



**HAL**  
open science

# Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity

Mila Nikolova

► **To cite this version:**

Mila Nikolova. Relationship between the optimal solutions of least squares regularized with L0-norm and constrained by k-sparsity. 2014. hal-00944006v1

**HAL Id: hal-00944006**

**<https://hal.science/hal-00944006v1>**

Preprint submitted on 10 Feb 2014 (v1), last revised 27 Aug 2017 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Relationship between the optimal solutions of least squares regularized with $\ell_0$ -norm and constrained by k-sparsity

Mila NIKOLOVA

CMLA, CNRS, ENS Cachan, 61 Avenue du Président Wilson, 94230 Cachan, France  
Email: nikolova@cmla.ens-cachan.fr

**Abstract.** Given an  $M \times N$  real-valued matrix  $A$  with  $M < N$  and a data-vector  $d$ , consider that  $d$  must be expressed as a linear combination of a small number of basis vectors taken from  $A$ . Two popular options to find the sought-after sparse solution are (1) to minimize the least-squares *regularized* with the counting function  $\|\cdot\|_0$  (called usually the  $\ell_0$ -norm) via a trade-off parameter  $\beta > 0$ , and (2) to solve the least-squares *constrained* by k-sparsity, i.e.  $\|\cdot\|_0 \leq k$ . This work provides an exhaustive description of the relationship between the optimal solutions of these two nonconvex (combinatorial) optimization problems. Small-size exact numerical tests give a flavour of the meaning of the obtained theoretical results.

**Keywords:**  $\ell_0$ -regularization; k-sparsity constraint; quasi-equivalence between nonconvex problems; optimal solution analysis; parameter selection; sparse signal recovery; under-determined linear systems.

## 1 Introduction

The recovery of sparse objects (e.g., signals, images) or representations  $u \in \mathbb{R}^N$  using a few basis vectors from incomplete and possibly inaccurate data  $d \in \mathbb{R}^N$  is a tremendously growing topic, especially with the recent progress in compressed sensing [13, 8, 15, 7, 36, 17]. The most natural way to measure sparsity is the counting function  $\|\cdot\|_0$ , called usually the  $\ell_0$ -norm

$$\|u\|_0 := \#\{i \in \{0, 1, \dots, N\} : u[i] \neq 0\}, \quad (1.1)$$

where  $\#\$  denotes cardinality and  $u[i]$  stands for the  $i$ th components of  $u$ . We consider a frame (a dictionary)  $A \in \mathbb{R}^{M \times N}$  with  $M < N$  where  $M$  and  $N$  are fixed.

Two popular options to find a sparse solution  $\hat{u}$  are defined by the following optimization problems:

- the  $\|\cdot\|_0$ -regularised problem where one looks for minimizers of  $\mathcal{F}_\beta : \mathbb{R}^N \rightarrow \mathbb{R}$  given by

$$(\mathcal{R}_\beta) \quad \mathcal{F}_\beta(u) := \|Au - d\|_2^2 + \beta\|u\|_0, \quad \beta > 0, \quad (1.2)$$

where  $\beta > 0$  is a regularization parameter;

- the k-sparsity constrained minimization problem defined by

$$(\mathcal{C}_k) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|_2^2, \quad \text{subject to } \|u\|_0 \leq k, \quad (1.3)$$

where  $k$  is a fixed integer. As usual, a vector  $u$  is said to be k-sparse if  $\|u\|_0 \leq k$ .

Finding an optimal solution of these problems is NP-hard in general [12, 35]. We note that recent advances in stochastic optimization can or could make these tasks feasible [34].

In the literature these problems are often considered as somehow “equivalent”; see, e.g. [4, p. 631]. The goal of this work is to clarify the connection between *the optimal solutions of  $(\mathcal{R}_\beta)$  and of  $(\mathcal{C}_k)$* . More precisely, we look for guarantees that *for a  $k$  and some values  $\beta > 0$ ,  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  have exactly the same optimal set*. Since  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are nonconvex problems, our goal is highly non trivial.

## 1.1 Related work

The amount of papers dealing with problems  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  is huge. We present a brief summary that helps to position the goals of our work.

**On algorithms.** The solutions of  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are usually approximated by greedy pursuit, relaxation of the  $\|\cdot\|_0$  penalty often combined with nonconvex minimization [18, 25, 19, 10, 22], as well as direct optimization [29, 33, 1]. Tropp and Wright [36] gave a comprehensive overview, mainly focused on greedy pursuits and convex relaxation. These algorithms require strong assumptions on  $A$ , e.g., RIP or bounds on  $\text{spark}(A)$ , and on the sparsity of the solution. Iterative thresholding algorithms has become quite popular after the local convergence results of Blumensath and Davies [4], further expanded by the authors in [5, 6]. Recent results has enabled the RIP assumption to be lighten [26, 3, 9].

Problem  $(\mathcal{R}_\beta)$  is a particular case of an objective whose global minimizer is computed in finite time with high probability by the stochastic continuation algorithm conceived by Robini, Lachal and Magnin [32] and refined by Robini and Magnin in [33]. Recently, Robini and Reissman [34] extended the methodology to general (combinatorial) objectives and gave results on the probability for global convergence versus the running time. So [34] can be adapted to solve optimally problems  $(\mathcal{C}_k)$  as well.

**Some applications.** Problem  $(\mathcal{R}_\beta)$  has been widely considered for subset selection [29, 4], model selection [25], variable selection [23], feature selection [30, 18], signal and image reconstruction [21, 19, 10, 37, 14]. Problem  $(\mathcal{C}_k)$  involves a natural sparse coding constraint; it is well known as the  $k$ -best term approximation model [13, 11]. It has been used for low-rank matrix decomposition [2], sparse inverse problems [6], dictionary learning [16], among others.

**Comparison between  $\|\cdot\|_0$ -related problems.** Here are few references. Fung and Mangasarian [20] consider the problem: minimize  $\|u\|_p$  subject to  $Au = d$ ,  $Bu \geq b$  and  $\|u\|_\infty \leq 1$ , where  $p \in [0, 1)$ , and  $B$  and  $b$  are a matrix and a vector, respectively. They prove that the  $\|u\|_0$ -problem is equivalent to the  $\|u\|_p$ -problem for a sufficiently small  $p > 0$ . Malgouyres and Nikolova [27] focus on the approximation performances of the problem: minimize  $\|u\|_0$  subject to  $\|Au - d\| \leq \tau$ . They give a geometrical description of the data sets  $d \in \mathbb{R}^M$  yielding an optimal solution of  $(\mathcal{C}_k)$ .

## 1.2 Our goals and contributions

In this work we provide an exhaustive description of the connection between *the optimal solutions of the two nonconvex, combinatorial problems*  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$ , stated in (1.2) and (1.3), respectively. This goal is achieved using simple mathematical tools. Our main results are summarized below.

- There is an integer  $L \leq M$  (dependent on  $A$  and on  $d$ ) for which it holds that: (a) For any  $k \leq L$ , any optimal solution  $\hat{u}$  of  $(\mathcal{C}_k)$  obeys  $\|\hat{u}\|_0 = k$  and the columns of  $A$  indexed by the support of  $\hat{u}$  are linearly independent; (b) For any  $\beta > 0$ , any global minimizer  $\hat{u}$  of  $\mathcal{F}_\beta$  verifies  $\|\hat{u}\|_0 \leq L$ .
- We exhibit critical values  $\beta_k$ ,  $k \in \{0, \dots, L\}$ , and we prove that for any  $k$ ,  $(\mathcal{R}_\beta) \forall \beta \in (\beta_k, \beta_{k-1})$  and  $(\mathcal{C}_k)$  have the same optimal set *if and only if*  $\{\beta_k\}$  is strictly decreasing. But for any  $k \leq L - 1$  and  $\beta = \beta_k$ , the optimal set of  $(\mathcal{R}_\beta)$  is the union of the (disjoint) optimal sets of  $(\mathcal{C}_k)$  and  $(\mathcal{C}_{k+1})$ . Thus when  $\{\beta_k\}$  is strictly decreasing, we can say that  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are *quasi-equivalent*.
- It is shown that  $\{\beta_k\}$  *can be oscillating*, relying on  $A$  or  $d$ . Then there is a subset  $\{k_n\}$  so that for any  $n$  and for any  $\beta > 0$ , any global minimizer  $\hat{u}$  of  $\mathcal{F}_\beta$  is not an optimal solution of  $(\mathcal{C}_{k_n})$  and  $\|\hat{u}\|_0 \neq k_n$ . In such a case, only *partial quasi-equivalence* can hold: for  $k \notin \{k_n\}$ , there are  $\{\beta'_k, \beta'_{k-1}\}$  and  $\{\bar{b}, \underline{b}\}$  so that  $(\mathcal{R}_\beta)$  for any  $\beta \in (\beta'_k, \beta'_{k-1})$  and  $(\mathcal{C}_k)$  share the same optimal set if  $\bar{b} \geq \beta'_{k-1} > \beta'_k \geq \underline{b}$ .
- The union of all global minimizers of  $\mathcal{F}_\beta$  for all  $\beta > 0$  is *included* in the union of all optimal solutions of  $(\mathcal{C}_k)$  for  $k \in \{0, \dots, L\}$ .

## 1.3 Outline of the paper

Results relating the optimality conditions for  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are derived in section 2. In particular, useful properties on the optimal sets of  $(\mathcal{C}_k)$  are exhibited in subsection 2.2. Necessary and sufficient conditions for quasi-equivalence between problems  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  are provided in section 3. Otherwise, partial equivalence occurs which is also examined. Some facts on the critical parameter values are given in section 4. In section 5, small-size exact numerical tests illustrate the main results of the paper.

## 1.4 Main notation and definitions

To simplify the notation, the  $\ell_2$ -norm is *systematically* denoted by

$$\|\cdot\| := \|\cdot\|_2 .$$

Let  $n$  be any positive integer. The identity operator on  $\mathbb{R}^n$  is denoted by  $I_n$ . We denote by  $\mathbb{I}_n$  and  $\mathbb{I}_n^0$  the *totally and strictly ordered* index sets

$$\mathbb{I}_n := (\{1, \dots, n\}, <) \quad \text{and} \quad \mathbb{I}_n^0 := (\{0, 1, \dots, n\}, <) , \quad (1.4)$$

where the symbol  $<$  stands for the natural order of integers. Thus any subset  $\omega \subseteq \mathbb{I}_n$  is also totally and strictly ordered. Without this precision, the expressions in (1.6) and (1.7) below are ambiguous.

For any  $u \in \mathbb{R}^N$ , the *support*  $\sigma(u)$  of  $u$  is defined by

$$\sigma(u) := \left\{ i \in \mathbb{I}_N : u[i] \neq 0 \right\} \subseteq \mathbb{I}_N . \quad (1.5)$$

The  $i$ th column in a matrix  $A \in \mathbb{R}^{M \times N}$  is denoted by  $a_i$ . For a matrix  $A \in \mathbb{R}^{M \times N}$  and a vector  $u \in \mathbb{R}^N$ , with any  $\omega \subseteq \mathbb{I}_N$ , we associate the *submatrix*  $A_\omega$  and the *subvector*  $u_\omega$  given by

$$A_\omega := (a_{\omega[1]}, \dots, a_{\omega[\#\omega]}) \in \mathbb{R}^{M \times \#\omega} , \quad (1.6)$$

$$u_\omega := (u[\omega[1]], \dots, u[\omega[\#\omega]]) \in \mathbb{R}^{\#\omega} , \quad (1.7)$$

respectively, as well as the zero padding operator  $Z_\omega : \mathbb{R}^{\#\omega} \rightarrow \mathbb{R}^N$  given by

$$u = Z_\omega(u_\omega) \quad u[i] = \begin{cases} 0 & \text{if } i \notin \omega , \\ u_\omega[k] & \text{for the unique } k \text{ such that } \omega[k] = i. \end{cases} \quad (1.8)$$

Using (1.5) and the notation in (1.6)-(1.7), for any  $u \in \mathbb{R}^N$  we have

$$\omega \in \mathbb{I}_N \text{ and } \omega \supseteq \sigma(u) \quad \Rightarrow \quad Au = A_\omega u_\omega \quad \text{and} \quad u = Z_\omega(u_\omega) . \quad (1.9)$$

The superscript “ $T$ ” denotes transposed. For definiteness, we set  $A_\omega^T := (A_\omega)^T$  and  $A_\omega^{-1} := (A_\omega)^{-1}$ .

**Example 1** Let  $A = (a_1, a_2, a_3, a_4)$  and  $u = (1, 0, 1, 0)^T$ . Then  $\sigma := \sigma(u) = \{1, 3\}$ . Let  $\omega = \{1, 2, 3\} \supset \sigma$ . Clearly,  $Au = A_\sigma u_\sigma = A_\omega u_\omega = a_1 + a_3$ .

**Remark 1** For  $(\mathcal{C}_k)$  we have to consider also two trivial cases:  $k = 0$  because  $\mathcal{F}_\beta$  always has a strict (local) minimum at  $\hat{u} = 0$  [31, Lemma 2.2] and  $k = M$  since  $\mathcal{F}_\beta$  can have strict (local) minimizers  $\tilde{u}$  with  $\|\tilde{u}\|_0 = M$  [31, Proposition 3.9]. According to the value of  $\beta$ ,  $\hat{u}$  or  $\tilde{u}$  can be global minimizers of  $\mathcal{F}_\beta$ .

In order to unify the presentation, we adopt the definitions:  $A_\emptyset := [] \in \mathbb{R}^{M \times 0}$  and  $\text{rank}(A_\emptyset) := 0$ . Then all claims hold when  $k = 0$  for  $(\mathcal{C}_k)$  and when the optimal set of  $(\mathcal{R}_\beta)$  is null.

We shall consider the standard hypothesis on  $A$  stated below.

**H 1** *The matrix  $A \in \mathbb{R}^{M \times N}$  satisfies  $\text{rank}(A) = M < N$ .*

It is systematically assumed that  $d \neq 0$  to prevents from considering other trivial cases.

## 2 Preliminary results

In this section we derive simple tests relating the optimality conditions for  $(\mathcal{R}_\beta)$  and for  $(\mathcal{C}_k)$ . This task needs a few developments. Some results on  $(\mathcal{R}_\beta)$  obtained in [31] are used. We shall often refer to the constrained quadratic optimization problem stated in subsection 2.1.

## 2.1 A constrained quadratic optimization problem

Given  $d \in \mathbb{R}^M$  and  $\omega \subseteq \mathbb{I}_N$ , problem  $(\mathcal{P}_\omega)$  reads as:

$$(\mathcal{P}_\omega) \quad \min_{u \in \mathbb{R}^N} \|Au - d\|^2, \quad \text{subject to } u[i] = 0 \quad \forall i \in \mathbb{I}_N \setminus \omega. \quad (2.1)$$

Using the notation in (1.6), (1.7) and (1.8), the following equivalence is obvious and useful:

$$\left[ \hat{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \right] \Leftrightarrow \left[ \hat{u} = Z_\omega(\hat{u}_\omega) \text{ and } \hat{u}_\omega \text{ solves } \min_{v \in \mathbb{R}^{\#\omega}} \|A_\omega v - d\|^2 \right]. \quad (2.2)$$

The optimality conditions for the quadratic problem on the right side of (2.2) amount to solving the normal equation relevant to  $A_\omega$ , i.e. to finding a  $v \in \mathbb{R}^{\#\omega}$  satisfying  $A_\omega^T A_\omega v = A_\omega^T d$ . Such a  $v$  always exists; see e.g., [28]. Therefore for any  $d \in \mathbb{R}^M$  and for any  $\omega \subset \mathbb{I}_N$  problem  $(\mathcal{P}_\omega)$  has solutions.

Note that any  $\hat{u}$  solving  $(\mathcal{P}_\omega)$  for some  $\omega \in \mathbb{I}_N$  is a (local) minimizer of  $\mathcal{F}_\beta$  [31, Proposition 2.3].

## 2.2 On the optimal solutions of problem $(\mathcal{C}_k)$

The set of the supports of all  $k$ -sparse vectors reads as

$$\Theta_k := \bigcup_{n=0}^k \left\{ \omega \subset \mathbb{I}_N : \#\omega = n \right\}. \quad (2.3)$$

Using  $(\mathcal{P}_\omega)$  and this notation, for any  $k$ , problem  $(\mathcal{C}_k)$  in (1.3) equivalently reads as

$$(\mathcal{C}_k) \quad \begin{aligned} \|A\hat{u} - d\|^2 &= \min_{\omega \in \Theta_k} \left\{ \|A\tilde{u} - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k \right\} \\ &= \min_{\omega \in \Theta_k} \left\{ \|A_\omega \tilde{u}_\omega - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k \right\}. \end{aligned} \quad (2.4)$$

We first verify the existence of optimal solution of  $(\mathcal{C}_k)$ .

**Lemma 1** For any  $k \in \mathbb{I}_N^0$ , the optimal set of problem  $(\mathcal{C}_k)$  is nonempty.

**Proof.** Consider the equivalent formulation of  $(\mathcal{C}_k)$  in (2.4). For  $k \in \mathbb{I}_N^0$ , define  $\theta^\omega \in \mathbb{R}_+$  as

$$\theta^\omega := \|A\tilde{u} - d\|^2, \quad \text{where } \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Theta_k.$$

The facts that  $\#\Theta_k$  is finite and that  $(\mathcal{P}_\omega)$  has solutions for any  $\omega \in \Theta_k \subset \mathbb{I}_N$  ensure that the set  $\{\theta^\omega : \omega \in \Theta_k\}$  is nonempty and finite. Then the number  $\hat{\theta} := \min\{\theta^\omega : \omega \in \Theta_k\}$  is uniquely defined and there exists  $\hat{u}$  such that  $\|A\hat{u} - d\|^2 = \hat{\theta}$ ; so  $\hat{u}$  is an optimal solution of  $(\mathcal{C}_k)$ .  $\square$

By Lemma 1, for any  $k \in \mathbb{I}_N^0$  the optimal value  $\theta_d(k)$  of problem  $(\mathcal{C}_k)$  is well defined:

$$\boxed{\theta_d(k) := \|A\hat{u} - d\|^2 \quad \text{where } \hat{u} \text{ is an optimal solution of } (\mathcal{C}_k).} \quad (2.5)$$

The optimal set of any problem  $(\mathcal{C}_k)$  is denoted by

$$\boxed{\hat{\mathcal{C}}_k := \left\{ \hat{u} \in \mathbb{R}^N : \hat{u} \text{ is an optimal solution of } (\mathcal{C}_k) \right\}.} \quad (2.6)$$

If  $(\mathcal{C}_k)$  has a unique solution  $\hat{u}$ , then clearly  $\hat{\mathcal{C}}_k = \{\hat{u}\}$ .

Below we prove that the sequence  $\{\theta_d(k)\}_k$  is indeed decreasing.

**Lemma 2** Let  $d \in \mathbb{R}^M$ . Then  $\theta_d(0) = \|d\|^2$  and

$$\theta_d(k-1) \geq \theta_d(k) \quad \forall k \geq 1. \quad (2.7)$$

If H1 holds, then  $\theta_d(M) = 0$  and  $\theta_d(k) = 0$  for any  $k > M$ .

**Proof.** It is obvious that  $\theta_d(0) = \|d\|^2$ . Let  $\hat{u} \in \widehat{\mathcal{C}}_k$  for  $k \in \mathbb{I}_N$ . Set  $\hat{\sigma} := \sigma(\hat{u})$ . From (2.4) one has

$$\theta_d(k) = \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 \leq \|A_{\omega}u_{\omega} - d\|^2 \quad \forall \omega \in \Theta_n \quad \forall u_{\omega} \in \mathbb{R}^{\#\omega}, \quad \forall n \leq k. \quad (2.8)$$

Consequently,  $\theta_d(k) \leq \theta_d(k-n)$  for any  $n \in \mathbb{I}_k$ , which proves (2.7). From H1, there exists  $\omega \subset \mathbb{I}_N$  such that  $\text{rank}(A_{\omega}) = M = \#\omega$ . Then  $\hat{u}_{\omega} = (A_{\omega})^{-1}d$  leads to  $\|A_{\omega}\hat{u}_{\omega} - d\|^2 = 0 = \theta_d(M)$ . This, combined with (2.7), shows that  $\theta_d(k) = 0$  for any  $k > M$ .  $\square$

We verify that if  $\hat{u} \in \widehat{\mathcal{C}}_k$  is an unconstrained solution, then  $\hat{u}$  solves exactly  $Au = d$ .

**Lemma 3** Let H1 hold. For  $k \in \mathbb{I}_M$ , assume that  $(\mathcal{C}_k)$  has an optimal solution  $\hat{u}$  obeying

$$\|\hat{u}\|_0 = k - n \quad \text{for } n \geq 1. \quad (2.9)$$

Then

$$A\hat{u} = d. \quad (2.10)$$

Further,  $\hat{u} \in \widehat{\mathcal{C}}_m$  and  $\theta_d(m) = 0$  for any  $m \geq k - n$ .

**Proof.** Set  $\hat{\sigma} := \sigma(\hat{u})$  then  $\#\hat{\sigma} = k - n$ . Recalling that  $A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A\hat{u}$ , define  $z \in \mathbb{R}^M$  by

$$z := A\hat{u} - d = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d. \quad (2.11)$$

Assume that (2.10) fails, i.e.

$$z \neq 0. \quad (2.12)$$

Since by (2.4)  $\hat{u}$  solves  $(\mathcal{P}_{\hat{\sigma}})$ , one has  $A_{\hat{\sigma}}^T(A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d) = 0$  which combined with (2.11) leads to  $A_{\hat{\sigma}}^T z = 0$ . Select a  $\tilde{\sigma} \subseteq \hat{\sigma}$  yielding  $\text{rank}(A_{\tilde{\sigma}}) = r := \text{rank}(A_{\hat{\sigma}})$ ; recall that  $r \leq k - n < M$ . By H1, there is  $\omega \in \mathbb{I}_N$  such that  $\omega \supseteq \tilde{\sigma}$  and  $\text{rank}(A_{\omega}) = M = \#\omega$ . Then  $A_{\omega}^T z \neq 0$  by (2.12). So there is  $m \in \omega \setminus \hat{\sigma}$  obeying

$$\langle a_m, z \rangle \neq 0.$$

Clearly,  $a_m$  is linearly independent of  $\text{range}(A_{\hat{\sigma}})$ . One has

$$\hat{u}_m := \frac{\langle a_m, z \rangle}{\|a_m\|^2} \Rightarrow \hat{u}_m \langle a_m, z \rangle > 0. \quad (2.13)$$

Using the definition of  $z$  in (2.11), one derives

$$\begin{aligned} \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} + a_m\hat{u}_m - d\|^2 - \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 &= \|z + a_m\hat{u}_m\|^2 - \|z\|^2 = \|a_m\|^2\hat{u}_m^2 - 2\hat{u}_m\langle z, a_m \rangle \\ [ \text{by (2.13)} ] &= \frac{\hat{u}_m}{\|a_m\|^2} (\langle a_m, z \rangle - 2\langle a_m, z \rangle) = -\frac{\hat{u}_m \langle a_m, z \rangle}{\|a_m\|^2} < 0. \end{aligned}$$

Hence any optimal solution  $\hat{u}$  of  $(\mathcal{C}_k)$  satisfies  $\|\hat{u}\|_0 > \#\hat{\sigma} = k - n$ . But this contradicts (2.9); hence the assumption in (2.12) fails. This proves (2.10). Then  $\hat{u} \in \widehat{\mathcal{C}}_{k-n}$  since  $\theta_d(k-n) = 0$ . By Lemma 2, for any  $m \geq n - k$  one has  $\theta_d(m) = 0$  and  $\hat{u} \in \widehat{\mathcal{C}}_m$ .  $\square$

**Corollary 1** *Let H1 hold. For some  $k \in \mathbb{I}_M^0$ , suppose that  $\hat{u} \in \hat{C}_k$ . Then*

$$\theta_d(k) > 0 \quad \Rightarrow \quad \|\hat{u}\|_0 = k .$$

**Proof.** The case  $k = 0$  is trivial. Focus on  $k \geq 1$ . If  $\|\hat{u}\|_0 = k - n$  for  $n \in \mathbb{I}_k$ , Lemma 3 shows that  $\theta_d(k) = 0$ . This contradicts the fact that  $\theta_d(k) > 0$ . It follows that  $\|\hat{u}\|_0 = k$ .  $\square$

Given  $A$  satisfying H1 and  $d \neq 0$ , we introduce the constant

$$\boxed{L := \max \{k \in \mathbb{I}_M : \theta_d(k) > 0\} + 1 .} \quad (2.14)$$

Since  $\theta_d(\cdot)$  is decreasing (Lemma 2), the constant  $L$  is uniquely defined. From the same lemma,  $\theta_d(M) = 0$ , hence  $L \leq M$ . Note that the constant  $L$  relies on  $A$  or  $d$ .

**Remark 2** One has  $L \leq M - 1$  if  $d = Au$  for  $\|u\|_0 \leq M - 1$ . Then  $d$  belongs to a subspace of  $\mathbb{R}^M$  of dimension  $\|u\|_0$  which is a negligible subset of  $\mathbb{R}^M$ . Usual noisy data range on the whole  $\mathbb{R}^M$  and  $L = M$ .

**Lemma 4** *Let H1 be satisfied and  $L$  read as in (2.14). If  $\hat{u}$  is an optimal solution of  $(C_L)$ , then*

$$\theta_d(L) = 0 \quad \text{and} \quad \|\hat{u}\|_0 = L .$$

**Proof.** From (2.14),  $\theta_d(L) = \|A\hat{u} - d\|^2 = 0$ . If  $\|\hat{u}\|_0 = L - n$  for  $n \geq 1$ , then  $\hat{u}$  also satisfies  $\hat{u} \in \hat{C}_{L-n}$  and  $\theta_d(L - n) = 0$  by Lemma 3. But this contradicts the definition of  $L$ . Hence  $\|\hat{u}\|_0 = L$ .  $\square$

The result on the optimal solutions of  $(C_k)$  below is important.

**Lemma 5** *Let H1 hold. For  $k \in \mathbb{I}_L^0$ , suppose that  $\hat{u} \in \hat{C}_k$ . Set  $\hat{\sigma} := \sigma(\hat{u})$ . Then*

$$\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma} . \quad (2.15)$$

**Proof.** If  $k = 0$ , (2.15) holds. The proof is by contradiction. Suppose that (2.15) fails for  $k \geq 1$ , i.e.,

$$\text{rank}(A_{\hat{\sigma}}) \leq \#\hat{\sigma} - 1 . \quad (2.16)$$

The rank-nullity theorem [28] entails that  $\dim \ker(A_{\hat{\sigma}}) = \#\hat{\sigma} - \text{rank}(A_{\hat{\sigma}}) \geq 1$ . We can take an arbitrary  $v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \setminus \{0\}$ , set  $v := Z_{\hat{\sigma}}(v_{\hat{\sigma}})$  and select an  $i \in \hat{\sigma}$  obeying  $v[i] \neq 0$ . Define  $\tilde{u}$  by

$$\tilde{u} := \hat{u} - \hat{u}[i] \frac{v}{v[i]} .$$

Clearly,  $\tilde{u}[i] = 0$  and  $\hat{u}[i] \neq 0$ , so  $\tilde{\sigma} := \sigma(\tilde{u}) \subsetneq \hat{\sigma}$ , which leads to

$$\|\tilde{u}\|_0 \leq \|\hat{u}\|_0 - 1 . \quad (2.17)$$

From  $v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \in \ker(A_{\hat{\sigma}})$  one has  $A\hat{u} = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}} \left( \hat{u}_{\hat{\sigma}} - v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \right) = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A_{\tilde{\sigma}}\tilde{u}_{\tilde{\sigma}} = A\tilde{u}$ . Then

$$\theta_d(k) = \|A\hat{u} - d\|^2 = \|A\tilde{u} - d\|^2 . \quad (2.18)$$

Consequently,  $\tilde{u}$  is also an optimal solution of  $(\mathcal{C}_k)$  which by (2.17) satisfies

$$\|\tilde{u}\|_0 = k - n \quad \text{for } n := \|\hat{u}\|_0 - \|\tilde{u}\|_0 \geq 1 .$$

From Lemma 3,  $\tilde{u} \in \hat{\mathcal{C}}_{k-n}$  and  $\theta_d(k-n) = 0$ . This result contradicts the fact that by (2.14), one has  $\theta_d(L-n) > 0$  for any  $n \geq 1$ . Hence the assumption in (2.16) fails which proves the lemma.  $\square$

**Remark 3** The algorithm aimed at solving  $(\mathcal{C}_k)$  proposed in [4] was shown in [4, Lemma 6] to produce, under certain conditions, solutions that fulfill the necessary optimality condition in (2.15).

Based on Lemma 5, we focus on all  $M$ -row full column rank submatrices of  $A$ . All supports  $\omega \subset \mathbb{I}_N$  corresponding to such submatrices are described below:

$$\Omega_k := \left\{ \omega \subset \mathbb{I}_N : \#\omega = k = \text{rank}(A_\omega) \right\} \quad \forall k \in \mathbb{I}_M^0 \quad \text{and} \quad \Omega := \bigcup_{k=0}^M \Omega_k . \quad (2.19)$$

**Theorem 1** *Let H1 be satisfied and  $L$  read according to (2.14). Then*

$$\boxed{\hat{u} \text{ is an optimal solution of } (\mathcal{C}_k) \text{ for } k \in \mathbb{I}_L^0 \Rightarrow \|\hat{u}\|_0 = k \text{ and } \sigma(\hat{u}) \in \Omega_k .} \quad (2.20)$$

Further, if  $\hat{u} \in \hat{\mathcal{C}}_L$ , then  $\hat{u} \in \bigcap_{k=L}^N \hat{\mathcal{C}}_k$ .

**Proof.** For any  $k \in \mathbb{I}_L^0$ , Lemma 5 shows that  $\sigma(\hat{u}) \in \Omega_k$ . If  $k = L$ ,  $\|\hat{u}\|_0 = L$  from Lemma 4. Let  $k \leq L - 1$ . From the definition of  $L$  in (2.14),  $\theta_d(k) > 0$ , which by Corollary 1 shows that  $\|\hat{u}\|_0 = k$ . The last claim follows directly from Lemmas 2 and Lemma 4.  $\square$

We note that problem  $(\mathcal{C}_M)$  is unconstrained and has  $\#\Omega_M$  optimal solutions.

**Remark 4** Let H1 hold. By Theorem 1, for any  $k \in \mathbb{I}_M^0$ , problem  $(\mathcal{C}_k)$  equivalently reads as

$$(\mathcal{C}_k) \quad \min_{\omega \in \Omega_k} \left\{ \|A\tilde{u} - d\|^2 : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega_k \right\} . \quad (2.21)$$

For a given  $k$ , the formulation of  $(\mathcal{C}_k)$  in (2.4) suggests that getting an optimal solution of  $(\mathcal{C}_k)$  needs to compare the values of  $\#\Theta_k = \sum_{n=0}^k \#\Theta_n$  different solutions of  $(\mathcal{P}_\omega)$ . The new formulation in (2.21) shows that in fact, the values of only  $\#\Omega_k \leq \#\Theta_k$  solutions of  $(\mathcal{P}_\omega)$ , for  $\omega \in \Omega_k$ , must be compared. Correspondingly, the optimal value of  $(\mathcal{C}_k)$ , introduced in (2.5), satisfies

$$\theta_d(k) = \min \left\{ \|A_\omega u_\omega - d\|^2 : \omega \in \Omega_k \right\} \quad \forall k \in \mathbb{I}_M^0 .$$

Another direct and useful consequence of Theorem 1 is stated below.

**Corollary 2** *Let H1 hold. Then  $\hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_n = \emptyset$  for all  $(k, n) \in \mathbb{I}_L^0$  such that  $k \neq n$ .*

In words, if  $\hat{u}$  solves optimally  $(\mathcal{C}_k)$  for  $k \leq L$ , then  $\hat{u}$  cannot be an optimal solution of  $(\mathcal{C}_n)$  for any  $n \neq k$  obeying  $n \leq L$ .

### 2.3 Problem $(\mathcal{P}_\omega)$ – a bridge between problems $(\mathcal{R}_\beta)$ and $(\mathcal{C}_k)$

We recall that a (local) minimizer  $\hat{u}$  of  $\mathcal{F}_\beta$  is strict if there is a neighborhood  $\mathcal{O} \subset \mathbb{R}^N$ , containing  $\hat{u}$ , such that  $\mathcal{F}_\beta(\hat{u}) < \mathcal{F}_\beta(u)$  for any  $u \in \mathcal{O}$ . Obviously, such a minimizer  $\hat{u}$  is isolated.

Using the definition of  $\Omega$  in (2.19), we introduce the set  $U \subset \mathbb{R}^N$  given by

$$U = \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega\} . \quad (2.22)$$

**Lemma 6** *Let  $d \in \mathbb{R}^M$  and  $\beta > 0$ . Then*

$$\hat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_\beta \iff \hat{u} \in U . \quad (2.23)$$

**Proof.** From [31, Theorem 3.2],  $\hat{u}$  is a strict (local) minimizers of  $\mathcal{F}_\beta$  if and only if  $\hat{u}$  solves  $\mathcal{P}_{\hat{\sigma}}$  for  $\hat{\sigma} := \sigma(\hat{u}) \in \Omega$ ; hence  $\hat{u} \in U$ . And [31, Corollary 3.3] tells us that if  $\hat{u}$  solves  $(\mathcal{P}_\omega)$  for  $\omega \in \Omega$  (i.e., if  $\hat{u} \in U$ ) then  $\hat{u}$  is a strict (local) minimizer of  $\mathcal{F}_\beta$ . The proof is complete.  $\square$

Now we partition  $U$  in (2.22) according to

$$U = \bigcup_{k=0}^M U_k \quad \text{where} \quad U_k := \bigcup_{\omega \in \Omega} \{\tilde{u} \in \mathbb{R}^N : \tilde{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega \text{ and } \|\tilde{u}\|_0 = k\} \quad \forall k \in \mathbb{I}_M^0 . \quad (2.24)$$

**Lemma 7** *Let H1 be satisfied. Then  $\hat{C}_k \subset U_k, \forall k \in \mathbb{I}_L^0$ .*

**Proof.** Let  $\hat{u} \in \hat{C}_k$  for  $k \in \mathbb{I}_L^0$ . Set  $\hat{\sigma} := \sigma(\hat{u})$ . The formulation of  $(\mathcal{C}_k)$  in (2.21), combined with Theorem 1 shows that  $\hat{u}$  solves  $(\mathcal{P}_\omega)$  for  $\omega := \hat{\sigma} \in \Omega_k \subset \Omega$  and that  $\|\hat{u}\|_0 = k$ . Hence  $\hat{u} \in U_k$ .  $\square$

**Lemma 8** *Let H1 be satisfied and let  $\beta > 0$ .*

(i) *Let  $\hat{u} \in \hat{C}_k$  for  $k \in \mathbb{I}_L^0$ . Then*

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k \quad \forall \hat{u} \in \hat{C}_k ; \quad (2.25)$$

$$\mathcal{F}_\beta(\tilde{u}) > \mathcal{F}_\beta(\hat{u}) \quad \forall \tilde{u} \in U_k \setminus \hat{C}_k . \quad (2.26)$$

(ii) *Let  $\hat{u} \in \hat{C}_L$ . If  $L \leq M - 1$ , then*

$$n \in \{L + 1, \dots, M\} \text{ and } \tilde{u} \in U_n \implies \mathcal{F}_\beta(\tilde{u}) > \theta_d(L) + \beta L = \mathcal{F}_\beta(\hat{u}) . \quad (2.27)$$

**Proof.** (i). Using that  $k \in \mathbb{I}_L^0$ , Theorem 1 shows that  $\|\hat{u}\|_0 = k$ . This leads to (2.25). If  $\tilde{u} \in U_k \setminus \hat{C}_k$ , then  $\|\tilde{u}\|_0 = k$  from the definition of  $U_k$  and  $\tilde{u}$  is not an optimal solution of  $(\mathcal{C}_k)$ , hence

$$\tilde{u} \in U_k \setminus \hat{C}_k \implies \|A\tilde{u} - d\|^2 > \theta_d(k) .$$

(ii). The definition of  $L$  in (2.14) entails that  $\theta_d(n) = \theta_d(L) = 0, \forall n \geq L + 1$ . Therefore

$$n \geq L + 1 \text{ and } \tilde{u} \in U_n \implies \mathcal{F}_\beta(\tilde{u}) = \|A\tilde{u} - d\|^2 + \beta n > \theta_d(L) + \beta L . \quad \square$$

## 2.4 Joint optimality conditions for $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

We shall use the following result on  $\mathcal{F}_\beta$ :

**Theorem 2** (Nikolova, [31, Theorem 4.4]) *Let  $d \in \mathbb{R}^M$  and  $\beta > 0$ . Then*

(i)  $\mathcal{F}_\beta$  has a global minimizer: there exists  $\hat{u} \in \mathbb{R}^N$  such that  $\min_{u \in \mathbb{R}^N} \mathcal{F}_\beta(u) = \mathcal{F}_\beta(\hat{u}) \leq \mathcal{F}_\beta(u)$ ,  $\forall u \in \mathbb{R}^N$ ;

(ii) If  $\hat{u}$  is a global minimizer of  $\mathcal{F}_\beta$ , then  $\hat{u}$  is a strict minimizer of  $\mathcal{F}_\beta$ , i.e.  $\sigma(\hat{u}) \in \Omega$ .

The optimal set of an objective  $\mathcal{F}_\beta$  with regularization parameter  $\beta > 0$  is denoted by

$$\hat{\mathbb{F}}_\beta := \left\{ \hat{u} \in \mathbb{R}^N : \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for a given } \beta > 0 \right\}. \quad (2.28)$$

Our first result relating the optimal sets of  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  is given below.

**Proposition 1** *Let H1 hold and let  $\beta > 0$ . Then*

$$\hat{u} \in \hat{\mathbb{F}}_\beta \implies \hat{u} \in \hat{\mathbb{C}}_k \text{ for } k := \|\hat{u}\|_0 \in \mathbb{I}_L^0. \quad (2.29)$$

**Proof.** From Theorem 2(ii),  $\hat{u}$  is a strict minimizer of  $\mathcal{F}_\beta$ . Then  $\hat{u} \in U$  by Lemma 6 and  $\sigma(\hat{u}) \in \Omega$ . Set  $k := \|\hat{u}\|_0$ . By the definition of  $U_k$  in (2.24),  $\hat{u} \in U_k$ . Further,  $k \leq L$  because otherwise  $\hat{u} \notin \hat{\mathbb{F}}_\beta$  by Lemma 8(ii). And Lemma 8(i) shows that if  $\hat{u} \in \hat{\mathbb{F}}_\beta$ , then  $\hat{u} \notin U_k \setminus \hat{\mathbb{C}}_k$ . Therefore  $\hat{u} \in \hat{\mathbb{C}}_k$ .  $\square$

We denote by  $\hat{\mathbb{C}}$  the collection of all optimal solutions of problems  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_L^0$ :

$$\hat{\mathbb{C}} := \bigcup_{k=0}^L \hat{\mathbb{C}}_k, \quad (2.30)$$

where all  $\hat{\mathbb{C}}_k$ 's read as in (2.6). Likewise,  $\hat{\mathbb{F}}$  is the set of all global minimizers of  $\mathcal{F}_\beta$  for all  $\beta > 0$ :

$$\hat{\mathbb{F}} := \bigcup_{\beta > 0} \hat{\mathbb{F}}_\beta. \quad (2.31)$$

With these notation, the next claim is an important direct consequence of Proposition 1.

**Theorem 3** *Let H1 hold. Then  $\hat{\mathbb{F}} \subset \hat{\mathbb{C}}$ .*

**Proof.** Let  $\hat{u} \in \bigcup_{\beta > 0} \hat{\mathbb{F}}_\beta$  for some  $\beta > 0$ , hence  $\hat{u} \in \hat{\mathbb{F}}$ . By Proposition 1 and (2.30),  $\hat{u} \in \hat{\mathbb{C}}$ .  $\square$

**Remark 5** By [31, Corollary 3.3], each solution of problem  $(\mathcal{P}_\omega)$ , for any  $\omega \in \Omega$ , is a strict (local) minimizer of  $\mathcal{F}_\beta$ . So  $\mathcal{F}_\beta$  typically has numerous strict (local) minimizers. Among them, only those that solve optimally problems  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_L^0$  could be global minimizers of  $\mathcal{F}_\beta$  for some  $\beta > 0$ .

Next we give the main tool to compare the optimal sets of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ .

**Theorem 4** *Let H1 hold, and let  $k \in \mathbb{I}_L^0$  and  $\beta > 0$ . Suppose that  $\hat{u} \in \hat{\mathbb{C}}_k$  and that*

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) > 0 \quad \forall \bar{u} \in \hat{\mathbb{C}} \setminus \hat{\mathbb{C}}_k. \quad (2.32)$$

Then

$$u \in \mathbb{R}^N \setminus \hat{\mathbb{C}}_k \implies \mathcal{F}_\beta(u) > \mathcal{F}_\beta(\hat{u}) \quad \forall \hat{u} \in \hat{\mathbb{C}}_k; \quad (2.33)$$

in other words, any  $\hat{u} \in \hat{\mathbb{C}}_k$  is a global minimizer of  $\mathcal{F}_\beta$ .

**Proof.** By (2.25) in Lemma 8(i) one has

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k \quad \forall \hat{u} \in \hat{\mathcal{C}}_k. \quad (2.34)$$

The proof is conducted by contradiction. So suppose that (2.33) fails, that is

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\tilde{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some } \tilde{u} \in \mathbb{R}^N \setminus \hat{\mathcal{C}}_k. \quad (2.35)$$

From Theorem 2(i),  $\mathcal{F}_\beta$  has a global minimizer, say  $\bar{u} \in \hat{\mathcal{F}}_\beta$ , which obeys  $\bar{u} \in \hat{\mathcal{C}}$  by Theorem 3. Hence

$$u \in \mathbb{R}^N \Rightarrow \mathcal{F}_\beta(u) \geq \mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\tilde{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some } \bar{u} \in \hat{\mathcal{C}}. \quad (2.36)$$

- Let  $\bar{u} \in \hat{\mathcal{C}}_k$ . From (2.34) and (2.36) one has  $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u}) = \mathcal{F}_\beta(\bar{u})$ , hence  $\tilde{u} \in \hat{\mathcal{F}}_\beta$ . By (2.35) and Theorem 3 it follows that  $\tilde{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k$ . Hence

$$\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some } \tilde{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k.$$

This result contradicts (2.32). Hence the assumption in (2.35) fails.

- Otherwise  $\bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k$ . This, combined with (2.36), leads to

$$\mathcal{F}_\beta(\hat{u}) \geq \mathcal{F}_\beta(\bar{u}) \quad \forall \hat{u} \in \hat{\mathcal{C}}_k \quad \text{for some } \bar{u} \in \hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_k. \quad (2.37)$$

The last result contradicts the condition in (2.32), hence the assumption in (2.35) fails.  $\square$

### 3 The same optimal set for $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$

In this section we develop our findings on the connection between the optimal solutions of problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$ . We shall use the previously obtained Corollary 2, Lemma 8, and Theorems 3 and 4.

#### 3.1 Critical parameter values

**Remark 6** Let  $\hat{u} \in \hat{\mathcal{C}}_k$  and let  $\bar{u} \in \hat{\mathcal{C}}_{k+1}$ . The equality  $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\bar{u})$  equivalently reads as

$$\mathcal{F}_\beta(\hat{u}) = \theta_d(k) + \beta k = \theta_d(k+1) + \beta(k+1) = \mathcal{F}_\beta(\bar{u}),$$

by Lemma 8(i). Then for  $\beta := \beta_k := \theta_d(k) - \theta_d(k+1)$  one finds  $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\bar{u})$ .

For  $d \in \mathbb{R}^M$  given, this remark suggests the following set of  $M+1$  constants to be introduced:

$$\boxed{\beta_k := \theta_d(k) - \theta_d(k+1)}, \quad \forall k \in \mathbb{I}_M^0. \quad (3.1)$$

A few facts on  $\{\beta_k\}$  are given next.

**Remark 7** Let H1 hold. Using Lemma 2,  $\beta_k \geq 0$  for  $\forall k \in \mathbb{I}_M^0$ . Recall that  $d \neq 0$ .

(a) From H1, there is  $n \in \mathbb{N}$  such that  $\langle a_n, d \rangle \neq 0$ . Set  $\hat{u}_n = \arg \min_{v \in \mathbb{R}} \|a_n v - d\|^2 \in \mathbb{R}$ . Clearly,

$$\hat{u}_n = \frac{\langle a_n, d \rangle}{\|a_n\|^2} \neq 0 \quad \text{and} \quad \theta_d(1) \leq \|a_n \hat{u}_n - d\|^2 = \|d\|^2 - \frac{\langle a_n, d \rangle^2}{\|a_n\|^2} < \theta_d(0) = \|d\|^2 .$$

Then (3.1) yields

$$\beta_0 := \theta_d(0) - \theta_d(1) > 0 . \quad (3.2)$$

To unify the notation, we set  $\beta_{-1} := +\infty$ .

(b) From the definition of  $L$  in 2.14,  $\theta_d(L-1) > 0$  and  $\theta_d(k) = 0$  if  $k \geq L$ . So (3.1) leads to

$$\beta_{L-1} > 0 \quad \text{and} \quad \beta_L = 0 . \quad (3.3)$$

And if  $L < M$ , one has  $\beta_k = 0$  for any  $k \in \{L+1, \dots, M\}$ .

The next claim gives a flavor on the role of  $\{\beta_k\}$  as introduced in (3.1).

**Lemma 9** *Let H1 hold. Assume that  $\mathcal{F}_\beta$  has a global minimum at  $\hat{u} \in \hat{\mathcal{C}}_k$ . Then  $\beta \in [\beta_k, \beta_{k-1}]$ .*

**Proof.** Let  $\bar{u} \in \hat{\mathcal{C}}_{k+1}$  and  $\tilde{u} \in \hat{\mathcal{C}}_{k-1}$ . Since  $\hat{u} \in \hat{\mathcal{F}}_\beta$  for some  $\beta > 0$ , the following conditions must hold:

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 \quad \text{and} \quad \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\hat{u}) \geq 0 . \quad (3.4)$$

Combining this with (2.25) in Lemma 8(i) leads to

$$\begin{aligned} \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k+1) - \theta_d(k) + \beta = -\beta_k + \beta \geq 0 , \\ \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k-1) - \theta_d(k) - \beta = \beta_{k-1} - \beta \geq 0 . \end{aligned} \quad (3.5)$$

Hence  $\beta$  must satisfy  $\beta_k \leq \beta \leq \beta_{k-1}$ . □

The relation below is quite practical:

$$\beta_k + \beta_{k+1} + \dots + \beta_{k+n} = \theta_d(k) - \theta_d(k+n+1) , \quad \forall k \in \mathbb{I}_{M-1}^0 , \quad \forall n \in \mathbb{I}_{M-k-1}^0 . \quad (3.6)$$

Indeed,  $\sum_{m=k}^{k+n} \beta_m = \theta_d(k) - \theta_d(k+1) + \theta_d(k+1) - \theta_d(k) + \dots + \theta_d(k+n-1) - \theta_d(k+n) + \theta_d(k+n) - \theta_d(k+n+1)$ . For  $n = 0$ , (3.6) amounts to the definition of  $\beta_k$  in (3.1).

### 3.2 Particular cases

We exhibit a set of values for  $\beta$  so that  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  have the same optimal set.

**Theorem 5** *Let H1 hold and  $L$  be defined according to (2.14). For  $k \in \mathbb{I}_L^0$  assume that*

$$\bar{b}_{k-1} := \min_{n=1}^k \beta_{k-n} \geq \beta_{k-1} > \beta_k \geq \max_{n=0}^{L-k} \beta_{k+n} =: \underline{b}_k , \quad (3.7)$$

where all  $\beta_k$  read as in (3.1). Statements (i) and (ii) below are equivalent:

(i)  $\hat{u}$  is an optimal solution of  $(\mathcal{C}_k)$  ;

(ii)  $\hat{u}$  is a global minimizer of  $\mathcal{F}_\beta$  for  $\beta \in (\beta_k, \beta_{k-1})$ .

**Proof.** We break the proof into two parts.

[(i)  $\Rightarrow$  (ii)]. To prove the claim, we test the sufficient condition for global minimizer of  $\mathcal{F}_\beta$  in Theorem 4. Recall that by Lemma 8(i), if  $\tilde{u} \in \widehat{C}_n$  for  $n \in \mathbb{I}_L^0$ , then  $\mathcal{F}_\beta(\tilde{u}) = \theta_d(n) + \beta n$ . Let  $\hat{u} \in \widehat{C}_k$ .

• Let  $k \in \mathbb{I}_L$  and let  $\bar{u} \in \widehat{C}_{k-n}$  for  $n \in \mathbb{I}_k$ . Then

$$\begin{aligned} \forall n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k-n) - \theta_d(k) - n\beta \\ & \quad [ \text{by (3.6)} ] = \beta_{k-n} + \cdots + \beta_{k-1} - n\beta \\ & = \sum_{m=1}^n (\beta_{k-m} - \beta) \end{aligned} \quad (3.8)$$

$$[ \beta < \beta_{k-1} \leq \bar{b}_{k-1} \text{ by (3.7)} ] > 0. \quad (3.9)$$

• Consider that  $k \in \mathbb{I}_{L-1}^0$  and let  $\bar{u} \in \widehat{C}_{k+n}$  for  $n \in \mathbb{I}_{L-k}$ . Then

$$\begin{aligned} \forall n \in \mathbb{I}_{L-k} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k+n) - \theta_d(k) + n\beta \\ & \quad [ \text{by (3.6)} ] = -\beta_{k+n-1} - \cdots - \beta_k + n\beta \\ & = \sum_{m=0}^{n-1} (\beta - \beta_{k+m}) \end{aligned} \quad (3.10)$$

$$[ \beta > \beta_k \geq \underline{b}_k \text{ by (3.7)} ] > 0. \quad (3.11)$$

Inserting (3.11) and (3.9) in (2.32) in Theorem 4 shows that  $\hat{u} \in \widehat{F}_\beta$ .

[(ii)  $\Rightarrow$  (i)]. Since  $\hat{u} \in \widehat{F}$ , Theorem 3 shows that  $\hat{u} \in \widehat{C}$ , i.e. that  $\hat{u} \in \widehat{C}_n$  for some  $n \in \mathbb{I}_L^0$  which depends on  $\beta$ . We will show that any  $\bar{u} \in \widehat{C}_n$ ,  $n \neq k$ , cannot be global minimizer of  $\mathcal{F}_\beta$  for  $\beta \in (\beta_k, \beta_{k-1})$ .

Let  $\hat{u} \in \widehat{C}_k$ . A necessary condition for  $\mathcal{F}_\beta$  to have a global minimum at  $\bar{u} \in \widehat{C}_n$ ,  $n \neq k$  is that

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \leq 0 \quad \text{for some } \bar{u} \in \left( \bigcup_{n=1}^k \widehat{C}_{k-n} \right) \cup \left( \bigcup_{n=1}^{L-k} \widehat{C}_{k+n} \right). \quad (3.12)$$

• Using (3.8), the necessary condition (3.12) for  $\bar{u} \in \bigcup_{n=1}^k \widehat{C}_{k-n}$  reads as

$$\bar{u} \in \widehat{C}_{k-n} \quad n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = \sum_{m=1}^n (\beta_{k-m} - \beta) \leq 0.$$

According to this inequality, combined with (3.7),  $\beta$  must fulfill

$$\beta \geq \frac{1}{n} \sum_{m=1}^n \beta_{k-m} \geq \beta_{k-1}. \quad (3.13)$$

• From (3.10), the necessary condition (3.12) for  $\bar{u} \in \bigcup_{n=1}^{L-k} \widehat{C}_{k+n}$  is equivalent to

$$\bar{u} \in \widehat{C}_{k+n} \quad n \in \mathbb{I}_{L-k} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = \sum_{m=0}^{n-1} (\beta - \beta_{k+m}) \leq 0.$$

This inequality, combined with (3.7), show that  $\beta$  must satisfy

$$\beta \leq \frac{1}{n} \sum_{m=0}^{n-1} \beta_{k+m} \leq \beta_k . \quad (3.14)$$

Combining (3.14) and (3.13) yields  $\beta_{k-1} \leq \beta \leq \beta_k$  which contradicts (3.7).  $\square$

Let us slightly modify the condition in (3.7):

**Lemma 10** *Let H1 hold and let  $k \in \mathbb{I}_{L-1}$ . For  $p \in \{0, \dots, k-1\}$  assume that*

$$\bar{b}_{k-p-1} := \min_{n=1}^{k-p} \beta_{k-p-n} \geq \beta_{k-p-1} > \beta_{k-p} = \dots = \beta_k > \beta_{k+1} \geq \max_{n=1}^{L-k} \beta_{k+n} =: \underline{b}_{k+1} . \quad (3.15)$$

(i) *If  $\beta = \beta_k$  then  $\widehat{F}_\beta = \bigcup_{n=k-p}^{k+1} \widehat{C}_n$  where  $\widehat{C}_n \cap \widehat{C}_m = \emptyset$  for any  $m \neq n$  obeying  $(m, n) \in \{k-p, \dots, k+1\}$ .*

(ii) *If  $p \geq 1$  then  $\mathcal{F}_\beta$  can have a global minimum at  $\widehat{u} \in \widehat{C}_k$  only if  $\beta = \beta_k$ .*

**Proof.** Let  $\bar{u} \in \widehat{C}_{k-n}$  for  $n \in \mathbb{I}_k$ . By (3.8) and (3.15) we have  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = 0$  for any  $n \in \{1, \dots, p\}$ , while  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) > 0$  for any  $n \in \{p+1, \dots, k-p\}$ . Let now  $\bar{u} \in \widehat{C}_{k+n}$  for  $n \in \mathbb{I}_{L-k}$ . From (3.10) and (3.15),  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) = 0$  for  $n = 1$  whereas  $\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) > 0$  for any  $n \in \{2, \dots, L-k\}$ . Hence  $\mathcal{F}_\beta$  has a global minimizer for any  $\widehat{u} \in \widehat{C}_{k-p} \cup \dots \cup \widehat{C}_{k+1}$ . The rest of claim (i) follows from Theorem 3. By Lemma 9,  $\widehat{u} \in \widehat{C}_k$  can satisfy  $\widehat{u} \in \widehat{F}_\beta$  only if  $\beta_k \leq \beta \leq \beta_{k-1}$ . Since  $p \geq 1$ , we find  $\beta = \beta_k = \beta_{k-1}$ .  $\square$

According to the facts on  $\beta_k$  delivered in section 4, the case  $\beta_k = \beta_{k-1}$  should be quite exceptional. Another possibility is that  $\mathcal{F}_\beta$  can never have global minimizers with a desired  $\ell_0$ -norm.

**Lemma 11** *Let H1 be satisfied. Suppose that there are  $k \in \mathbb{I}_{L-1}^0$  and  $m \in \mathbb{I}_{L-k-1}^0$  such that*

$$n \in \{0, \dots, m\} \quad \Rightarrow \quad f_k(n) := \frac{1}{n+1} \sum_{p=0}^n \beta_{k-1+p} < \beta_{k+n} . \quad (3.16)$$

*Then any  $\widehat{u} \in \widehat{F}$  satisfies  $\widehat{u} \in \widehat{C} \setminus \bigcup_{n=0}^m \widehat{C}_{k+n}$  and  $\|\widehat{u}\|_0 \notin \{k, \dots, k+m\}$ .*

**Proof.** Observe that by (3.6),

$$f_k(n) = \frac{\theta_d(k-1) - \theta_d(k+n)}{n+1} .$$

The proof is by contradiction. For  $n \leq m$ , assume that  $\mathcal{F}_\beta$  has a global minimizer  $\widehat{u}$  satisfying  $\widehat{u} \in \widehat{C}_{k+n}$ . Let  $\bar{u} \in \widehat{C}_{k+n+1}$  and let  $\tilde{u} \in \widehat{C}_{k-1}$ . Then the necessary conditions below must be satisfied:

$$\begin{aligned} \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\widehat{u}) &= \theta_d(k+n+1) - \theta_d(k+n) + \beta = -\beta_{k+n} + \beta \geq 0 , \\ \mathcal{F}_\beta(\tilde{u}) - \mathcal{F}_\beta(\widehat{u}) &= \theta_d(k-1) - \theta_d(k+n) - (n+1)\beta = (n+1)f_k(n) - (n+1)\beta \geq 0 . \end{aligned}$$

One obtains  $\beta_{k+n} \leq \beta \leq f_k(n)$ , which contradicts (3.16). This proves the lemma.  $\square$

Since  $\beta_0 > 0$  and  $\beta_L = 0$  (Remark 7), if (3.16) occurs, the sequence  $\{\beta_k\}$  defined in (3.1) oscillates.

### 3.3 Necessary and sufficient conditions for quasi-equivalence

**Theorem 6** *Let H1 hold and  $\mathbb{L}$  be defined according to (2.14). Then*

$$k \in \mathbb{I}_{\mathbb{L}}^0 \quad \Rightarrow \quad \left[ \hat{u} \in \hat{\mathcal{C}}_k \iff \hat{u} \in \hat{\mathcal{F}}_{\beta} \text{ for } \beta \in (\beta_k, \beta_{k-1}) \right] \quad (3.17)$$

if and only if

$$\beta_0 > \beta_1 > \cdots > \beta_{\mathbb{L}-1} > \beta_{\mathbb{L}} . \quad (3.18)$$

**Proof.** Let (3.18) hold. Then the condition on  $\beta_k$  given in (3.7), Theorem 5 is satisfied for any  $k$ . Applying Theorem 5 for any  $k \in \mathbb{I}_{\mathbb{L}}^0$  proves (3.17).

From Remark 7, see (3.3),  $\beta_{\mathbb{L}-1} > \beta_{\mathbb{L}} = 0$  in all cases. To see that (3.18) is necessary for all other values of  $k$ , we argue by contradiction. Suppose there is  $k \in \mathbb{I}_{\mathbb{L}-1}$  so that  $\beta_{k-1} \leq \beta_k$ .

- The case  $\beta_{k-1} < \beta_k$  corresponds to  $m = 0$  in (3.16), Lemma 11. So (3.17) fails by this lemma.
- Let  $\beta_{k-1} = \beta_k$ . Using Lemma 10 for  $p = 1$  yields  $\hat{\mathcal{F}}_{\beta} = \hat{\mathcal{C}}_{k-1} \cup \hat{\mathcal{C}}_k \cup \hat{\mathcal{C}}_{k+1}$  with  $\hat{\mathcal{C}}_{k-1} \cap \hat{\mathcal{C}}_k = \hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_{k+1} = \emptyset$ . So (3.17) fails.  $\square$

Naturally, the boundary cases  $\beta = \beta_k$  for  $k \in \mathbb{I}_{\mathbb{L}-1}$  arise the curiosity.

**Proposition 2** *Let H1 hold. Assume that (3.18) is satisfied. Let  $k \in \mathbb{I}_{\mathbb{L}-1}$ . Then*

$$\beta = \beta_k \quad \Rightarrow \quad \hat{\mathcal{F}}_{\beta} = \hat{\mathcal{C}}_k \cup \hat{\mathcal{C}}_{k+1} \quad \text{where} \quad \hat{\mathcal{C}}_k \cap \hat{\mathcal{C}}_{k+1} = \emptyset .$$

**Proof.** Applying Lemma 10 for  $p = 0$  proves the proposition.  $\square$

Under the condition in (3.18), we can say that problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_{\beta})$  are *quasi-equivalent*: we have a partition  $\mathbb{R}_+ = \bigcup_{k=0}^{\mathbb{L}} (\beta_{k-1}, \beta_k]$  so that

- For every  $k \in \mathbb{I}_{\mathbb{L}}^0$ , problems  $(\mathcal{C}_k)$  and  $(\mathcal{R}_{\beta})$  for  $\beta \in (\beta_{k-1}, \beta_k)$  have exactly the same optimal set;
- For each  $\beta \in \{\beta_k\}_{k=0}^{\mathbb{L}-1}$  there is no equivalence since Proposition 2 holds.

### 3.4 Partial quasi-equivalence

**Remark 8** The condition in (3.18) reads as

$$\beta_{k-1} > \beta_k \quad \forall k \in \mathbb{I}_{\mathbb{L}-1} \quad \Leftrightarrow \quad \theta_d(k) < \frac{1}{2} (\theta_d(k-1) + \theta_d(k+1)) \quad \forall k \in \mathbb{I}_{\mathbb{L}-1} .$$

Its realization seems unstable, relying on  $A$  or  $d$ ; see Lemmas 10 and 11, and the tests in section 5.

**Theorem 7** *Let H1 hold. For  $k \geq 1$  and  $m \in \{k+2, \dots, \mathbb{L}\}$ , assume that the following holds:*

- (a) *If  $\bar{u} \in \bigcup_{n=k+1}^{m-1} \hat{\mathcal{C}}_n$  then for any  $\beta > 0$ ,  $\bar{u}$  is not a global minimizer of  $\mathcal{F}_{\beta}$ ;*

(b) The critical parameter value  $\bar{\beta}_{k,m}$  given by

$$\bar{\beta}_{k,m} := \frac{\theta_d(k) - \theta_d(m)}{m - k} \quad (3.19)$$

satisfies

$$\bar{b}_{k-1} := \min_{n=1}^k \beta_{k-n} \geq \beta_{k-1} > \bar{\beta}_{k,m} > \beta_m \geq \max_{n=0}^{L-m} \beta_{m+n} =: \underline{b}_m. \quad (3.20)$$

Then one has the following equivalences:

$$(i) \quad \hat{u} \in \widehat{C}_m \iff \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\beta_m, \bar{\beta}_{k,m});$$

$$(ii) \quad \hat{u} \in \widehat{C}_k \iff \hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\bar{\beta}_{k,m}, \beta_{k-1}).$$

**Proof.** We present only the proof of (i), since (ii) is proven likewise. It is split in two parts.

[  $\Rightarrow$  ] Let  $\hat{u} \in \widehat{C}_m$  and let  $\beta \in (\beta_m, \bar{\beta}_{k,m})$ .

• Let  $\bar{u} \in \widehat{C}_{k-n}$  for  $n \geq 0$ . From (3.19) it follows that

$$\begin{aligned} n = 0 \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k) - \theta_d(m) - (m - k)\beta \\ &= (m - k)(\bar{\beta}_{k,m} - \beta) \end{aligned} \quad (3.21)$$

$$[ \text{ by (3.20) } ] > 0. \quad (3.22)$$

Consider next that  $n \geq 1$ .

$$\begin{aligned} \forall n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) &= \theta_d(k - n) - \theta_d(m) - (m - k + n)\beta \\ &= \theta_d(k - n) - \theta_d(k) + \theta_d(k) - \theta_d(m) - (m - k)\beta - n\beta \end{aligned}$$

$$[ \text{ by (3.19) or by (3.6) } ] = (m - k)(\bar{\beta}_{k,m} - \beta) + \theta_d(k - n) - \theta_d(k) - n\beta$$

$$[ \text{ by (3.8) } ] = (m - k)(\bar{\beta}_{k,m} - \beta) + \sum_{p=1}^n (\beta_{k-p} - \beta) \quad (3.23)$$

$$[ \text{ by (3.20) } ] > 0. \quad (3.24)$$

• Let  $\bar{u} \in \widehat{C}_{m+n}$  for  $n \geq 1$ .

$$\forall n \in \mathbb{I}_{L-m} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = \theta_d(m + n) - \theta_d(m) + n\beta$$

$$[ \text{ by (3.10) or by (3.6) } ] = \sum_{p=0}^{n-1} (\beta - \beta_{m+p}) \quad (3.25)$$

$$[ \text{ by (3.20) } ] > 0. \quad (3.26)$$

From (3.22), (3.24) and (3.26),  $\hat{u} \in \widehat{F}_\beta$  according to Theorem 4.

[  $\Leftarrow$  ] Let  $\beta \in (\beta_m, \bar{\beta}_{k,m})$  and let  $\hat{u} \in \widehat{C}_m$ . By Theorem 3 and assumption (a),  $\widehat{F} \subset \widehat{C} \setminus \bigcup_{n=k+1}^{m-1} \widehat{C}_n$ .

Suppose that  $\mathcal{F}_\beta$  has a global minimum at  $\bar{u} \in \widehat{C}_p$  for  $p \neq m$ . Then necessarily

$$\mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) \leq 0 \quad \text{for some } \bar{u} \in \left( \bigcup_{n=0}^k \widehat{C}_{k-n} \right) \cup \left( \bigcup_{n=1}^{L-m} \widehat{C}_{m+n} \right). \quad (3.27)$$

- We examine (3.27) for  $\bar{u} \in \bigcup_{n=0}^k \widehat{\mathcal{C}}_{k-n}$ . Combining (3.21) and (3.27) yields

$$\bar{u} \in \widehat{\mathcal{C}}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = (m-k)(\bar{\beta}_{k,m} - \beta) \leq 0 .$$

Then  $\beta$  must satisfy

$$\beta \geq \bar{\beta}_{k,m} . \quad (3.28)$$

Further, using (3.23) and (3.27) lead to

$$\bar{u} \in \widehat{\mathcal{C}}_{k-n} \quad n \in \mathbb{I}_k \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = (m-k)\bar{\beta}_{k,m} + \sum_{p=1}^n \beta_{k-p} - (m-k+n)\beta \leq 0 .$$

This inequality, combined with (3.20) shows that

$$\beta \geq \frac{1}{m-k+n} \left( (m-k)\bar{\beta}_{k,m} + \sum_{p=1}^n \beta_{k-p} \right) \geq \frac{(m-k)\bar{\beta}_{k,m} + n\beta_{k-1}}{m-k+n} > \bar{\beta}_{k,m} . \quad (3.29)$$

- Using (3.25), the condition (3.27) for  $\bar{u} \in \bigcup_{n=1}^{L-m} \widehat{\mathcal{C}}_{m+n}$  is equivalent to

$$\bar{u} \in \widehat{\mathcal{C}}_{m+n} \quad n \in \mathbb{I}_{L-m} \quad \Rightarrow \quad \mathcal{F}_\beta(\bar{u}) - \mathcal{F}_\beta(\hat{u}) = n\beta - \sum_{p=0}^{n-1} \beta_{m+p} \leq 0 .$$

By this inequality and (3.20),  $\beta$  must fulfill

$$\beta \leq \frac{1}{n} \sum_{p=0}^{n-1} \beta_{m+p} \leq \beta_m . \quad (3.30)$$

Combining the results in (3.28), (3.29) and (3.30) with assumption (a) shows that  $\mathcal{F}_\beta$  for  $\beta \in (\beta_m, \bar{\beta}_{k,m})$  cannot have a global minimum at  $\bar{u} \in \widehat{\mathcal{C}}_p$  if  $p \in \{0, \dots, k\} \cup \{m+1, \dots, L\}$ .  $\square$

**Remark 9** Likewise Remark 6, for  $\beta = \bar{\beta}_{k,m}$  one has  $\mathcal{F}_\beta(\hat{u}) = \mathcal{F}_\beta(\tilde{u})$ , so if (a)-(b) hold, one finds  $\widehat{\mathcal{F}}_\beta = \widehat{\mathcal{C}}_k \cup \widehat{\mathcal{C}}_m$  with  $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_m = \emptyset$ ; see Corollary 2 and Proposition 2.

**Example 2** Let  $\beta_{k-1} < \beta_k$  for  $k \in \mathbb{I}_{L-1}$ . Then by Lemma 11 for any  $\beta > 0$ , any global minimizer  $\hat{u}$  of  $\mathcal{F}_\beta$  obeys  $\hat{u} \notin \widehat{\mathcal{C}}_k$  and hence  $\|\hat{u}\|_0 \neq k$  by Theorem 1. Assume that

$$\min_{n=2}^k \beta_{k-n} \geq \beta_{k-2} > \bar{\beta}_{k-1,k+1} := \frac{\theta_d(k-1) - \theta_d(k+1)}{2} > \beta_{k+1} \geq \max_{n=1}^{L-k} \beta_{k+n} . \quad (3.31)$$

Here  $\bar{\beta}_{k-1,k+1}$  is calculated according to (3.19). Theorems 7 and 1 show that

$$\hat{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\beta_{k+1}, \bar{\beta}_{k-1,k+1}) \Leftrightarrow \hat{u} \in \widehat{\mathcal{C}}_{k+1} \text{ and } \|\hat{u}\|_0 = k+1 ,$$

$$\tilde{u} \text{ is a global minimizer of } \mathcal{F}_\beta \text{ for } \beta \in (\bar{\beta}_{k-1,k+1}, \beta_{k-2}) \Leftrightarrow \tilde{u} \in \widehat{\mathcal{C}}_{k-1} \text{ and } \|\tilde{u}\|_0 = k-1 .$$

For  $\beta = \bar{\beta}_{k-1,k+1}$  one has  $\widehat{\mathcal{F}}_\beta = (\mathcal{C}_{k-1}) \cup (\mathcal{C}_{k+1})$ .

## 4 On the critical parameter values

From Theorem 1 and Theorem 3, any optimal solution  $\hat{u}$  of problems  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  in (1.2) and (2.4) is a solutions of problem  $(\mathcal{P}_\omega)$  for  $\omega := \sigma(\hat{u}) \in \Omega$ . Then  $\hat{u}$  is of the form

$$\hat{u} = Z_\omega(\hat{u}_\omega) \quad \text{where} \quad \hat{u}_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d. \quad (4.1)$$

**Remark 10** The optimal solutions  $\hat{u}$  of  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  should in general be sensitive to noise since their non-null part  $\hat{u}_{\sigma(\hat{u})}$  minimizes a non-regularized least-squares objective relying on  $A_{\sigma(\hat{u})}$  and  $d$ .

With any  $\omega \in \Omega$  we associate a matrix  $B_\omega$  whose columns are an orthonormal basis for  $A_\omega$ , that is

$$B_\omega^T B_\omega = I_{\#_\omega} \quad \text{and} \quad \text{range}(B_\omega) = \text{range}(A_\omega). \quad (4.2)$$

The orthogonal projector  $\Pi_\omega$  onto the subspace spanned by the columns of  $A_\omega$  reads as

$$\Pi_\omega = B_\omega B_\omega^T. \quad (4.3)$$

For  $\omega \in \Omega$ , one has  $\Pi_\omega = A_\omega (A_\omega^T A_\omega)^{-1} A_\omega^T$ ; see, e.g., [28]. Then

$$\hat{u} \in \hat{\mathcal{C}}_k \quad \text{for } k \in \mathbb{I}_L^0 \quad \text{and} \quad \hat{\sigma} := \sigma(\hat{u}) \quad \Rightarrow \quad \theta_d(k) = \|(I - \Pi_{\hat{\sigma}})d\|^2. \quad (4.4)$$

Using the definition of  $\Omega_k$  in (2.19), we introduce the subsets of  $\mathbb{R}^M$  given below:

$$E_k := \bigcup_{\omega \in \Omega_k} \ker(A_\omega^T) \quad \text{and} \quad G_k := \bigcup_{\omega \in \Omega_k} \text{range}(A_\omega). \quad (4.5)$$

Clearly,  $G_M = \mathbb{R}^M$  and  $E_M = \emptyset$  by H1.

**Lemma 12** *Let H1 hold and  $d \neq 0$ .*

(i) *Let  $k \in \{2, \dots, M-1\}$ . Then*

$$d \in \mathbb{R}^M \setminus E_k \quad \Rightarrow \quad \theta_d(k-1) > \theta_d(k).$$

(ii) *Let  $k \in \{1, \dots, M-1\}$ . Then*

$$d \in \mathbb{R}^M \setminus G_k \quad \Leftrightarrow \quad \theta_d(k) > 0.$$

**Proof.** (i) Let  $\hat{u} \in \hat{\mathcal{C}}_{k-1}$ . Set  $\hat{\sigma} := \sigma(\hat{u})$ . From Theorem 1  $\hat{\sigma} \in \Omega_{k-1}$  and then

$$\exists n \in \mathbb{I}_N \setminus \hat{\sigma} \quad \text{such that} \quad \omega := \hat{\sigma} \cup \{n\} \in \Omega_k.$$

Obviously,  $\theta_d(k) \leq T(k) := \|(I - \Pi_\omega)d\|^2$ . Using the notation in (4.2), it follows that (see, e.g. [24])

$$\exists b_k \in \text{range}(A_\omega) \quad \text{such that} \quad B_\omega = [B_{\hat{\sigma}} \mid b_k].$$

By (4.3), one has  $\Pi_\omega = B_{\hat{\sigma}}B_{\hat{\sigma}}^T + b_k b_k^T$ . Therefore

$$\theta_d(k-1) - \theta_d(k) \geq \theta_d(k-1) - T(k) = d^T (\Pi_\omega - \Pi_{\hat{\sigma}}) d = \langle b_k, d \rangle^2.$$

From  $d \in \mathbb{R}^M \setminus E_k$  we have  $d \notin (\text{range}(A_\omega))^\perp$ . Hence  $\langle b_k, d \rangle^2 > 0$ .

(ii) Let  $d \in \mathbb{R}^M \setminus G_k$  and let  $\hat{u} \in \hat{C}_k$ . Set  $\hat{\sigma} := \sigma(\hat{u})$ . From Theorem 1,  $\hat{\sigma} \in \Omega_k$ . Since  $(I - \Pi_{\hat{\sigma}})$  is the orthogonal projector onto  $(\text{range}(A_{\hat{\sigma}}))^\perp$ , having  $d \notin \text{range}(A_\omega)$  leads to

$$\theta_d(k) = \|A\hat{u} - d\|^2 = \|(I - \Pi_{\hat{\sigma}})d\|^2 > 0.$$

Conversely, let  $\theta_d(k) > 0$ . By Theorem 1, the support of any optimal solution of  $(\mathcal{C}_k)$  belongs to  $\Omega_k$ . To get a contradiction, suppose that  $d \in G_k$ : then there is  $\omega \in \Omega_k$  such that  $d \in \text{range}(A_\omega)$  and for  $u_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d$  one has  $\|A_\omega u_\omega - d\|^2 = \|(I - \Pi_\omega)d\|^2 = 0$ .  $\square$

Observe that the condition on  $d$  in Lemma 12(i) is a strong sufficient condition.

**Proposition 3** *Let H1 hold and  $d \in \mathbb{R}^M \setminus (E_2 \cup G_{L-1})$ . Then the critical parameters  $\beta_k$  in (3.1) obey*

$$\beta_k > 0 \quad \forall k \in \mathbb{I}_{L-1}^0 \quad \text{and} \quad \beta_k = 0 \quad \forall k \geq L.$$

**Proof.** Let  $k \in \mathbb{I}_{L-1}$ . Let  $\omega \in \Omega_k$ , i.e.,  $\text{rank}(A_\omega) = k$ . Then for any  $\bar{\omega} \subset \omega$  such that  $\#\bar{\omega} = k-1$  one has  $\bar{\omega} \in \Omega_{k-1}$ . Since  $\text{range}(A_{\bar{\omega}}) \subset \text{range}(A_\omega)$ , one has  $(\mathbb{R}^M \setminus G_{k-1}) \supsetneq (\mathbb{R}^M \setminus G_k)$  by (4.5). Hence

$$d \in \mathbb{R}^M \setminus G_{L-1} \quad \Rightarrow \quad \theta_d(k) > 0 \quad \forall k \in \mathbb{I}_{L-1}^0$$

and  $\theta_d(L) = 0$  by (2.14). From  $\ker(A_\omega^T) \subset \ker(A_{\bar{\omega}}^T)$ , one has  $(\mathbb{R}^M \setminus E_k) \supsetneq (\mathbb{R}^M \setminus E_{k-1})$ ; see (4.5). This, combined with Lemma 12(i) and the definition of  $\beta_k$  in (3.1) shows that

$$d \in \mathbb{R}^M \setminus E_2 \quad \Rightarrow \quad \beta_{k-1} = \theta_d(k-1) - \theta_d(k) > 0 \quad \forall k \in \{2, \dots, L\}.$$

And  $\beta_0 > 0$  by Remark 7; see (3.2).  $\square$

**Remark 11** The subset  $E_2$  is a finite union of vector subspaces of dimension  $M-2$ . Likewise,  $G_{L-1}$  is a finite union of vector subspaces of dimension  $L-1 \leq M-1$ . Hence  $E_2 \cup G_{L-1}$  is closed in  $\mathbb{R}^M$  and its Lebesgue measure in  $\mathbb{R}^M$  is null. So data located in  $E_2 \cup G_{L-1}$  are *highly* exceptional. Conversely,  $\mathbb{R}^M \setminus (E_2 \cup G_{L-1})$  is open and dense in  $\mathbb{R}^M$ , so we usually have data living in this subset.

## 5 Numerical tests

Here we summarize the outcome of a series of experiments corresponding to different matrices  $A \in \mathbb{R}^{M \times N}$  for  $(M, N) \in \{(5, 10), (10, 20)\}$ , original vectors  $u_o \in \mathbb{R}^N$  and data samples  $d = Au_o(+\text{noise})$ . *All results were calculated using an exhaustive combinatorial search.* In all experiments we carried out, the following facts were observed:

- All claims concerning  $(\mathcal{C}_k)$  (Lemma 3, Theorem 1, Proposition 1, Theorem 3) were confirmed.
- For noisy data every time we had  $L = M$ ; see Remark 2.
- We always had  $\beta_k > 0, \forall k \in \mathbb{I}_{L-1}^0$  and  $\beta_L = 0$ , which is the general case by Proposition 3.
- For any original  $u_o$  there are many  $A$  and  $d = Au_o(+\text{noise})$ , and for any  $A$  there are many  $d$  so that the critical parameter sequence  $\{\beta_k\}$  defined in (3.1) is either strictly decreasing or oscillating. In all cases the conditions for partial or quasi-equivalence (Theorems 5, 6 and 7) were corroborated.
- The numerical instabilities, evoked in Remark 10, were always present.

Next we present in detail three experiments for  $(M, N) = (5, 10)$  where

$$A = \begin{pmatrix} 13.94 & 16.36 & 4.88 & -3.09 & -15.42 & 1.31 & -3.18 & -12.13 & -4.26 & -10.09 \\ 7.06 & -6.48 & -9.07 & -8.37 & -2.72 & -17.42 & -5.83 & -3.81 & 3.87 & -1.80 \\ 11.63 & 6.73 & -4.75 & -6.28 & 3.42 & 6.68 & -1.64 & 13.23 & 9.03 & -20.27 \\ -7.54 & 12.74 & -6.66 & 5.01 & 4.84 & 8.98 & -9.35 & 3.85 & 7.18 & 4.09 \\ 3.22 & -10.40 & -5.02 & 16.70 & 9.53 & -5.49 & 11.88 & -3.62 & 17.36 & 7.34 \end{pmatrix}$$

$$u_o = (0 \ 4 \ 0 \ 0 \ 0 \ 9 \ 0 \ 0 \ 3 \ 0)^T. \quad (5.1)$$

The entries of  $A$  follow a nearly normal distribution with variance 10. One has  $\text{rank}(A) = M = 5$ , so H1 holds. Problem  $(\mathcal{C}_M)$  is unconstrained with  $\#\Omega_M = 252$  optimal solutions; none of them is shown.

**Noise-free data** Data is given by

$$d = Au_o = (64.45 \ -171.09 \ 114.13 \ 153.32 \ -38.93)^T. \quad (5.2)$$

Since data are noise-free and  $\|u_o\|_0 = 3$ , it is clear that  $\hat{u} := u_o$  is an optimal solution to problems

Table 1: The optimal set of  $(\mathcal{C}_k)$  for  $k \in \mathbb{I}_3^0$  where  $d$  is given in (5.2). For  $k = 3$ , one has exact recovery,  $\hat{u} = u_o$ ; see (5.1). The values of  $\beta_k$  and  $\bar{\beta}_{1,3}$  are given in (5.3) and (5.4), respectively.

k	Optimal solution of $(\mathcal{C}_k)$	Global min. of $\mathcal{F}_\beta$	$\theta_d(k)$
3	0 4 0 0 0 9 0 0 3 0	$\beta \in (0, \bar{\beta}_{1,3})$	0
2	0 <b>3.25</b> 0 0 0 <b>9.29</b> 0 0 0 0	<b>no</b>	3968
1	0 0 0 0 0 <b>11.76</b> 0 0 0 0	$\beta \in (\bar{\beta}_{1,3}, \beta_0)$	7745
0	0 0 0 0 0 0 0 0 0 0	$\beta > \beta_0$	$7.147 \times 10^4$

$(\mathcal{C}_3)$ ,  $(\mathcal{C}_4)$  and  $(\mathcal{C}_5)$  with  $\theta_d(3) = \theta_d(4) = \theta_d(5) = 0$ . So  $L = 3$ ; see (2.14). The other optimal values are seen in Table 1. By Theorem 3, any  $\hat{u} \in \hat{F}$  obeys  $\|\hat{u}\|_0 \leq 3$ . The critical parameters  $\{\beta_k\}$  in (3.1) are

$$\beta_0 = 6.373 \times 10^4, \quad \beta_1 = 3777, \quad \beta_2 = 3968, \quad \beta_3 = 0. \quad (5.3)$$

Since  $\beta_2 > \beta_1$ , Lemma 11 for  $m = 0$  tells us that any  $\hat{u} \in \hat{F}$  obeys  $\hat{u} \notin \hat{C}_3$  and  $\|\hat{u}\|_0 \neq 2$ . We compute  $\bar{\beta}_{1,3}$  according to (3.19) and verify that (3.20) is satisfied:

$$\beta_0 > \bar{\beta}_{1,3} := \frac{1}{2}(\theta_d(1) - \theta_d(3)) = 3872.46 > \beta_3. \quad (5.4)$$

By Theorem 7,  $\hat{C}_3 = \{ \hat{F}_\beta : \beta \in (\beta_3, \bar{\beta}_{1,3}) \}$ , as well as  $\hat{C}_1 = \{ \hat{F}_\beta : \beta \in (\bar{\beta}_{1,3}, \beta_0) \}$ .

**Noisy data 1.** Data are corrupted with nearly normal, centered, i.i.d. noise and SNR= 32.32 dB:

$$d = ( 69.13 \quad -171.95 \quad 113.74 \quad 150.27 \quad -36.09 )^T . \quad (5.5)$$

The optimal solutions are given in Table 2. Obviously,  $L = M = 5$ . The critical values  $\{\beta_k\}$  read

Table 2: The optimal value and solution of  $(C_k)$  for  $k \in \mathbb{I}_4$  where  $d$  is given in (5.5). The values of  $\beta_k$  and  $\bar{\beta}_{1,3}$  are given in (5.6) and (5.7), respectively.

k	$\theta_d(k)$	Optimal solution of $(C_k)$										Global min. of $\mathcal{F}_\beta$
4	0.0681	0	<b>4.40</b>	0	0	0	<b>8.71</b>	<b>0.54</b>	0	<b>2.95</b>	0	$\beta \in (\beta_4, \beta_3)$
3	36.31	0	<b>4.09</b>	0	0	0	<b>8.88</b>	0	0	<b>3.01</b>	0	$\beta \in (\beta_3, \bar{\beta}_{1,3})$
2	4039	0	<b>3.33</b>	0	0	0	<b>9.17</b>	0	0	0	0	<b>no</b>
1	8012	0	0	0	0	0	<b>11.71</b>	0	0	0	0	$\beta \in (\bar{\beta}_{1,3}, \beta_0)$
0	$7.117 \times 10^4$	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

$$\beta_0 = 6.315 \times 10^4 \quad \beta_1 = 3973 \quad \beta_2 = 4003 \quad \beta_3 = 36.25 \quad \beta_4 = 0.0681 \quad \beta_5 = 0 . \quad (5.6)$$

On has  $\beta_2 > \beta_1$ ; by Lemma 11 for  $m = 0$ , any  $\hat{u} \in \hat{F}$  verifies  $\hat{u} \notin \hat{C}_2$  and  $\|\hat{u}\|_0 \neq 2$ . Using Theorem 7,

$$\beta_0 > \bar{\beta}_{1,3} := \frac{1}{2}(\theta_d(1) - \theta_d(3)) = 3987.6848 > \beta_3 > \beta_4 > \beta_5 , \quad (5.7)$$

so  $\hat{C}_3 = \{ \hat{F}_\beta : \beta \in (\beta_3, \bar{\beta}_{1,3}) \}$  and  $\hat{C}_1 = \{ \hat{F}_\beta : \beta \in (\bar{\beta}_{1,3}, \beta_0) \}$ . Further,  $\hat{C}_5 = \{ \hat{F}_\beta : \beta \in (\beta_4, 0) \}$ .

**Noisy data 2.** The noise is nearly normal, centered, i.i.d., SNR= 25.74 dB:

$$d = ( 66.67 \quad -169.08 \quad 101.56 \quad 149.38 \quad -39.50 )^T . \quad (5.8)$$

The critical values  $\{\beta_k\}$  introduced in (3.1) read as

Table 3: The optimal values and solutions of  $(C_k)$  for  $k \in \mathbb{I}_4$  where  $d$  is given in (5.8). Here  $\{\beta_k\}$  is strictly decreasing, see (5.9), so  $(\mathcal{R}_\beta)$  and  $(C_k)$  are quasi-equivalent.

k	$\theta_d(k)$	Optimal solution of $(C_k)$										Global min. of $\mathcal{F}_\beta$
4	0.02588	0	<b>8.54</b>	0	0	<b>4.49</b>	<b>4.90</b>	<b>2.73</b>	0	0	0	$\beta \in (\beta_4, \beta_3)$
3	72.76	0	<b>3.93</b>	0	0	0	<b>8.70</b>	0	0	<b>2.63</b>	0	$\beta \in (\beta_3, \beta_2)$
2	3110	0	<b>3.27</b>	0	0	0	<b>8.95</b>	0	0	0	0	$\beta \in (\beta_2, \beta_1)$
1	6935	0	0	0	0	0	<b>11.44</b>	0	0	0	0	$\beta \in (\beta_1, \beta_0)$
0	$6.7 \times 10^4$	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

$$\beta_0 = 6.029 \times 10^4 \quad \beta_1 = 3825 \quad \beta_2 = 3037 \quad \beta_3 = 72.73 \quad \beta_4 = 0.02588 \quad \beta_5 = 0 . \quad (5.9)$$

The sequence  $\{\beta_k\}$  is strictly decreasing, so  $(\mathcal{R}_\beta)$  and  $(C_k)$  are quasi-equivalent (see Theorem 6).

## 6 Conclusions and future work

The obtention of a complete description of the relationship between the optimal solutions of nonconvex problems is in general unapparent task. We have fully clarified the relationship between the optimal solutions of least-squares regularized by  $\|\cdot\|_0$  via a parameter  $\beta > 0$  (problem  $(\mathcal{R}_\beta)$ ) and constrained by  $k$ -sparsity (problem  $(\mathcal{C}_k)$ ). New facts on the optimal solutions of  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  were given. They can help the refinement or the conception of algorithms. We exhibited critical values  $\{\beta_k\}$  so that  $(\mathcal{C}_k)$  and  $(\mathcal{R}_\beta)$  for  $\beta \in (\beta_k, \beta_{k-1})$  share exactly the same optimal set. But full equivalence (in a Lagrangian sense, as for convex problems) is impossible because for these critical values there is no equivalence. Further, there can be subsets of integers  $\{k_n\}$  so that *for any  $n$ , any optimal solution  $\hat{u}$  of  $(\mathcal{R}_\beta)$  for any  $\beta > 0$  fulfills  $\|\hat{u}\|_0 \neq k_n$ , whereas any optimal solution  $\tilde{u}$  of  $\mathcal{C}_{k_n}$  obeys  $\|\tilde{u}\|_0 = k_n$* . All these  $\{\beta_k\}$  and  $\{k_n\}$  are sensitive to  $A$  or to  $d$ . Our comparative results can clarify a proper choice between models  $(\mathcal{R}_\beta)$  and  $(\mathcal{C}_k)$  in applications. Extensions to penalties of the form  $\|Du\|_0$  for  $D$  a linear operator, or to low rank matrix recovery, seem important.

## Acknowledgments

Our work was supported by the “FMJH Program Gaspard Monge in optimization and operation research”, and by the support to this program from EDF.

## References

- [1] H. ATTOUCH, J. BOLTE, AND B. F. SVAITER, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized gausseidel methods*, Math. Program., 137 (2013).
- [2] E. VAN DEN BERG AND M. P. FRIEDLANDER, *Sparse optimization with least-squares constraints*, SIAM J. Optim., 21 (2011), pp. 1201–1229.
- [3] D. BLANCHARD, C. CARTIS, J. TANNER, AND A. THOMPSON, *Phase transitions for greedy sparse approximation algorithms*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 188–203.
- [4] T. BLUMENSATH AND M. DAVIES, *Iterative thresholding for sparse approximations*, J. Fourier Anal. Appl., 14 (2008), pp. 629–654.
- [5] T. BLUMENSATH AND M. DAVIES, *Iterative hard thresholding for compressed sensing*, Appl. Comput. Harmon. Anal., 27 (2009), pp. 265–274.
- [6] T. BLUMENSATH AND M.E. DAVIES, *Normalized iterative hard thresholding: Guaranteed stability and performance*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 298–309.

- [7] A. M. BRUCKSTEIN, D. L. DONOHO, AND M. ELAD, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev., 51 (2009), pp. 34–81.
- [8] E. CANDÈS, J. ROMBERG, AND T. TAO, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information*, IEEE Trans. Inform. Theory, 52 (2006), pp. 489–509.
- [9] C. CARTIS AND A. THOMPSON, *A new and improved quantitative recovery analysis for iterative hard thresholding algorithms in compressed sensing*, arXiv:1309.5406, (2013).
- [10] E. CHOUZENOUX, A. JEZIERSKA, J.-C. PESQUET, AND H. TALBOT, *A majorize-minimize subspace approach for  $\ell_2 - \ell_0$  image regularization*, SIAM J. Imaging Sci., 6 (2013), pp. 563–591.
- [11] A. COHEN, W. DAHMIEN, AND R. A. DEVORE, *Compressed sensing and best  $k$ -term approximation*, J. Amer. Math. Soc., 22 (2009), pp. 211–231.
- [12] G. DAVIS, S. MALLAT, AND M. AVELLANEDA, *Adaptive greedy approximations*, Constructive approximation, 13 (1997), pp. 57–98.
- [13] R. A. DEVORE, *Nonlinear approximation*, Acta Numerica, 7 (1998).
- [14] B. DONG AND Y. ZHANG, *An efficient algorithm for  $\ell_0$  minimization in wavelet frame based image restoration*, J. Sci. Comput., 54 (2013).
- [15] D.L. DONOHO, *Compressed sensing*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.
- [16] M. ELAD AND M. AHARON, *Image denoising via learned dictionaries and sparse representation*, in Computer Vision and Pattern Recognition, vol. 1, IEEE, 2006, pp. 895–900.
- [17] Y. C. ELDAR AND G. KUTYNIOK, *Compressed Sensing: Theory and Applications*, Cambridge Univ. Press, 2012.
- [18] J. FAN AND R. LI, *Statistical challenges with high dimensionality: feature selection in knowledge discovery*, in Proc. Intern. Congr. Math., vol. 3, Eur. Math. Soc., Zürich, 2006, pp. 595–622.
- [19] M. FORNASIER AND R. WARD, *Iterative thresholding meets free-discontinuity problems*, Found. Comput. Math., 10 (2010), pp. 527–567.
- [20] G. M. FUNG AND O. L. MANGASARIAN, *Equivalence of minimal  $\ell_0$ - and  $\ell_p$ -norm solutions of linear equalities, inequalities and linear programs for sufficiently small  $p$* , J. Optim. Theory Appl., 151 (2011), pp. 1–10.
- [21] J. HAUPT AND R. NOWAK, *Signal reconstruction from noisy random projections*, IEEE Trans. Inform. Theory, 52 (2006), pp. 4036–4048.

- [22] Y. JIAO, B. JIN, AND X. LU, *A primal dual active set algorithm for a class of nonconvex sparsity optimization*, arXiv:1310.1147v1, (2013).
- [23] Y. LIU AND Y. WU, *Variable selection via a combination of the  $l_0$  and  $l_1$  penalties*, J. Comp. Graph. Stat., 16 (2007), pp. 782–798.
- [24] D.G. LUENBERGER, *Optimization by Vector Space Methods*, Wiley, J., New York, 1 ed., 1969.
- [25] J. LV AND Y. FAN, *A unified approach to model selection and sparse recovery using regularized least squares*, The Annals of Statistics, 37 (2009).
- [26] A. MALEKI AND D. L. DONOHO, *Optimally tuned iterative reconstruction algorithms for compressed sensing*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 330–341.
- [27] F. MALGOUYRES AND M. NIKOLOVA, *Average performance of the sparsest approximation using a general dictionary*, Numer. Funct. Anal. Optim., 32 (2011), pp. 768–805.
- [28] C. D. MEYER, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.
- [29] A. J. MILLER, *Subset Selection in Regression*, Chapman and Hall, London, U.K., 2 ed., 2002.
- [30] J. NEUMANN, C. SCHÖRR, AND G. STEIDL, *Combined SVM-based Feature Selection and classification*, Machine Learning 61, 2005, pp. 129–150.
- [31] M. NIKOLOVA, *Description of the minimizers of least squares regularized with  $l_0$ -norm. uniqueness of the global minimizer*, SIAM J. Imaging Sci., 6 (2013), pp. 904–937.
- [32] M.C. ROBINI, A. LACHAL, AND I.E. MAGNIN, *A stochastic continuation approach to piecewise constant reconstruction*, IEEE Trans. Image Process., 16 (2007), pp. 2576–2589.
- [33] M. C. ROBINI AND I. E. MAGNIN, *Optimization by stochastic continuation*, SIAM J. Imaging Sci., 3 (2010), pp. 1096–1121.
- [34] M. C. ROBINI AND P.-J. REISSMAN, *From simulated annealing to stochastic continuation: a new trend in combinatorial optimization*, J. Global Optim., 56 (2013), pp. 185–215.
- [35] J.A. TROPP, *Just relax: convex programming methods for identifying sparse signals in noise*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1030–1051.
- [36] J. TROPP AND S. J. WRIGHT, *Computational methods for sparse solution of linear inverse problems*, Proc. IEEE, 98 (2010), pp. 948–958.
- [37] Y. ZHANG, B. DONG, AND Z. LU,  *$l_0$  minimization of wavelet frame based image restoration*, Mathematics of Computation, 82 (2013), pp. 995–1015.