Whitham averaged equations and modulational stability of periodic traveling waves of a hyperbolic-parabolic balance law

Blake Barker, Mathew A. Johnson, Pascal Noble, Luis Miguel Miguel Rodrigues, Kevin Zumbrun

Whitham averaged equations and modulational stability of periodic traveling waves of a hyperbolic-parabolic balance law

BLAKE BARKER[*]   MATHEW A. JOHNSON[†]   PASCAL NOBLE[‡]
L.MIGUEL RODRIGUES[§]   KEVIN ZUMBRUN[¶]

September 2, 2010

Keywords: Periodic traveling waves; St. Venant equations; Spectral stability; Nonlinear stability.

2000 MR Subject Classification: 35B35.

Abstract

In this note, we report on recent findings concerning the spectral and nonlinear stability of periodic traveling wave solutions of hyperbolic-parabolic systems of balance laws, as applied to the St. Venant equations of shallow water flow down an incline. We begin by introducing a natural set of spectral stability assumptions, motivated by considerations from the Whitham averaged equations, and outline the recent proof yielding nonlinear stability under these conditions. We then turn to an analytical and numerical investigation of the verification of these spectral stability assumptions. While spectral instability is shown analytically to hold in both the Hopf and homoclinic limits, our numerical studies indicates spectrally stable periodic solutions of intermediate period. A mechanism for this moderate-amplitude stabilization is proposed in terms of numerically observed "metastability" of the the limiting homoclinic orbits.

1   Introduction

Nonclassical viscous conservation or balance laws arise in many areas of mathematical modeling including the analysis of multiphase fluids or solid mechanics. Such equations are known to exhibit a wide variety of traveling wave phenomena such as homoclinic or heteroclinc solutions, corresponding to the standard pulse and front or shock type solutions, respectively, as well as solutions which are spatially periodic. Historically, a great deal of effort has been applied to understanding the time-evolutionary stability of the homoclinic/heteroclinic solutions of such equation, and their spectral and nonlinear stability theories are well understood (in general). In contrast, until recently the analogous stability theories of the periodic counterparts have received relatively little attention. The goal of this paper is to present recent progress towards the understanding of periodic traveling-wave solutions within the context of a particular physically interesting hyperbolic–parabolic system of second order PDE's which we describe below. We begin, however, by briefly recalling the known theory in the case of a strictly parabolic system of conservation laws.

There has been a great deal of recent progress towards the understanding of the stability properties of periodic traveling waves of viscous strictly parabolic systems of conservation laws of the form

$$(1.1) \qquad\qquad u_t + \nabla \cdot f(u) = \Delta u, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^n;$$

see [JZ4, JZ2, OZ1, OZ2, OZ3, OZ4, Se1]. In particular, using delicate analysis of the resolvent of the linearized operator it has been shown that any periodic traveling wave solution of (1.1) that is spectrally stable with respect to localized (L^2) perturbations is (time-evolutionary) nonlinearly stable in L^p or H^s for appropriate values of p and s. In fact, such solutions are asymptotically stable (in an appropriate sense) for dimensions $d \geq 2$, while they are only nonlinearly bounded stable for dimensions $d = 1$. However, while these results are mathematically satisfying, up to now no example of a spectrally stable periodic solution of equations of the form (1.1) has yet been found. In fact, in dimension one it was shown in [OZ1] by rigorous Evans function computations that such solutions cannot exist for certain model systems of form (1.1) admitting a Hamiltonian structure, due to the existence of a spectral dichotomy.

Though these isolated results may seem discouraging, it should be noted that the explicit examples of form (1.1) considered so far have possibly had too much or the wrong sort of structure to admit stable periodic solution. In particular, it may very well be the case that by considering more exotic potentials[1] it may be possible to find a stable periodic solution even within the class of examples studied in [OZ1]. And, for higher dimensional ($n \geq 3$) or more general systems, one may expect to find richer behavior, including, possibly, stable periodic waves. Whether or not this occurs for systems of form (1.1) remains an interesting open problem. However, within the slightly wider class of nonstrictly parabolic systems of balance laws, it has recently been shown that stable periodic waves *can indeed occur*, which leads us to the investigations presented in this paper.

---

[1] In particular, potentials for which the period is decreasing with respect to amplitude in some regime.

A particular class of examples of more general (non-strictly parabolic) balance laws which have been recently seen (numerically) to admit stable periodic traveling wave solutions is given by the generalized St. Venant equations, which in Eulerian coordinates take the form

(1.2)
$$h_t + (hu)_x = 0,$$
$$(hu)_t + (h^2/2F + hu^2)_x = h - u|u|^{r-1}/h^s + \nu(hu_x)_x,$$

where $1 \leq r \leq 2$ and $0 \leq s \leq 2$. Unlike (1.1), the St. Venant equations are a second order hyperbolic–parabolic system of PDE's in balance law form (due to the non-differentiated source term in the parabolic part). Equations of this form arise naturally when approximating shallow water flow on an inclined ramp, in which case h represents the height of the fluid, u the velocity average with respect to height, ν is a nondimensional viscosity equal to the inverse of the Reynolds number, and F is the Froude number, which here is the square of the ratio between the speed of the fluid and the speed of gravity waves[2]. Further, the term $u|u|^{r-1}/h^s$ models turbulent friction along the bottom surface and x measures longitudinal distance along the ramp. Finally, we point out that the form of the viscosity term $\nu(hu_x)_x$ is motivated by the formal derivations from the Navier Stokes equations with free surfaces; other choices are obviously available and are sometimes used in the literature [HC]. Typical choices for the parameters (r, s) are $r \in \{1, 2\}$ and $s \in \{0, 1, 2\}$; see [BM, N1, N2] and references therein. The choice $(r, s) = (2, 0)$ is considered in detail in the works [N1, N2, JZN, BJRZ].

Periodic and solitary traveling waves, known as roll waves, are well-known to appear as solutions of (1.2), generated by competition between gravitational force and friction along the bottom. Such patterns have been used to model phenomena in several areas of the engineering literature, including landslides, river and spillway flow, and the topography of sand dunes and sea beds and their stability properties have been much studied numerically, experimentally, and by formal asymptotics; see [BM] and references therein. However, until very recently, there was no rigorous linear (as opposed to spectral, or normal modes) or nonlinear stability theory for these waves.

For the physically relevant system (1.2), it turns out that we are able to perform a complete spectral, linear, and nonlinear stability analysis of the associated periodic traveling wave solutions. Indeed, although the abstract nonlinear stability theory of [JZ2], developed for equations of form (1.1) does not apply directly to the St. Venant equations due to its hyperbolic-parabolic nature and source terms, a suitable modification of this theory can be made to establish that spectral stability of a given periodic traveling wave implies nonlinear bounded stability. This theory will be outlined briefly in the proceeding sections. The interested reader is invited to find more details in [JZN, N1, N2, NR, BJNRZ].

The outline of this paper is as follows. In the next subsection, Section 1.1, we briefly review the existence theory for the periodic traveling wave solutions of the generalized St. Venant equations (1.2). In particular, we derive the Hopf bifurcation conditions which guarantee the bifurcation of a family of periodic orbits from the equilibrium solution. Then,

---

[2]In particular, the Froude number F depends on the angle of inclination of the incline.

in Section 2, we outline the known stability theory for the periodic solutions found in Section 1.1. The main result of this section is that, under some "natural" spectral stability assumptions, the given periodic wave is nonlinearly stable under the PDE dynamics in an appropriate sense. The aforementioned spectral stability assumptions are motivated through consideration of the associated Whitham averaged system and its hyperbolicity, i.e. local well-posedness. The details of the nonlinear stability proof are beyond the scope of the current presentation and hence only an outline of the argument is given; the interested reader is referred to [JZN] for details.

After establishing that (an appropriate sense of) spectral stability implies nonlinear stability, we then turn our attention in Section 3 to the verification of the spectral stability assumptions, restricting our attention to the commonly studied case $(r, s) = (2, 0)$ in (1.2). We begin by looking in the Hopf and homoclinic limits, as these limits are amenable to direct analysis.

A striking feature of the particular example chosen, corresponding to $(r, s) = (2, 0)$ in (1.2), is that, in the region of existence of periodic orbits, *all constant solutions are spectrally unstable* with respect to frequencies in a neighborhood of the origin. At a philosophical level, this means that, for this model, stability can only come about through dynamical effects having to do with the variation of the underlying wave, and cannot be understood from a "frozen-coefficients" point of view. To us, this makes it a particularly interesting and illuminating example from a phenomenological point of view.

At a practical level, it means that periodic waves sufficiently close to either the Hopf equilibrium or bounding homoclinic orbit must be spectrally unstable, as follows by standard continuity arguments. Thus, stable periodic waves, if they exist, must be found within intermediate amplitudes bounded away from either of these simplifying limits.

We pursue this line of investigation by conducting a numerical study of the spectrum of the intermediate amplitude waves found in Section 3.2 and we find numerically that there indeed exist periodic waves between the Hopf and homoclinic orbits which are spectrally stable, and hence are nonlinearly stable by the main theorem in Section 2. We then conclude with a brief conclusion and discussion of the future directions of this project.

1.1 Periodic Traveling Waves

We begin with a brief discussion of the traveling wave solutions of the generalized St. Venant equations. Following [JZN] we restrict ourselves to positive velocities $u > 0$ and consider (1.2) in Lagrangian coordinates[3]

$$
(1.3) \qquad
\begin{aligned}
\tau_t - u_x &= 0, \\
u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau^{s+1}u^r + \nu(\tau^{-2}u_x)_x,
\end{aligned}
$$

---

[3]These coordinates are of course equivalent to the Eulerian formulation presented in (1.2), and stability in one formulation is clearly equivalent to stability in the other. Moreover, we point out that while the Eulerian formulation is possibly more physically transparent, the Lagrangian formulation is more analytically convenient.

where $\tau := h^{-1}$ and now the variable x denotes a Lagrangian marker, rather than a physical location. Notice that since the equation (1.2) models waves propagating down a ramp, there is no loss in enforcing the restriction $u > 0$[4]. In this coordinate frame, a traveling wave solution of (1.3) is a solution which is stationary in an appropriate moving coordinate frame of the form $x - st$, where $s \in \mathbb{R}$ is the wavespeed. That is, they take the form

$$U(x,t) = \bar{U}(x - st),$$

where $\bar{U}(\cdot) = (\bar{\tau}(\cdot), \bar{u}(\cdot))$ is a solution of the ODE

(1.4)
$$-c\bar{\tau}' - \bar{u}' = 0,$$
$$-c\bar{u}' + ((2F)^{-1}\bar{\tau}^{-2})' = 1 - \bar{\tau}^{s+1}\bar{u}^r + \nu(\bar{\tau}^{-2}\bar{u}')'.$$

Integration of the first equation yields $\bar{u} = \bar{u}(\tau; q, s) := q - c\bar{\tau}$, where q is the corresponding integration constant. Substitution of this identity into the second equation yields the second-order scalar profile equation for the function $\bar{\tau}$:

(1.5)
$$c^2\bar{\tau}' + ((2F)^{-1}\bar{\tau}^{-2})' = 1 - \bar{\tau}^{s+1}(q - c\bar{\tau})^r - c\nu(\bar{\tau}^{-2}\bar{\tau}')'.$$

The orbits of (1.5) can be studied by simple phase plane analysis. In particular, we find that the equilibrium solutions τ_0 satisfy the algebraic identity

$$\tau_0^{s+1}(q - c\tau_0)^r = 1.$$

Considering homoclinic solutions then, as in [BJRZ], under an appropriate normalization we can assume $\tau_0 = 1$. Here, however, we are interested in the periodic orbits of the profile ODE (1.5) which are easily seen not to exist in the case $c = 0$. Indeed, in that case (1.5) reduces to the first order scalar ODE

(1.6)
$$\bar{\tau}' = F\bar{\tau}^3(\bar{\tau}^{s+1}q^r - 1),$$

which clearly has no nontrivial solutions with $\bar{\tau} > 0$. The existence of periodic orbits for non-zero values of c was considered in [N1, N2], and are generically seen to emerge from a Hopf bifurcation from the equilibrium state generating a family of periodic orbits which terminate into the bounding homoclinic. The conditions for a Hopf bifurcation to occur can be derived from straightforward Fourier analysis as in [BJRZ]. Indeed, simply notice that the linearization of the profile ODE (1.5) about an equilibrium solution τ_0 (with $q = u_0 + c\tau_0$) is given by

$$\left(\frac{s+1}{\tau_0} - \frac{cr}{u_0}\right)\tau + (c^2 - c_s^2)\tau' + \frac{c\nu\tau''}{\tau_0^2} = 0,$$

where we have used the relation $\tau_0^{s+1}u_0^r = 1$. Taking the Fourier transform, it follows that the Fourier frequency k must satisfy the polynomial equation

$$\frac{s+1}{\tau_0} - \frac{cr}{u_0} + ik(c^2 - c_s^2) - \frac{c\nu k^2}{\tau_0^2} = 0.$$

---

[4]However, we must always remember to discard any spurious solutions for which u may become negative.

Evidently, such a $k \in \mathbb{R}$ exists if and only if $c = c_s$ and $\left(\frac{s+1}{r}\right) \frac{u_0}{\tau_0} = \left(\frac{s+1}{r}\right) \tau_0^{-(r+s+1)/r} > c_s$, in which case the solutions are $\pm k_H$ with $k_H \neq 0$. These translate then to the *Hopf bifurcation conditions*

$$(1.7) \qquad c = c_s = \frac{\tau_0^{-3/2}}{\sqrt{F}} \quad \text{and} \quad \left(\frac{s+1}{r}\right) \tau_0^{-(r+s+1)/r} > c_s.$$

When $(r, s) = (2, 0)$, which is the case considered in [JZN], this reduces to $F > 4$. This is in agreement with the experiments of [N2, BJNRZ], which indicate that when $(r, s) = (2, 0)$ and $F > 4$ there exists a smooth family of periodic orbits of (1.5) parametrized by the period, which increase in amplitude as the period is increased and finally approaching a limiting homclinic orbit as the period tends to infinity.

As far as the general existence theory is concerned, we notice that periodic orbits of (1.5) correspond to values $(X, c, q, b) \in \mathbb{R}^5$, where X, c, and q denote the period, constant of integration, and wave speed, respectively, and $b = (b_1, b_2)$ denotes the initial values of (τ, τ') at $x = 0$ or, equivalently, at $x = X$. Furthermore, in the spirit of [Se1, OZ3, OZ4, JZ2, JZ3], we make the following general assumptions:

(H1) $\bar{\tau} > 0$, so that all terms in (1.4) are C^{K+1}, $K \geq 4$.

(H2) The map $H : \mathbb{R}^5 \to \mathbb{R}^2$ taking $(X, c, q, b) \mapsto (\tau, \tau')(X, c, b; X) - b$ is full rank at $(\bar{X}, \bar{c}, \bar{b})$, where $(\tau, \tau')(\cdot; \cdot)$ is the solution operator of (1.5).

By the Implicit Function Theorem, then, conditions (H1)–(H2) imply that the set of periodic solutions in the vicinity of $\bar{U}$ form a smooth 3-dimensional manifold $\{\bar{U}^\beta(x - \alpha - c(\beta)t)\}$, with $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^2$.

The goal of our analysis is to understand the modulational stability, i.e. the spectral and nonlinear time evolutionary stability with respect to small *localized* initial perturbations. We begin by considering the Whitham averaged system corresponding to the dynamical version of (1.4), which yields a necessary condition for spectral stability. These considerations lead to a natural set of spectral stability assumptions, and under these assumptions we outline the recent nonlinear stability theory developed in [JZN]. With this theory in place, we numerically study the spectrum of the linearized operator for various values of the turbulent parameters (r, s). In particular, we are able to numerically find a spectrally stable periodic traveling wave solution of (1.3). Recall that, up till now, the existence of such a stable solution was not at all clear from the known examples, for example found in [OZ1].

2   Analytical Results

In this section, we review as briefly as possible the known analytical stability and instability results concerning the periodic traveling wave solutions of (1.3). While this theory is similar to that developed for the parabolic conservation laws (1.1) developed in [OZ4, JZ2, JZ4], the extension to the current case involves a number of subtle technical issues associated with lack of parabolicity and nonconservative form. In particular, the presence of non-divergence

source terms in (1.3) requires a more detailed analysis of the associated Green function since there are no derivatives to enhance decay. This is handled by an observation made from the structure of this Whitham averaged equations that of all the modulations the wave can undergo under low-frequency perturbation, modulations in translation dominate. This serves as motivation for a decomposition of the linearized solution operator and allows us to prove the time-asymptotic convergence of the underlying periodic profile to an appropriate modulation of itself. The precise statement of this result is the main focus of this section.

We begin our study by analyzing the spectral stability of a periodic traveling wave solution of (1.3). To this end, notice that (1.3) can be written in the abstract form

$$(2.1) \qquad\qquad U_t + f(U)_x = (B(U)U_x)_x + g(U)$$

and linearizing (1.3) about $\bar{U}(\cdot)$, we obtain

$$(2.2) \qquad\qquad v_t = Lv := (\partial_x B \partial_x - \partial_x A + C)v,$$

where the coefficients

$$(2.3) \qquad A := df(\bar{U}) - (dB(\bar{U})(\cdot))\bar{U}_x = \begin{pmatrix} -c & -1 \\ -\bar{\tau}^{-3}(F^{-1} - 2\nu\bar{u}_x) & -c \end{pmatrix},$$

$$B := B(\bar{U}) = \begin{pmatrix} 0 & 0 \\ 0 & \nu\bar{\tau}^{-2} \end{pmatrix}, \quad C := dg(\bar{U}) = \begin{pmatrix} 0 & 0 \\ -\bar{u}^2 & -2\bar{u}\bar{\tau} \end{pmatrix}$$

are periodic functions of x. As the underlying solution $\bar{U}$ depends on x only, equation (2.2) is clearly autonomous in time. Seeking separated solutions of the form $v(x,t) = e^{\lambda t}v(x)$, it is clear that the stability of $\bar{U}$ requires a detailed analysis of the linearized operator L. In particular, we say the underlying periodic traveling wave is spectrally stable provided the linearized operator L has no spectrum in the unstable right half plane $\Re(\lambda) > 0$. However, the analysis of the spectrum of L is made exceedingly difficult by the following two facts: first, as the coefficients of L are X-periodic, Floquet theory implies that the spectrum is purely continuous and hence any spectral instability of the underlying periodic wave must come from the essential spectrum; secondly, by the translation invariance of (1.3) it is known that $\bar{U}'$ is an eigenfunction of L corresponding to $\lambda = 0$ and hence the essential spectrum intersect the imaginary axis in at least one point. The first issue is dealt with here by conducting a numerical study of the spectrum as opposed to a analytical spectral stability study.

The second issue on the other hand actually gives us a starting point for our spectral stability study. Since it is possible that the spectral curve through the origin might pass through to the unstable half plane, a natural place to begin our study is to analyze the spectrum of the linearized operator in a neighborhood of the origin $\lambda = 0$ in the spectral plane. Physically, instability/stability in a neighborhood of the origin corresponds to the underlying wave being spectrally stable to long-wavelength perturbations, i.e. to slow modulations of the traveling wave profile. Thus, we can analyze the long-wavelength stability of a periodic traveling wave by using a well-developed (formal) physical theory for dealing

with such stability problems known as Whitham theory. In the next section, we summarize recent results concerning the application of Whitham theory to the current situation and its rigorous verification through the use of Evans function techniques. This will lead to an analytically necessary condition for spectral stability and hence to a natural set of stability assumptions similar to those proposed by Schnieder in the context of reaction-diffusion and related pattern-formation systems [S1, S2, S3].

2.1   Whitham averaging and spectral instability

Very recently a necessary condition for the spectral stability of periodic traveling wave solutions of the generalized St. Venant equations in Eulerian coordinates (1.2) has been derived by a novel relation between the Evans function and the corresponding linearized Whitham averaged system proposed by Serre [Se1]; see [NR] for complete details. In particular, the authors show that the linearized dispersion relation obtained from the leading order asymptotics of the Evans function near the origin can be derived formally through a slow modulation (WKB) approximation yielding the Whitham averaged system. It follows that the formal homogenization procedures introduced by Whitham [W] and Serre [Se1] yields a necessary condition for the spectral stability of the underlying periodic traveling wave. Here, we briefly review this procedure and its implication for spectral instability of the periodic waves of (1.2), and hence of (1.3).

As a first step, we let $\varepsilon > 0$ be a small perturbation parameter and introduce a set of slow-variables $(x, t) = (\varepsilon X, \varepsilon T)$. In these slow variables, we search for a solution of (1.2) of the form

$$(h, u)(X, T) = (h^0, u^0)\left(X, T; \frac{\phi(X, T)}{\varepsilon}\right) + \varepsilon(h^1, u^1)\left(X, T; \frac{\phi(X, T)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2),$$

where $\mathbb{R} \ni y \mapsto (h^j, u^j)(X, T; y)$ are unknown 1-periodic functions. It follows then that the local period of oscillation is $\varepsilon / \partial_X \phi$, where we assume the unknown phase *a priori* satisfies $\partial_X \phi \neq 0$. Plugging this expansion into rescaled version of (1.2) and collecting like powers of ε results in a hierarchy of consistency conditions which must hold. At the lowest order $\mathcal{O}(\varepsilon^{-1})$, we find that the functions (h^0, u^0) satisfy the corresponding rescaled profile ODE with wavespeed s in the variable ωy, where

$$s = -\frac{\phi_T}{\phi_X}, \quad \text{and} \quad \omega = \phi_T.$$

Furthermore, notice then that ω denotes the local frequency and $k = \phi_X$ the local wave number of the modulated wave. It follows then that (h^0, u^0) can be chosen to agree with a given periodic traveling wave solution of (1.2) in the variable y.

Continuing, collecting the $\mathcal{O}(1)$ terms yields the mass conservation law

$$\partial_y \left(\omega(k, \bar{q})h^1 - ku^1\right) = -\left(\partial_T h^0 + \partial_X u^0\right),$$

which has a solution if and only if the the right hand side has zero spatial average over a period, i.e. if and only if

$$(2.4) \qquad \partial_T \left(M(X,T) \right) + \partial_X \left(cM(X,T) - q \right) = 0,$$

where $M(X,T) := \int_0^1 h^0(X,T,y)dy$ denotes the corresponding mass functional, c the wave speed, and $q = ch^0 - u^0$ the corresponding integration constant. Together with the consistency condition

$$(2.5) \qquad \partial_X k(X,T) + \partial_X (k(X,T)c(X,T)) = 0$$

these equations form a closed first order linear system of partial differential equations known as the Whitham averaged equations[5].

The linear system (2.4)-(2.5) is seen to be of evolutionary type provided that the non-degeneracy condition

$$(2.6) \qquad \partial_q M(X,T) \neq 0,$$

holds. This condition is discussed in both the small-amplitude and small-viscosity regimes in [NR]. Furthermore, the local well-posedness of this linear system is equivalent with its local hyperbolicity, i.e. the fact that the dispersion relation $\Delta(\lambda, \nu)$

$$\Delta(\lambda, \nu) := \det \left(\lambda \frac{\partial(k, M)}{\partial(c, q)} - \nu \frac{\partial(kc, cM - q)}{\partial(c, q)} \right) = 0,$$

with $(\lambda, \nu) \in \mathbb{C} \times i\mathbb{R}$ and where all arguments are evaluated at the underlying periodic wave (h^0, u^0), corresponding to the linearization about the underlying wave (h^0, u^0), has all real roots. Notice, in particular, that $\Delta(\lambda, \nu)$ is a homogeneous quadratic polynomial in λ and ν.

While hyperbolicity of the Whitham averaged system can heuristically be related to its stability to long-wavelength perturbations, a rigorous proof of this fact has only been recently given in [NR] through the use of the Evans function, which we briefly recall here. Writing the linearization of (1.2) about a given X-periodic traveling wave solution as the first order system

$$Y' = A(\lambda)Y,$$

the Evans function $D(\lambda, \sigma)$ is defined for $(\lambda, \sigma) \in \mathbb{C} \times S^1$ via

$$D(\lambda, e^\nu) = \det \left(\Psi(X; \lambda) - \sigma \mathbb{I}_3 \right),$$

where $\Psi(\cdot; \lambda)$ denotes the fundamental solution matrix, normalized so that $\Psi(0; \lambda) = \mathbb{I}_3$, evaluated at the period point X. In particular, notice that λ belongs to the spectrum of the

---

[5]Notice that these are not the true Whitham equations for (1.2). Indeed, the Whitham equations are the inherently nonlinear equations arising at order $\mathcal{O}(1)$ from substituting the above expansion in the rescaled version of (1.2). Upon averaging these nonlinear equations over one spatial period, however, one arrives at the given linear system; hence its naming as the Whitham *averaged* equations.

linearized operator if and only if $D(\lambda, \sigma) = 0$ for some $\sigma \in S^1$, and hence spectral stability of the underlying wave is equivalent to the condition that $D(\lambda, \sigma)$ does not vanish for any $\Re(\lambda) > 0$ and $\sigma \in S^1$. Notice, however, that $D(0, 1) = 0$ by translation invariance, and hence the spectrum must touch the imaginary axis. Whether or not the periodic wave is spectrally stable in a neighborhood of the origin is connected to the hyperbolicity of the Whitham averaged system through the following theorem.

Theorem 1 (Noble & Rodrigues [NR]). *Let $\bar{U}$ be a periodic traveling wave solution of* (1.2) *such that the non-degeneracy condition* (2.6) *holds. Then in the limit* $(\lambda, \nu) \to (0, 0)$ *the following asymptotic relation holds:*

$$D(\lambda, e^{\nu}) = \Gamma \Delta(\lambda, \nu) + \mathcal{O}((|\lambda| + |\nu|)^3)$$

for some non-zero constant Γ.

That is, the dispersion relation $\Delta(\lambda, \nu)$ agrees to leading order with the Evans function in a neighborhood of the origin. Recalling that Δ is a homogeneous in the variables λ and ν, introduction of the projective coordinate $z = \frac{\lambda}{\nu}$ reduces the dispersion relation to the quadratic polynomial

$$(2.7) \qquad\qquad\qquad\qquad \Delta(z, 1) = 0,$$

whose roots z_1, z_2 are distinct so long as the corresponding discriminate is non-zero. Under this assumption, the implicit function theorem applies in a neighborhood of $z = z_j$, $\kappa = 0$ and hence, in terms of the original spectral variable λ there are two spectral branches

$$(2.8) \qquad\qquad\qquad\qquad \lambda_j = z_j \nu + \mathcal{O}(\nu^2).$$

Thus, if the Whitham system is hyperbolic, corresponding to $z_j \in \mathbb{R}$, then the two spectral branches emerge from the origin tangent to the imaginary axis. This is clearly a necessary condition for spectral stability. On the other hand, failure of hyperbolicity of the Whitham system implies that the z_j have non-zero imaginary part and hence the corresponding spectral branches must emerge from the origin and enter the unstable half plane, immediately yielding spectral instability of the underlying wave.

Similar results concerning the spectral verification of the Whitham averaged equations have also been derived in the viscous conservation law setting [OZ3, Se1]. Notice however, that while hyperbolicity of the Whitham system is necessary for the spectral stability of a given periodic traveling wave solution of (1.2), it may not be sufficient. Indeed, hyperbolicity of the Whitham system is a first order condition, implying agreement of the spectrum near the origin along lines. Thus, the Whitham system will be hyperbolic so long as the spectral curve is tangent to the imaginary axis at the origin, whether or not the spectral curve then proceeds to the stable or unstable half planes. Nevertheless these considerations lead us to a natural set of spectral stability assumptions, which in the next section we show implies nonlinear stability of the underlying wave.

2.2 Bloch decomposition and spectral stability conditions

A particularly useful way to analyze the continuous spectrum of the linearized operator L is to decompose the problem into a continuous family of eigenvalue problems through the use of a Bloch decomposition. To this end, a straightforward application of Floquet theory implies that the L^2 spectrum of the linearized operator L is purely continuous and corresponds to the union of the L^∞ eigenvalues of the operator L taken with boundary conditions $v(x + X) = e^{i\kappa}v(x)$ for all $x \in \mathbb{R}$, where $\kappa \in [-\pi, \pi)$ is referred to as the Floquet exponent. In particular, it follows that $\lambda \in \sigma(L)$ if and only if the spatially periodic spectral problem

$$(2.9) \qquad\qquad Lv = \lambda v$$

admits a uniformly bounded eigenfunction of the form $v(x) = e^{i\xi x}w(x)$, where w is X-periodic. Substitution of this Ansatz into (2.9) motivates the use of the Fourier-Bloch decomposition of the spectral problem.

Following [G] then, we define a one-parameter family of linear operators, referred to as the Bloch operators, via

$$L_\xi := e^{-i\xi x}Le^{i\xi x}, \quad \xi \in [-\pi, \pi)$$

operating on $L^2_{\mathrm{per}}([0, X])$, the space of X-periodic square integrable functions. The spectrum of L is then seen to be given by the union of the spectra of the Bloch-operators. Furthermore, since the domain $[0, X]$ is compact the operators L_ξ, for each fixed ξ, have discrete spectrum in $L^2_{\mathrm{per}}([0, X])$ and hence, by continuity of the spectrum, the spectra of L may be described by the union of countable many continuous surfaces.

Continuing, taking without loss of generality $X = 1$, we recall that any localized function $v \in L^2(\mathbb{R})$ admits an inverse Bloch-Fourier representation

$$v(x) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x}\hat{v}(\xi, x)d\xi$$

where the functions $\hat{v}(\xi, \cdot) = \sum_{j\in\mathbb{Z}} e^{2\pi ijx}\hat{v}(\xi + 2\pi j)$ belongs to $L^2_{\mathrm{per}}([0, X])$ for each ξ, where here $\hat{v}(\cdot)$ denotes with a slight abuse of notation the usual Fourier transform of the function v in the spatial variable x. By Parseval's identity it is seen that the Bloch-Fourier transformation $v(x) \to \hat{v}(\xi, x)$ is an isometry of $L^2(\mathbb{R})$, i.e.

$$\|v\|_{L^2(\mathbb{R})} = \int_{-\pi}^{\pi}\int_{0}^{X} |\hat{v}(\xi, x)|^2 dx \ d\xi =: \|\hat{g}\|_{L^2(\xi;L^2(x))}.$$

Furthermore, this transformation is readily seen to diagonalize the periodic-coefficient linearized operator L, yielding the inverse Bloch-Fourier transform representation

$$e^{Lt}v(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x}e^{L_\xi t}\hat{g}(\xi, x)d\xi$$

effectively relating the behavior of the linearized system to that of the diagonal operator L_ξ.

Together with the long-wavelength stability analysis in the previous section, we now state our main spectral stability assumptions in terms of the diagonal Bloch operators L_ξ.

(D1) $\sigma(L_\xi) \subset \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$ for all $\xi \neq 0$.

(D2) There exists a constant $\theta > 0$ such that $\Re\sigma(L_\xi) \leq -\theta|\xi|^2$ for all $|\xi| \ll 1$.

(D3') $\lambda = 0$ is an eigenvalue of L_0 of multiplicity exactly two.[6]

Notice that assumption $(D1)$ implies weak hyperbolicity of the Whitham averaged system, while $(D2)$ corresponds to "diffusivity" of the large-time ($\sim$ small frequency) behavior of the linearized operator L. Moreover, $(D3')$ holds generically and can be directly verified through the use of the Evans function arguments as in [N1]. Finally, we point out that $(D1)-(D3')$ are balance law analogues of the spectral assumptions introduced by Schneider for reaction-diffusion equations [S1, S2, S3].

Furthermore, we make the following non-degeneracy hypothesis:

(H3) The roots z_j of (2.7) are distinct.

(H4) The eigenvalue 0 of L_0 is non-semisimple, i.e. $\dim\ker(L_0) = 1$.

Conditions $(H1)-(H4)$ generically imply that (D2) hold[7]. Moreover, (H3) corresponds to strict hyperbolicity of the Whitham averaged system, and implies the analyticity of the spectrum in a neighborhood of the origin. Specifically, since $\Delta(0,1) \neq 0$, as is readily seen in [NR], it follows that the roots z_j of (2.7) are non-zero and distinct by (H3) and hence relation (2.8) and standard spectral perturbation theory [K] implies the spectral curves $\lambda_j = \lambda_j(\nu) = \lambda_j(i\xi)$ are analytic functions of ξ in a neighborhood of $\xi = 0$. Finally, notice that assumptions $(D3')$ and $(H4)$ imply the existence of a Jordan block at $(\lambda, \xi) = (0, 0)$. In particular, we have the following spectral preparation result.

Lemma 2.1 ([JZN]). *Assuming (H1)–(H4), (D1), and (D3'), the eigenvalues $\lambda_j(\xi)$ of L_ξ are analytic functions and the Jordan structure of the zero eigenspace of L_0 consists of a 1-dimensional kernel and a single Jordan chain of height 2, where the left kernel of L_0 is spanned by the constant function $\tilde{f} \equiv (1,0)^T$, and $\bar{u}'$ spans the right eigendirection lying at the base of the Jordan chain. Moreover, for $|\xi|$ sufficiently small, there exist right and left eigenfunctions $q_j(\xi, \cdot)$ and $\tilde{q}_j(\xi, \cdot)$ of L_ξ associated with $\lambda_j(\xi)$ which are analytic in ξ for in a neighborhood of $\xi = 0$. Furthermore, $\langle \tilde{q}_j, q_k \rangle = \delta_j^k$*

Remark 2.2. *Notice that the results of Lemma 2.1 are somewhat unexpected since, in general, eigenvalues bifurcating from a non-trivial Jordan block typically do so in a nonanalytic fashion, rather being expressed in a Puiseux series in fractional powers of ξ. The fact that*

---

[6]Note that the zero eigenspace of L_0, corresponding to variations along the three-dimensional manifold of periodic solutions in directions for which the period does not change [Se1, JZ2], is at least two-dimensional by the linearized existence theory and assumption (H2).

[7]This amounts to nonvanishing b_j in the Taylor series expansion $\lambda_j(\xi) = -iz_j\xi - b_j\xi^2 + o(|\xi|^2)$ guaranteed by Lemma 2.1 given $(H1)-(H4)$, $(D1)$, and $(D3')$.

analyticity prevails in our situation is a consequence of the very special structure of the left and right generalized null-spaces of the unperturbed operator L_0, and the special forms of the equations considered.

From the standpoint of obtaining a nonlinear stability result, the existence of the non-trivial Jordan block over the translation mode suggests that one can not expect traditional orbital asymptotic stability of the original periodic traveling wave in any standard L^p or H^s norm; see [OZ2]. Nevertheless, following the ideas of [JZ2] we are able to prove nonlinear asymptotic stability to an appropriate modulation of the original wave, and hence an L^∞ stability result for the underlying wave. The technical details driving this nonlinear stability argument are beyond the scope of what we wish to discuss here; the interested reader can see [JZ2, JZN]. However, for a sense of completeness we recall here the general outline of the argument in the next section.

2.3 Nonlinear stability: a guided tour

Here, we wish to recall the basic ideas behind the nonlinear stability of periodic traveling wave solutions of the St. Venant equations (1.2). For technical reasons, we find it essential to utilize the Lagrangian formulation (1.3) throughout this analysis. To begin, let $\bar{U}(x)$ denote a periodic traveling wave solution of (1.4) and let $\tilde{U}(x,t)$ denote any other solution of (1.3). Our goal is to prove that if $\tilde{U}(x,0)$ is sufficiently close to $\bar{U}(x)$ in a suitable norm, then it remains close for all future times $t > 0$. To this end, define the nonlinear perturbation variable

$$(2.10) \qquad v(x) := \tilde{U}(x + \psi(x,t)) - \bar{U}(x),$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a modulation function to be chosen later. Our starting point is the following observation: by a direct computation and Taylor expansion, the nonlinear residual (2.10) is seen to satisfy

$$(\partial_t - L)\, v = (\partial_t - L)\, \bar{U}'(x)\psi - Q_x + T + P + R_x + \partial_t S,$$

where

$$P = (0,1)^T O(|v|(|\psi_{xt}| + |\psi_{xx}| + |\psi_{xxx}|),$$
$$Q := f(\tilde{U}(x + \psi(x,t),t)) - f(\bar{U}(x)) - df(\bar{U}(x))v = \mathcal{O}(|v|^2),$$
$$T := (0,1)^T \left((\tilde{U}(x + \psi(x,t),t))^2 - (\bar{U}(x))^2 - \bar{U}(x))v \right) = (0,1)^T \mathcal{O}(|v|^2),$$
$$R := v\psi_t + v\psi_{xx} + (\bar{U}_x + v_x)\frac{\psi_x^2}{1 + \psi_x}, \quad \text{and}$$
$$S = O(|v|(|\psi_x|).$$

Letting $G(x, t; y)$ denote the Green function of (2.9) then, an application of Duhamel's formula implies the nonlinear residual must satisfy the integral equation

$$v(x, t) = \psi(x, t)\bar{U}'(x) + \int_{-\infty}^{\infty} G(x, t; y)v_0(y)\, dy$$

(2.11)

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, t - s; y)(-Q_y + T + R_y + S_t)(y, s)\, dy\, ds.$$

In order to obtain control over v in a given H^s norm then, we seek to obtain pointwise bounds on the Green function G and an appropriate expression for ψ for which (2.11) becomes susceptible to an iteration argument. Since, as expected, the low-frequency behavior of the solution operator near the neutral eigenvalue $\lambda = 0$ is the most difficult to control, we decompose the solution operator $S(t) = e^{Lt}$ corresponding to the linearized operator L into high and low-frequency components.

To this end, standard spectral perturbation theory [K] implies that the total eigenprojection $P(\xi)$ onto the eigenspace of L_ξ associated with the eigenvalues $\lambda_j(\xi)$ described in (2.8) is well-defined and analytic in ξ for $|\xi|$ sufficiently small, since these (by discreteness of the spectra of L_ξ) are separated at $\xi = 0$ from the rest of the spectrum of L_0. Choosing an appropriate cut-off function $\phi(\xi)$ supported in a small neighborhood of the origin and identically one in a slightly smaller neighborhood, we split $S(t)$ into a low-frequency part

$$S^I(t)u_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x}\phi(\xi)P(\xi)e^{L_\xi t}\hat{u}_0(\xi, x)d\xi$$

and the associated high-frequency part

$$S^{II}(t)u_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x}(1 - \phi(\xi))P(\xi)e^{L_\xi t}\hat{u}_0(\xi, x)d\xi.$$

We begin by analyzing S^{II}. Fairly routine semigroup estimates [He, Pa] imply the bounds

$$\left\|e^{L_\xi t}g\right\|_{L^2([0,X])} \lesssim e^{-\theta t}\|g\|_{L^2([0,X])}$$
$$\left\|\partial_x e^{L_\xi t}g\right\|_{L^2([0,X])} \lesssim e^{-\theta t}t^{-1/2}\|g\|_{L^2([0,X])}$$
$$\left\|e^{L_\xi t}\partial_x g\right\|_{L^2([0,X])} \lesssim e^{-\theta t}t^{-1/2}\|g\|_{L^2([0,X])}$$

for all times $t > 0$ and some constant $\theta > 0$. Since the Bloch-Fourier transform is an isometry of L^2, it follows by standard L^p interpolation and Sobolev's inequality that there exists a constant $\theta > 0$ such that

$$\|S^{II}(t)\partial_x^l g\|_{L^p(\mathbb{R})} \lesssim e^{-\theta t}t^{-l/t}\|g\|_{L^2(\mathbb{R})}.$$

for all $0 \le l \le 2$ and, similarly, one obtains an estimate on terms of the form $\|S^{II}(t)\partial_t^m g\|_{L^2(\mathbb{R})}$.

Next, we seek analogous bounds on the low-frequency portion of the solution operator. This however is complicated by the presence of spectral curves which touch the imaginary axis at the origin, and hence a more delicate analysis is necessary. We begin by denoting by

$$G^I(x, t; y) := S^I(t)\delta_y(x)$$

the Green kernel associated with S^I. Furthermore, recalling Lemma 2.1, for $|\xi|$ sufficiently small we denote by $q_j(x, \xi)$ and $\tilde{q}_j(x, \xi)$ the right and left eigenfunctions of the Bloch operator L_ξ, respectively, associated with the spectral curves $\lambda_j(\xi)$ bifurcating from the $(\xi, \lambda_j(\xi)) = (0, 0)$ state and we enforce the normalization condition $\langle \tilde{q}_j(\cdot, \xi), q_k(\cdot, \xi) \rangle_{L^2([0,X])} = \delta_j^k$. It follows then that we can express the low-frequency Green function as

$$G^I(x, t; y) = \left(\frac{1}{2\pi} \right) \int_{\mathbb{R}} e^{i\xi(x-y)} \phi(\xi) \sum_{j=1}^2 e^{\lambda_j(\xi)t} q_j(\xi, x) \tilde{q}_j(\xi, y)^* d\xi,$$

where $*$ denotes the complex conjugate transpose. Notice that this Bloch expansion for the Green kernel is analogous to using a Fourier representation in the constant coefficient case. Similarly as in the constant–coefficient case, we may read off decay from the spectral representation using the following generalization of the Hausdorff–Young inequality.

Lemma 2.3 (Generalized Hausdorff–Young inequality [JZ2]).

(2.12) $$\|u\|_{L^p(x)} \le \|\hat{u}\|_{L^q(\xi; L^p(0,X))}, \quad \text{for } q \le 2 \le p \quad \text{and } \frac{1}{p} + \frac{1}{q} = 1$$

Proof. Relation (2.12) holds in the extremal cases $p = 2$ and $p = \infty$ by Parseval's identity and the triangle inequality, respectively. This generalized version of the Hausdorff–Young inequality then holds for all stated pairs (p, q) by a generalized version of Riesz-Thorin interpolation theorem; see Appendix A of [JZ2] for more details. □

In order to analyze the decay properties of the kernel G^I in t, we notice that while assumption (D2) implies $e^{\lambda_j(\xi)t} \lesssim e^{-\theta|\xi|^2 t}$ for all $t > 0$, this does not immediately yield decay since the presence of the Jordan block (guaranteed by assumptions (D3') and (H4)) implies $q_1(\xi)\tilde{q}_1(\xi) \sim \xi^{-1}$, where $q_1(0)(x) = \bar{U}'(x)$ corresponds to the translation mode. Following this intuition, we find that the terms in the Bloch expansion of the Green kernel G^I not associated with the Jordan block near $\xi = 0$ decay in $L^p(x)$ as $\|e^{-\theta\xi^2 t}\|_{L^q(\xi)} \lesssim t^{-\frac{1}{2}(1-1/p)}$, i.e. at the rate of a heat kernel, while the portion of the kernel associated with the Jordan block decays in $L^\infty(x)$ as $\|\xi^{-1} e^{-\theta\xi^2 t}\|_{L^1(\xi)} \lesssim 1$. These bounds, derived from the Bloch-norm Hausdorff–Young inequality discussed above should be compared with those found by weighted-energy estimate methods of Schneider [S1].

Using the above $L^p_x \to L^q_\xi L^p_z$ bounds then it follows that the low-frequency Green kernel G^I can be decomposed as

$$G^I(x, y; t) = \bar{U}'(x)e(x, t; y) + \tilde{G}^I(x, t; y)$$

where the residual $\tilde{G}^I$ and amplitude e satisfy the bounds

$$\sup_{y \in \mathbb{R}} \|\tilde{G}^I(\cdot, t; y)\|_{L^p(x)} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)}$$

$$\sup_{y \in \mathbb{R}} \|e(\cdot, t; y)\|_{L^p(x)} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)}$$

for all $2 \le p \le \infty$. Furthermore, it can be shown that the derivatives of these functions decay in time even faster according to the variable and order of differentiation. These estimate can be used to control the "free-evolution" type terms appearing in the integral equation (2.11). To control the integrals associated with the (implicit) source terms, arguments like those outlined above can be used to obtain the estimates

$$\left\| \int_{\mathbb{R}} \tilde{\partial}_y^r G(\cdot, t; y) f(y) dy \right\|_{L^p(x)} \lesssim (1+t)^{-\frac{1}{2}(1/q - 1/p) - r/2} \|f\|_{L^q \cap H^1}$$

$$\left\| \int_{\mathbb{R}} \partial_{x,y,t} e(\cdot, t; y) f(y) dy \right\|_{L^p(x)} \lesssim (1+t)^{-\frac{1}{2}(1/q - 1/p)} \|f\|_{L^q}$$

where $1 \le q \le 2 \le p \le \infty$ and $r = 0, 1$.

With these preparations, we return to (2.11) and define

$$\psi(x,t) := -\int_0^t \int_{\mathbb{R}} e(x, t-s; y) \left(-Q_y + R_y + \left(\partial_t + \partial_y^2 \right) S \right) dy \, ds$$

and note that this choice cancels the "bad" term $\bar{U}'(x)e$ in the decomposition $G^I = \bar{U}'e + \tilde{G}^I$. Furthermore, using (2.11) this choice results in a closed system in the variables (v, ψ_x, ψ_t), where now v satisfies

$$v(x,t) = \int_0^t \int_{\mathbb{R}} \tilde{G}^I(x, t-s; y) \left(-Q_y + R_y + \left(\partial_t + \partial_y^2 \right) S \right) dy \, ds,$$

and

$$\psi_{x,t}(x,t) = \int_0^t \int_{\mathbb{R}} e_{x,t}(x, t-s; y) \left(-Q_y + R_y + \left(\partial_t + \partial_y^2 \right) S \right) dy \, ds.$$

Recalling then that $(Q, R, S) = \mathcal{O}(|v, \psi_{x,t}|^2)$ and $\tilde{G}^I$ and $e_{x,t}$ decay at Gaussian rates in $L^p(x)$, nonlinear stability follows by a direct iteration (contraction mapping) argument as in the more well familiar viscous shock case (where the profile decays to constant solutions). As with the linearized bounds derived above, these cancelation computations, carried out in the physical variables, should be compared to the cancelation computations carried out in the frequency domain by Schneider for the reaction diffusion case. In particular, we arrive at the main theorem of [JZN].

Theorem 2. *Assuming (H1)–(H4) and (D1)–(D3'), let $\bar{U} = (\bar{\tau}, \bar{u})$ be a traveling-wave solution of* (1.3) *satisfying the derivative condition*

$$(2.13) \qquad\qquad\qquad \nu \bar{u}_x < F^{-1}.$$

Then, for some $C > 0$ and $\psi \in W^{K,\infty}(x,t)$, K as in (H1),

$$\|\tilde{U} - \bar{U}(\cdot - \psi - ct)\|_{L^p}(t) \leq C(1+t)^{-\frac{1}{2}(1-1/p)}\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0},$$

(2.14) $$\|\tilde{U} - \bar{U}(\cdot - \psi - ct)\|_{H^K}(t) \leq C(1+t)^{-\frac{1}{4}}\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0},$$

$$\|(\psi_t, \psi_x)\|_{W^{K+1,p}} \leq C(1+t)^{-\frac{1}{2}(1-1/p)}\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0},$$

and

(2.15) $$\|\tilde{U} - \bar{U}(\cdot - ct)\|_{L^\infty}(t), \ \|\psi(t)\|_{L^\infty} \leq C\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}$$

for all $t \geq 0$, $p \geq 2$, for solutions $\tilde{U}$ of (1.3) with $\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}$ sufficiently small. In particular, $\bar{U}$ is nonlinearly bounded $L^1 \cap H^K \to L^\infty$ stable.

Theorem 2 asserts not only bounded $L^1 \cap H^K \to L^\infty$ stability, a very weak notion of stability, but also asymptotic convergence of $\tilde{U}$ to the space-time modulated wave $\bar{U}(x - \psi(x,t))$. The fact that we don't get decay to $\bar{U}$ is the fundamental reason why we had to factor out the translation mode from the low-frequency Green kernel G^I: indeed, the nonlinear (source) terms in (2.11) can not be considered as asymptotically negligible without asymptotic decay of the modulation ψ to zero. After factoring out the translation mode, however, it is found that the perturbation v decays to zero as ψ_x, which, thanks to the diffusive nature of the linearized operator, decays at a rate $t^{1/2}$ (in L^2) faster than ψ. Finally, we also note that the derivative condition (2.13) is effectively an upper bound on the amplitude of the underlying periodic wave and is a technical condition needed to control the H^K norm of the perturbation v in terms of the L^2 norm; this technical detail is beyond the scope of our current presentation, and interested readers are referred to [JZN]. It should be noted, however, that (2.13) is satisfied when either the wave amplitude or viscosity coefficient ν is sufficiently small, and is seen to be satisfied for *all* roll-waves computed numerically in [N2] and here in Section 3.2.

Our next goal is to verify, at least numerically, the spectral stability conditions necessary in Theorem 2 to conclude nonlinear (bounded) stability of the underlying periodic wave $\bar{U}$.

3 Spectral Stability

Now that we have established that the spectral stability of a given periodic traveling wave solution of (1.3) implies nonlinear stability in the sense of Theorem 2, we continue our investigation by analyzing the spectral stability question. To begin, we restrict to the commonly studied case $(r, s) = (2, 0)$ and depict in Figure 1(a) a typical phase portrait for the corresponding profile ODE (1.5) in the τ, τ' variables. In this case, all periodic orbits arise through a Hopf bifurcation, corresponding to minimun period $X \approx 3.9$, as c is decreased through the critical wavespeed c_s, as depicted in Figure 1(b). An interesting feature here is that the upper stability boundary, corresponding to the bold orbit of greatest amplitude, is nearly indistinguishable from the limiting homoclinic orbit in both shape, by Figure 1(a), and speed, by Figure 1(b).
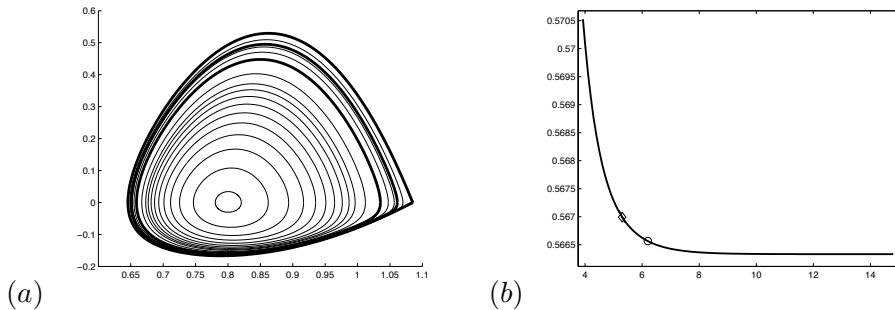
Figure 1: (a) A typical phase portrait depicting a family of periodic orbits parameterized by the wave speed c generated through a Hopf bifurcation at the enclosed equilibrium solution. The inner and outer most bold orbits correspond to the lower (period $X \approx 5.3$) and upper ($X \approx 20.6$) stability boundaries, while the bold orbit in between corresponds to the stable periodic traveling wave solution ($X \approx 6.2$) depicted in Figure 5. (b) A plot of the period X versus the wavespeed c of the corresponding periodic traveling wave. Notice that all periodic orbits sufficiently close to the bounding homoclinic have approximately the same wavespeed and hence locally resemble in both shape and speed the limiting homoclinic wave. The diamond denotes the lower stability boundary, while the circle signifies the stable solution depicted in Figure 5.

The goal of this section is to attempt to find a spectrally stable solution of the St. Venant equation

$$(3.1) \qquad \begin{aligned} \tau_t - u_x &= 0, \\ u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned}$$

considered here again in Lagrangian coordinates, There are two natural limits in which the spectrum of the corresponding linearized operator L seems amenable to direct analysis. The first is a small-amplitude, i.e. Hopf, limit as one approaches the enclosed equilibrium solution, while the other corresponds to the large-period limit as the periodic wave approaches the bounding homoclinic in phase space. In the next section, we recall recent results of [BJRZ] concerning stability in these distinguished limits.

3.1 Hopf and homoclinic limits

We begin our search for stable periodic traveling wave solutions of (3.1) by considering the small amplitude limit in which one approaches the enclosed equilibrium solution. More generally, we consider the stability of the equilibrium solutions, which satisfy the relation $\tau_0 u_0^2 = 1$. To this end, we recall from [BJRZ] that the linearization of (3.1) about an equilibrium solution (τ_0, u_0) satisfying $\tau_0 u_0^2 = 1$ has as an associated dispersion relation

between the eigenvalue λ of the linearized operator L and the Fourier frequency k given by

$$\lambda^2 + \left[\frac{r}{u_0} - 2ick + \frac{\nu k^2}{\tau_0^2}\right]\lambda + ik\left[\frac{s+1}{\tau_0} - \frac{cr}{u_0} + ik(c^2 - c_s^2) - \frac{c\nu k^2}{\tau_0^2}\right] = 0.$$

Notice that the eigenvalues corresponding to frequency $k = 0$ are $\lambda = -u_0^{-1}$, which remains negative for $|k| \ll 1$, and $\lambda = 0$. To determine the behavior of the $\lambda = 0$ for small nonzero k, we Taylor expand the dispersion relation about $(\lambda, k) = (0, 0)$ with $\lambda = \lambda(k)$ and find that the spectral curve $\lambda(k)$ must satisfy

$$\lambda'(0) = -i\left[\frac{u_0}{2\tau_0} - c\right],$$

indicating stability, while

$$\frac{1}{2}\lambda''(0) = \frac{u_0}{2}\left[\left(i\lambda'(0) + c\right)^2 - c_s^2\right] = \frac{u_0}{2}\left[\left(\frac{u_0}{2\tau_0}\right)^2 - c_s^2\right].$$

Recalling the Hopf bifurcation conditions (1.7), we find that if the equilibrium solution τ_0 corresponds to a Hopf bifurcation point of the profile ODE (1.5) then we must have $\lambda''(0) > 0$, yielding instability of the equilibrium (Hopf) point. In particular, we see that in the regime of existence of periodic waves, i.e. $F > 4$, *all* constant solutions are spectrally unstable. Therefore, since the spectrum of the linearized operator L changes continuously as we nearby periodic orbits we conclude that all periodic traveling wave solutions of (3.1) solutions must be spectrally unstable in the small amplitude limit. This is verified numerically in Figure 2(a).

Next, we turn to the large-period limit as the periodic orbits approach the bounding homoclinic profile in phase space. In this case, we can use the same arguments as in the Hopf limit in order to determine the stability of the limiting end state of the homoclinic orbit, which recall determines the essential spectrum of the homoclinic [He]. It follows that the limiting endstate is spectrally unstable in a neighborhood of the origin whenever the Hopf bifurcation condition $F > 4$ hold, and this is numerically verified in Figure 2(b). Therefore, we should not expect to find any spectrally stable periodic waves in the homoclinic limit.

It follows that any spectrally stable periodic traveling wave of (3.1), if one exists, must be of intermediate amplitude. In particular, due to the complicated nature of the linearized operator and the fact that we can not rely on perturbation techniques from a particular limit, our analysis now turns over to a numerical study. Before continuing, however, we wish to give a heuristic argument, reconciling physically observed stability with this analytically-demonstrated instability, why one still might expect the existence of stable periodic waves of intermediate amplitude despite the instability of the Hopf and homoclinic limiting states. If one considers the stability of the limiting homoclinic profiles more carefully, it can be found by rigorous Evans function calculations[8] that there exist homoclinic orbits having

---

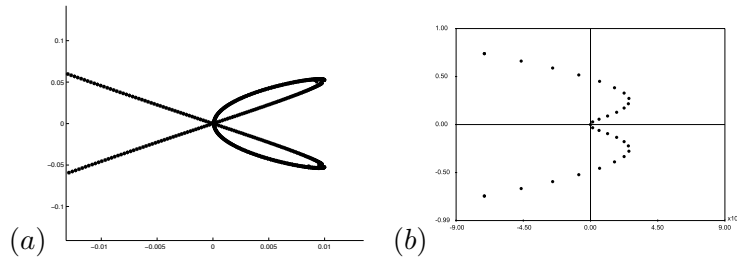[8]More precisely, numerical Evans function computations with rigorous error bounds; see [BJRZ].

Figure 2: (a) The essential spectrum for the unstable constant solution at the Hopf bifurcation point is depicted.(b) The essential spectrum of the bounding homoclinic is shown. Both of these spectral plots, as well as the rest presented throughout this paper, are of $\Re(\lambda)$ vs. $\Im(\lambda)$ and were generated using the SpectrUW package developed at the University of Washington [DK], which is designed to find the essential spectrum of linear operators with periodic coefficients by using Fourier-Bloch decompositions and Galerkin truncation; see [CuD, CDKK, DK] for further information and details concerning convergence.

unstable essential spectrum, as predicted from the preceding discussion, and *stable* point spectrum. As a result, we find that the associated homoclinic orbit stabilizes perturbations across dynamic parts of the wave, i.e. where the gradient varies nontrivially, reflecting the stable point spectrum of the wave, while the portion of the wave near the limiting constant endstate amplifies the perturbation, reflecting the unstable essential spectrum; see Figure 3. Accordingly, one encounters an interesting "metastability" mechanism where the stable point spectrum induces a stabilizing effect on a closely spaced array of solitary waves, since the unstable constant endstates would have little effect due to the "closeness" of the array. This leads one to a notion of the "dynamic stability" of a solitary wave, which is essentially the spectrum of an appropriately periodically extended version of the original homoclinic; this issue is discussed in more detail in [BJRZ]. Heuristically, then, considering a closely spaced array of solitary solutions as a periodic orbit we are led to the possibility of finding spectrally stable periodic waves away from either the homoclinic or Hopf limits. In the next section, we numerically verify this heuristic by presenting numerical computations indicating the existence of a stable periodic solution to the St. Venant equations (1.3). These numerical stability results are formal in the sense that we do not present any error bounds or rigorous high-frequency asymptotics precluding the existence of unstable spectrum sufficiently far from the origin; in a future paper [BJNRZ] we hope to present such bounds and hence make the formal numerical arguments here rigorous. For the purposes of this article, however, our formal numerical investigation will suffice.

3.2   Numerical study

In this section, we use the SpectrUW package, which relies on Fourier-Bloch decompositions and Galerkin truncation to numerically evaluate the spectrum of linear operators with
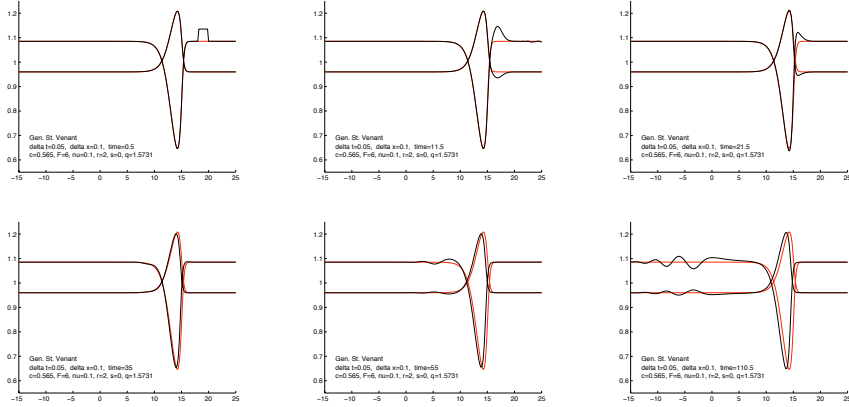
Figure 3: Time evolution of square wave perturbation of a homoclinic orbit possessing stable point spectrum and unstable essential spectrum. Here $u_- = 0.96$, $q = u_- + c/u_-^2$, $\nu = 0.1$, $r = 2$, $s = 0$, and $F = 6$. Notice that the perturbation decays across the region of the profile where the gradient varies non-trivially and grows near the unstable limiting constant states and is eventually convected to minus infinity.

periodic coefficients, to numerically compute the spectrum of the periodic orbits depicted in Figure 1. As described in the previous section, we expect the solutions near the Hopf and homoclinic cycles to have unstable essential spectrum. Nevertheless, the metastability of the limiting homoclinic profile suggests that waves of intermediate period may be spectrally stable, and hence be nonlinearly stable by Theorem 2. In this investigation then, we animate the spectrum as the period X is increased, or equivalently the wave speed c is decreased, from the Hopf period $X \approx 3.9$ to $X = 29.9$, which corresponds to a periodic orbit seemingly very close to the homoclinic in phase space. The results of this study are shown in Figure 4 and indeed seem to indicate a region of stable periodic orbits. Indeed, it seems that the unstable small-amplitude periodic waves eventually stabilize in a neighborhood of the origin as the period is increased and then are later destabilized by the essential spectrum crossing the imaginary axis at non-trivial complex conjugate points as the period is increased further.

We see then that there seems to be a regime of stability in which periodic orbits with particular intermediate periods are spectrally stable solutions of (3.1). A spatial plot in the original physical coordinates ($h = \tau^{-1}$ vs. x) of a periodic roll-wave in this stable regime is depicted in Figure 5. In [BJNRZ], high frequency asymptotics have been obtained which make this numerical evidence rigorous by proving that any spectral instability must occur within a specified compact region of the complex plane. Furthermore, for the seemingly stable spectra depicted in Figure 4 it is verified in [BJNRZ] through the development of rigorous error bounds that the corresponding waves are indeed spectrally stable as solutions of (3.1).

Finally, we make some remarks concerning the various instabilities present in Figure
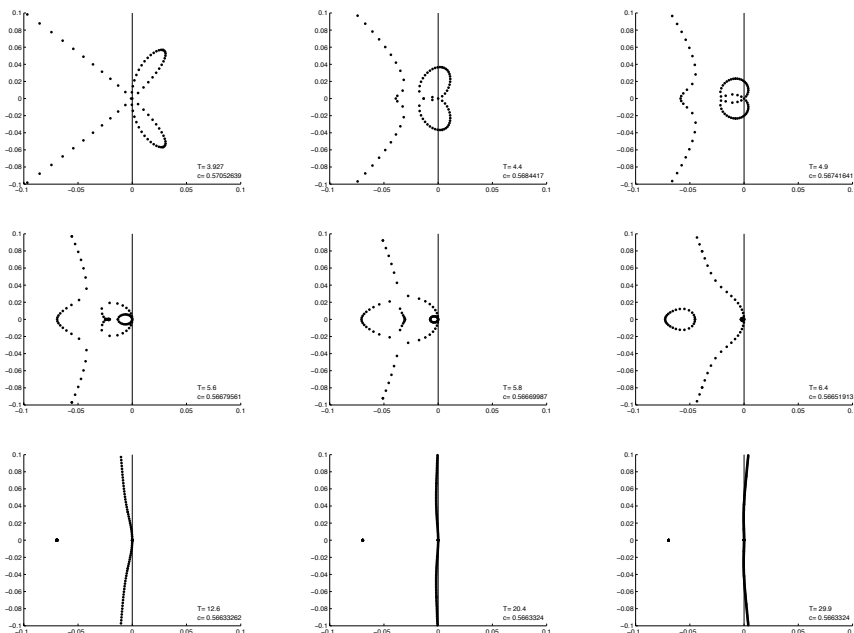
Figure 4: Evolution of spectra as period, here denoted as $X = T$, and wave speed, c, vary. Here $u_- = 0.96$, $q = u_- + c/u_-^2$, $\nu = 0.1$, $r = 2$, $s = 0$, and $F = 6$. Starting in the top left picture and running from left to right and from top to bottom, we see the evolution of the spectra from the Hopf bifurcation at $T \approx 3.9$ to a wave seemingly near the the homoclinic in phase space with period $T \approx 29.9$.
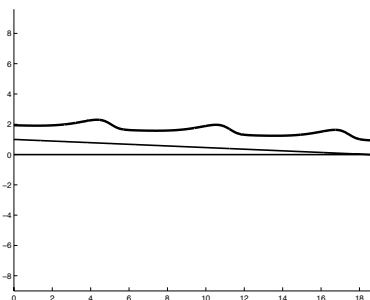

Figure 5: A numerically stable periodic wave of the St. Venant equation 3.1, plotted in the original physical coordinates ($h = \tau^{-1}$ vs. x). This particular wave has period $X \approx 6.2$, and corresponds to the bold orbit in Figure 1(a) between the upper and lower stability boundaries, and whose period is designated by the circle in Figure 1(b) . In particular, notice that the corresponding wave speed, as depicted in Figure 1(b), is close to the limiting homoclinic speed.

4 and their relation to the hyperbolicity of the associated Whitham averaged system. To begin, we recall that by the recent work [NR] hyperbolicity of this Whitham system is necessary for spectral stability; see Theorem 1. The lack of sufficiency in this theorem is associated with the fact that it is only a *first order* verification. Hence, Theorem 1 essentially states that hyperbolicity of the Whitham averaged system is equivalent to the spectrum of the associated linearized spectral problem being tangent to the imaginary axis at the origin, which is clearly a necessary condition for stability but not sufficient[9]. As an example, notice that the first three spectral plots, ordered from left to right and top to bottom, in Figure 4 are spectrally unstable in a neighborhood of the origin due to lack of hyperbolicity of the associated Whitham averaged system. The remaining six spectral plots are seemingly associated with hyperbolic Whitham averaged systems, but we see instability arising for sufficiently large period due to an essential instability occurring away from the origin. Thus, as expected, hyperbolicity of this Whitham equation is only a *local* condition for spectral stability, in the sense it only detects instabilities in a neighborhood of the origin. In particular, the lower stability boundary, occurring with period within 0.1 of $X = 5.3$ is marked by the hyperbolic Whitham criterion, while the upper stability boundary, occuring around $X = 20.6$ is not.

Notice, however, that in the final spectral plot the wave seems to stabilize in a neighborhood of the origin. While this seems to be a general phenonamon for periodic waves where the period is not "too" large, tentative numerical experiments indicate that for periodic waves with with very large periods the spectrum seems to destabilize in a neighborhood of the origin; in particular, the spectrum eventually seems to resemble that of Figure 2(b) for the limiting homoclinic. This seems to suggest that, although the Whitham averaged system is hyperbolic, the wave is spectrally *unstable* in a neighborhood of the origin. Readers should be warned, however, that a stabilization effect near the origin may still occur, but we may not be able to see it due to a low-resolution of the spectral plot. Furthermore, it seems to be quite difficult to numerically generate periodic orbits with very large period and hence we had to resort to periodically extending a homoclinic orbit for these numerics. Nevertheless, these experiments seem to suggest that it may be possible for a periodic traveling wave solution of the St. Venant equations (3.1) with sufficiently large period to have a hyperbolic Whitham averaged system but be spectrally unstable to long-wavelength perturbations.

Continuing, we should note that using a Bloch-wave expansion, two of the authors of the present paper have recently been able to rigorously validate the *second order* Whitham expansion [NR], showing that this second order Whitham system determines the convexity of the spectrum near the origin: a clearly more refined feature than the first order verification provided by hyperbolicity. Thus, it may be possible to numerically verify the existence of periodic waves where the spectrum leaves the origin at second order and moves into

---

[9]This is in contrast to the dispersive Hamiltonian case, such as the generalized Korteweg-de Vries or nonlinear Schrodinger equations, where the Hamiltonian structure of the linearized operator implies the stability spectrum is invariant with respect to reflections across the imaginary axis. In that case, generically it can be shown that hyperbolicity of the Whitham equation is *equivalent* with spectral stability of the underlying wave *in a neighborhood of the origin*. See, for example, [JZ1].

the unstable half plane by analyzing the associated second order Whitham system. This system is however considerably more complicated than its first order counterpart discussed in Section 2.1 and it is not immediately clear how to "compute" the necessary information from this system for a given numerically generated solution.

4 Conclusions & Discussion

In this note, we have considered both analytical and numerical aspects of the stability of periodic roll-wave solutions of the generalized St. Venant equations. In particular, we reviewed known results concerning the nonlinear stability of such solutions and then proceeded to numerically investigate the necessary spectral stability assumptions in the nonlinear stability theorem. To this end, we utilized the SpectrUW package developed at the University of Washington and, formally, we made the case for the existence of a spectrally, and hence nonlinearly, stable periodic traveling wave solution of the governing St. Venant equation. This stands in contrast to the fact that periodic solutions near either the Hopf equilibrium solution or the bounding homoclinic solutions are unstable. By briefly outlining the heuristic of the "dynamic stability" of the bounding homoclinic wave, however, we were able to give a (possibly general) explanation of how an equation with unstable solitary waves can admit stable periodic waves solutions.

This concept of "metastability" of solitary waves has been considered also by Pego, Schneider, and Uecker [PSU] in the context of the related fourth-order diffusive Kuramoto–Sivashinsky model

$$u_t + \partial_x^4 u + \partial_x^2 u + \frac{\partial_x u^2}{2} = 0,$$

which has been proposed as an alternative model for thin film flow down an inclined ramp. Therein, the authors analyze the time-asymptotic behavior of solutions of this equation, and conclude that they are dominated by trains of solitary pulses. As such, the mechanism of stable "dynamic spectrum" seems to provide a partial answer for how a train of solitary pulses can stabilize the convective instabilities shed from their neighbors.

In general, however, it seems that the mechanism by which an equation with an unstable solitary wave can admit stable periodic waves is not completely clear, although we suspect that it is closely tied with the notion of the "dynamic stability" of the limiting homoclinic profile. In particular, to prove a general theorem describing this mechanism at a quantitative level seems outside the realm of current methods and remains an interesting open problem.

Finally, we wish to emphasize again that the numerical evidence for stability presented in Section 3.2 are formal in the sense that they are lacking the error estimates and high-frequency asymptotics/energy estimates necessary to preclude the existence of unstable spectrum outside the window of our computation. In a future paper [BJNRZ], we will carry out these details and provide rigorous numerics which indicate the existence of a spectrally stable periodic traveling wave solution of the St. Venant equation (3.1).

Evans function computations performed in this paper were carried out using the STABLAB package developed by Jeffrey Humpherys with help of the first and last authors.

References

[BM]   N.J. Balmforth and S. Mandre, *Dynamics of roll waves,* J. Fluid Mech. 514 (2004), 1–33.

[BJNRZ]   B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Spectral stability of periodic viscous roll waves*, in preparation.

[BJRZ]   B. Baker, M. Johnson, M. Rodrigues, and K. Zumbrun, *Metastability of Solitary Roll Wave Solutions of the St. Venant Equations with Viscosity,* preprint (2010).

[CuD]   C. Curtis and B. Deconick, *On the convergence of Hill's method*, Mathematics of computation 79 (2010), 169–187.

[CDKK]   J. D. Carter, B. Deconick, F. Kiyak, and J. Nathan Kutz, *SpectrUW: a laboratory for the numerical exploration of spectra of linear operators*, Mathematics and Computers in Simulation 74 (2007), 370–379.

[DK]   Bernard Deconinck and J. Nathan Kutz, *Computing spectra of linear operators using Hill's method.* J. Comp. Physics 219 (2006), 296-321.

[G]   R. Gardner, *On the structure of the spectra of periodic traveling waves*, J. Math. Pures Appl. 72 (1993), 415-439.

[He]   D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981).

[HC]   S.-H. Hwang and H.-C. Chang, *Turbulent and inertial roll waves in inclined film flow,* Phys. Fluids 30 (1987), no. 5, 1259–1268.

[JZ1]   M. Johnson and K. Zumbrun, *Rigorous Justification of the Whitham Modulation Equations for the Generalized Korteweg-de Vries Equation,* Studies in Applied Mathematics, 125 no. 1 (2010), 69-89.

[JZ4]   M. Johnson and K. Zumbrun, *Nonlinear stability and asymptotic behavior of periodic traveling waves of multidimensional viscous conservation laws in dimensions one and two*, preprint (2009).

[JZ2]   M. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling waves of viscous conservation laws in the generic case*, J. Diff. Eq. 249 no. 5 (2010), 1213-1240.

[JZ3]   M. Johnson and K. Zumbrun, *Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction diffusion equations,* preprint (2010).

[JZN]   M. Johnson, K. Zumbrun, and P. Noble, *Nonlinear stability of viscous roll waves,* preprint (2010).

[K]   T. Kato, *Perturbation theory for linear operators*, Springer–Verlag, Berlin Heidelberg (1985).

[N1]   P. Noble, *On the spectral stability of roll waves,* Indiana Univ. Math. J. 55 (2006), 795–848.

[N2]   P. Noble, *Linear stability of viscous roll waves,* Comm. Partial Differential Equations 32 no. 10-12 (2007), 1681–1713.

[NR]   P. Noble and L. M. Rodrigues, *Whitham's equations for modulated roll-waves in shallow flows,* preprint (2010).

[OZ1]   M. Oh and K. Zumbrun, *Stability of periodic solutions of viscous conservation laws with viscosity- 1. Analysis of the Evans function*, Arch. Ration. Mech. Anal. 166 no. 2 (2003), 99–166.

[OZ2]   M. Oh and K. Zumbrun, *Stability of periodic solutions of viscous conservation laws with viscosity- Pointwise bounds on the Green function*, Arch. Ration. Mech. Anal. 166 no. 2 (2003), 167–196.

[OZ3]   M. Oh, and K. Zumbrun, *Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions*, Journal for Analysis and its Applications, 25 (2006), 1–21.

[OZ4]   M. Oh, and K. Zumbrun, *Stability and asymptotic behavior of traveling-wave solutions of viscous conservation laws in several dimensions*,Arch. Ration. Mech. Anal., 196 no. 1 (2010), 1–20. Erratum: Arch. Ration. Mech. Anal., 196 no. 1 (2010), 21–23.

[Pa]   A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, 44, Springer-Verlag, New York-Berlin, (1983) viii+279 pp. ISBN: 0-387-90845-5.

[PSU]   R. Pego, H. Schneider, and H. Uecker, *Long-time persistence of Korteweg-de Vries solitons as transient dynamics in a model of inclined film flow,* Proc. Royal Soc. Edinburg 137A (2007) 133–146.

[S1]   G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions– abstract theorem and higher space dimensions*, Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.

[S2]   G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation,* (English. English summary) Comm. Math. Phys. 178 no. 3 (1996), 679–702.

[S3]   G. Schneider, *Nonlinear stability of Taylor vortices in infinite cylinders,* Arch. Rat. Mech. Anal. 144 no. 2 (1998), 121–200.

[Se1]   D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis*, Comm. Partial Differential Equations 30 no. 1-3 (2005), 259–282.

[W]   G. B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.