First exit time from a bounded interval for pseudo-processes driven by the equation $\partial/\partial t = (- - 1)^{N--1}\partial^{2N}/\partial x^{2N}$

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First exit time from a bounded interval for pseudo-processes driven by the equation $\partial/\partial t = (-1)^{N-1}\partial^{2N}/\partial x^{2N}$

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Abstract

Let N be an integer greater than 1. We consider the pseudo-process $X = (X_t)_{t\geq 0}$ driven by the high-order heat-type equation $\partial/\partial t = (-1)^{N-1}\partial^{2N}/\partial x^{2N}$. Let us introduce the first exit time τ_{ab} from a bounded interval (a, b) by X ($a, b \in \mathbb{R}$) together with the related location, namely $X_{\tau_{ab}}$.

In this paper, we provide a representation of the joint pseudo-distribution of the vector $(\tau_{ab}, X_{\tau_{ab}})$ by means of some determinants. The method we use is based on a Feynman-Kac-like functional related to the pseudo-process X which leads to a boundary value problem. In particular, the pseudo-distribution of $X_{\tau_{ab}}$ admits a fine expression involving famous Hermite interpolating polynomials.

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1 Introduction

Let N be an integer greater than 1 and set $\kappa_N = (-1)^{N-1}$. We consider the pseudo-process $(X_t)_{t\geq 0}$ driven by the high-order heat-type equation $\partial/\partial t = \kappa_N \partial^{2N}/\partial x^{2N}$, the so-called pseudo-Brownian motion. This is the pseudo-Markov process with independent and stationary increments, associated with the *signed* heat-type kernel $p(t; x)$ which is the elementary solution of the foregoing equation. The kernel $p(t; x)$ is characterized by its Fourier transform:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}ux}\, p(t; x)\, \mathrm{d}x = \mathrm{e}^{-tu^{2N}}.$$

We define the related transition kernel as $p(t; x, y) = p(t; x - y)$ for any time $t > 0$ and any real numbers x, y, which represents the pseudo-probability that the pseudo-process started at x is in state y at time t. In symbols,

$$\mathbb{P}_x\{X_t \in \mathrm{d}y\} = p(t; x, y)\, \mathrm{d}y.$$

The $\mathbb{P}_x$, $x \in \mathbb{R}$, define a family of *signed* measures whose total mass equals one:

$$\mathbb{P}_x\{X_t \in \mathbb{R}\} = \int_{-\infty}^{+\infty} p(t; x, y)\, \mathrm{d}y = 1.$$

The transition kernel $p(t; x, y)$ satisfies the backward and forward Kolmogorov equations

$$\frac{\partial p}{\partial t}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial x^{2N}}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial y^{2N}}(t; x, y).$$

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To be more precise, let us recall that the pseudo-Markov process $(X_t)_{t\geq 0}$ is defined according to the usual chain rule: for any positive integer n, for any times $t_1, \ldots, t_n$ such that $0 < t_1 < \cdots < t_n$ and any real numbers $x_1, \ldots, x_n$, and setting $t_0 = 0$, $x_0 = x$,

$$\mathbb{P}_x\{X_{t_1} \in \mathrm{d}x_1, \ldots, X_{t_n} \in \mathrm{d}x_n\} = \left(\prod_{k=1}^{n} p(t_k - t_{k-1}; x_{k-1}, x_k)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{1.1}$$

In particular, by setting $T_t\phi(x) = \mathbb{E}_x[\phi(X_t)]$ for any time t, any real number x and any bounded $\mathcal{C}^{2N}$-function ϕ, the family $(T_t)_{t\geq 0}$ is a semi-group of operators whose infinitesimal generator $\mathcal{G}$ is given by

$$\mathcal{G}\phi(x) = \lim_{h\to 0^+} \frac{1}{h}\left[\,\mathbb{E}_x[\phi(X_h)] - \phi(x)\right] = \kappa_{_N}\, \phi^{(2N)}(x). \tag{1.2}$$

Above and throughout the paper, for any non-negative integer ℓ, $\phi^{(\ell)}$ stands for the derivative of ϕ of order ℓ.

The very notion of pseudo-process in a general framework goes back to Daletskii and Fomin in 1965 ([7]). The reader can find an extensive literature on the particular case of pseudo-Brownian motion. For instance, let us quote the works of Beghin, Cammarota, Hochberg, Krylov, Lachal, Nakajima, Nikitin, Nishioka, Orsingher, Ragozina ([2] to [6], [8, 9], [11] to [23]) and the references therein. These papers deal with several functionals related to pseudo-Brownian motion: sojourn time in a bounded or not interval, first overshooting time of a *single* level, maximum or minimum up to a fixed time... Let us mention also other interesting works : one dealing with high-order Schrödinger-type equation $\partial/\partial t = \mathrm{i}\,\partial^{2N}/\partial x^{2N}$ which is related to the so-called Feynman-Kac measure [1], as well as [10] in which the authors develop an alternative and more probabilistic approach to pseudo-processes.

In [13, 14], we obtained the pseudo-distribution of the first overshooting time of a single threshold, together with the corresponding location at this time. In symbols, if τ_a denotes the first overshooting time of a fixed level a (upwards or downwards), we derived the joint pseudo-distribution of the couple (τ_a, X_{τ_a}). Therein, we used an extension of famous Spitzer's identity. In [13, 14] and, in the particular case $N = 2$, in [20, 21], the authors observed a curious fact concerning the pseudo-distribution of X_{τ_a}: it is a linear combination of the Dirac distribution and its successive derivatives (in the sense of Schwartz distributions):

$$\mathbb{P}_x\{X_{\tau_a} \in \mathrm{d}z\}/\mathrm{d}z = \sum_{k=0}^{N-1} \frac{(a-x)^k}{k!}\, \delta_a^{(k)}(z). \tag{1.3}$$

The quantity $\delta_a^{(k)}$ is to be understood as the functional acting on test functions ϕ according as $\langle \delta_a^{(k)}, \phi\rangle = (-1)^k \phi^{(k)}(a)$. Formula (1.3) says that the overshoot through level a should be actually concentrated at a. The appearance of the Schwartz-Dirac distribution δ_a together with its successive derivatives can be interpreted by means of "multipoles" in reference to electric dipoles as in [13, 14] and, for $N = 2$, in [20, 21]. In particular, therein, δ_a and δ_a' are respectively named "monopole" and "dipole". We refer the reader to [22] for a detailed account on monopoles and dipoles. An explanation of this curious fact should be found in considering a linear pseudo-random walk with $2N$ consecutive neighbours around each sites. Indeed, after suitably normalizing such a walk, the neighbours cluster into a single site and form a multipole; see the draft [17].

Till now, the first exit time from a bounded interval, or, equivalently, the first overshooting time of a *double* threshold has not yet been considered. This is the purpose of this work.

Let us introduce the first exit time from (a, b) (a, b being real numbers such that $a < b$) for $(X_t)_{t\geq 0}$:

$$\tau_{ab} = \inf\{t \geq 0 : X_t \notin (a, b)\}$$

with the usual convention that $\inf \varnothing = +\infty$. In this paper, we tackle the problem of finding the pseudo-distribution related to the double threshold: we provide a representation for the joint pseudo-distribution of the couple $(\tau_{ab}, X_{\tau_{ab}})$. This representation involves some determinants; this is the object of Theorems 1 and 2. For the location $X_{\tau_{ab}}$, we have the following counterpart to (1.3); see Theorem 4:

$$\mathbb{P}_x\{X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}z = \sum_{k=0}^{N-1} (-1)^k H_k^-(x)\, \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k^+(x)\, \delta_b^{(k)}(z) \tag{1.4}$$

where the functions $H_k^{\pm}$, $0 \leq k \leq N-1$, are the classical Hermite interpolating polynomials of degree $(2N-1)$ related to points a and b satisfying

$$(H_k^-)^{(\ell)}(a) = (H_k^+)^{(\ell)}(b) = \delta_{k\ell}, \quad (H_k^+)^{(\ell)}(a) = (H_k^-)^{(\ell)}(b) = 0, \quad 0 \leq \ell \leq N-1.$$

Above, the quantity $\delta_{k\ell}$ denotes the usual Kronecker symbol: if $k = \ell$, $\delta_{k\ell} = 1$, else $\delta_{k\ell} = 0$. They explicitly write as

$$H_k^-(x) = \left(\frac{b-x}{b-a}\right)^N \frac{(x-a)^k}{k\,!} \sum_{\ell=0}^{N-k-1} \binom{\ell+N-1}{\ell} \left(\frac{x-a}{b-a}\right)^\ell,$$
$$H_k^+(x) = \left(\frac{x-a}{b-a}\right)^N \frac{(x-b)^k}{k\,!} \sum_{\ell=0}^{N-k-1} \binom{\ell+N-1}{\ell} \left(\frac{b-x}{b-a}\right)^\ell.$$

In particular, we can deduce from (1.4) the "ruin pseudo-probabilities", that is, the pseudo-probabilities of overshooting one level (a or b) before the other one; see Corollary 3.

These results have been announced without any proof in a survey on pseudo-Brownian motion, [16], after a conference held in Madrid (IWAP 2010).

Throughout the paper, the function φ denotes any $(N-1)$ times differentiable function.

2 Feynman-Kac functional

We start from the following fact: in [13, 14], we first obtained the pseudo-distribution of the couple $(\sup_{0\leq s\leq t} X_s, X_t)$ by making use of an extension of Spitzer's identity. From this, we deduced that of the couple (τ_a, X_{τ_a}) and we made the observation that, for any $\lambda \geq 0$ and any $(N-1)$ times differentiable bounded function φ, the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x\left(\mathrm{e}^{-\lambda\tau_a}\varphi(X_{\tau_a})\,\mathbb{1}_{\{\tau_a<+\infty\}}\right)$ solves the boundary value problem

$$\begin{cases} \kappa_{_N}\Phi^{(2N)}(x) = \lambda\,\Phi(x), \quad x \in (-\infty, a) \text{ (or } x \in (a, +\infty)), \\ \Phi^{(k)}(a) = \varphi^{(k)}(a) \quad \text{for } k \in \{0, 1, \ldots, N-1\}. \end{cases} \tag{2.1}$$

So, we state the heuristic that an analogous boundary value problem should hold for the Feynman-Kac functional related to τ_{ab}. The results obtained here through this approach coincide with limiting results deduced from a suitable pseudo-random walk studied in [17]. Moreover, when taking the limit as a goes to $-\infty$ or b goes to $+\infty$ in the present results, we retrieve the pseudo-distribution of (τ_a, X_{τ_a}) obtained in [14]. So, these observations comfort us in our heuristic. Actually, our purpose in this work is essentially concentrated in calculating the pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$.

As pointed out in several works on pseudo-processes, pseudo-Brownian motion is properly defined only on the set of dyadic times and ad-hoc definitions should be taken for computing certain functionals of this pseudo-process depending on a continuous set of times; see, e.g., [14] and, in the particular case $N = 2$, [21]. Roughly speaking, the dense subset of dyadic times is appropriate because of the usual property that for any $n \in \mathbb{N}$, $\{k/2^n, k \in \mathbb{N}\} \subset \{k/2^{n+1}, k \in \mathbb{N}\}$. Indeed, this latter permits to view the pseudo-process $(X_t)_{t\geq 0}$ as an informal limit of the family of step-processes $(X_{n,t})_{t\geq 0}$ defined according to the following sampling procedure:

$$X_{n,t} = \sum_{k=0}^{\infty} 1_{[k/2^n,(k+1)/2^n)}(t) X_{k/2^n}.$$

For each fixed $n \in \mathbb{N}$, the sequence $(X_{k/2^n})_{k\in\mathbb{N}}$ can be correctly defined thanks to (1.1). But the fact that $\int_{-\infty}^{+\infty} |p(t;x)|\,\mathrm{d}x > 1$ prevent us from applying the classical extension theorem of Kolmogorov for finding *a priori* a σ-additive measure on the usual space of right-continuous functions on $[0, +\infty)$ which have left-hand limits, measure whose finite projections would yield the finite-dimensional pseudo-distributions of the sequence $(X_{k/2^n})_{k\in\mathbb{N}}$.

For our concern, we set

$$\tau_{ab,n} = \frac{1}{2^n} \min\{k \in \mathbb{N} : X_{k/2^n} \notin (a, b)\}$$

and, for $x \in (a,b)$,

$$\Phi_n(x) = \mathbb{E}_x\Big(\mathrm{e}^{-\lambda\tau_{ab,n}}\varphi(X_{\tau_{ab,n}})\,\mathbb{1}_{\{\tau_{ab,n}<+\infty\}}\Big).$$

Then, we define the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x\Big(\mathrm{e}^{-\lambda\tau_{ab}}\varphi(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big)$ as the limit

$$\Phi(x) \overset{\text{def}}{=} \lim_{n\to+\infty}\Phi_n(x)$$

and we state below the analogue to (2.1).

Heuristic. *For any $\lambda \geq 0$ and any $(N-1)$ times differentiable bounded function φ, the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x\Big(\mathrm{e}^{-\lambda\tau_{ab}}\varphi(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big)$ solves the boundary value problem*

$$\begin{cases} \kappa_N\Phi^{(2N)}(x) = \lambda\,\Phi(x), \quad x \in (a,b), \\ \Phi^{(k)}(a) = \varphi^{(k)}(a) \quad \textit{and} \quad \Phi^{(k)}(b) = \varphi^{(k)}(b) \quad \textit{for } k \in \{0,1,\ldots,N-1\}. \end{cases} \tag{2.2}$$

3 Joint pseudo-distribution of $\big(\tau_{ab}, X_{\tau_{ab}}\big)$

In this section, we solve boundary value problem (2.2) in order to derive the joint pseudo-probability of $\big(\tau_{ab}, X_{\tau_{ab}}\big)$. In this way, if we choose $\varphi(x) = \mathrm{e}^{\mathrm{i}\mu x}$, $\mu \in \mathbb{R}$, we first obtain its Laplace-Fourier transform. Actually, the results we derived hold true for any $(N-1)$ times differentiable function φ.

Let us introduce the $(2N)$th roots of κ_N: $\theta_\ell = \mathrm{e}^{\mathrm{i}\frac{2\ell+N-1}{2N}\pi}$, $1 \leq \ell \leq 2N$. We have $\theta_\ell^{2N} = \kappa_N$. For any complex number z, we set $e_\lambda^z = \mathrm{e}^{\lambda^{1/(2N)}z}$.

Theorem 1. *The Feynman-Kac functional related to $\big(\tau_{ab}, X_{\tau_{ab}}\big)$ admits the following representation:*

$$\mathbb{E}_x\Big(\mathrm{e}^{-\lambda\tau_{ab}}\varphi(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big) = \sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\frac{\Delta_k^-(\lambda;x)}{\Delta(\lambda)}\,\varphi^{(k)}(a) + \sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\frac{\Delta_k^+(\lambda;x)}{\Delta(\lambda)}\,\varphi^{(k)}(b) \tag{3.1}$$

where the quantities $\Delta(\lambda)$ and $\Delta_k^{\pm}(\lambda;x)$ are the determinants below:

$$\Delta(\lambda) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \theta_1\, e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}\, e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots \\ \theta_1^{N-1}\, e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{N-1}\, e_\lambda^{\theta_{2N} a} \\ \cdots & \cdots & \cdots \\ e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \theta_1\, e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}\, e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots \\ \theta_1^{N-1}\, e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{N-1}\, e_\lambda^{\theta_{2N} b} \end{vmatrix}$$

and

$$\Delta_k^-(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{2N} a}\\ \vdots & & \vdots\\ \theta_1^{k-1}e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{k-1}e_\lambda^{\theta_{2N} a}\\ e_\lambda^{\theta_1 x} & \cdots & e_\lambda^{\theta_{2N} x}\\ \theta_1^{k+1}e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{k+1}e_\lambda^{\theta_{2N} a}\\ \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} a}\\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{2N} b}\\ \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} b}\end{vmatrix},\quad \Delta_k^+(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{2N} a}\\ \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} a}\\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{2N} b}\\ \vdots & & \vdots\\ \theta_1^{k-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{k-1}e_\lambda^{\theta_{2N} b}\\ e_\lambda^{\theta_1 x} & \cdots & e_\lambda^{\theta_{2N} x}\\ \theta_1^{k+1}e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{k+1}e_\lambda^{\theta_{2N} b}\\ \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} b}\end{vmatrix}.$$

The functions $x\mapsto\Delta_k^\pm(\lambda;x)$, $0\le k\le N-1$, are the solutions of the boundary value problems

$$\begin{cases}(\Delta_k^-)^{(2N)}(\lambda;x)=\kappa_N\lambda\,\Delta_k^-(\lambda;x),\\ (\Delta_k^-)^{(\ell)}(\lambda;a)=\delta_{k\ell}\,\lambda^{\ell/(2N)}\Delta(\lambda),\ (\Delta_k^-)^{(\ell)}(\lambda;b)=0\quad \textit{for } \ell\in\{0,\dots,N-1\},\end{cases}$$

$$\begin{cases}(\Delta_k^+)^{(2N)}(\lambda;x)=\kappa_N\lambda\,\Delta_k^+(\lambda;x),\\ (\Delta_k^+)^{(\ell)}(\lambda;a)=0,\ (\Delta_k^+)^{(\ell)}(\lambda;b)=\delta_{k\ell}\,\lambda^{\ell/(2N)}\Delta(\lambda)\quad \textit{for } \ell\in\{0,\dots,N-1\}.\end{cases}$$

Proof. The solution of linear boundary value problem (2.2) has the form $\Phi(x)=\sum_{\ell=1}^{2N}\alpha_\ell\,e_\lambda^{\theta_\ell x}$ where the coefficients α_ℓ, $1\le\ell\le 2N$, satisfy the linear system below:

$$\begin{cases}\displaystyle\sum_{\ell=1}^{2N}\theta_\ell^k\,e_\lambda^{\theta_\ell a}\,\alpha_\ell=\lambda^{-\frac{k}{2N}}\varphi^{(k)}(a),\quad 0\le k\le N-1,\\ \displaystyle\sum_{\ell=1}^{2N}\theta_\ell^k\,e_\lambda^{\theta_\ell b}\,\alpha_\ell=\lambda^{-\frac{k}{2N}}\varphi^{(k)}(b),\quad 0\le k\le N-1.\end{cases}\tag{3.2}$$

This system can be solved by using Cramer's formulae:

$$\alpha_\ell=\frac{\Delta_\ell(\lambda,\varphi)}{\Delta(\lambda)},\quad 1\le\ell\le 2N,$$

where $\Delta(\lambda)$ is the determinant displayed in Theorem 1 and $\Delta_\ell(\lambda,\varphi)$ is the determinant deduced from $\Delta(\lambda)$ by replacing its ℓth column by the right-hand side of (3.2), that is

$$\Delta_\ell(\lambda,\varphi)=\begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1} a} & \varphi(a) & e_\lambda^{\theta_{\ell+1} a} & \cdots & e_\lambda^{\theta_{2N} a}\\ \theta_1 e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}e_\lambda^{\theta_{\ell-1} a} & \lambda^{-\frac{1}{2N}}\varphi'(a) & \theta_{\ell+1}e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}e_\lambda^{\theta_{2N} a}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1}e_\lambda^{\theta_{\ell-1} a} & \lambda^{-\frac{N-1}{2N}}\varphi^{(N-1)}(a) & \theta_{\ell+1}^{N-1}e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} a}\\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1} b} & \varphi(b) & e_\lambda^{\theta_{\ell+1} b} & \cdots & e_\lambda^{\theta_{2N} b}\\ \theta_1 e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}e_\lambda^{\theta_{\ell-1} b} & \lambda^{-\frac{1}{2N}}\varphi'(b) & \theta_{\ell+1}e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}e_\lambda^{\theta_{2N} b}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \theta_1^{N-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1}e_\lambda^{\theta_{\ell-1} b} & \lambda^{-\frac{N-1}{2N}}\varphi^{(N-1)}(b) & \theta_{\ell+1}^{N-1}e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N} b}\end{vmatrix}.$$

The determinant $\Delta_\ell(\lambda,\varphi)$ can be expanded with respect to its ℓth column:

$$\Delta_\ell(\lambda,\varphi)=\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\Delta_{k\ell}^-(\lambda)\,\varphi^{(k)}(a)+\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\Delta_{k\ell}^+(\lambda)\,\varphi^{(k)}(b)$$

with

$$\Delta^{-}_{k\ell}(\lambda) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1} a} & 0 & e_\lambda^{\theta_{\ell+1} a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k-1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{k-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k-1} e_\lambda^{\theta_{2N} a} \\ \theta_1^{k} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k} e_\lambda^{\theta_{\ell-1} a} & 1 & \theta_{\ell+1}^{k} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k} e_\lambda^{\theta_{2N} a} \\ \theta_1^{k+1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k+1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{k+1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k+1} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1} b} & 0 & e_\lambda^{\theta_{\ell+1} b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} b} & 0 & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}$$

$$= \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1} a} & 0 & e_\lambda^{\theta_{\ell+1} a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k-1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{k-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k-1} e_\lambda^{\theta_{2N} a} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \\ \theta_1^{k+1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k+1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{k+1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k+1} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} a} & 0 & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1} b} & 0 & e_\lambda^{\theta_{\ell+1} b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} b} & 0 & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}$$

$$= \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1} a} & e_\lambda^{\theta_{\ell} a} & e_\lambda^{\theta_{\ell+1} a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k-1} e_\lambda^{\theta_{\ell-1} a} & \theta_{\ell}^{k-1} e_\lambda^{\theta_{\ell} a} & \theta_{\ell+1}^{k-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k-1} e_\lambda^{\theta_{2N} a} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \\ \theta_1^{k+1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{k+1} e_\lambda^{\theta_{\ell-1} a} & \theta_{\ell}^{k+1} e_\lambda^{\theta_{\ell} a} & \theta_{\ell+1}^{k+1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{k+1} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} a} & \theta_{\ell}^{N-1} e_\lambda^{\theta_{\ell} a} & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \hdashline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1} b} & e_\lambda^{\theta_{\ell} b} & e_\lambda^{\theta_{\ell+1} b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} b} & \theta_{\ell}^{N-1} e_\lambda^{\theta_{\ell} b} & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}$$

and, in the same manner,

$$\Delta_{k\ell}^{+}(\lambda)=\begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1}a} & e_\lambda^{\theta_\ell a} & e_\lambda^{\theta_{\ell+1}a} & \cdots & e_\lambda^{\theta_{2N}a}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1}e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1}e_\lambda^{\theta_{\ell-1}a} & \theta_\ell^{N-1}e_\lambda^{\theta_\ell a} & \theta_{\ell+1}^{N-1}e_\lambda^{\theta_{\ell+1}a} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N}a}\\ \hline e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1}b} & e_\lambda^{\theta_\ell b} & e_\lambda^{\theta_{\ell+1}b} & \cdots & e_\lambda^{\theta_{2N}b}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{k-1}e_\lambda^{\theta_{\ell-1}b} & \theta_\ell^{k-1}e_\lambda^{\theta_\ell b} & \theta_{\ell+1}^{k-1}e_\lambda^{\theta_{\ell+1}b} & \cdots & \theta_{2N}^{k-1}e_\lambda^{\theta_{2N}a}\\ 0 & \cdots & 0 & 1 & 0 & \cdots & \\ \theta_1^{k+1}e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{k+1}e_\lambda^{\theta_{\ell-1}b} & \theta_\ell^{k+1}e_\lambda^{\theta_\ell b} & \theta_{\ell+1}^{k+1}e_\lambda^{\theta_{\ell+1}b} & \cdots & \theta_{2N}^{k+1}e_\lambda^{\theta_{2N}a}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1}e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1}e_\lambda^{\theta_{\ell-1}b} & \theta_\ell^{N-1}e_\lambda^{\theta_\ell b} & \theta_{\ell+1}^{N-1}e_\lambda^{\theta_{\ell+1}b} & \cdots & \theta_{2N}^{N-1}e_\lambda^{\theta_{2N}b} \end{vmatrix}.$$

With these settings at hand, we can write the solution of (2.2):

$$\begin{aligned}\Phi(x)&=\sum_{\ell=1}^{2N}\alpha_\ell\, e_\lambda^{\theta_\ell x}=\sum_{\ell=1}^{2N}\frac{\Delta_\ell(\lambda,\varphi)}{\Delta(\lambda)}\,e_\lambda^{\theta_\ell x}\\ &=\frac{1}{\Delta(\lambda)}\left[\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\left(\sum_{\ell=1}^{2N}\Delta_{k\ell}^{-}(\lambda)\,e_\lambda^{\theta_\ell x}\right)\varphi^{(k)}(a)+\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\left(\sum_{\ell=1}^{2N}\Delta_{k\ell}^{+}(\lambda)\,e_\lambda^{\theta_\ell x}\right)\varphi^{(k)}(b)\right]\\ &=\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\frac{\Delta_k^{-}(\lambda;x)}{\Delta(\lambda)}\,\varphi^{(k)}(a)+\sum_{k=0}^{N-1}\lambda^{-\frac{k}{2N}}\frac{\Delta_k^{+}(\lambda;x)}{\Delta(\lambda)}\,\varphi^{(k)}(b)\end{aligned}$$

with

$$\Delta_k^{-}(\lambda;x)=\sum_{\ell=1}^{2N}\Delta_{k\ell}^{-}(\lambda)\,e_\lambda^{\theta_\ell x},\quad \Delta_k^{+}(\lambda;x)=\sum_{\ell=1}^{2N}\Delta_{k\ell}^{+}(\lambda)\,e_\lambda^{\theta_\ell x}. \tag{3.3}$$

We immediately see that equalities (3.3) are the expansions of the determinants displayed in Theorem 1 with respect to their $(k-1)$th raw and $(k+N-1)$th raw respectively. Formula (3.1) is proved.

Finally, it is easy to check the boundary value problems satisfied by the functions $x\mapsto\Delta_k^{\pm}(\lambda;x)$ by using elementary rules on differentiating a determinant. In particular for, e.g., Δ_k^{-}, the determinants defining $(\Delta_k^{-})^{(\ell)}(\lambda;a)$, $\ell\in\{0,\dots,N-1\}\backslash\{k\}$, and $(\Delta_k^{-})^{(\ell)}(\lambda;b)$, $\ell\in\{0,\dots,N-1\}$, have two identical rows, thus they vanish. The determinant $(\Delta_k^{-})^{(k)}(\lambda;a)$ has the same rows as $\Delta(\lambda)$ up to the multiplicative factor $\lambda^{k/(2N)}$ for its kth row, then it coincides with $\lambda^{k/(2N)}\Delta(\lambda)$. The proof of Theorem 1 is finished. □

Now, by eliminating the function φ in (3.1), we get the following result which should be understood in the sense of Schwartz distributions:

$$\begin{aligned}&\mathbb{E}_x\left(\mathrm{e}^{-\lambda\tau_{ab}}\mathbb{1}_{\{\tau_{ab}<+\infty\}},X_{\tau_{ab}}\in\mathrm{d}z\right)/\mathrm{d}z\\ &\qquad=\sum_{k=0}^{N-1}(-1)^k\lambda^{-\frac{k}{2N}}\frac{\Delta_k^{-}(\lambda;x)}{\Delta(\lambda)}\,\delta_a^{(k)}(z)+\sum_{k=0}^{N-1}(-1)^k\lambda^{-\frac{k}{2N}}\frac{\Delta_k^{+}(\lambda;x)}{\Delta(\lambda)}\,\delta_b^{(k)}(z)\end{aligned} \tag{3.4}$$

from which we derive the following representation for the pseudo-distribution of $\left(\tau_{ab},X_{\tau_{ab}}\right)$.

Theorem 2. *The joint pseudo-distribution of $\left(\tau_{ab},X_{\tau_{ab}}\right)$ admits the following representation:*

$$\mathbb{P}_x\{\tau_{ab}\in\mathrm{d}t,X_{\tau_{ab}}\in\mathrm{d}z\}/\mathrm{d}t\,\mathrm{d}z=\sum_{k=0}^{N-1}(-1)^k I_k^{-}(t;x)\,\delta_a^{(k)}(z)+\sum_{k=0}^{N-1}(-1)^k I_k^{+}(t;x)\,\delta_b^{(k)}(z) \tag{3.5}$$

where the functions $I_k^{\pm}(t;x)$, $0\le k\le N-1$, are characterized by their Laplace transforms:

$$\int_0^\infty I_k^{\pm}(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t=\lambda^{-\frac{k}{2N}}\frac{\Delta_k^{\pm}(\lambda;x)}{\Delta(\lambda)}.$$

They are also characterized by the boundary value problems

$$\begin{cases} \dfrac{\partial I_k^-}{\partial t}(t;x) = \kappa_N \dfrac{\partial^{2N} I_k^-}{\partial^{2N} x}(t;x) \\ \dfrac{\partial^\ell I_k^-}{\partial^\ell x}(t;a) = \delta_{k\ell},\ \dfrac{\partial^\ell I_k^-}{\partial^\ell x}(t;b) = 0 \quad \textit{for } \ell \in \{0,1,\dots,N-1\}, \end{cases}$$

$$\begin{cases} \dfrac{\partial I_k^+}{\partial t}(t;x) = \kappa_N \dfrac{\partial^{2N} I_k^+}{\partial^{2N} x}(t;x) \\ \dfrac{\partial^\ell I_k^+}{\partial^\ell x}(t;a) = 0,\ \dfrac{\partial^\ell I_k^+}{\partial^\ell x}(t;b) = \delta_{k\ell} \quad \textit{for } \ell \in \{0,1,\dots,N-1\}. \end{cases}$$

The boundary value problems satisfied by the functions $I_k^\pm$, $0 \le k \le N-1$, come from those satisfied by the functions $\Delta_k^\pm$ displayed in Theorem 1. The only details we have to check are that $I_k^\pm(t;x)$ goes to 0 as t tends to 0^+ and that $I_k^\pm(t;x)$ is bounded as t tends to $+\infty$ (in order to have $\int_0^\infty (\partial/\partial t)\, I_k^\pm(t;x)\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t = \lambda \int_0^\infty I_k^\pm(t;x)\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t$) which can be deduced from the fact that their Laplace transforms go to 0 exponentially quickly as λ goes to $+\infty$ and are bounded as λ goes to 0^+. These facts are proved in Appendix A; see (A.2) and (A.5).

Remark 1. The functions $I_k^\pm$, $0 \le k \le N-1$, are real-valued. Indeed, observing that the complex numbers θ_ℓ, $1 \le \ell \le 2N$, are conjugate two by two, it is easily seen that the determinants contain conjugate columns two by two, so they are real numbers. More precisely, conjugating $\theta_1,\dots,\theta_N,\theta_{N+1},\dots,\theta_{2N}$ respectively yields $\theta_N,\dots,\theta_1,\theta_{2N},\dots,\theta_{N+1}$. Therefore, conjugating the determinants Δ and $\Delta_{k\ell}^\pm$ boils down to interchanging their 1st and Nth columns, their 2nd and $(N-1)$th columns, …, their $(N+1)$th and $(2N)$th columns, their $(N+2)$th and $(2N-1)$th columns, and so on. In this way, we perform an even number of transpositions and we retrieve the original determinants: $\overline{\Delta} = \Delta$ and $\overline{\Delta_{k\ell}^\pm} = \Delta_{k\ell}^\pm$, proving that they are real numbers.

Moreover, the functions I_k^+ and I_k^- are related according to the identity $I_k^+(t;x) = (-1)^k\, I_k^-(t;a+b-x)$ as it can be seen by proving the same identity concerning their Laplace transforms; see (A.1) in Appendix A.

Remark 2. Let us compute the limit of (3.4) as b tends towards $+\infty$. To this aim, we find that

$$\frac{\Delta_k^-(\lambda;x)}{\Delta(\lambda)} \underset{b\to+\infty}{\longrightarrow} \sum_{\ell=1}^N \alpha_{k\ell}\, \mathrm{e}_\lambda^{\theta_\ell\,(x-a)} \quad \text{and} \quad \frac{\Delta_k^+(\lambda;x)}{\Delta(\lambda)} \underset{b\to+\infty}{\longrightarrow} 0 \tag{3.6}$$

with

$$\alpha_{k\ell} = \frac{1}{\det(V)} \begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{\ell-1} & 0 & \theta_{\ell+1} & \cdots & \theta_N \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{\ell-1}^{k-1} & 0 & \theta_{\ell+1}^{k-1} & \cdots & \theta_N^{k-1} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \theta_1^{k+1} & \cdots & \theta_{\ell-1}^{k+1} & 0 & \theta_{\ell+1}^{k+1} & \cdots & \theta_N^{k+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{\ell-1}^{N-1} & 0 & \theta_{\ell+1}^{N-1} & \cdots & \theta_N^{N-1} \end{vmatrix}.$$

The coefficients $\alpha_{k\ell}$ are characterized by the identity

$$\sum_{k=0}^{N-1} \alpha_{k\ell}\, x^k = \frac{1}{\det(V)} \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{\ell-1} & x & \theta_{\ell+1} & \cdots & \theta_N \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{\ell-1}^{N-1} & x^{N-1} & \theta_{\ell+1}^{N-1} & \cdots & \theta_N^{N-1} \end{vmatrix} = \prod_{\substack{1\le k\le N \\ k\ne\ell}} \left(\frac{x-\theta_k}{\theta_\ell-\theta_k}\right) \tag{3.7}$$

as it is easily seen by appealing to the well-known Vandermonde determinant $\det(V) = \prod_{1\le i<j\le N} (\theta_j - \theta_i)$. Notice that polynomial (3.7) is nothing but an elementary Lagrange interpolating polynomial related to the numbers θ_i, $1 \le i \le N$. The details of these limiting results being cumbersome, we postpone them to Appendix A.

In regards to (3.4), (3.5) and (3.6), we conclude that, for $x > a$,

$$\lim_{b\to+\infty} \mathbb{P}_x\{\tau_{ab} \in \mathrm{d}t, X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}t\,\mathrm{d}z = \sum_{k=0}^{N-1} (-1)^k K_k(t;x)\,\delta_a^{(k)}(z)$$

where K_k is the function whose Laplace transform is given by

$$\int_0^\infty K_k(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \lambda^{-\frac{k}{2N}} \sum_{\ell=1}^{N} \alpha_{k\ell}\,\mathrm{e}^{\theta_\ell\, \sqrt[2N]{\lambda}\,(x-a)}.$$

We retrieve at the limit the pseudo-distribution of (τ_a, X_{τ_a}) related to the first overshooting time of level a displayed in [14], formula (5.15).

4 Pseudo-distribution of τ_{ab}

By applying the Schwartz distribution (3.5) to the test function 1, we immediately extract the pseudo-distribution of τ_{ab}: $\mathbb{P}_x\{\tau_{ab} \in \mathrm{d}t\}/\mathrm{d}t = I_0^-(t;x) + I_0^+(t;x)$ that we state as follows.

Theorem 3. *The pseudo-distribution of τ_{ab} is given either by one of both formulae below:*

$$\mathbb{P}_x\{\tau_{ab} \in \mathrm{d}t\}/\mathrm{d}t = I(t;x), \quad \mathbb{P}_x\{\tau_{ab} \le t\} = J(t;x)$$

with

$$\int_0^\infty I(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{\Delta_0^+(\lambda;x) + \Delta_0^-(\lambda;x)}{\Delta(\lambda)}, \quad \int_0^\infty J(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{1}{\lambda}\,\frac{\Delta_0^+(\lambda;x) + \Delta_0^-(\lambda;x)}{\Delta(\lambda)}.$$

Let us introduce the up-to-date minimum and maximum functionals of X:

$$m_t = \min_{s\in[0,t]} X_s, \quad M_t = \max_{s\in[0,t]} X_s.$$

It is plain that the functionals m_t, M_t and time τ_{ab} are related according as $a < m_t \le M_t < b \iff \tau_{ab} > t$. Then $\mathbb{P}_x\{a < m_t \le M_t < b\} = 1 - \mathbb{P}_x\{\tau_{ab} \le t\}$.

Corollary 1. *The joint pseudo-distribution of (m_t, M_t) is given by*

$$\mathbb{P}_x\{a < m_t \le M_t < b\} = 1 - J(t;x)$$

and its Laplace transform with respect to t writes

$$\int_0^\infty \mathbb{P}_x\{a < m_t \le M_t < b\}\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{1}{\lambda}\,\frac{\Delta(\lambda) - \Delta_0^+(\lambda;x) - \Delta_0^-(\lambda;x)}{\Delta(\lambda)}.$$

5 Pseudo-distribution of $X_{\tau_{ab}}$

In this part, we focus on the exit location of X at time τ_{ab} whose pseudo-distribution admits a remarkable expression by means of Hermite interpolating polynomials whose expressions are displayed in the introduction.

Theorem 4. *The pseudo-distribution of the exit location $X_{\tau_{ab}}\mathbb{1}_{\{\tau_{ab}<+\infty\}}$ is given, in the sense of Schwartz distributions, by*

$$\mathbb{P}_x\{X_{\tau_{ab}} \in \mathrm{d}z, \tau_{ab} < +\infty\}/\mathrm{d}z = \sum_{k=0}^{N-1} (-1)^k H_k^-(x)\,\delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k^+(x)\,\delta_b^{(k)}(z). \tag{5.1}$$

Proof. We directly solve boundary value problem (2.2) in the case where $\lambda = 0$ therein. Namely, by setting $\Psi(x) = \mathbb{E}_x\Big(\varphi(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big)$,

$$\begin{cases} \Psi^{(2N)}(x) = 0, \quad x \in (a,b), \\ \Psi^{(k)}(a) = \varphi^{(k)}(a) \text{ and } \Psi^{(k)}(b) = \varphi^{(k)}(b) \quad \text{for } k \in \{0,1,\dots,N-1\}. \end{cases}$$

It is clear that Ψ is the polynomial of degree not greater than $(2N-1)$ whose derivatives at a and b up to order $(N-1)$ are the given numbers $\varphi^{(k)}(a)$ and $\varphi^{(k)}(b)$, $0 \leq k \leq N-1$. It can be written as a linear combination of the Hermite interpolating fundamental polynomials $H_k^{\pm}$, $0 \leq k \leq N-1$, displayed in Theorem 4 as follows: for any test functions φ,

$$\mathbb{E}_x\Big(\varphi(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big) = \sum_{k=0}^{N-1} H_k^-(x)\,\varphi^{(k)}(a) + \sum_{k=0}^{N-1} H_k^+(x)\,\varphi^{(k)}(b). \tag{5.2}$$

Formula (5.1) is nothing but (5.2) rephrased by means of Schwartz distributions. □

Remark 3. Formula (5.2) yields for $\varphi = H_k^{\pm}$, $0 \leq k \leq N-1$, that $\mathbb{E}_x\Big(H_k^{\pm}(X_{\tau_{ab}})\,\mathbb{1}_{\{\tau_{ab}<+\infty\}}\Big) = H_k^{\pm}(x)$.

Remark 4. *By letting b tend to $+\infty$, we see that $H_k^+(x)$ tends to 0 while $H_k^-(x)$ tends to $(x-a)^k/(k\,!)$. Hence, we find that*

$$\lim_{b\to+\infty} \mathbb{P}_x\{X_{\tau_{ab}} \in \mathrm{d}z, \tau_{ab} < +\infty\}/\mathrm{d}z = \sum_{k=0}^{N-1} \frac{(a-x)^k}{k!}\,\delta_a^{(k)}(z).$$

We retrieve at the limit the pseudo-distribution (1.3) of the location X_{τ_a} of X at the first overshooting time of level a, which is displayed in [14], formula (5.18).

Corollary 2. *Time τ_{ab} is $\mathbb{P}_x$-almost surely finite in the sense that*

$$\mathbb{P}_x\{\tau_{ab} < +\infty\} = 1.$$

Because of this, in the sequel of the paper, we shall omit the condition $\tau_{ab} < +\infty$ when considering the pseudo-random variable $X_{\tau_{ab}}$. Actually, let us recall that, in the framework of signed measures, if A is a set of $\mathbb{P}_x$-measure 1, it does *not* entail that for any set B that $\mathbb{P}_x(A \cap B) = \mathbb{P}_x(B)$ contrarily to the case of ordinary probability.

Proof. The pseudo-probability $\mathbb{P}\{\tau_{ab} < +\infty\}$ can be deduced from (5.1) by choosing $\varphi = 1$. Indeed, we have that

$$\begin{aligned} H_0^-(x) &= \left(\frac{b-x}{b-a}\right)^N \sum_{\ell=0}^{N-1} \binom{\ell+N-1}{\ell} \left(\frac{x-a}{b-a}\right)^{\ell} \\ &= \frac{(b-x)^N}{(b-a)^{2N-1}} \sum_{\ell=0}^{N-1} \binom{\ell+N-1}{\ell} (x-a)^{\ell}(b-a)^{N-1-\ell}. \end{aligned}$$

By writing the term $(b-a)^{N-1-\ell}$ as

$$(b-a)^{N-1-\ell} = [(x-a)+(b-x)]^{N-1-\ell} = \sum_{k=0}^{N-1-\ell} \binom{N-1-\ell}{k} (x-a)^k (b-x)^{N-1-k-\ell},$$

it follows that

$$\begin{aligned} H_0^-(x) &= \frac{1}{(b-a)^{2N-1}} \sum_{\substack{0\leq\ell\leq N-1 \\ 0\leq k\leq N-1-\ell}} \binom{N-1-\ell}{k}\binom{\ell+N-1}{\ell} (x-a)^{k+\ell}(b-x)^{2N-1-k-\ell} \\ &= \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{N-1} \left[\sum_{\ell=0}^{m} \binom{\ell+N-1}{\ell}\binom{N-1-\ell}{m-\ell}\right] (x-a)^m (b-x)^{2N-1-m}. \end{aligned}$$

By using the elementary identity $\sum_{\ell=0}^{n} \binom{\ell+p}{\ell}\binom{n+q-\ell}{n-\ell} = \binom{n+p+q+1}{n}$ which comes from the equality $(1+u)^{-p}(1+u)^{-q} = (1+u)^{-p-q}$ together with the expansion, e.g., for p, $(1+u)^{-p} = \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\ell+p-1}{\ell}u^{\ell}$, we get that

$$\sum_{\ell=0}^{m} \binom{\ell+N-1}{\ell}\binom{N-1-\ell}{m-\ell} = \binom{2N-1}{m}.$$

As a byproduct,

$$H_0^-(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m}.$$

Similarly,

$$H_0^+(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=N}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m}$$

and we immediately deduce that

$$\mathbb{P}\{\tau_{ab} < +\infty\} = H_0^-(x) + H_0^+(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m} = 1.$$

□

Let us introduce the first down- and up-overshooting times of the single thresholds a and b for $(X_t)_{t\geq 0}$:

$$\tau_a^- = \inf\{t \geq 0 : X_t < a\}, \quad \tau_b^+ = \inf\{t \geq 0 : X_t > b\}.$$

The famous problem of the ruin of the gambler in the context of pseudo-Brownian motion consists in computing the pseudo-probability of overshooting one level (a or b) before the other one. For instance, we have that

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = \mathbb{P}_x\{X_{\tau_{ab}} \leq a\}.$$

Hence, in view of formula (5.1), we obtain the following result.

Corollary 3. *The "ruin" pseudo-probabilities related to pseudo-Brownian motion are given by*

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^-(x), \quad \mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^+(x).$$

In the corollary below, we provide a way for computing the pseudo-moments of $X_{\tau_{ab}}$.

Corollary 4. *Let P be a polynomial and R the remainder of the Euclidean division of $P(x)$ by $(x-a)^N(x-b)^N$. We have that*

$$\mathbb{E}_x[P(X_{\tau_{ab}})] = R(x).$$

In particular, the pseudo-moments of $X_{\tau_{ab}}$ are given, for any $p \in \{0, 1, \ldots, 2N-1\}$, by

$$\mathbb{E}_x[(X_{\tau_{ab}})^p] = x^p$$

and for any positive integer p, by setting $c_n = \sum_{k=0}^n \binom{N+k-1}{k}\binom{N+n-1-k}{n-k} a^k\, b^{n-k}$, by

$$\begin{aligned}
\mathbb{E}_x[(X_{\tau_{ab}})^{2N+p}] &= x^{2N+p} - \left(\sum_{n=0}^{p} c_{p-n} x^n\right)(x-a)^N(x-b)^N \\
&= x^{2N+p} - \Big[x^p + N(a+b)\, x^{p-1} \\
&\qquad + \left(\tfrac{1}{2}\, N(N+1)(a^2+b^2) + N^2 ab\right) x^{p-2} + \cdots\Big](x-a)^N(x-b)^N.
\end{aligned}$$

For instance,

$$\begin{aligned}
\mathbb{E}_x[(X_{\tau_{ab}})^{2N}] &= x^{2N} - (x-a)^N(x-b)^N, \\
\mathbb{E}_x[(X_{\tau_{ab}})^{2N+1}] &= x^{2N+1} - [x + N(a+b)](x-a)^N(x-b)^N, \\
\mathbb{E}_x[(X_{\tau_{ab}})^{2N+2}] &= x^{2N+2} - \Big[x^2 + N(a+b)\, x + \left(\tfrac{1}{2}\, N(N+1)(a^2+b^2) + N^2 ab\right)\Big](x-a)^N(x-b)^N.
\end{aligned}$$

Proof. Let us introduce the quotient Q of the Euclidean division of $P(x)$ by $(x-a)^N(x-b)^N$: we have $P(x) = Q(x)(x-a)^N(x-b)^N + R(x)$. The polynomial R is of degree not greater than $(2N-1)$. Since a and b are roots of the polynomial $P(x) - R(x) = Q(x)(x-a)^N(x-b)^N$ with a

multiplicity not less than N, the successive derivatives of $P - R$ up to order $(N-1)$ vanish at a and b. Therefore, by (5.2), we deduce that

$$\mathbb{E}_x\Big[Q(X_{\tau_{ab}})(X_{\tau_{ab}}-a)^N(X_{\tau_{ab}}-b)^N\Big]=0$$

and then

$$\mathbb{E}_x[P(X_{\tau_{ab}})]=\mathbb{E}_x[R(X_{\tau_{ab}})].$$

Since the polynomial R is of degree not greater than $(2N-1)$, we can write the decomposition

$$R(x)=\sum_{k=0}^{N-1}R^{(k)}(a)\,H_k^-(x)+\sum_{k=0}^{N-1}R^{(k)}(b)\,H_k^+(x).$$

Therefore, appealing to Remark 3, we obtain that

$$\mathbb{E}_x[P(X_{\tau_{ab}})]=\mathbb{E}_x[R(X_{\tau_{ab}})]=R(x)=P(x)-Q(x)(x-a)^N(x-b)^N.$$

Next, we compute the quotient Q when $P(x)=x^{2N+p}$:

$$\begin{aligned}\frac{x^{2N+p}}{(x-a)^N(x-b)^N}&=x^p\left(1-\frac{a}{x}\right)^{-N}\left(1-\frac{b}{x}\right)^{-N}\\&=x^p\left(\sum_{k=0}^{\infty}\binom{N+k-1}{k}\frac{a^k}{x^k}\right)\left(\sum_{\ell=0}^{\infty}\binom{N+\ell-1}{\ell}\frac{b^\ell}{x^\ell}\right)\\&=\sum_{k=0}^{\infty}c_nx^{p-n}=\sum_{k=0}^{p}c_{p-n}x^n+\sum_{n=1}^{\infty}\frac{c_{n+p}}{x^n}\end{aligned}$$

where

$$c_n=\sum_{\substack{k,\ell\geq 0\\ k+\ell=n}}\binom{N+k-1}{k}\binom{N+\ell-1}{\ell}a^k\,b^\ell=\sum_{k=0}^{n}\binom{N+k-1}{k}\binom{N+n-1-k}{n-k}a^k\,b^{n-k}.$$

Then, the quotient of x^{2N+p} by $(x-a)^N(x-b)^N$ is equal to $\sum_{k=0}^n c_nx^{p-n}$. In particular,

$$c_0=1,\ c_1=N(a+b),\ c_2=\frac{1}{2}N(N+1)(a^2+b^2)+N^2ab.$$

□

Remark 5. By (5.2), we easily get that

$$\mathbb{E}_x\Big[(X_{\tau_{ab}}-b)^p\mathbb{1}_{\{\tau_b^+<\tau_a^-\}}\Big]=\begin{cases}p\,!\,H_p^+(x) & \text{if } p\leq N,\\ 0 & \text{if } p\geq N+1.\end{cases}$$

This formula suggests the following interpretation of Hermite polynomials in terms of pseudo-Brownian motion: for $p\in\{0,\dots,N-1\}$,

$$H_p^+(x)=\frac{1}{p\,!}\,\mathbb{E}_x\Big[(X_{\tau_{ab}}-b)^p\mathbb{1}_{\{\tau_b^+<\tau_a^-\}}\Big].$$

6 The case $N=2$

For $N=2$, pseudo-Brownian motion is the so-called biharmonic-pseudo-process. In this case, the settings write $\theta_1=\mathrm{e}^{\mathrm{i}\,3\pi/4}$, $\theta_2=\mathrm{e}^{\mathrm{i}\,5\pi/4}=\overline{\theta_1}$, $\theta_3=\mathrm{e}^{\mathrm{i}\,7\pi/4}$, $\theta_4=\mathrm{e}^{\mathrm{i}\,\pi/4}=\overline{\theta_3}$, and, by setting $\nu=\lambda/4$,

$$\Delta(\lambda)=\begin{vmatrix}e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a}\\ \theta_1\,e_\lambda^{\theta_1 a} & \theta_2\,e_\lambda^{\theta_2 a} & \theta_3\,e_\lambda^{\theta_3 a} & \theta_4\,e_\lambda^{\theta_4 a}\\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b}\\ \theta_1\,e_\lambda^{\theta_1 b} & \theta_2\,e_\lambda^{\theta_2 b} & \theta_3\,e_\lambda^{\theta_3 b} & \theta_4\,e_\lambda^{\theta_4 b}\end{vmatrix},$$

$$\Delta_0^-(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x}\\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a}\\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b}\\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b}\end{vmatrix},\quad \Delta_1^-(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a}\\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x}\\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b}\\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b}\end{vmatrix},$$

$$\Delta_0^+(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a}\\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a}\\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x}\\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b}\end{vmatrix},\quad \Delta_1^+(\lambda;x)=\begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a}\\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a}\\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b}\\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x}\end{vmatrix}.$$

Elementary computations yield that

$$\Delta(\lambda)=4\left[\cosh\big(2\sqrt[4]{\nu}\,(b-a)\big)+\cos\big(2\sqrt[4]{\nu}\,(b-a)\big)-2\right].$$

Let us expand, e.g., $\Delta_0^-(\lambda;x)$ with respect to its first row:

$$\Delta_0^-(\lambda;x)=c_1\,e_\lambda^{\theta_1 x}+c_2\,e_\lambda^{\theta_2 x}+c_3\,e_\lambda^{\theta_3 x}+c_4\,e_\lambda^{\theta_4 x}$$

where c_1,c_2,c_3,c_4 are the cofactors of $\Delta_0^-(\lambda;x)$ related to the first row. Straightforward (but cumbersome) computations yield that $c_2=\overline{c_1}$ and $c_4=\overline{c_3}$ and

$$\begin{aligned}c_1&=(1-\mathrm{i})\,\mathrm{e}^{\sqrt[4]{\nu}\,((2b-a)-\mathrm{i}a)}+(1+\mathrm{i})\,\mathrm{e}^{\sqrt[4]{\nu}\,(a-\mathrm{i}(2b-a))}-2\,\mathrm{e}^{(1-\mathrm{i})\sqrt[4]{\nu}\,a},\\ c_3&=(1-\mathrm{i})\,\mathrm{e}^{-\sqrt[4]{\nu}\,((2b-a)-\mathrm{i}a)}+(1+\mathrm{i})\,\mathrm{e}^{-\sqrt[4]{\nu}\,(a-\mathrm{i}(2b-a))}-2\,\mathrm{e}^{-(1-\mathrm{i})\sqrt[4]{\nu}\,a}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\Delta_0^-(\lambda;x)&=2\,\Re\big(c_1\,e_\lambda^{\theta_1 x}+c_3\,e_\lambda^{\theta_3 x}\big)\\ &=2\Big[\mathrm{e}^{\sqrt[4]{\nu}\,(x-a)}\cos\big(\sqrt[4]{\nu}\,(x+a-2b)\big)+\mathrm{e}^{-\sqrt[4]{\nu}\,(x-a)}\cos\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\\ &\quad+\mathrm{e}^{\sqrt[4]{\nu}\,(x-a)}\sin\big(\sqrt[4]{\nu}\,(x+a-2b)\big)-\mathrm{e}^{-\sqrt[4]{\nu}\,(x-a)}\sin\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\\ &\quad-2\,\mathrm{e}^{\sqrt[4]{\nu}\,(x-a)}\cos\big(\sqrt[4]{\nu}\,(x-a)\big)-2\,\mathrm{e}^{-\sqrt[4]{\nu}\,(x-a)}\cos\big(\sqrt[4]{\nu}\,(x-a)\big)\\ &\quad+\mathrm{e}^{\sqrt[4]{\nu}\,(x+a-2b)}\cos\big(\sqrt[4]{\nu}\,(x-a)\big)+\mathrm{e}^{-\sqrt[4]{\nu}\,(x+a-2b)}\cos\big(\sqrt[4]{\nu}\,(x-a)\big)\\ &\quad-\mathrm{e}^{\sqrt[4]{\nu}\,(x+a-2b)}\sin\big(\sqrt[4]{\nu}\,(x-a)\big)+\mathrm{e}^{-\sqrt[4]{\nu}\,(x+a-2b)}\sin\big(\sqrt[4]{\nu}\,(x-a)\big)\Big]\end{aligned}$$

which simplifies by means of hyperbolic functions into

$$\begin{aligned}\Delta_0^-(\lambda;x)&=4\left[\cosh\big(\sqrt[4]{\nu}\,(x-a)\big)\cos\big(\sqrt[4]{\nu}\,(x+a-2b)\big)+\sinh\big(\sqrt[4]{\nu}\,(x-a)\big)\sin\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\right.\\ &\quad+\cosh\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\cos\big(\sqrt[4]{\nu}\,(x-a)\big)-\sinh\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\sin\big(\sqrt[4]{\nu}\,(x-a)\big)\\ &\quad\left.-2\cosh\big(\sqrt[4]{\nu}\,(x-a)\big)\cos\big(\sqrt[4]{\nu}\,(x-a)\big)\right].\end{aligned}$$

Quite similar computations yield that

$$\begin{aligned}\Delta_1^-(\lambda;x)&=4\left[\cosh\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\sin\big(\sqrt[4]{\nu}\,(x-a)\big)+\sinh\big(\sqrt[4]{\nu}\,(x-a)\big)\cos\big(\sqrt[4]{\nu}\,(x+a-2b)\big)\right.\\ &\quad\left.-\cosh\big(\sqrt[4]{\nu}\,(x-a)\big)\sin\big(\sqrt[4]{\nu}\,(x-a)\big)-\sinh\big(\sqrt[4]{\nu}\,(x-a)\big)\cos\big(\sqrt[4]{\nu}\,(x-a)\big)\right].\end{aligned}$$

The determinants Δ_0^+ and Δ_1^+ can be immediately deduced from Δ_0^- and Δ_1^- by interchanging the roles of a and b as it can be seen upon interchanging certain rows therein. We obtain that

$$\begin{aligned}\Delta_0^+(\lambda;x)&=4\left[\cosh\big(\sqrt[4]{\nu}\,(x-b)\big)\cos\big(\sqrt[4]{\nu}\,(x+b-2a)\big)+\sinh\big(\sqrt[4]{\nu}\,(x-b)\big)\sin\big(\sqrt[4]{\nu}\,(x+b-2a)\big)\right.\\ &\quad+\cosh\big(\sqrt[4]{\nu}\,(x+b-2a)\big)\cos\big(\sqrt[4]{\nu}\,(x-b)\big)-\sinh\big(\sqrt[4]{\nu}\,(x+b-2a)\big)\sin\big(\sqrt[4]{\nu}\,(x-b)\big)\\ &\quad\left.-2\cosh\big(\sqrt[4]{\nu}\,(x-b)\big)\cos\big(\sqrt[4]{\nu}\,(x-b)\big)\right],\\ \Delta_1^+(\lambda;x)&=4\left[\cosh\big(\sqrt[4]{\nu}\,(x+b-2a)\big)\sin\big(\sqrt[4]{\nu}\,(x-b)\big)+\sinh\big(\sqrt[4]{\nu}\,(x-b)\big)\cos\big(\sqrt[4]{\nu}\,(x+b-2a)\big)\right.\\ &\quad\left.-\cosh\big(\sqrt[4]{\nu}\,(x-b)\big)\sin\big(\sqrt[4]{\nu}\,(x-b)\big)-\sinh\big(\sqrt[4]{\nu}\,(x-b)\big)\cos\big(\sqrt[4]{\nu}\,(x-b)\big)\right].\end{aligned}$$

Now, formula (3.5) reads

$$\mathbb{P}_x\{\tau_{ab} \in \mathrm{d}t, X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}t\,\mathrm{d}z = I_0^-(t;x)\,\delta_a(z) + I_1^-(t;x)\,\delta_a'(z) + I_0^+(t;x)\,\delta_b(z) + I_1^+(t;x)\,\delta_b'(z)$$

where the functions $I_0^\pm$ and $I_1^\pm$ are characterized by

$$\int_0^\infty I_0^\pm(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{\Delta_0^\pm(\lambda;x)}{\Delta(\lambda)}, \quad \int_0^\infty I_1^\pm(t;x)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{1}{\sqrt[4]{\lambda}}\frac{\Delta_1^\pm(\lambda;x)}{\Delta(\lambda)}.$$

Concerning the pseudo-distribution of the exit location $X_{\tau_{ab}}$, it is given by

$$\mathbb{P}_x\{X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}z = H_0^-(x)\,\delta_a(z) - H_1^-(x)\,\delta_a'(z) + H_0^+(x)\,\delta_b(z) - H_1^+(x)\,\delta_b'(z)$$

with

$$H_0^-(x) = \frac{(x-b)^2(2x-3a+b)}{(b-a)^3}, \quad H_1^-(x) = \frac{(x-a)(x-b)^2}{(b-a)^2},$$
$$H_0^+(x) = -\frac{(x-a)^2(2x+a-3b)}{(b-a)^3}, \quad H_1^+(x) = \frac{(x-a)^2(x-b)}{(b-a)^2}.$$

When the pseudo-process starts at the middle of the interval $[a,b]$, we obtain the following expressions for the determinants of interest: by setting $L = (b-a)/2$,

$$\Delta(\lambda) = 32\left[\cosh^2\big(\sqrt[4]{\nu}\,L\big)\sinh^2\big(\sqrt[4]{\nu}\,L\big) - \cos^2\big(\sqrt[4]{\nu}\,L\big)\sin^2\big(\sqrt[4]{\nu}\,L\big)\right],$$

$$\begin{aligned}\Delta_0^-\left(\lambda;\frac{a+b}{2}\right) = \Delta_0^+\left(\lambda;\frac{a+b}{2}\right) &= 4\Big[\cosh\big(\sqrt[4]{\nu}\,L\big)\cos\big(\sqrt[4]{\nu}\,L\big)\Big(\cosh^2\big(\sqrt[4]{\nu}\,L\big) + \cos^2\big(\sqrt[4]{\nu}\,L\big) - 2\Big)\\ &\quad + \sinh\big(\sqrt[4]{\nu}\,L\big)\sin\big(\sqrt[4]{\nu}\,L\big)\Big(\cosh^2\big(\sqrt[4]{\nu}\,L\big) - \cos^2\big(\sqrt[4]{\nu}\,L\big)\Big)\Big],\\ \Delta_1^-\left(\lambda;\frac{a+b}{2}\right) = -\Delta_1^+\left(\lambda;\frac{a+b}{2}\right) &= 4\Big[\cosh\big(\sqrt[4]{\nu}\,L\big)\sin\big(\sqrt[4]{\nu}\,L\big)\sinh^2\big(\sqrt[4]{\nu}\,L\big)\\ &\quad - \sinh\big(\sqrt[4]{\nu}\,L\big)\cos\big(\sqrt[4]{\nu}\,L\big)\sin^2\big(\sqrt[4]{\nu}\,L\big)\Big].\end{aligned}$$

Hence, in this case, we have the following symmetric expression:

$$\mathbb{P}_{\frac{a+b}{2}}\{\tau_{ab} \in \mathrm{d}t, X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}t\,\mathrm{d}z = I_0^+\left(t;\frac{a+b}{2}\right)(\delta_a(z) + \delta_b(z)) + I_1^+\left(t;\frac{a+b}{2}\right)(\delta_b'(z) - \delta_a'(z)).$$

Moreover,

$$H_0^-\left(\lambda;\frac{a+b}{2}\right) = H_0^+\left(\lambda;\frac{a+b}{2}\right) = \frac{1}{2}, \quad H_1^-\left(\lambda;\frac{a+b}{2}\right) = -H_1^+\left(\lambda;\frac{a+b}{2}\right) = \frac{L}{4}.$$

Then,

$$\mathbb{P}_{\frac{a+b}{2}}\{X_{\tau_{ab}} \in \mathrm{d}z\}/\mathrm{d}z = \frac{1}{2}(\delta_a(z) + \delta_b(z)) + \frac{b-a}{8}(\delta_b'(z) - \delta_a'(z)).$$

A Appendix

A.1 Asymptotics of $\Delta(\lambda)$ and $\Delta_k^{\pm}(\lambda;x)$ as b tends to $+\infty$

In this appendix, we check limits (3.6). By factorizing the ℓth column of the determinant $\Delta(\lambda)$ by $\mathrm{e}_\lambda^{\theta_\ell\, a}$ for each $\ell \in \{1,\dots,2N\}$ and observing that $\sum_{\ell=1}^{2N}\theta_\ell = 0$, we find that

$$\Delta(\lambda) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \cdots & \cdots & \cdots \\ \mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(b-a)} \\ \theta_1\,\mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \theta_{2N}\,\mathrm{e}_\lambda^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_1^{N-1}\,\mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \theta_{2N}^{N-1}\,\mathrm{e}_\lambda^{\theta_{2N}(b-a)} \end{vmatrix}.$$

We separate $\Delta(\lambda)$ into four squared blocks as follows:

$$\Delta(\lambda) = \begin{vmatrix} V & \vdots & \tilde{V} \\ \cdots & \cdots & \cdots \\ W(\lambda) & \vdots & \tilde{W}(\lambda) \end{vmatrix}$$

with

$$V = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_N \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_{N+1} & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix},$$

$$W(\lambda) = \begin{pmatrix} \mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \mathrm{e}_\lambda^{\theta_N(b-a)} \\ \theta_1\,\mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \theta_N\,\mathrm{e}_\lambda^{\theta_N(b-a)} \\ \vdots & & \vdots \\ \theta_1^{N-1}\,\mathrm{e}_\lambda^{\theta_1(b-a)} & \cdots & \theta_N^{N-1}\mathrm{e}_\lambda^{\theta_N(b-a)} \end{pmatrix},$$

$$\tilde{W}(\lambda) = \begin{pmatrix} \mathrm{e}_\lambda^{\theta_{N+1}(b-a)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(b-a)} \\ \theta_{N+1}\,\mathrm{e}_\lambda^{\theta_{N+1}(b-a)} & \cdots & \theta_{2N}\,\mathrm{e}_\lambda^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1}\,\mathrm{e}_\lambda^{\theta_{N+1}(b-a)} & \cdots & \theta_{2N}^{N-1}\mathrm{e}_\lambda^{\theta_{2N}(b-a)} \end{pmatrix}.$$

Due to the fact that $\Re(\theta_\ell) < 0$ for $\ell \in \{1,\dots,N\}$ and $\Re(\theta_\ell) > 0$ for $\ell \in \{N+1,\dots,2N\}$, it may be easily seen by using an expansion by blocks of type $N \times N$ that the leading terms of $\Delta(\lambda)$ are obtained by performing the product of the determinants of both diagonal blocks V and $\tilde{W}(\lambda)$, namely:

$$\Delta(\lambda) \underset{b\to+\infty}{\sim} \det(V) \times \det(\tilde{W}(\lambda)).$$

Similarly, we decompose $\Delta_k^-(\lambda;x)$ into

$$\Delta_k^-(\lambda;x) = \begin{vmatrix} V_k(\lambda;x) & \vdots & \tilde{V}_k(\lambda;x) \\ \cdots & \cdots & \cdots \\ W(\lambda) & \vdots & \tilde{W}(\lambda) \end{vmatrix}$$

with

$$V_k(\lambda;x)=\begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_N \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_N^{k-1} \\ \mathrm{e}_\lambda^{\theta_1(x-a)} & \cdots & \mathrm{e}_\lambda^{\theta_N(x-a)} \\ \theta_1^{k+1} & \cdots & \theta_N^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix}, \quad \tilde{V}_k(\lambda;x)=\begin{pmatrix} 1 & \cdots & 1 \\ \theta_{N+1} & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{N+1}^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \mathrm{e}_\lambda^{\theta_{N+1}(x-a)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(x-a)} \\ \theta_{N+1}^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix}.$$

We can easily see that, for $x \in (a,b)$,

$$\Delta_k^-(\lambda;x) \underset{b\to+\infty}{\sim} \det(V_k(\lambda;x)) \times \det(\tilde{W}(\lambda)).$$

As a byproduct, we get the first limit of (3.6):

$$\frac{\Delta_k^-(\lambda;x)}{\Delta(\lambda)} \underset{b\to+\infty}{\longrightarrow} \frac{\det(V_k(\lambda;x))}{\det(V)}.$$

By expanding the determinant of $V_k(\lambda;x)$ with respect to its kth row, we obtain that

$$\frac{\det(V_k(\lambda;x))}{\det(V)} = \sum_{\ell=1}^{N} \alpha_{k\ell}\, \mathrm{e}_\lambda^{\theta_\ell\,(x-a)}$$

where the coefficients $\alpha_{k\ell}$ are explicitly written in Remark 2.

Next, concerning the determinant $\Delta_k^+(\lambda;x)$, by factorizing the ℓth column by $\mathrm{e}_\lambda^{\theta_\ell\, b}$ for each $\ell \in \{1,\dots,2N\}$, using the identity $\sum_{\ell=1}^{2N}\theta_\ell = 0$ and permuting the kth and $(N+k)$th rows for each $k \in \{1,\dots,N\}$, we get that

$$\Delta_k^+(\lambda;x) = (-1)^N \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \mathrm{e}_\lambda^{\theta_1(x-b)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(x-b)} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \mathrm{e}_\lambda^{\theta_1(a-b)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(a-b)} \\ \vdots & & \vdots \\ \theta_1^{N-1}\,\mathrm{e}_\lambda^{\theta_1(a-b)} & \cdots & \theta_{2N}^{N-1}\,\mathrm{e}_\lambda^{\theta_{2N}(a-b)} \end{vmatrix}.$$

As previously, we decompose $\Delta_k^+(\lambda;x)$ into

$$\Delta_k^+(\lambda;x) = (-1)^N \begin{vmatrix} Y_k(\lambda;x) & \vdots & \tilde{Y}_k(\lambda;x) \\ \cdots & \cdots & \cdots \\ Z(\lambda) & \vdots & \tilde{Z}(\lambda) \end{vmatrix} = \begin{vmatrix} \tilde{Y}_k(\lambda;x) & \vdots & Y_k(\lambda;x) \\ \cdots & \cdots & \cdots \\ \tilde{Z}(\lambda) & \vdots & Z(\lambda) \end{vmatrix}$$

with

$$Y_k(\lambda;x)=\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_N^{k-1} \\ \mathrm{e}_\lambda^{\theta_1(x-b)} & \cdots & \mathrm{e}_\lambda^{\theta_N(x-b)} \\ \theta_1^{k+1} & \cdots & \theta_N^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix},\quad \tilde{Y}_k(\lambda;x)=\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_{N+1}^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \mathrm{e}_\lambda^{\theta_{N+1}(x-b)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(x-b)} \\ \theta_{N+1}^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix},$$

$$Z(\lambda)=\begin{pmatrix} \mathrm{e}_\lambda^{\theta_1(a-b)} & \cdots & \mathrm{e}_\lambda^{\theta_N(a-b)} \\ \theta_1\,\mathrm{e}_\lambda^{\theta_1(a-b)} & \cdots & \theta_N\,\mathrm{e}_\lambda^{\theta_N(a-b)} \\ \vdots & & \vdots \\ \theta_1^{N-1}\,\mathrm{e}_\lambda^{\theta_1(a-b)} & \cdots & \theta_N^{N-1}\mathrm{e}_\lambda^{\theta_N(a-b)} \end{pmatrix},$$

$$\tilde{Z}(\lambda)=\begin{pmatrix} \mathrm{e}_\lambda^{\theta_{N+1}(a-b)} & \cdots & \mathrm{e}_\lambda^{\theta_{2N}(a-b)} \\ \theta_{N+1}\,\mathrm{e}_\lambda^{\theta_{N+1}(a-b)} & \cdots & \theta_{2N}\,\mathrm{e}_\lambda^{\theta_{2N}(a-b)} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1}\,\mathrm{e}_\lambda^{\theta_{N+1}(a-b)} & \cdots & \theta_{2N}^{N-1}\mathrm{e}_\lambda^{\theta_{2N}(a-b)} \end{pmatrix}.$$

Finally, by remarking that $\theta_{N+\ell}=-\theta_\ell$ for any $\ell\in\{1,\dots,N\}$, we derive that

$$\Delta_k^+(\lambda;x)=(-1)^k\begin{vmatrix} U_k(\lambda;x) & \vdots & \tilde{U}_k(\lambda;x) \\ \cdots & \cdots & \cdots \\ W(\lambda) & \vdots & \tilde{W}(\lambda) \end{vmatrix}$$

where the matrices $U_k(\lambda;x)$ and $\tilde{U}_k(\lambda;x)$ are deduced from $V_k(\lambda;x)$ and $\tilde{V}_k(\lambda;x)$ by changing $(x-a)$ into $(b-x)$, that is, $U_k(\lambda;x)=V_k(\lambda;a+b-x)$ and $\tilde{U}_k(\lambda;x)=\tilde{V}_k(\lambda;a+b-x)$. As a byproduct, we derive the identity

$$\Delta_k^+(\lambda;x)=(-1)^k\Delta_k^-(\lambda;a+b-x) \tag{A.1}$$

which is evoked in Remark 1. Thanks to an expansion by blocks, we can see that, for $x\in(a,b)$,

$$\Delta_k^+(\lambda;x)\underset{b\to+\infty}{\sim}(-1)^k\det(V_k(\lambda;a+b-x))\times\det(\tilde{W}(\lambda))=o[\det(\tilde{W}(\lambda))].$$

From this, we deduce the second limit of (3.6):

$$\frac{\Delta_k^+(\lambda;x)}{\Delta(\lambda)}\underset{b\to+\infty}{\longrightarrow}0.$$

A.2 Asymptotics of $\Delta(\lambda)$ and $\Delta_k^\pm(\lambda;x)$ as λ tends to 0^+ or $+\infty$

The procedure depicted in the previous subparagraph can be carried out *mutatis mutandis* in the case where λ tends to $+\infty$. This yields the following limiting result:

$$\frac{\Delta_k^\pm(\lambda;x)}{\Delta(\lambda)}\underset{\lambda\to+\infty}{\longrightarrow}0, \tag{A.2}$$

the rate of convergence being exponential. Then $I_k^\pm(t;x)\underset{t\to0^+}{\longrightarrow}0$. Below, we examine the case where λ tends to 0^+.

A.2.1 Asymptotics of $\Delta(\lambda)$ as λ tends to 0^+

Set $c = \lambda^{1/(2N)}(b-a)$. The number c tends to 0. We expand the exponentials lying in $\Delta(\lambda)$ into power series: for $k \in \{0, \dots, N-1\}$,

$$\theta_\ell^k \, \mathrm{e}_\lambda^{\theta_k(b-a)} = \sum_{i=0}^{\infty} \theta_\ell^{i+k} \, \frac{c^i}{i!} = \sum_{i=k}^{\infty} \theta_\ell^i \, \frac{c^{i-k}}{(i-k)!}.$$

Then,

$$\Delta(\lambda) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \sum_{i=0}^{\infty} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=1}^{\infty} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=1}^{\infty} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N-1}^{\infty} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N-1}^{\infty} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

By multilinearity, we see that the terms including a power of θ_ℓ less than N can be discarded (for these terms, the corresponding determinant has two or more identical rows, thus it vanishes). Hence, the determinant $\Delta(\lambda)$ does not change if we only keep the sums $\sum_{i=N}^{\infty} \theta_\ell^i \, c^{i-k}/(i-k)!$:

$$\Delta(\lambda) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \sum_{i=N}^{\infty} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=N}^{\infty} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N}^{\infty} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

By multilinearity, we can rewrite $\Delta(\lambda)$ as

$$\Delta(\lambda) = \sum_{\substack{i_1,\dots,i_N \geq N \\ i_1,\dots,i_N \text{ all distinct}}} \frac{c^{i_1+(i_2-1)+\cdots+(i_N-N+1)}}{i_1!(i_2-1)!\cdots(i_N-N+1)!} \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \theta_1^{i_1} & \cdots & \theta_{2N}^{i_1} \\ \theta_1^{i_2} & \cdots & \theta_{2N}^{i_2} \\ \vdots & & \vdots \\ \theta_1^{i_N} & \cdots & \theta_{2N}^{i_N} \end{vmatrix}.$$

Because of the conditions on the indices $i_1, \dots, i_N$, the least power of c is not less than N^2: indeed, the indices being distinct and not less than N, we have $i_1 + i_2 + \cdots + i_N \geq N + (N+1) + \cdots + (2N-1)$ or, equivalently, $i_1 + (i_2 - 1) + \cdots + (i_N - N + 1) \geq N^2$. Moreover, if an index is greater than $(2N-1)$, say $i_N \geq 2N$, then $i_1 + i_2 + \cdots + i_{N-1} + i_N \geq N + (N+1) + \cdots + (2N-2) + (2N)$, that is, $i_1 + (i_2 - 1) + \cdots + (i_N - N + 1) \geq N^2 + 1$. In words, the term c^{N^2} is obtained at

most for the indices not greater than $(2N-1)$. Consequently, we see that the terms of the sums corresponding to i greater than $(2N-1)$ can be neglected when c tends to 0, namely:

$$\Delta(\lambda) \underset{c\to 0^+}{=} \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \sum_{i=N}^{2N-1} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=N}^{2N-1} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N}^{2N-1} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix} + o\left(c^{N^2}\right)$$

We observe that the matrix lying in the foregoing determinant can be factorized into the product of the two following matrices:

$$A_1 = \begin{pmatrix} I & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & B \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{2N-1} & \cdots & \theta_{2N}^{2N-1} \end{pmatrix}$$

where I and O are respectively the unit and zero matrices of type $N \times N$, and

$$B = \begin{pmatrix} \frac{c^N}{N!} & \frac{c^{N+1}}{(N+1)!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ \frac{c^{N-1}}{(N-1)!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ c & \frac{c^2}{2!} & \cdots & \frac{c^N}{N!} \end{pmatrix}.$$

We can decompose B into $C_1 \tilde{B} C_2$ where C_1 and C_2 are the diagonal matrices with $c^N, c^{N-1}, \ldots, c$ and $1, c, \ldots, c^{N-1}$ as diagonal terms respectively, and

$$\tilde{B} = \begin{pmatrix} \frac{1}{N!} & \frac{1}{(N+1)!} & \cdots & \frac{1}{(2N-1)!} \\ \frac{1}{(N-1)!} & \frac{1}{N!} & \cdots & \frac{1}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{2!} & \cdots & \frac{1}{N!} \end{pmatrix}.$$

Hence, all this discussion plainly entails that

$$\Delta(\lambda) \underset{c\to 0^+}{\sim} \det(A_1) \times \det(A_2) = \det(A_2) \times \det(\tilde{B}) \times \det(C_1) \times \det(C_2) = constant \times c^{N^2}$$

where the constant does not vanish, or, by means of the variable λ,

$$\Delta(\lambda) \underset{\lambda\to 0^+}{\sim} constant \times \lambda^{N/2}. \tag{A.3}$$

A.2.2 Asymptotics of $\Delta_k^{\pm}(\lambda; x)$ as λ tends to 0^+

A similar analysis can be carried out in the case of the determinant $\Delta_k^{\pm}(\lambda; x)$. Recall that $c = \lambda^{1/(2N)}(b-a)$ and set $\gamma = \lambda^{1/(2N)}\,(x-a)$. The numbers c and γ tend to 0 as λ tends to 0^+. E.g.,

for $\Delta_k^-(\lambda;x)$, we have that

$$\Delta_k^{\pm}(\lambda;x)=\begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \sum_{i=0}^{\infty}\theta_1^i\frac{\gamma^i}{i!} & \cdots & \sum_{i=0}^{\infty}\theta_{2N}^i\frac{\gamma^i}{i!} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \sum_{i=0}^{\infty}\theta_1^i\frac{c^i}{i!} & \cdots & \sum_{i=0}^{\infty}\theta_{2N}^i\frac{c^i}{i!} \\ \sum_{i=1}^{\infty}\theta_1^i\frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=1}^{\infty}\theta_{2N}^i\frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N-1}^{\infty}\theta_1^i\frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N-1}^{\infty}\theta_{2N}^i\frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

As previously, this determinant remains unchanged by removing the terms related to the indices $0,1,\ldots,k-1,k+1,\ldots,N-1$ in each sum. Moreover, for obtaining an asymptotics when c,γ tend to 0 (actually c and γ have the same order of growth when λ tends to 0), it is enough to keep the terms related to the indices not greater than $(2N-1)$. Then, by setting $I_k=\{k\}\cup\{N,N+1,\ldots,2N-1\}$,

$$\Delta_k^-(\lambda;x)\underset{c,\gamma\to 0^+}{=}\begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \sum_{i\in I_k}\theta_1^i\frac{\gamma^i}{i!} & \cdots & \sum_{i\in I_k}\theta_{2N}^i\frac{\gamma^i}{i!} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hdashline \sum_{i\in I_k}\theta_1^i\frac{c^i}{i!} & \cdots & \sum_{i\in I_k}\theta_{2N}^i\frac{c^i}{i!} \\ \sum_{\substack{i\in I_k\\ i\geq 1}}\theta_1^i\frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{\substack{i\in I_k\\ i\geq 1}}\theta_{2N}^i\frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{\substack{i\in I_k\\ i\geq N-1}}\theta_1^i\frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{\substack{i\in I_k\\ i\geq N-1}}\theta_{2N}^i\frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}+o\left(\gamma^k c^{N^2}\right).$$

We observe that the matrix lying in the above determinant is the product of $\tilde{A}_1$ by A_2 where

- $\tilde{A}_1=\begin{pmatrix} I_k & \vdots & O_{2,k} \\ \cdots & \cdots & \cdots \\ O_{1,k} & \vdots & B \end{pmatrix}$;
- I_k is the diagonal matrix of type $N\times N$ with diagonal terms equal to 1 except for the $(k+1)$th which is $\gamma^k/k!$;
- $O_{1,k}$ is the matrix of type $N\times N$ with all terms equal to 0 except for the $(k+1)$th column which is made of $c^k/k!,c^{k-1}/(k-1)!,\ldots,c,1,0,\ldots,0$;
- $O_{2,k}$ is the matrix of type $N\times N$ with all terms equal to 0 except for the $(k+1)$th row which is made of $\gamma^N/N!,\gamma^{N+1}/(N+1)!,\ldots,\gamma^{2N-1}/(2N-1)!$.

The determinant of $\tilde{A}_1$ remains unchanged by interchanging its $(k+1)$th and Nth columns and its $(k+1)$th and $(k+1)$th rows. This yields that

$$\det(\tilde{A}_1) = \left|\begin{array}{cccc:ccc} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\gamma^k}{k!} & \frac{\gamma^N}{N!} & \cdots & \frac{\gamma^{2N-1}}{(2N-1)!} \\ \hdashline 0 & \cdots & 0 & \frac{c^k}{k!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ 0 & \cdots & 0 & \frac{c^{k-1}}{(k-1)!} & \frac{c^{N-1}}{(N-1)!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \frac{c^{N-k}}{(N-k)!} & \cdots & \frac{c^{2N-k-1}}{(2N-k-1)!} \\ 0 & \cdots & 0 & 0 & \frac{c^{N-k-1}}{(N-k-1)!} & \cdots & \frac{c^{2N-k-2}}{(2N-k-2)!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & c & \cdots & \frac{c^N}{N!} \end{array}\right|$$

$$= \left|\begin{array}{cccc} \frac{\gamma^k}{k!} & \frac{\gamma^N}{N!} & \cdots & \frac{\gamma^{2N-1}}{(2N-1)!} \\ \frac{c^k}{k!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ \frac{c^{k-1}}{(k-1)!} & \frac{c^{N-1}}{(N-1)!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{c^{N-k}}{(N-k)!} & \cdots & \frac{c^{2N-k-1}}{(2N-k-1)!} \\ 0 & \frac{c^{N-k-1}}{(N-k-1)!} & \cdots & \frac{c^{2N-k-2}}{(2N-k-2)!} \\ \vdots & \vdots & & \vdots \\ 0 & c & \cdots & \frac{c^N}{N!} \end{array}\right|.$$

By expanding this last determinant with respect to its first row, it is not difficult to see that $\det(\tilde{A}_1) = O\left(\gamma^k c^{N^2}\right)$ (recall that c and γ have the same order of growth when λ tends to 0). Therefore, in terms of the variable λ,

$$\Delta_k^-(\lambda; x) \underset{\lambda \to 0^+}{=} O\big(\lambda^{k/(2N)+N/2}\big) \tag{A.4}$$

and the same holds for $\Delta_k^+(\lambda; x)$. Finally, by (A.3) and (A.4), we derive that

$$\frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \underset{\lambda \to 0^+}{=} O\big(\lambda^{k/(2N)}\big). \tag{A.5}$$

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