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The Phan-Thien and Tanner Model Applied to Thin Film Spherical Coordinates: Applications for Lubrication of Hip Joint Replacement

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The Phan-Thien and Tanner (PTT) model is one of the most widely used rheological models. It can properly describe the common characteristics of viscoelastic non-Newtonian fluids. There is evidence that synovial fluid in human joints, which also lubricates artificial joints, is viscoelastic. Modeling the geometry of the total hip replacement, the PTT model is applied in spherical coordinates for a thin confined fluid film. A modified Reynolds equation is developed for this geometry. Several simplified illustrative problems are solved. The effect of the edge boundary condition on load-carrying normal stress is discussed. Solutions are also obtained for a simple squeezing flow. The effect of both the relaxation time and the PTT shear parameter is to reduce the load relative to a Newtonian fluid with the same viscosity. This implies that the Newtonian model is not conservative and may overpredict the load capacity. The PTT theory is a good candidate model to use for joint replacement lubrication. It is well regarded and derivable from molecular considerations. The most important non-Newtonian characteristics can be described with only three primary material parameters.

Keywords: hip, prosthesis, lubrication, non-Newtonian

1 Introduction

Synovial fluid is a polymeric fluid and, as such, is likely to exhibit non-Newtonian viscoelastic flow properties. These properties may include an elastic effect, a shear thinning effect, and a normal stress effect. Evidence has been reported decades ago by King [1] and Balazs and Gibbs [2] and was more recently summarized and discussed by Fung [3]. In these references, the viscoelastic dynamic storage modulus G' (proportional to fluid elasticity) and loss modulus G'' (proportional to viscosity) are measured. If the synovial fluid were purely viscous, one would observe that $G'=0$, but instead these references show that G' and G'' are of the same order of magnitude, from which one could infer significant fluid elastic effects. Apparently, it remains a controversy as to whether or not these viscoelastic effects are significant, but it would certainly seem worthy of study.

There are some recent experimental studies on the non-Newtonian and viscoelastic properties of synovial fluid. In 2005, Oda and Sugishita [4] attributed its excellent lubrication properties in the natural joint to viscoelastic properties. Szwajczak [5] in 2004 reported on studies of hyaluronic acid in which natural and artificial solutions have exhibited viscoelastic properties. Krause et al. [6] performed rheological measurements on synovial fluid and found significant effects. Finally, Oates et al. [7] investigated the role of rheology in the behavior of a bovine synovial fluid.

The importance of viscoelastic effects in natural human joint lubrication was perhaps first discussed by Mow and Ateshian [8]. However, nearly all existing studies of synovial lubrication in artificial joints are performed with purely viscous fluid models. Purely viscous fluids may be non-Newtonian in exhibiting viscos-

ity shear thinning, but they do not exhibit normal stress and time dependence—stress is always in phase with strain rate.

Basic concepts, such as squeeze film lubrication, elastohydrodynamic lubrication (EHL), and boundary lubrication, have contributed to the many possible theories of joint lubrication. Mow et al. [9] developed a theory that considered several complex factors, including the dynamics of non-Newtonian synovial fluid flow and its interaction with the cartilage surface. A description of the thin spherical film lubrication geometry for the hip prosthesis for rigid surfaces has been presented by Meyer and Tichy [10]. Several recent articles have appeared concerning EHL as applied to the modeling of hip implants (see Liu et al. [11] and Udofia and Jin [12]).

Various modes of lubrication occur under different operating conditions and occur simultaneously in most situations during various cycles of articulation. For instance, Mabuchi and Sasada [13] used a standard Newtonian EHL approach, but only the squeezing effect is considered. This approach has been slightly modified by Hlaváček [14] using a piecewise power law lubricant in steady rolling motion. Mazzucco et al. [15] illustrated the importance of the steady shear viscosity and linear viscoelastic properties in fluid film lubrication of a joint replacement prosthesis. Bou-Saïd and Kane [16] studied non-Newtonian lubrication approaches to knee joint lubrication.

Additional knowledge on SF rheology in hip implants may be useful for the development of prosthesis materials and designs and for a better understanding of the tribological characteristics of replacement bearings for the human body. If we accept the premise of the viscoelastic nature of synovial fluid, there is certainly no consensus as to which model to use. This paper is not the proper forum for a thorough discussion of the pros and cons of the numerous models extant. However, the PTT model [17] is often considered for polymer solutions. Some advantages are that the model describes the major features of non-Newtonian flow in both shear and extension (effective viscosity variation, normal stress,

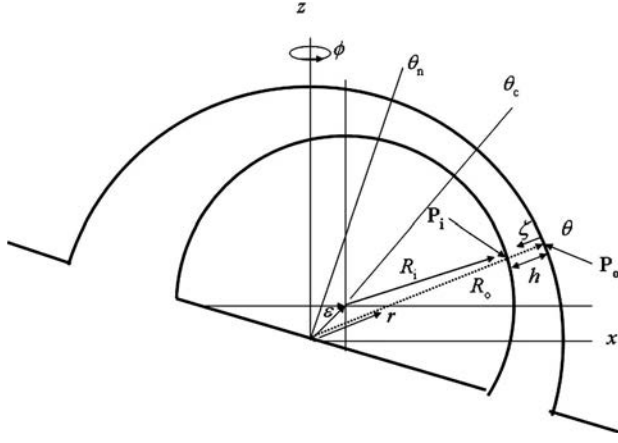


Fig. 1 Schematic of eccentric sphere geometry

time dependence, and overshoot), the time scales involved can be superimposed, and the model can be derived from the molecular scale network theory. There are several existing studies of the PTT model applied to lubrication (see Bou-Said and Ehret [18], and Akyildiz and Bellot [19]).

2 Analysis

The notation of Bird et al. [20] is used throughout this paper. The PTT model can be expressed as

$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i, \quad Z_i = 1 - \varepsilon \lambda_i \text{tr} \boldsymbol{\tau}_i / \eta_i \quad \text{or} \quad Z_i = \exp(-\varepsilon \lambda_i \text{tr} \boldsymbol{\tau}_i / \eta_i) \quad (1)$$

$$Z_i \boldsymbol{\tau}_i + \lambda_i \boldsymbol{\tau}_{i(1)} + \frac{\xi}{2} \lambda_i (\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\tau}_i + \boldsymbol{\tau}_i \cdot \dot{\boldsymbol{\gamma}}) = -\eta_i \dot{\boldsymbol{\gamma}}$$

Extra stress $\boldsymbol{\tau}$ is described as a superposition of partial stresses. The rate of strain is denoted as $\dot{\boldsymbol{\gamma}}$, and the subscript on the extra stress (1) indicates the upper convected (contravariant) first derivative. Relaxation time λ , viscosity η , and the function Z are described by a spectrum,

$$\lambda = \lambda_1, \lambda_2, \dots, \quad \eta = \eta_1, \eta_2, \dots, \quad Z = Z_1, Z_2, \dots$$

The symbols ε and ξ denote flow parameters, which are generally curve fit to experimental data, where tr denotes the trace. The parameter ε does not play a significant role in shear-dominated flow (for simple shear, $\text{tr} \boldsymbol{\tau}_i = 0$). Likewise, ξ is not influential in elongational flow. An advantage of the PTT model is that by independent curve-fit adjustment of the two parameters, the model can accurately portray both shear and elongational flow (a common drawback of other differential models). The empirical stress function Z allows the model to accurately portray Trouton elongational viscosity data. The Maxwell fluid is obtained if $\varepsilon = \xi = 0$.

We use the standard definition of spherical coordinates given by Bird et al. The acetabular cup center is considered to be the stationary origin (see Fig. 1). The outer radius $r_p = R_o$ is measured from the origin to the point in question on the cup surface. The inner radius $r_{p_i}(\theta, \phi) \neq R_i$ is measured from the origin to the point along the outer radius where it intersects the femur (inner) head. The location of the head is specified by its eccentricity e and the pair of angles denoting the direction of the line of centers (θ_c, ϕ_c). The direction z is normal to the ground, and the direction of walking is y (not shown), into the plane of the paper. The head and cup are hemispheres, and the normal location is θ_n, ϕ_n .

We now apply the thin film approximation in which the difference between the inner and outer undeformed radii is $c = R_o - R_i$, $c \ll R_o \approx R$. The eccentricity ratio is given by $\varepsilon = e/R$. For these conditions, the film thickness is given by

$$h(\theta, \phi, t) = c[1 - \varepsilon(t)(\cos \theta \cos \theta_c + \sin \theta \sin \theta_c \cos \phi \cos \phi_c + \sin \theta \sin \theta_c \sin \phi \sin \phi_c)] \quad (2)$$

where t is time. For a thin film of fluid confined between two nearly spherical surfaces (possibly nonconcentric), we have the following orders of magnitude:

$$\frac{\partial}{\partial r} \sim \frac{1}{c} \ll \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \frac{1}{r} \frac{\partial}{\partial \phi}, \quad r \sim R, \quad v_r \ll v_\theta, v_\phi \quad (3)$$

In the analysis below, we assume that we are working with one of the spectral equations of Eq. (1) or, equivalently, a one time scale (sometimes called the one mode version of the PTT equation). For notational simplicity, the numerical subscripts are omitted. Applying the orders of magnitude of Eq. (3) to the PTT model (Eq. (1)), we obtain the following equations for the unsteady stresses:

$$\begin{aligned} \lambda \frac{\partial \tau_{r\theta}}{\partial t} + Z \tau_{r\theta} &= -\frac{\partial v_\theta}{\partial r} \left[\eta + \lambda \left(-\tau_{rr} + \frac{\xi}{2} (\tau_{rr} + \tau_{\theta\theta}) \right) \right] \\ \lambda \frac{\partial \tau_{r\phi}}{\partial t} + Z \tau_{r\phi} &= -\frac{\partial v_\phi}{\partial r} \left[\eta + \lambda \left(-\tau_{rr} + \frac{\xi}{2} (\tau_{rr} + \tau_{\phi\phi}) \right) \right] \\ \lambda \frac{\partial \tau_{rr}}{\partial t} + Z \tau_{rr} &= -\frac{\partial v_\theta}{\partial r} [\lambda \xi \tau_{r\theta}] - \frac{\partial v_\phi}{\partial r} [\lambda \xi \tau_{r\phi}] \end{aligned} \quad (4)$$

$$\lambda \frac{\partial \tau_{\theta\theta}}{\partial t} + Z \tau_{\theta\theta} = -\frac{\partial v_\theta}{\partial r} [\lambda (\xi - 2) \tau_{r\theta}]$$

$$\lambda \frac{\partial \tau_{\phi\phi}}{\partial t} + Z \tau_{\phi\phi} = -\frac{\partial v_\phi}{\partial r} [\lambda (\xi - 2) \tau_{r\phi}]$$

The initial and boundary conditions are indicated below. For the time being, we assume that the initial condition is at rest. In principle, any sort of motion can be described, but to eliminate undue complexity, we restrict ourselves to combinations of pure squeezing in the direction normal to the ground $\dot{\varepsilon}_z$ and pure rotation in the direction parallel to the ground and normal to the direction of walking ω_x . At the femur head (inner moving sphere) and at the acetabular cup (outer fixed sphere), we have, respectively,

$$t = 0: v_\theta = v_\phi = v_r = 0$$

$$r = R - h(\theta, \phi, t): v_\theta = -\omega_x R \sin \phi, \quad v_\phi = -\omega_x R \cos \theta \cos \phi \quad (5)$$

$$v_r = -\frac{\partial h}{\partial t} = -\dot{\varepsilon}_z(t) c \cos \theta$$

$$r = R: v_\theta = v_\phi = v_r = 0$$

The $\{r, \theta, \phi\}$ equations of motion are given by

$$0 = \frac{\partial \pi_{rr}}{\partial r} \quad (6)$$

$$0 = \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{R} \left(\frac{\partial \pi_{rr}}{\partial \theta} + \frac{\partial}{\partial \theta} (\tau_{\theta\theta} - \tau_{rr}) + \cot \theta (\tau_{\theta\theta} - \tau_{\phi\phi}) \right)$$

$$0 = \frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{R \sin \theta} \left(\frac{\partial \pi_{rr}}{\partial \phi} + \frac{\partial}{\partial \phi} (\tau_{\phi\phi} - \tau_{rr}) \right)$$

We have used the thin film assumption (Eq. (3)), neglecting body forces and fluid inertia, as normally done. A time dependence also arises through the radial velocity boundary condition $v_r = -\partial h / \partial t$. The continuity equation for spherical thin films can be expressed as

$$0 = \frac{\partial v_r}{\partial r} + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (7)$$

In viscoelastic flows, “pressure” has no direct physical meaning. For incompressible flow, the total stress $\boldsymbol{\pi}$ exerts the physical force, the extra stress $\boldsymbol{\tau}$ is the portion of the total stress that can be determined by the flow kinematics and the rheological model, and the pressure p is simply the difference between the two. In this case, the stress component that carries the load is the *total* stress π_{rr} ,

$$\begin{aligned} F_z &= \int_S \boldsymbol{\delta}_z \cdot (\mathbf{n} \cdot \boldsymbol{\pi}) dS = \int_S (\cos \theta \pi_{rr} - \sin \theta \tau_{r\theta}) dS \\ &\equiv \int_S \cos \theta \pi_{rr} dS \\ \pi_{rr}(\theta, \phi, t) &= p(r, \theta, \phi, t) + \tau_{rr}(r, \theta, \phi, t) \end{aligned} \quad (8)$$

The normal vector to the bearing surface \mathbf{n} is the spherical coordinate unit vector in the radial direction $\boldsymbol{\delta}_r$. From the thin film scaling (Eq. (3)) and Eq. (6), we can see that the normal stress π_{rr} is much greater than the shear stress, $\pi_{rr} \sim (R/c)\tau_{r\theta}$, just as pressure is much greater than shear stress in Newtonian lubrication.

3 Goal of the Paper and the Complexity of Finding Solutions

Even in the case of rigid surfaces, the difficulties in obtaining numerical solutions to the equations are enormous. Equations (6) and (7) must be solved simultaneously with Eq. (4). Consider a simple physical motion impact normal to the ground after jumping. Because the prosthesis is not aligned with the ground, the film thickness is a function of two spatial coordinates, θ and ϕ , as well as time. Variation with radial position r across the film is required in the constitutive equations (Eq. (4)); thus, a very simple human movement requires a full three-dimensional unsteady solution of nine coupled equations to find $\pi_{rr}(r, \theta, \phi, t)$ and thus determine the load.

Below, we make several simplifying assumptions in the governing equations and attempt to reduce the complexity by attacking highly idealized geometrical configurations. We only have chosen highly idealized model problems that reduce to ordinary differential equations. The results illustrate some basic characteristics of the behavior. In that spirit, we use the assumption of rigid surfaces, which may be applicable in some circumstances to metal-on-metal implants. Our goal is to examine the general suitability of the PTT model in joint lubrication, rather than to model a particular prosthesis. Thus, one can better assess the feasibility of more costly and realistic numerical studies.

Recall that the Reynolds equation (modified or otherwise) is quasisteady. The cross-film coordinate, in this case r , is eliminated. Time dependence is introduced through the squeeze film term, but stress or pressure variation is exactly in phase with surface velocity $\partial h / \partial t$. In deriving a modified Reynolds equation, we must assume that the fluid mechanics are quasisteady; i.e., the time derivatives of Eq. (4) can be neglected. If so, the stresses of Eq. (4) can be solved algebraically and substituted in the equations of motion (Eq. (6)). Similarly, the time dependent terms of the Navier–Stokes equations are neglected in the development of the classical Newtonian Reynolds equation.

It is known that for the PTT model, one can set $Z=1$ (or $\varepsilon=0$) without greatly affecting the shear flow behavior. If $\partial / \partial t \sim \dot{\varepsilon}_z$, then quasisteady conditions exist if $\lambda \dot{\varepsilon}_z \ll 1$. The product $\lambda \dot{\varepsilon}_z$ is a sort of Deborah number De . The quasisteady assumption to develop a modified Reynolds equation is restrictive in the sense that certain viscoelastic time-dependent phenomena cannot be described, such as overshoot and relaxation.

In illustrative examples below, for simplicity, we will consider a form of pure squeezing. We assume that the hemispherical cup is oriented parallel to the ground ($\theta_n = \phi_n = 0$) and thus the film thickness (Eq. (2)) becomes $h = c(1 - \varepsilon_z \cos \theta)$, with $\partial h / \partial t = c \dot{\varepsilon}_z$. We set

the walking rotation $\omega_x = 0$. Of course, such an orientation of the acetabular cup does not occur in practice. The load-carrying normal stress becomes simply $\pi_{rr} = \pi_{rr}(\theta)$, and only an ordinary differential equation need be solved. The time dependence becomes parametric, directly proportional to $\dot{\varepsilon}_z$. In a very qualitative way, this problem simulates the impact after jumping.

There is no correspondingly simple way to simulate walking (sliding of the surfaces as opposed to squeezing). In Eq. (5), if we set $\dot{\varepsilon}_z = 0$ but $\omega_x \neq 0$ (walking, no jumping), a ϕ dependence is introduced in the surface velocities. Thus, $\pi_{rr} = \pi_{rr}(\theta, \phi)$, and a partial differential equation must be solved.

4 Approximations for a Modified PTT Reynolds Equation

Setting relaxation time $\lambda = 0$ and the PTT stress factor $Z = 1$, the Newtonian velocity field can be found by straightforward but complicated manipulations,

$$v_\theta^N = \frac{R-r}{h} (\omega_x R \sin \phi) + [(R-r)^2 - (R-r)h] \frac{1}{2R\eta} \frac{\partial p^N}{\partial \theta} \quad (9)$$

$$\begin{aligned} v_\phi^N &= \frac{R-r}{h} (-\omega_x R \cos \theta \cos \phi) \\ &+ [(R-r)^2 - (R-r)h] \frac{1}{2R\eta \sin \theta} \frac{\partial p^N}{\partial \phi} \end{aligned}$$

Also, by the standard development, the Newtonian Reynolds equation is

$$\begin{aligned} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{h^3}{12R^2\eta} \frac{\partial p^N}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\csc^2 \theta \frac{h^3}{12R^2\eta} \frac{\partial p^N}{\partial \phi} \right) \\ = -\dot{\varepsilon}_z c \cos \theta + \frac{1}{2} \omega_x \left(\sin \phi \frac{\partial h}{\partial \theta} + \cos \phi \cot \theta \frac{\partial h}{\partial \phi} \right) \end{aligned} \quad (10)$$

Let us now consider quasisteady conditions, dropping the time derivative term of the constitutive equations (Eq. (5)), as discussed above. We now seek an approximate solution based on the Newtonian velocity field,

$$\tau_{r\theta}^N = -\eta \frac{\partial v_\theta^N}{\partial r}, \quad \tau_{r\phi}^N = -\eta \frac{\partial v_\phi^N}{\partial r} \quad (11)$$

For $Z=1$, the approximate normal stresses can be found algebraically and then averaged across the film to obtain

$$\begin{aligned} \tau_{\theta\theta} &\approx -\lambda(\xi-2) \frac{\partial v_\theta^N}{\partial r} \tau_{r\theta}^N \approx \eta \lambda (\xi-2) \overline{\partial_r v_\theta^2} \\ \tau_{\phi\phi} &\approx -\lambda(\xi-2) \frac{\partial v_\phi^N}{\partial r} \tau_{r\phi}^N \approx \eta \lambda (\xi-2) \overline{\partial_r v_\phi^2} \end{aligned} \quad (12)$$

$$\tau_{rr} \approx -\lambda \xi \frac{\partial v_\theta^N}{\partial r} \tau_{r\theta}^N - \lambda \xi \frac{\partial v_\phi^N}{\partial r} \tau_{r\phi}^N \approx \eta \lambda \xi \overline{\partial_r v_\theta^2} + \eta \lambda \xi \overline{\partial_r v_\phi^2}$$

Substituting Eq. (12) into Eq. (5), let us rewrite cross-film averaged versions of the θ and ϕ equations of motion (Eq. (6)) as

$$\begin{aligned} 0 &= \frac{1}{R} \frac{\partial \pi_{rr}}{\partial \theta} + \frac{1}{R} a_\theta(\theta, \phi) - b_\theta(\theta, \phi) \frac{\partial^2 v_\theta}{\partial r^2} \\ 0 &= \frac{1}{R \sin \theta} \frac{\partial \pi_{rr}}{\partial \phi} + \frac{1}{R \sin \theta} a_\phi(\theta, \phi) - b_\phi(\theta, \phi) \frac{\partial^2 v_\phi}{\partial r^2} \end{aligned} \quad (13)$$

where, after considerable manipulation,

$$\begin{aligned} a_\theta &= -\eta \lambda \cot \theta (2 - \xi) (\overline{\partial_r v_\theta^2} - \overline{\partial_r v_\phi^2}) - \eta \lambda [4 \overline{\partial_r v_\theta \partial_r v_\theta} + 2 \xi \overline{\partial_r v_\phi \partial_r v_\phi}] \\ a_\phi &= -\eta \lambda [4 \overline{\partial_r v_\phi \partial_r v_\phi} + 2 \xi \overline{\partial_r v_\theta \partial_r v_\theta}] \end{aligned}$$

$$b_\theta = \eta \left[1 - \lambda^2 \xi (2 - \xi) \left(\overline{\partial_r v_\theta^2} + \frac{1}{2} \overline{\partial_r v_\phi^2} \right) \right] \quad (14)$$

$$b_\phi = \eta \left[1 - \lambda^2 \xi (2 - \xi) \left(\overline{\partial_r v_\phi^2} + \frac{1}{2} \overline{\partial_r v_\theta^2} \right) \right]$$

with

$$\begin{aligned} \overline{\partial_r v_\theta^2} &= \frac{1}{h} \int_{R-h}^R \left(\frac{\partial v_\theta^N}{\partial r} \right)^2 dr, & \overline{\partial_r v_\phi^2} &= \frac{1}{h} \int_{R-h}^R \left(\frac{\partial v_\phi^N}{\partial r} \right)^2 dr \\ \overline{\partial_r v_\theta \partial_r v_\theta} &= \frac{1}{h} \int_{R-h}^R \frac{\partial v_\theta^N}{\partial r} \frac{\partial^2 v_\theta^N}{\partial r \partial \theta} dr, & \overline{\partial_r v_\theta \partial_r v_\phi} &= \frac{1}{h} \int_{R-h}^R \frac{\partial v_\theta^N}{\partial r} \frac{\partial^2 v_\theta^N}{\partial r \partial \phi} dr \end{aligned} \quad (15)$$

$$\overline{\partial_r v_\phi \partial_r v_\phi} = \frac{1}{h} \int_{R-h}^R \frac{\partial v_\phi^N}{\partial r} \frac{\partial^2 v_\phi^N}{\partial r \partial \theta} dr, \quad \overline{\partial_r v_\phi \partial_r v_\theta} = \frac{1}{h} \int_{R-h}^R \frac{\partial v_\phi^N}{\partial r} \frac{\partial^2 v_\phi^N}{\partial r \partial \phi} dr$$

Following an identical procedure to the classical derivation of Reynolds equation, a PTT modified Reynolds equation is thus obtained,

$$\begin{aligned} & \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{h^3}{12R^2 b_\theta} \frac{\partial \pi_{rr}}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\csc^2 \theta \frac{h^3}{12R^2 b_\phi} \frac{\partial \pi_{rr}}{\partial \phi} \right) \\ &= -\dot{\epsilon}_z c \cos \theta + \frac{1}{2} \omega_x \left(\sin \phi \frac{\partial h}{\partial \theta} + \cos \phi \cot \theta \frac{\partial h}{\partial \phi} \right) \\ &+ \frac{h^3}{12R^2 b_\theta b_\phi} \left(a_\theta b_\phi \cot \theta + b_\theta \csc^2 \theta \frac{\partial a_\phi}{\partial \phi} - a_\phi b_\theta \frac{1}{b_\phi} \csc^2 \theta \frac{\partial b_\phi}{\partial \phi} \right. \\ &\left. + 3a_\phi b_\theta \csc^2 \theta \frac{1}{h} \frac{\partial h}{\partial \phi} + b_\phi \frac{\partial a_\theta}{\partial \theta} - a_\theta b_\phi \frac{1}{b_\theta} \frac{\partial b_\theta}{\partial \theta} + 3a_\theta b_\phi \frac{1}{h} \frac{\partial h}{\partial \theta} \right) \end{aligned} \quad (16)$$

The specification of proper edge boundary conditions is a controversial topic. This issue in lubrication flows was originally noted by Tanner [21] and discussed in more detail by Sawyer and Tichy [22]. In the case of an open-ended contact, none of the boundary condition alternatives is entirely satisfactory. There are several such choices. The following is the easiest to apply:

$$\theta = \theta_n: \pi_{rr} = p_{\text{atm}} \quad (17)$$

However, there is no obvious physical argument in its favor. Likewise, there is no particular logic in setting the pressure at the film edge equal to ambient pressure. Another alternative would be to balance the normal stress in the flow direction $\pi_{\theta\theta}$ against the ambient pressure imposed about the periphery of the the head-cup interface,

$$\theta = \theta_n: \pi_{\theta\theta} = \pi_{rr} + (\tau_{\theta\theta} - \tau_{rr}) = p_{\text{atm}} \quad (18)$$

5 One-Dimensional Contact: Pure Squeezing Flow Formulation

Let us consider the case of a one-dimensional contact defined by

$$\omega_x = \theta_c = \phi_c = \frac{\partial}{\partial \phi} = v_\phi = 0, \quad h = c(1 - \epsilon \cos \theta) \quad (19)$$

All of the stresses and the velocities v_θ and v_ϕ are thus functions of (at most) r and θ , as well as, perhaps, of time t .

We now solve the governing equations (Eqs. (5)–(8)) for one-dimensional steady squeezing. At the instant considered, the sphere is assumed to be centered, $\epsilon=0$, and the location in question is the film edge where $\theta=\pi/2$. As previously discussed, “steady” implies that the $\partial/\partial t$ terms of Eq. (5) are small. This is probably the simplest possible set of conditions that allows for non-uniformity, i.e., variation of velocity and stress over the film surface.

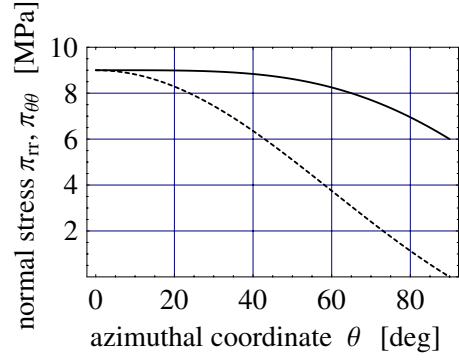


Fig. 2 Effect of edge boundary condition—profile of radial and azimuthal normal stress. Azimuthal normal stress equals ambient pressure at film edge. Simple steady squeezing flow $\epsilon = 0.0$. Relaxation time $\lambda = 1$ s; PTT parameter $\xi^* = 0.0$ (Maxwell fluid). Dashed line: normal stress in flow direction $\pi_{\theta\theta}$. Solid line: load carrying normal stress π_{rr} .

Measurements of the linear viscoelastic properties of synovial fluid are reported in the literature as mentioned above (the elastic storage modulus G' and viscous loss modulus G'' as functions of frequency ω). However, measurements of viscosity and relaxation time as functions of shear rate are far less common. Noting that for the Maxwell model $\eta = (G'^2 + G''^2) / (G'' \omega \Delta x / H)$ and $\lambda = G' / (G'' \omega)$, where $\Delta x / H$ represents a ratio of oscillation amplitude to film thickness, we can make some crude estimates of viscosity and relaxation time at the kinematic conditions.

At high shear rates, large non-Newtonian normal stresses are predicted. To illustrate, consider Eqs. (4) in steady conditions with $Z=1$ and $v_\phi=0$, i.e., one-dimensional bearing conditions. By algebraic manipulation for small ξ , we find $\tau_{r\theta} = -\eta \dot{\gamma} (1 - 2\xi \lambda^2 \dot{\gamma}^2)$. Thus, we define a PTT parameter $\xi^* = \xi (\lambda \dot{\gamma}_0)^2$, which approximately represents the strength of a correction to a purely viscous behavior.

We use the following parameter values: viscosity $\eta = 1$ Pa s, relaxation time $\lambda = 0.1$ s, PPT parameters $\xi^* = 0.0$ and 0.1 , $\epsilon = 0.0$, and shear rate $\dot{\gamma}_0 = \dot{\epsilon} R / c = 1000$.

The approximation of Eq. (15) is given by

$$\overline{\partial_r v_\theta^2} = 3 \frac{R^2}{c^2} \frac{\dot{\epsilon}_z^2 \sin^2 \theta}{(1 - \epsilon \cos \theta)^4} \quad (20)$$

We now solve the modified PTT Reynolds equation for the one-dimensional spherical bearing case in squeezing flow. Reynolds equation simplifies to

$$\begin{aligned} & \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{h^3}{12R^2 b_\theta} \frac{\partial \pi_{rr}}{\partial \theta} \right) \\ &= \dot{\epsilon}_z c \cos \theta + \frac{h^3}{12R^2 b_\theta} \left(a_\theta \cot \theta + \frac{\partial a_\theta}{\partial \theta} - a_\theta \frac{1}{b_\theta} \frac{\partial b_\theta}{\partial \theta} + 3a_\theta \frac{1}{h} \frac{\partial h}{\partial \theta} \right) \end{aligned} \quad (21)$$

and we use the boundary conditions

$$\theta = 0: \frac{\partial \pi_{rr}}{\partial \theta} = 0 \quad (22)$$

$$\theta = \pi/2: \pi_{rr} = p_{\text{atm}} \quad \text{or} \quad \pi_{\theta\theta} = \pi_{rr} - 2\lambda \eta \overline{\partial_r v_\theta^2} = p_{\text{atm}}$$

6 One-Dimensional Contact: Pure Squeezing Flow Results

First, we illustrate the effect of the edge boundary conditions in Fig. 2. We show the load-carrying (radial) total normal stress π_{rr}

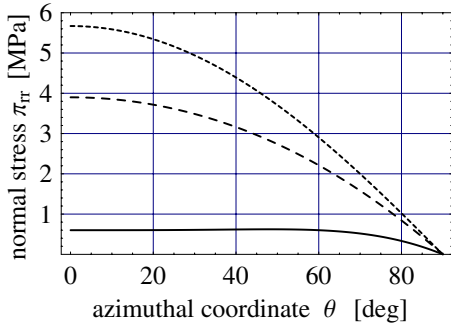


Fig. 3 Effect of relaxation time—profile of radial (load carrying) normal stress. Simple squeezing flow $\epsilon=0.0$. PTT parameter $\xi^*=0.1$. Solid line: $\lambda=1$ s; long dashed line: $\lambda=0.5$ s; short dashed line: Newtonian, $\lambda=0.0$ s.

and the normal stress in the flow direction $\pi_{\theta\theta}$. The PTT and kinematic parameter values follow: viscosity $\eta=1$ Pa s, relaxation time $\lambda=1.0$ s, PPT parameters $\xi^*=0.0$ and $\epsilon=0.0$, and shear rate $\dot{\gamma}_0=\dot{\epsilon}R/c=1000$ s⁻¹. The normal stress in the flow direction is balanced against the ambient pressure at the end of the film, $\theta=\pi/2$. Note that the load-carrying normal stress is much greater and is effectively prestressed at the edge.

The effect of relaxation time is shown in Fig. 3. The load-carrying normal stress π_{rr} is shown, and it is set to ambient at the edge of the film. As above, the sphere is assumed to be centered, with viscosity $\eta=1$ Pa s and shear rate $\dot{\gamma}_0=\dot{\epsilon}R/c=1000$ s⁻¹. For this boundary condition, the effect of increasing relaxation time is to decrease total normal stress and load-carrying capacity. The short-dashed curve is Newtonian. The normal stress is also much more uniform across the bearing surface in the viscoelastic case. Clearly, the effect of viscoelasticity in this case is to decrease load relative to the equivalent Newtonian fluid.

The same set of conditions as in the previous figure is shown in Fig. 4 (except as noted), but now the normal stress in the flow direction $\pi_{\theta\theta}$ is set to zero at the edge of the film. Note the stressed state of π_{rr} at the edge $\theta=\pi/2$ and that the viscoelasticity now apparently increases load capacity. This ambiguous outcome is still the current state of affairs and awaits experimental resolution.

The effect of the PTT shear parameter ξ^* is shown in Fig. 5. The various parameter values are the same as for the previous figures, except relaxation time $\lambda=1$ s. The effect of the PTT shear parameter is to significantly reduce the load, largely through a shear thinning decrease in effective viscosity.

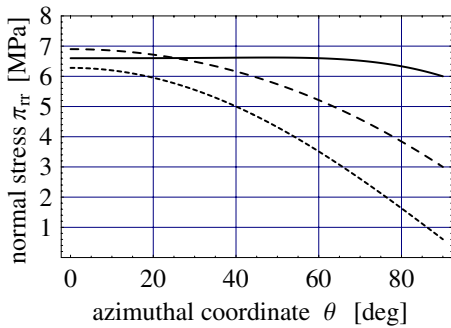


Fig. 4 Effect of relaxation time—profile of radial (load carrying) normal stress. Normal stress $\pi_{\theta\theta}$ set to ambient at film edge. Simple squeezing flow $\epsilon=0.0$. PTT parameter $\xi^*=0.1$. Solid line: $\lambda=1$ s; long dashed line: $\lambda=0.5$ s; short dashed line: $\lambda=0.1$ s.

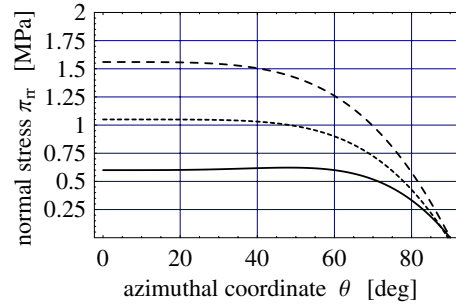


Fig. 5 Effect of PTT shear parameter ξ^* —profile of radial (load carrying) normal stress. Simple squeezing flow $\epsilon=0.0$. Relaxation time $\lambda=1$ s. Solid line: $\xi^*=0.1$; short dashed line: $\xi^*=0.08$; long dashed line: $\xi^*=0.05$.

Finally, we show the effect of instantaneous eccentricity ratio. As the femoral head approaches the cup surface, total normal stress rises dramatically (see Fig. 6). The conditions are the same as for the previous figures and as noted in the caption. For the quasisteady conditions of a modified Reynolds equation, transition from the lower (solid) curve to the upper (long dash) curve illustrates time dependence as the head approaches the cup.

7 Conclusions

We have attempted to examine the suitability of the PTT viscoelastic fluid model to describe the behavior of synovial fluid in joint replacement prostheses. The properties of synovial fluid have been measured elsewhere and found to be viscoelastic, thus the need to determine an appropriate model. Both the PTT model and the geometry and kinematics of the hip joint are extremely complex—even in the case of rigid surfaces. Thus, we have examined the behavior of the PTT fluid in a number of simplified idealized problems to judge its overall effectiveness as a candidate for a more complete numerical approach. These model problems all reduce to ordinary differential equations, while the real-world problems are unsteady and three dimensional. We make no claim that we are presently offering a complete model of the lubrication total hip replacement prosthesis.

The PTT model in thin film spherical coordinates portrays shear thinning of the effective viscosity, time-dependent behavior such as relaxation and overshoot, and normal stress behavior. Both relaxation and the PTT shear parameter tend to decrease load in steady conditions, consistent with other viscoelastic lubrication studies. However, there is uncertainty as to the proper edge boundary conditions, and a different condition may cause quite different results.

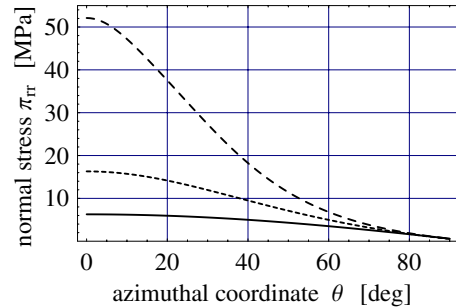


Fig. 6 Effect of eccentricity—profile of radial (load carrying) normal stress. Simple squeezing flow. Relaxation time $\lambda=0.1$ s. PTT parameter $\xi^*=0.1$. Solid line: $\epsilon=0.0$; short dashed line: $\epsilon=0.5$; long dashed line: $\epsilon=0.8$.

At present, there is a very cloudy picture as to the status of viscoelasticity in prosthesis lubrication. However, it is certainly possible that synovial fluid in an artificial joint may exhibit significant non-Newtonian effects. Only experimental studies can resolve many of the existing uncertainties. However, the PTT model is flexible and robust and must be considered a leading contender as a model for synovial fluid in these applications once better data are obtained as to the real behavior of these complex and vitally important devices.

Figure 3 shows that for certain conditions viscoelasticity significantly reduces the load-carrying normal stress relative to a Newtonian fluid of equivalent viscosity. Thus, the Newtonian model may overpredict the forces developed to separate the bearing surfaces. Therefore, in the design sense, the Newtonian viscous fluid is not a conservative choice for fluid model.

References

- [1] King, R. G., 1966, "A Rheological Measurement of Three Synovial Fluids," *Rheol. Acta*, **5**, pp. 41–44.
- [2] Balazs, E. A., and Gibbs, D. A., 1970, "The Rheological Properties and Biological Functions of Hyaluronic Acid," *Chemistry and Molecular Biology of the Intercellular Matrix*, E. A. Balazs, ed., Academic, New York, Vol. 3, pp. 1241–1253.
- [3] Fung, Y. C., 1993, *Biomechanics: Mechanical Properties of Living Tissues*, 2nd ed., Springer, New York, pp. 234–238.
- [4] Oda, Y., and Sugishita, J., 2005, "The Spinnability Effect Under Lubrication (Part 1). Lubrication System of Natural Joint," *Journal of Japanese Society of Tribologists*, **50**(4), pp. 319–326.
- [5] Szwajczak, E., 2004, "Rheological Properties of Aqueous Solutions of Biopolymeric Hyaluronan," *Proc. SPIE*, **5565**, pp. 397–402.
- [6] Krause, W. E., Oates, K. M. N., and Colby, R. H., 2004, "Hyaluronan's Role in a Synovial Fluid: A Rheological Investigation," *Transactions: Seventh World Biomaterials Congress*, p. 147.
- [7] Oates, K. M. N., Krause, W. E., and Colby, R. H., 2002, "Using Rheology to Probe the Mechanism of Joint Lubrication: Polyelectrolyte/Protein Interactions in Synovial Fluid," *Mater. Res. Soc. Symp. Proc.*, **711**, pp. 53–58.
- [8] Mow, V. C., and Ateshian, G. A., 1989, "Lubrication and Wear of Diarthrodial Joints," *Basic Orthopaedic Biomechanics*, 2nd ed., V. C. Mow and W. C. Hayes, eds., Lippincott-Raven, Philadelphia.
- [9] Mow, V. C., Ateshian, G. A., and Spilker, R. L., 1993, "Biomechanics of Diarthrodial Joints: A Review of Twenty Years of Progress," *J. Biomech.*, **115**, pp. 460–467.
- [10] Meyer, D. M., and Tichy, J. A., 1999, "Lubrication Model of an Artificial Hip Joint: Pressure Profile Vs. Inclination Angle," *ASME J. Tribol.*, **129**(3), pp. 492–498.
- [11] Liu, F., Jin, Z. M., Hirt, F., Rieker, C., Roberts, P., and Grigoris, P., 2005, "Transient Elastohydrodynamic Lubrication Analysis of Metal-on-Metal Hip Implant Under Simulated Walking Conditions," *J. Biomech.*, **39**(5), pp. 905–914.
- [12] Udofia, I. J., and Jin, Z. M., 2003, "Elastohydrodynamic Lubrication Analysis of Metal-on-Metal Hip-Resurfacing Prosthesis," *J. Biomech.*, **36**(4), pp. 537–544.
- [13] Mabuchi, K., and Sasada, T., 1990, "Numerical Analysis of Elastohydrodynamic Squeeze Film Lubrication of Total Hip Prostheses," *Wear*, **140**, pp. 1–16.
- [14] Hlaváček, M., 1997, "A Central Film Thickness Formula for Elastohydrodynamic Lubrication of Cylinders With Soft Incompressible Coatings and a Non-Newtonian Piecewise Power-Law in Steady Rolling Motion," *Wear*, **205**, pp. 20–27.
- [15] Mazzucco, D., McKinley, G., Scott, R. D., and Spector, M., 2002, "Rheology of Joint Fluid in Total Knee Arthroplasty Patients," *J. Orthop. Res.*, **20**, pp. 1157–1163.
- [16] Bou-Saïd, B., and Kane, M., 2002, "Approach and Tribological Comprehension of the Contact on the Level of the Knee," *Leeds-Lyon Symposium*, D. Dowson, M. Priest, G. Dalmaz, and A. A. Lubrecht, eds., Elsevier, New York, pp. 351–362.
- [17] Phan-Thien, N., and Tanner, R. I., 1977, "New Constitutive Equation Derived From Network Theory," *J. Non-Newtonian Fluid Mech.*, **2**, pp. 353–365.
- [18] Bou-Saïd, B., and Ehret, P., 1996, "Boundary Element Method for Transient Viscoelastic Flow: The MPTT Model," *STLE Tribol. Trans.*, **392**, pp. 314–321.
- [19] Akyildiz, F. T., and Bellot, H., 2002, "Viscoelastic Lubrication With Phan-Thien–Tanner Fluid (PTT)," *J. Non-Newtonian Fluid Mech.*, **3**, pp. 288–291.
- [20] Bird, R. B., Armstrong, R. C., and Hassager, O., 1987, *Dynamics of Polymeric Liquids. Vol. 1, Fluid Mechanics*. Wiley, New York.
- [21] Tanner, R. I., 1967, "Increase in Bearing Loads Due to Large Normal Stress," *ASME J. Appl. Mech.*, **363**, pp. 634–635.
- [22] Sawyer, G. W., and Tichy, J. A., 1998, "Non-Newtonian Lubrication With the Second Order Fluid," *ASME J. Tribol.*, **120**(3), pp. 622–627.