > 0$, \bar{u} is not a global minimizer of \mathcal{F}_{β} ;*

(b) The critical parameter value $\bar{\beta}_{k,m}$ given by

$$\bar{\beta}_{k,m} := \frac{\theta_d(k) - \theta_d(m)}{m - k} \quad (3.19)$$

satisfies

$$\bar{b}_{k-1} := \min_{n=1}^k \beta_{k-n} \geq \beta_{k-1} > \bar{\beta}_{k,m} > \beta_m \geq \max_{n=0}^{L-m} \beta_{m+n} =: \underline{b}_m. \quad (3.20)$$

Then one has the following equivalences:

$$(i) \quad \hat{u} \in \hat{C}_m \iff \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\beta_m, \bar{\beta}_{k,m});$$

$$(ii) \quad \hat{u} \in \hat{C}_k \iff \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\bar{\beta}_{k,m}, \beta_{k-1}).$$

Proof. We present only the proof of (i), since (ii) is proven likewise. It is split in two parts.

[\Rightarrow] Let $\hat{u} \in \hat{C}_m$ and let $\beta \in (\beta_m, \bar{\beta}_{k,m})$.

• Let $\bar{u} \in \hat{C}_{k-n}$ for $n \geq 0$. From (3.19) it follows that

$$\begin{aligned} n = 0 \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k) - \theta_d(m) - (m - k)\beta \\ &= (m - k)(\bar{\beta}_{k,m} - \beta) \end{aligned} \quad (3.21)$$

$$[\text{ by (3.20) }] > 0. \quad (3.22)$$

Consider next that $n \geq 1$.

$$\begin{aligned} \forall n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k - n) - \theta_d(m) - (m - k + n)\beta \\ &= \theta_d(k - n) - \theta_d(k) + \theta_d(k) - \theta_d(m) - (m - k)\beta - n\beta \\ [\text{ by (3.19) or by (3.6) }] &= (m - k)(\bar{\beta}_{k,m} - \beta) + \theta_d(k - n) - \theta_d(k) - n\beta \\ [\text{ by (3.8) }] &= (m - k)(\bar{\beta}_{k,m} - \beta) + \sum_{p=1}^n (\beta_{k-p} - \beta) \end{aligned} \quad (3.23)$$

$$[\text{ by (3.20) }] > 0. \quad (3.24)$$

• Let $\bar{u} \in \hat{C}_{m+n}$ for $n \geq 1$.

$$\begin{aligned} \forall n \in \mathbb{I}_{L-m} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(m + n) - \theta_d(m) + n\beta \\ [\text{ by (3.10) or by (3.6) }] &= \sum_{p=0}^{n-1} (\beta - \beta_{m+p}) \end{aligned} \quad (3.25)$$

$$[\text{ by (3.20) }] > 0. \quad (3.26)$$

From (3.22), (3.24) and (3.26), $\hat{u} \in \hat{F}_\beta$ according to Theorem 4.

[\Leftarrow] Let $\beta \in (\beta_m, \bar{\beta}_{k,m})$ and let $\hat{u} \in \hat{C}_m$. By Theorem 3 and assumption (a), $\hat{F} \subset \hat{C} \setminus \bigcup_{n=k+1}^{m-1} \hat{C}_n$.

Suppose that \mathcal{F}_β has a global minimum at $\bar{u} \in \hat{C}_p$ for $p \neq m$. Then necessarily

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \leq 0 \quad \text{for some } \bar{u} \in \left(\bigcup_{n=0}^k \hat{C}_{k-n} \right) \cup \left(\bigcup_{n=1}^{L-m} \hat{C}_{m+n} \right). \quad (3.27)$$

- We examine (3.27) for $\bar{u} \in \bigcup_{n=0}^k \widehat{C}_{k-n}$. Combining (3.21) and (3.27) yields

$$\bar{u} \in \widehat{C}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = (m-k)(\bar{\beta}_{k,m} - \beta) \leq 0 .$$

Then β must satisfy

$$\beta \geq \bar{\beta}_{k,m} . \quad (3.28)$$

Further, using (3.23) and (3.27) lead to

$$\bar{u} \in \widehat{C}_{k-n} \quad n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = (m-k)\bar{\beta}_{k,m} + \sum_{p=1}^n \beta_{k-p} - (m-k+n)\beta \leq 0 .$$

This inequality, combined with (3.20) shows that

$$\beta \geq \frac{1}{m-k+n} \left((m-k)\bar{\beta}_{k,m} + \sum_{p=1}^n \beta_{k-p} \right) \geq \frac{(m-k)\bar{\beta}_{k,m} + n\beta_{k-1}}{m-k+n} > \bar{\beta}_{k,m} . \quad (3.29)$$

- Using (3.25), the condition (3.27) for $\bar{u} \in \bigcup_{n=1}^{L-m} \widehat{C}_{m+n}$ is equivalent to

$$\bar{u} \in \widehat{C}_{m+n} \quad n \in \mathbb{I}_{L-m} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = n\beta - \sum_{p=0}^{n-1} \beta_{m+p} \leq 0 .$$

By this inequality and (3.20), β must fulfill

$$\beta \leq \frac{1}{n} \sum_{p=0}^{n-1} \beta_{m+p} \leq \beta_m . \quad (3.30)$$

Combining the results in (3.28), (3.29) and (3.30) with assumption (a) shows that \mathcal{F}_β for $\beta \in (\beta_m, \bar{\beta}_{k,m})$ cannot have a global minimum at $\bar{u} \in \widehat{C}_p$ if $p \in \{0, \dots, k\} \cup \{m+1, \dots, L\}$. \square

Remark 9 Likewise Remark 6, for $\beta = \bar{\beta}_{k,m}$ one has $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u})$, so if (a)-(b) hold, one finds $\widehat{F}_\beta = \widehat{C}_k \cup \widehat{C}_m$ with $\widehat{C}_k \cap \widehat{C}_m = \emptyset$; see Corollary 2 and Proposition 2.

Example 2 Let $\beta_{k-1} < \beta_k$ for $k \in \mathbb{I}_{L-1}$. Then by Lemma 11 for any $\beta > 0$, any global minimizer \hat{u} of \mathcal{F}_β obeys $\hat{u} \notin \widehat{C}_k$ and hence $\|\hat{u}\|_0 \neq k$ by Theorem 1. Assume that

$$\min_{n=2}^k \beta_{k-n} \geq \beta_{k-2} > \bar{\beta}_{k-1,k+1} := \frac{\theta_d(k-1) - \theta_d(k+1)}{2} > \beta_{k+1} \geq \max_{n=1}^{L-k} \beta_{k+n} . \quad (3.31)$$

Here $\bar{\beta}_{k-1,k+1}$ is calculated according to (3.19). Theorems 7 and 1 show that

$$\hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\beta_{k+1}, \bar{\beta}_{k-1,k+1}) \Leftrightarrow \hat{u} \in \widehat{C}_{k+1} \text{ and } \|\hat{u}\|_0 = k+1 ,$$

$$\tilde{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\bar{\beta}_{k-1,k+1}, \beta_{k-2}) \Leftrightarrow \tilde{u} \in \widehat{C}_{k-1} \text{ and } \|\tilde{u}\|_0 = k-1 .$$

For $\beta = \bar{\beta}_{k-1,k+1}$ one has $\widehat{F}_\beta = (\mathcal{C}_{k-1}) \cup (\mathcal{C}_{k+1})$.

4 On the critical parameter values

From Theorem 1 and Theorem 3, any optimal solution \hat{u} of problems (\mathcal{R}_β) and (\mathcal{C}_k) in (1.2) and (2.4) is a solutions of problem (\mathcal{P}_ω) for $\omega := \sigma(\hat{u}) \in \Omega$. Then \hat{u} is of the form

$$\hat{u} = Z_\omega(\hat{u}_\omega) \quad \text{where} \quad \hat{u}_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d. \quad (4.1)$$

Remark 10 The optimal solutions \hat{u} of (\mathcal{R}_β) and (\mathcal{C}_k) should in general be sensitive to noise since their non-null part $\hat{u}_{\sigma(\hat{u})}$ minimizes a non-regularized least-squares objective relying on $A_{\sigma(\hat{u})}$ and d .

With any $\omega \in \Omega$ we associate a matrix B_ω whose columns are an orthonormal basis for A_ω , that is

$$B_\omega^T B_\omega = I_{\# \omega} \quad \text{and} \quad \text{range}(B_\omega) = \text{range}(A_\omega). \quad (4.2)$$

The orthogonal projector Π_ω onto the subspace spanned by the columns of A_ω reads as

$$\Pi_\omega = B_\omega B_\omega^T. \quad (4.3)$$

For $\omega \in \Omega$, one has $\Pi_\omega = A_\omega (A_\omega^T A_\omega)^{-1} A_\omega^T$; see, e.g., [28]. Then

$$\hat{u} \in \hat{\mathcal{C}}_k \quad \text{for } k \in \mathbb{I}_L^0 \quad \text{and} \quad \hat{\sigma} := \sigma(\hat{u}) \quad \Rightarrow \quad \theta_d(k) = \|(I - \Pi_{\hat{\sigma}})d\|^2. \quad (4.4)$$

Using the definition of Ω_k in (2.19), we introduce the subsets of \mathbb{R}^M given below:

$$E_k := \bigcup_{\omega \in \Omega_k} \ker(A_\omega^T) \quad \text{and} \quad G_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega). \quad (4.5)$$

Clearly, $G_M = \mathbb{R}^M$ and $E_M = \emptyset$ by H1.

Lemma 12 *Let H1 hold and $d \neq 0$.*

(i) *Let $k \in \{2, \dots, M-1\}$. Then*

$$d \in \mathbb{R}^M \setminus E_k \quad \Rightarrow \quad \theta_d(k-1) > \theta_d(k).$$

(ii) *Let $k \in \{1, \dots, M-1\}$. Then*

$$d \in \mathbb{R}^M \setminus G_k \quad \Leftrightarrow \quad \theta_d(k) > 0.$$

Proof. (i) Let $\hat{u} \in \hat{\mathcal{C}}_{k-1}$. Set $\hat{\sigma} := \sigma(\hat{u})$. From Theorem 1 $\hat{\sigma} \in \Omega_{k-1}$ and then

$$\exists n \in \mathbb{I}_N \setminus \hat{\sigma} \quad \text{such that} \quad \omega := \hat{\sigma} \cup \{n\} \in \Omega_k.$$

Obviously, $\theta_d(k) \leq T(k) := \|(I - \Pi_\omega)d\|^2$. Using the notation in (4.2), it follows that (see, e.g. [24])

$$\exists b_k \in \text{range}(A_\omega) \quad \text{such that} \quad B_\omega = [B_{\hat{\sigma}} \mid b_k].$$

By (4.3), one has $\Pi_\omega = B_{\hat{\sigma}} B_{\hat{\sigma}}^T + b_k b_k^T$. Therefore

$$\theta_d(k-1) - \theta_d(k) \geq \theta_d(k-1) - T(k) = d^T (\Pi_\omega - \Pi_{\hat{\sigma}}) d = \langle b_k, d \rangle^2.$$

From $d \in \mathbb{R}^M \setminus E_k$ we have $d \notin (\text{range}(A_\omega))^\perp$. Hence $\langle b_k, d \rangle^2 > 0$.

(ii) Let $d \in \mathbb{R}^M \setminus G_k$ and let $\hat{u} \in \hat{C}_k$. Set $\hat{\sigma} := \sigma(\hat{u})$. From Theorem 1, $\hat{\sigma} \in \Omega_k$. Since $(I - \Pi_{\hat{\sigma}})$ is the orthogonal projector onto $(\text{range}(A_{\hat{\sigma}}))^\perp$, having $d \notin \text{range}(A_\omega)$ leads to

$$\theta_d(k) = \|A\hat{u} - d\|^2 = \|(I - \Pi_{\hat{\sigma}})d\|^2 > 0.$$

Conversely, let $\theta_d(k) > 0$. By Theorem 1, the support of any optimal solution of (\mathcal{C}_k) belongs to Ω_k . To get a contradiction, suppose that $d \in G_k$: then there is $\omega \in \Omega_k$ such that $d \in \text{range}(A_\omega)$ and for $u_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d$ one has $\|A_\omega u_\omega - d\|^2 = \|(I - \Pi_\omega)d\|^2 = 0$. \square

Observe that the condition on d in Lemma 12(i) is a strong sufficient condition.

Proposition 3 *Let H1 hold and $d \in \mathbb{R}^M \setminus (E_2 \cup G_{L-1})$. Then the critical parameters β_k in (3.1) obey*

$$\beta_k > 0 \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \text{and} \quad \beta_k = 0 \quad \forall k \geq L.$$

Proof. Let $k \in \mathbb{I}_{L-1}$. Let $\omega \in \Omega_k$, i.e., $\text{rank}(A_\omega) = k$. Then for any $\bar{\omega} \subset \omega$ such that $\sharp \bar{\omega} = k-1$ one has $\bar{\omega} \in \Omega_{k-1}$. Since $\text{range}(A_{\bar{\omega}}) \subset \text{range}(A_\omega)$, one has $(\mathbb{R}^M \setminus G_{k-1}) \supsetneq (\mathbb{R}^M \setminus G_k)$ by (4.5). Hence

$$d \in \mathbb{R}^M \setminus G_{L-1} \quad \Rightarrow \quad \theta_d(k) > 0 \quad \forall k \in \mathbb{I}_{L-1}^0$$

and $\theta_d(L) = 0$ by (2.14). From $\ker(A_\omega^T) \subset \ker(A_{\bar{\omega}}^T)$, one has $(\mathbb{R}^M \setminus E_k) \supsetneq (\mathbb{R}^M \setminus E_{k-1})$; see (4.5). This, combined with Lemma 12(i) and the definition of β_k in (3.1) shows that

$$d \in \mathbb{R}^M \setminus E_2 \quad \Rightarrow \quad \beta_{k-1} = \theta_d(k-1) - \theta_d(k) > 0 \quad \forall k \in \{2, \dots, L\}.$$

And $\beta_0 > 0$ by Remark 7; see (3.2). \square

Remark 11 The subset E_2 is a finite union of vector subspaces of dimension $M-2$. Likewise, G_{L-1} is a finite union of vector subspaces of dimension $L-1 \leq M-1$. Hence $E_2 \cup G_{L-1}$ is closed in \mathbb{R}^M and its Lebesgue measure in \mathbb{R}^M is null. So data located in $E_2 \cup G_{L-1}$ are *highly* exceptional. Conversely, $\mathbb{R}^M \setminus (E_2 \cup G_{L-1})$ is open and dense in \mathbb{R}^M , so we usually have data living in this subset.

5 Numerical tests

Here we summarize the outcome of a series of experiments corresponding to different matrices $A \in \mathbb{R}^{M \times N}$ for $(M, N) \in \{(5, 10), (10, 20)\}$, original vectors $u_o \in \mathbb{R}^N$ and data samples $d = Au_o(+\text{noise})$. All results were calculated using an exhaustive combinatorial search. In all experiments we carried out, the following facts were observed:

- All claims concerning (\mathcal{C}_k) (Lemma 3, Theorem 1, Proposition 1, Theorem 3) were confirmed.
- For noisy data every time we had $\mathbf{L} = \mathbf{M}$; see Remark 2.
- We always had $\beta_k > 0$, $\forall k \in \mathbb{I}_{\mathbf{L}-1}^0$ and $\beta_{\mathbf{L}} = 0$, which is the general case by Proposition 3.
- For any original u_o there are many A and $d = Au_o(+\text{noise})$, and for any A there are many d so that the critical parameter sequence $\{\beta_k\}$ defined in (3.1) is either strictly decreasing or oscillating. In all cases the conditions for partial or quasi-equivalence (Theorems 5, 6 and 7) were corroborated.
- The numerical instabilities, evoked in Remark 10, were always present.

Next we present in detail three experiments for $(\mathbf{M}, \mathbf{N}) = (5, 10)$ where

$$A = \begin{pmatrix} 13.94 & 16.36 & 4.88 & -3.09 & -15.42 & 1.31 & -3.18 & -12.13 & -4.26 & -10.09 \\ 7.06 & -6.48 & -9.07 & -8.37 & -2.72 & -17.42 & -5.83 & -3.81 & 3.87 & -1.80 \\ 11.63 & 6.73 & -4.75 & -6.28 & 3.42 & 6.68 & -1.64 & 13.23 & 9.03 & -20.27 \\ -7.54 & 12.74 & -6.66 & 5.01 & 4.84 & 8.98 & -9.35 & 3.85 & 7.18 & 4.09 \\ 3.22 & -10.40 & -5.02 & 16.70 & 9.53 & -5.49 & 11.88 & -3.62 & 17.36 & 7.34 \end{pmatrix}$$

$$u_o = (0 \ 4 \ 0 \ 0 \ 0 \ \mathbf{9} \ 0 \ 0 \ \mathbf{3} \ 0)^T. \quad (5.1)$$

The entries of A follow a nearly normal distribution with variance 10. One has $\text{rank}(A) = \mathbf{M} = 5$, so H1 holds. Problem $(\mathcal{C}_{\mathbf{M}})$ is unconstrained with $\sharp \Omega_{\mathbf{M}} = 252$ optimal solutions; none of them is shown.

Noise-free data Data is given by

$$d = Au_o = (64.45 \ -171.09 \ 114.13 \ 153.32 \ -38.93)^T. \quad (5.2)$$

Since data are noise-free and $\|u_o\|_0 = 3$, it is clear that $\hat{u} := u_o$ is an optimal solution to problems

Table 1: The optimal set of (\mathcal{C}_k) for $k \in \mathbb{I}_3^0$ where d is given in (5.2). For $k = 3$, one has exact recovery, $\hat{u} = u_o$; see (5.1). The values of β_k and $\bar{\beta}_{1,3}$ are given in (5.3) and (5.4), respectively.

k	Optimal solution of (\mathcal{C}_k)										Global min. of \mathcal{F}_{β}	$\theta_d(k)$
3	0	4	0	0	0	9	0	0	3	0	$\beta \in (0, \bar{\beta}_{1,3})$	0
2	0	3.25	0	0	0	9.29	0	0	0	0	no	3968
1	0	0	0	0	0	11.76	0	0	0	0	$\beta \in (\bar{\beta}_{1,3}, \beta_0)$	7745
0	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$	7.147×10^4

(\mathcal{C}_3) , (\mathcal{C}_4) and (\mathcal{C}_5) with $\theta_d(3) = \theta_d(4) = \theta_d(5) = 0$. So $\mathbf{L} = 3$; see (2.14). The other optimal values are seen in Table 1. By Theorem 3, any $\hat{u} \in \hat{\mathbf{F}}$ obeys $\|\hat{u}\|_0 \leq 3$. The critical parameters $\{\beta_k\}$ in (3.1) are

$$\beta_0 = 6.373 \times 10^4, \ \beta_1 = 3777, \ \beta_2 = 3968, \ \beta_3 = 0. \quad (5.3)$$

Since $\beta_2 > \beta_1$, Lemma 11 for $m = 0$ tells us that any $\hat{u} \in \hat{\mathbf{F}}$ obeys $\hat{u} \notin \hat{\mathbf{C}}_3$ and $\|\hat{u}\|_0 \neq 2$. We compute $\bar{\beta}_{1,3}$ according to (3.19) and verify that (3.20) is satisfied:

$$\beta_0 > \bar{\beta}_{1,3} := \frac{1}{2}(\theta_d(1) - \theta_d(3)) = 3872.46 > \beta_3. \quad (5.4)$$

By Theorem 7, $\hat{\mathbf{C}}_3 = \{ \hat{\mathbf{F}}_{\beta} : \beta \in (\beta_3, \bar{\beta}_{1,3}) \}$, as well as $\hat{\mathbf{C}}_1 = \{ \hat{\mathbf{F}}_{\beta} : \beta \in (\bar{\beta}_{1,3}, \beta_0) \}$.

Noisy data 1. Data are corrupted with nearly normal, centered, i.i.d. noise and SNR= 32.32 dB:

$$d = (69.13 \quad -171.95 \quad 113.74 \quad 150.27 \quad -36.09)^T . \quad (5.5)$$

The optimal solutions are given in Table 2. Obviously, $L = M = 5$. The critical values $\{\beta_k\}$ read

Table 2: The optimal value and solution of (\mathcal{C}_k) for $k \in \mathbb{I}_4$ where d is given in (5.5). The values of β_k and $\bar{\beta}_{1,3}$ are given in (5.6) and (5.7), respectively.

k	$\theta_d(k)$	Optimal solution of (\mathcal{C}_k)										Global min. of \mathcal{F}_{β}
4	0.0681	0	4.40	0	0	0	8.71	0.54	0	2.95	0	$\beta \in (\beta_4, \beta_3)$
3	36.31	0	4.09	0	0	0	8.88	0	0	3.01	0	$\beta \in (\beta_3, \bar{\beta}_{1,3})$
2	4039	0	3.33	0	0	0	9.17	0	0	0	0	no
1	8012	0	0	0	0	0	11.71	0	0	0	0	$\beta \in (\bar{\beta}_{1,3}, \beta_0)$
0	7.117×10^4	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

$$\beta_0 = 6.315 \times 10^4 \quad \beta_1 = 3973 \quad \beta_2 = 4003 \quad \beta_3 = 36.25 \quad \beta_4 = 0.0681 \quad \beta_5 = 0 . \quad (5.6)$$

On has $\beta_2 > \beta_1$; by Lemma 11 for $m = 0$, any $\hat{u} \in \hat{F}$ verifies $\hat{u} \notin \hat{C}_2$ and $\|\hat{u}\|_0 \neq 2$. Using Theorem 7,

$$\beta_0 > \bar{\beta}_{1,3} := \frac{1}{2}(\theta_d(1) - \theta_d(3)) = 3987.6848 > \beta_3 > \beta_4 > \beta_5 , \quad (5.7)$$

so $\hat{C}_3 = \{ \hat{F}_{\beta} : \beta \in (\beta_3, \bar{\beta}_{1,3}) \}$ and $\hat{C}_1 = \{ \hat{F}_{\beta} : \beta \in (\bar{\beta}_{1,3}, \beta_0) \}$. Further, $\hat{C}_5 = \{ \hat{F}_{\beta} : \beta \in (\beta_4, 0) \}$.

Noisy data 2. The noise is nearly normal, centered, i.i.d., SNR= 25.74 dB:

$$d = (66.67 \quad -169.08 \quad 101.56 \quad 149.38 \quad -39.50)^T . \quad (5.8)$$

The critical values $\{\beta_k\}$ introduced in (3.1) read as

Table 3: The optimal values and solutions of (\mathcal{C}_k) for $k \in \mathbb{I}_4$ where d is given in (5.8). Here $\{\beta_k\}$ is strictly decreasing, see (5.9), so (\mathcal{R}_{β}) and (\mathcal{C}_k) are quasi-equivalent.

k	$\theta_d(k)$	Optimal solution of (\mathcal{C}_k)										Global min. of \mathcal{F}_{β}
4	0.02588	0	8.54	0	0	4.49	4.90	2.73	0	0	0	$\beta \in (\beta_4, \beta_3)$
3	72.76	0	3.93	0	0	0	8.70	0	0	2.63	0	$\beta \in (\beta_3, \beta_2)$
2	3110	0	3.27	0	0	0	8.95	0	0	0	0	$\beta \in (\beta_2, \beta_1)$
1	6935	0	0	0	0	0	11.44	0	0	0	0	$\beta \in (\beta_1, \beta_0)$
0	6.7×10^4	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

$$\beta_0 = 6.029 \times 10^4 \quad \beta_1 = 3825 \quad \beta_2 = 3037 \quad \beta_3 = 72.73 \quad \beta_4 = 0.02588 \quad \beta_5 = 0 . \quad (5.9)$$

The sequence $\{\beta_k\}$ is strictly decreasing, so (\mathcal{R}_{β}) and (\mathcal{C}_k) are quasi-equivalent (see Theorem 6).

6 Conclusions and future work

The obtention of a complete description of the relationship between the optimal solutions of nonconvex problems is in general unapparent task. We have fully clarified the relationship between the optimal solutions of least-squares regularized by $\|\cdot\|_0$ via a parameter $\beta > 0$ (problem (\mathcal{R}_β)) and constrained by k -sparsity (problem (\mathcal{C}_k)). New facts on the optimal solutions of (\mathcal{C}_k) and (\mathcal{R}_β) were given. They can help the refinement or the conception of algorithms. We exhibited critical values $\{\beta_k\}$ so that (\mathcal{C}_k) and (\mathcal{R}_β) for $\beta \in (\beta_k, \beta_{k-1})$ share exactly the same optimal set. But full equivalence (in a Lagrangian sense, as for convex problems) is impossible because for these critical values there is no equivalence. Further, there can be subsets of integers $\{k_n\}$ so that *for any n , any optimal solution \hat{u} of (\mathcal{R}_β) for any $\beta > 0$ fulfills $\|\hat{u}\|_0 \neq k_n$, whereas any optimal solution \tilde{u} of \mathcal{C}_{k_n} obeys $\|\tilde{u}\|_0 = k_n$* . All these $\{\beta_k\}$ and $\{k_n\}$ are sensitive to A or to d . Our comparative results can clarify a proper choice between models (\mathcal{R}_β) and (\mathcal{C}_k) in applications. Extensions to penalties of the form $\|Du\|_0$ for D a linear operator, or to low rank matrix recovery, seem important.

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