



HAL
open science

The colourful simplicial depth conjecture

Pauline Sarrabezolles

► **To cite this version:**

| Pauline Sarrabezolles. The colourful simplicial depth conjecture. 2014. hal-00943550v1

HAL Id: hal-00943550

<https://hal.science/hal-00943550v1>

Preprint submitted on 7 Feb 2014 (v1), last revised 4 Mar 2014 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE EXACT VALUE OF COLOURFUL SIMPLICIAL DEPTH

PAULINE SARRABEZOLLES

ABSTRACT. Given $d+1$ sets of points, or colours, $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d , a *colourful simplex* is a set $T \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$, for $i = 1, \dots, d+1$. The colourful Carathéodory theorem states that, if $\mathbf{0}$ is in the convex hull of each \mathbf{S}_i , then there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. In 2006, Deza, Huang, Stephen, and Terlaky (*Colourful simplicial depth*, Discrete Comput. Geom., **35**, 597–604 (2006)) conjectured that, actually, when $|\mathbf{S}_i| = d+1$ for all $i = 1, \dots, d+1$, there are always at least $d^2 + 1$ colourful simplices containing $\mathbf{0}$ in their convex hulls. We prove this conjecture with the help of combinatorial objects called octahedral systems.

1. INTRODUCTION

A *colourful point configuration* is a collection of $d+1$ sets of points $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d . A *colourful simplex* is a subset T of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$. The colourful Carathéodory theorem, proved by Bárány in 1982 [1], states that, given a colourful point configuration $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d such that $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$, there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. In the same paper, Bárány uses this theorem combined with Tverberg's theorem to give a bound on simplicial depth. His argument motivated the following question: how many colourful simplices, at least, contain $\mathbf{0}$ in their convex hulls?

Let $\mu(d)$ denote the minimal number of colourful simplices containing $\mathbf{0}$ in their convex hulls over all colourful point configurations $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d such that $\mathbf{0} \in \text{conv}(\mathbf{S}_i)$ and $|\mathbf{S}_i| = d+1$ for $i = 1, \dots, d+1$. The colourful Carathéodory theorem states that $\mu(d) \geq 1$. The quantity $\mu(d)$ has been investigated by Deza, Huang, Stephen, and Terlaky [3]. They proved that $2d \leq \mu(d) \leq d^2 + 1$ and conjectured that $\mu(d) = d^2 + 1$. Later Bárány and Matoušek [2] proved the bound $\mu(d) \geq \max\left(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil\right)$ for $d \geq 3$, Stephen and Thomas [6] proved that $\mu(d) \geq \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, and Deza, Stephen, and Xie [4] showed that $\mu(d) \geq \left\lceil \frac{(d+1)^2}{2} \right\rceil$. More recently Deza, Meunier, and Sarrabezolles [5] improved the bound to $\frac{1}{2}d^2 + \frac{7}{2}d - 8$ for $d \geq 4$. This latter result was obtained with the help of a combinatorial generalization of the colourful point configurations suggested by Bárány and known as *octahedral systems*, see [4].

We use this combinatorial approach to prove the conjecture.

Theorem 1. *The equality $\mu(d) = d^2 + 1$ holds for every integer $d \geq 1$.*

The outline of the paper goes as follows. Section 2 is divided into two parts. First we define the octahedral systems and show their link with the colourful point configurations. Second, we introduce one of our main tools: the decomposition of an octahedral system over some elementary objects called umbrellas. Section 3 is devoted to the proof of Theorem 1.

Date: February 7, 2014.

Key words and phrases. Colourful Carathéodory Theorem, colourful simplicial depth, octahedral systems.

2. PRELIMINARIES

2.1. Octahedral systems. Let V_1, \dots, V_n be n pairwise disjoint finite sets, each of size $n \geq 2$. An *octahedral system* is a set $\Omega \subseteq V_1 \times \dots \times V_n$ satisfying the *parity condition*: the cardinality of $\Omega \cap (X_1 \times \dots \times X_n)$ is even if $X_i \subseteq V_i$ and $|X_i| = 2$ for all $i \in \{1, \dots, n\}$. We use the terminology of hypergraphs to describe an octahedral system: the sets V_i are the *classes*, the elements in V_i are the *vertices*, and the n -tuples of $V_1 \times \dots \times V_n$ are the *edges*. An edge whose i th component is a vertex $x \in V_i$ is *incident with the vertex x* , and conversely. A vertex x incident with no edges is *isolated*. A class V_i is *covered* if each vertex of V_i is incident with at least one edge. Finally, the set of edges incident with x is denoted by $\delta_\Omega(x)$ and the *degree of x* , denoted by $\deg_\Omega(x)$, refers to $|\delta_\Omega(x)|$.

The following lemma states that a nonempty octahedral system covers at least one class.

Lemma 1. *In every nonempty octahedral system, at least one class is covered.*

Proof. Consider an octahedral system $\Omega \subseteq V_1 \times \dots \times V_n$. Suppose that no classes are covered. There is at least one isolated vertex x_i in each V_i . Hence, if there were an edge (y_1, \dots, y_n) in Ω , then the parity condition would not be satisfied for $X_i = \{x_i, y_i\}$. \square

Given a colourful point configuration $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$, the Octahedron Lemma [2, 3] states that, for any $\mathbf{S}'_1 \subseteq \mathbf{S}_1, \dots, \mathbf{S}'_{d+1} \subseteq \mathbf{S}_{d+1}$, with $|\mathbf{S}'_1| = \dots = |\mathbf{S}'_{d+1}| = 2$, the number of colourful simplices generated by $\bigcup_{i=1}^{d+1} \mathbf{S}'_i$ and containing $\mathbf{0}$ in their convex hulls is even. The hypergraph over $V_1 \times \dots \times V_n$ where V_i is identified with \mathbf{S}_i and whose edges are identified with the colourful simplices containing $\mathbf{0}$ in their convex hulls is therefore an octahedral system. Furthermore, a strengthening of the colourful Carathéodory Theorem, given in [1], states that if $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$, then each point of the colourful point configuration is in some colourful simplices containing $\mathbf{0}$ in their convex hulls. Hence, an octahedral system Ω arising from such a colourful point configuration covers each class V_i .

2.2. Decompositions. The following proposition, proved in [5], states that the set of all octahedral systems is stable for the operation ‘‘symmetric difference’’.

Proposition 1. *Let Ω and Ω' be two octahedral systems over the same vertex set. $\Omega \Delta \Omega'$ is an octahedral system.*

Proof. Let $\Omega'' = \Omega \Delta \Omega'$. As Ω'' is a subset of $V_1 \times \dots \times V_n$, we simply check that the parity condition is satisfied. Consider $X_1 \subseteq V_1, \dots, X_n \subseteq V_n$ with $|X_i| = 2$ for $i = 1, \dots, n$. We have

$$|\Omega'' \cap (X_1 \times \dots \times X_n)| = |\Omega \cap (X_1 \times \dots \times X_n)| + |\Omega' \cap (X_1 \times \dots \times X_n)| - 2|\Omega \cap \Omega' \cap (X_1 \times \dots \times X_n)|.$$

All the terms of the sum are even, which allows to conclude. \square

We now present a family of specific octahedral systems we call *umbrellas*. An umbrella U is a set of the form $\{x^{(1)}\} \times \dots \times \{x^{(i-1)}\} \times V_i \times \{x^{(i+1)}\} \times \dots \times \{x^{(n)}\}$, with $x^{(j)} \in V_j$ for $j \neq i$. The class V_i covered by U is called its *colour*. $T = (x^{(1)}, \dots, x^{(i-1)}, x^{(i+1)}, \dots, x^{(n)})$ is its *transversal*. An umbrella is clearly an octahedral system on $V_1 \times \dots \times V_n$ and we have the following proposition.

Proposition 2. *Two umbrellas of the same colour have an edge in common if and only if they are equal.*

Proof. An umbrella is entirely determined by its colour V_i and its transversal T . Therefore, if two umbrellas of the same colour have an edge in common, they necessarily have the same transversal, which implies that they are equal. \square

It was proved in [5] that any octahedral system can be described as a symmetric difference of umbrellas. This decomposition is not unique. In this paper, we use a particular decomposition of an octahedral system to bound its cardinality.

Consider a nonempty octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$. Denote by i_1 the smallest $i \in \{1, \dots, n\}$ such that V_i is covered by Ω and order the vertices $\{x_1, \dots, x_n\}$ of V_{i_1} by increasing degree: $\deg_\Omega(x_1) \leq \cdots \leq \deg_\Omega(x_n)$. We define \mathcal{U} to be the set of umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1 and $W = \Delta_{U \in \mathcal{U}} U$. Let Ω_j be the set of all edges in $\Omega \Delta W$ incident with x_j . Formally,

$$\mathcal{U} = \{U : U \text{ umbrella of colour } V_{i_1} \text{ and } \delta_\Omega(x_1) \cap U \neq \emptyset\} \text{ and } \Omega_j = \delta_{\Omega \Delta W}(x_j).$$

In the remaining of the paper we refer to $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ as a *suitable decomposition*.

Lemma 2. *For any suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$, with $W = \Delta_{U \in \mathcal{U}} U$, we have*

- (i) $\Omega_j \cap \Omega_\ell = \emptyset$, for all $j \neq \ell$ (they have no edge in common),
- (ii) $\Omega = W \Delta \Omega_2 \Delta \cdots \Delta \Omega_n$,
- (iii) Ω_j is an octahedral system, for all $j \in \{2, \dots, n\}$,
- (iv) $\deg_\Omega(x_j) \geq \max(|\mathcal{U}|, |\Omega_j| - |\Omega_j \cap W|)$.
- (v) If V_i is not covered in Ω , then V_i is neither covered in $\Omega \Delta W$ nor in any Ω_j .

Proof. We first prove (i). The i_1 th component of any edge in Ω_j is x_j . Therefore, Ω_j and Ω_ℓ have no edge in common if $j \neq \ell$.

We then prove (ii). There are exactly $\deg_\Omega(x_1)$ umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1 . As W is the symmetric difference of these umbrellas, x_1 is isolated in $\Omega \Delta W$. Thus, $\Omega_2, \dots, \Omega_n$ form a partition of the edges in $\Omega \Delta W$ and $\Omega \Delta W = \Omega_2 \Delta \cdots \Delta \Omega_n$. Taking the symmetric difference of this equality with W we obtain $\Omega = W \Delta \Omega_2 \cdots \Delta \Omega_n$.

We now prove (iii). By definition, the Ω_j 's are subsets of $V_1 \times \cdots \times V_n$. It remains to prove that they satisfy the parity condition. Consider $X_i \subseteq V_i$ with $|X_i| = 2$ for $i = 1, \dots, n$. If X_{i_1} does not contain x_j , there are no edges in Ω_j induced by $X_1 \times \cdots \times X_n$. If X_{i_1} contains x_j , the edges in Ω_j induced by $X_1 \times \cdots \times X_n$ are the ones induced by $X_1 \times \cdots \times X_{i_1-1} \times \{x_j\} \times X_{i_1+1} \times \cdots \times X_n$. As x_1 is isolated in $\Omega \Delta W$, those edges are exactly the edges in $\Omega \Delta W$ induced by $X_1 \times \cdots \times X_{i_1-1} \times \{x_1, x_j\} \times X_{i_1+1} \times \cdots \times X_n$. According to Proposition 1, $\Omega \Delta W$ is an octahedral system, hence there is an even number of edges.

We prove (iv). We have $|\mathcal{U}| = \deg_\Omega(x_1) \leq \deg_\Omega(x_j)$ for $j = 1, \dots, n$. Furthermore, by definition of the symmetric difference, we have $(\Omega_2 \Delta \cdots \Delta \Omega_n) \setminus W \subseteq \Omega$. This inclusion becomes $(\Omega_2 \setminus W) \Delta \cdots \Delta (\Omega_n \setminus W) \subseteq \Omega$. As two Ω_j 's share no edges, considering the edges incident with x_j , we have $\Omega_j \setminus W \subseteq \delta_\Omega(x_j)$. We obtain

$$|\Omega_j| - |\Omega_j \cap W| \leq \deg_\Omega(x_j).$$

Finally to prove (v) it suffices to prove that a class V_i not covered in Ω remains not covered in $\Omega \Delta W$. Indeed, if a class is covered in an Ω_j , it is also covered in $\Omega \Delta W$. Consider V_i not covered in Ω . There is a vertex $x \in V_i$ incident with no edges in Ω . In particular, there are no edges in Ω incident with x_1 and x . Therefore, the umbrellas in \mathcal{U} , which are defined

by the edges incident with x_1 , contain no edges incident with x . Hence, x is isolated in $W = \Delta_{U \in \mathcal{U}} U$ and in Ω . Finally, x remains isolated in $\Omega \Delta W$. \square

Unlike the suitable decomposition of Ω , which is a decomposition over general octahedral systems, the decomposition given in the following lemma is over umbrellas.

Lemma 3. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$. There exists a set of umbrellas \mathcal{D} , such that $\Omega = \Delta_{U \in \mathcal{D}} U$ and such that the following implication holds:*

$$V_i \text{ is the colour of some } U \in \mathcal{D} \Rightarrow V_i \text{ is covered in } \Omega.$$

Proof. The proof works by induction on the number of covered classes in Ω . If no classes are covered, then, according to Lemma 1, Ω is empty.

Suppose now that k classes are covered, with $k \geq 1$, and consider a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of Ω . Denote by W the symmetric difference $W = \Delta_{U \in \mathcal{U}} U$. According to Proposition 1, W is an octahedral system, and so is $\Omega \Delta W$. There are strictly less covered classes in $\Omega \Delta W$ than in Ω . Indeed, in $\Omega \Delta W$, the class V_{i_1} is no longer covered, since x_1 is isolated, and according to (v) of Lemma 3, a class not covered in Ω remains not covered in $\Omega \Delta W$. By induction, there exists a set \mathcal{D}' of umbrellas such that $\Omega \Delta W = \Delta_{U \in \mathcal{D}'} U$, and such that if there is an umbrella of colour V_i in \mathcal{D}' , it implies that V_i is covered in $\Omega \Delta W$. As the umbrellas in \mathcal{D}' are not of colour V_{i_1} , we have $\mathcal{D}' \cap \mathcal{U} = \emptyset$. Therefore, $\Omega = (\Delta_{U \in \mathcal{U}} U) \Delta (\Delta_{U \in \mathcal{D}'} U)$ and the set $\mathcal{D} = \mathcal{U} \cup \mathcal{D}'$ satisfies the statement of the lemma. \square

3. PROOF OF THE MAIN RESULT

The following theorem gives a general lower bound on the cardinality of an octahedral system. Our main theorem is a corollary of it.

Theorem 2. *Let $\Omega \subseteq V_1 \times \cdots \times V_n$ be an octahedral system with $|V_1| = \dots = |V_n| = n$ and $n \geq 2$. If k classes among the V_i 's are covered, then*

$$|\Omega| \geq k(n - 2) + 2.$$

Before proving this theorem, we show how the main theorem can be deduced from it.

Proof of Theorem 1. The inequality $\mu(d) \leq d^2 + 1$ is proved in [3]. Let $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ be a colourful point configuration in \mathbb{R}^d . As explained in Section 2.1, the set $\Omega \subseteq V_1 \times \cdots \times V_{d+1}$, with $V_i = \mathbf{S}_i$ for $i = 1, \dots, d + 1$ and whose edges correspond to the colourful simplices containing $\mathbf{0}$ in their convex hulls, is an octahedral system. According to [1, Theorem 2.3.], this octahedral system covers all the classes. Applying Theorem 2 with $k = n = d + 1$ gives the lower bound: $\mu(d) \geq d^2 + 1$. \square

The remainder of the section is devoted to the proof of Theorem 2. The proof distinguishes two cases, corresponding to the following Propositions 3 and 4. We first prove these propositions.

Proposition 3. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ and a class V_i covered by Ω . If Ω can be written as a symmetric difference of umbrellas, none of them being of colour V_i , then $|\Omega| \geq n^2$.*

Proof. Let \mathcal{D} be a set of umbrellas such that there are no umbrellas of colour V_i in \mathcal{D} and $\Omega = \Delta_{U \in \mathcal{D}} U$. Denote by x_1, \dots, x_n the vertices of V_i , and by \mathcal{Q}_j the set of umbrellas of \mathcal{D} incident with x_j for $j = 1, \dots, n$. As \mathcal{D} does not contain any umbrellas of colour V_i , the

\mathcal{Q}_j 's are disjoint: no two \mathcal{Q}_j have an umbrella in common. Furthermore, denoting by Q_j the symmetric difference of the umbrellas in \mathcal{Q}_j , we have that Q_j is an octahedral system, according to Proposition 1, and that $\delta_\Omega(x_j) = Q_j$, $Q_j \neq \emptyset$, and $Q_j \cap Q_\ell = \emptyset$ for all $j \neq \ell$. A nonempty octahedral system necessarily covers one class according to Lemma 1, hence it contains at least n edges. Therefore, we have

$$|\Omega| = \sum_{j=1}^n \deg_\Omega(x_j) = \sum_{j=1}^n |Q_j| \geq n^2$$

□

Proposition 4. *Consider an octahedral system $\Omega \subseteq V_1 \times \dots \times V_n$ and a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of it. Denote by V_{i_1} the colour of the umbrellas in \mathcal{U} and suppose that each covered class V_i with $i \neq i_1$ is covered in at least one of the Ω_j . Choose a set $\mathcal{O} \subseteq \{\Omega_2, \dots, \Omega_n\}$, minimal for inclusion, such that each covered class V_i , with $i \neq i_1$, is covered in at least one of the Ω_j in \mathcal{O} . Denote by $\mathcal{P} \subseteq \mathcal{O}$ the subset of octahedral systems Ω_j in \mathcal{O} that are umbrellas. We have*

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

Proof. As usual, let $W = \Delta_{U \in \mathcal{U}} Y$. The number of edges in Ω is equal to $\sum_{j=1}^n \deg_\Omega(x_j)$. We bound $\deg_\Omega(x_j)$ by $|\mathcal{U}|$ if $\Omega_j \notin \mathcal{O}$ and by $|\Omega_j| - |\Omega_j \cap W|$ otherwise, see (iv) in Lemma 2. We obtain

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} (|\Omega_j| - |\Omega_j \cap W|).$$

We introduce a graph $G = (\mathcal{V}, \mathcal{E})$ defined as follows. We use the terminology *nodes* and *links* for G in order to avoid confusion with the vertices and edges of Ω . The nodes in \mathcal{V} are identified with the umbrellas in \mathcal{U} and the Ω_j 's in \mathcal{O} : $\mathcal{V} = \mathcal{U} \cup \mathcal{O}$. There is a link in \mathcal{E} between two nodes if the corresponding octahedral systems have an edge in common. G is bipartite: indeed, two umbrellas in \mathcal{U} are of the same colour V_{i_1} and, according to Proposition 2, they do not have an edge in common. According to Lemma 2, two Ω_j 's do not have an edge in common either.

For Ω_j in \mathcal{O} , we have $|\Omega_j \cap W| = \sum_{U \in \mathcal{U}} |\Omega_j \cap U| = \deg_G(\Omega_j)$. Note that here the degree is counted in G . The fact that the umbrellas in \mathcal{U} are disjoint proves the first equality. The second inequality is deduced from the facts that Ω_j has at most one edge in common with each umbrella in \mathcal{U} , the one incident with x_j , and that G is bipartite. We obtain the following bound

$$\begin{aligned} |\Omega| &\geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} (|\Omega_j| - \deg_G(\Omega_j)) \\ &= |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - \deg_G(\mathcal{O} \setminus \mathcal{P}) - \deg_G(\mathcal{P}). \end{aligned}$$

Again, for the equality, we use the fact that G is bipartite. The number of links in \mathcal{E} incident with a node in $\mathcal{O} \setminus \mathcal{P}$ is at most $|\mathcal{U}|$. Hence, $\deg_G(\mathcal{O} \setminus \mathcal{P}) \leq |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|)$. It remains to bound $\deg_G(\mathcal{P})$. Note that if U is an umbrella in \mathcal{P} , it is the only umbrella of its colour in \mathcal{P} , otherwise it would contradict the minimality of \mathcal{O} . We now prove that there are no cycles induced by $\mathcal{P} \cup \mathcal{U}$ in G .

Suppose there is such a cycle \mathcal{C} and consider an umbrella U of \mathcal{P} in this cycle. Denote its colour by V_i and its neighbours in \mathcal{C} by L and R . As G is simple, L and R are distinct. L and R are both in \mathcal{U} , and hence are of colour V_{i_1} and do not have an edge in common. Therefore $U \cap L$ and $U \cap R$ do not have an edge in common either, which implies that the i th component of the transversals of L and R are distinct. Note that two umbrellas adjacent in \mathcal{C} , both of colour distinct from V_i , have necessarily transversals with the same i th component. Hence there must be another umbrella of colour V_i in the path in \mathcal{C} between L and R not containing U . This is a contradiction since U is the only umbrella in \mathcal{P} of colour V_i .

The number of links in \mathcal{E} incident with \mathcal{P} is then at most $|\mathcal{P}| + |\mathcal{U}| - 1$. This allows us to conclude. \square

Proof of Theorem 2. Let $\Omega \subseteq V_1 \times \cdots \times V_n$ be an octahedral system with $|V_1| = \cdots = |V_n| = n$ and $n \geq 2$, and suppose that k classes V_{i_1}, \dots, V_{i_k} , with $i_1 < \cdots < i_k$, are covered in Ω . The proof works by induction on k .

If $k = 1$, then Ω must contain at least n edges for one class to be covered.

Assume now that $k > 1$. If $|\mathcal{U}| \geq n - 1$, then, according to (iv) of Lemma 2, $|\Omega| = \sum_{j=1}^n \deg_{\Omega}(x_j) \geq n|\mathcal{U}| \geq k(n - 2) + 2$ and we are done. Assume now that $|\mathcal{U}| \leq n - 2$. We consider a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of Ω and distinguish two cases.

Case 1: *One of the covered classes V_i , for $i \in \{i_2, \dots, i_k\}$, is not covered in any Ω_j .* Let V_i be a covered class in Ω not covered in any Ω_j . Applying Lemma 3 on Ω_j gives a set \mathcal{D}_j of umbrellas, all of colour distinct from V_i , such that $\Omega_j = \Delta_{U \in \mathcal{D}_j} U$. We obtain $\Omega = (\Delta_{U \in \mathcal{U}} U) \Delta (\Delta_{j=2}^n \Delta_{U \in \mathcal{D}_j} U)$. Thus, we can apply Proposition 3 which ensures that

$$|\Omega| \geq n^2 \geq k(n - 2) + 2.$$

Case 2: *Each covered class V_i is covered in at least one of the Ω_j for $i \in \{i_2, \dots, i_k\}$.* Applying Proposition 4, we obtain

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

It remains to bound $\sum_{\Omega_j \in \mathcal{O}} |\Omega_j|$. By induction, the cardinality of Ω_j is at least $k_j(n - 2) + 2$, where k_j is the number of covered classes in Ω_j . We have $k_j < k$ according to (v) of Lemma 3. This lower bound is not good enough for the $\Omega_j \notin \mathcal{P}$ such that $k_j = 1$. We explain now how to improve the lower bound for such Ω_j 's. Assume that Ω_j covers only one class and that $\Omega_j \notin \mathcal{P}$. According to Lemma 3, Ω_j can be written as a symmetric difference of distinct umbrellas of the same colour. According to Proposition 2, these umbrellas are pairwise disjoint and $|\Omega_j|$ is equal to n times the number of umbrellas in this decomposition. Since Ω_j is not an umbrella itself, otherwise Ω_j would have been in \mathcal{P} , there are at least two umbrellas in this decomposition. We denote by \mathcal{A} the octahedral systems in $\mathcal{O} \setminus \mathcal{P}$ covering only one class. We obtain

$$\sum_{\Omega_j \in \mathcal{O}} |\Omega_j| \geq \left(\sum_{\Omega_j \in \mathcal{O} \setminus \mathcal{A}} k_j \right) (n - 2) + 2|\mathcal{O} \setminus \mathcal{A}| + 2n|\mathcal{A}| \geq \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n - 2) + 2|\mathcal{O}| + n|\mathcal{A}|$$

We have thus

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n - 2) + 2|\mathcal{O}| + n|\mathcal{A}| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

Finally, we have

$$(1) \quad 2|\mathcal{O}| - |\mathcal{P}| - |\mathcal{A}| \leq \sum_{\Omega_j \in \mathcal{O}} k_j$$

$$(2) \quad k - 1 \leq \sum_{\Omega_j \in \mathcal{O}} k_j$$

Equation (1) is obtained by distinguishing the Ω_j with $k_j = 1$ from those with $k_j \geq 2$. Equation (2) results from the fact that each class V_{i_2}, \dots, V_{i_k} is covered in at least one Ω_j in \mathcal{O} . Thus,

$$\begin{aligned} |\Omega| &\geq |\mathcal{U}|(n - |\mathcal{O}|) + \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n - 2) + 2|\mathcal{O}| + |\mathcal{U}||\mathcal{A}| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1 \\ &\geq (k - 1)(n - 2) + 2|\mathcal{O}| - |\mathcal{P}| + 1 + \left(\sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}| \right) |\mathcal{U}| \end{aligned}$$

where we only used the inequalities $n \geq n - 2 \geq |\mathcal{U}|$ and (2). According to (1), the expression $\left(\sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}| \right)$ is nonnegative. Moreover, we have already noted that $|\mathcal{U}| = \deg_{\Omega}(x_1)$, which is at least 1. Therefore,

$$|\Omega| \geq (k - 1)(n - 2) + 2|\mathcal{O}| - |\mathcal{P}| + 1 + \sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}|.$$

Using (2) again, we obtain

$$|\Omega| \geq k(n - 2) + 2.$$

□

ACKNOWLEDGEMENT

The author thanks Antoine Deza for introducing her to the colourful simplicial depth conjecture and Frédéric Meunier for his thorough reading of the manuscript and his helpful comments.

REFERENCES

- [1] Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Mathematics*, 40: 141–152, 1982.
- [2] Imre Bárány and Jiří Matoušek. Quadratically many colorful simplices. *SIAM Journal on Discrete Mathematics*, 21:191–198, 2007.
- [3] Antoine Deza, Sui Huang, Tamon Stephen, and Tamás Terlaky. Colourful simplicial depth. *Discrete and Computational Geometry*, 35:597–604, 2006.

- [4] Antoine Deza, Tamon Stephen, and Feng Xie. More colourful simplices. *Discrete and Computational Geometry*, 45:272–278, 2011.
- [5] Antoine Deza, Frédéric Meunier, and Pauline Sarrabezolles. A combinatorial approach to colourful simplicial depth. *SIAM Journal on Discrete Mathematics*, 2014.
- [6] Tamon Stephen and Hugh Thomas. A quadratic lower bound for colourful simplicial depth. *Journal of Combinatorial Optimization*, 16:324–327, 2008.

P. SARRABEZOLLES, UNIVERSITÉ PARIS EST, CERMICS (ECOLE DES PONTS PARISTECH), 6-8 AVENUE BLAISE PASCAL, CITÉ DESCARTES, 77455 MARNE-LA-VALLÉE, CEDEX 2, FRANCE
E-mail address: pauline.sarrabezolles@cermics.enpc.fr