



**HAL**  
open science

# The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3

Olivier Rey

► **To cite this version:**

Olivier Rey. The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3. *Advances in Differential Equations*, 1999, 4, pp.581-616. hal-00943490

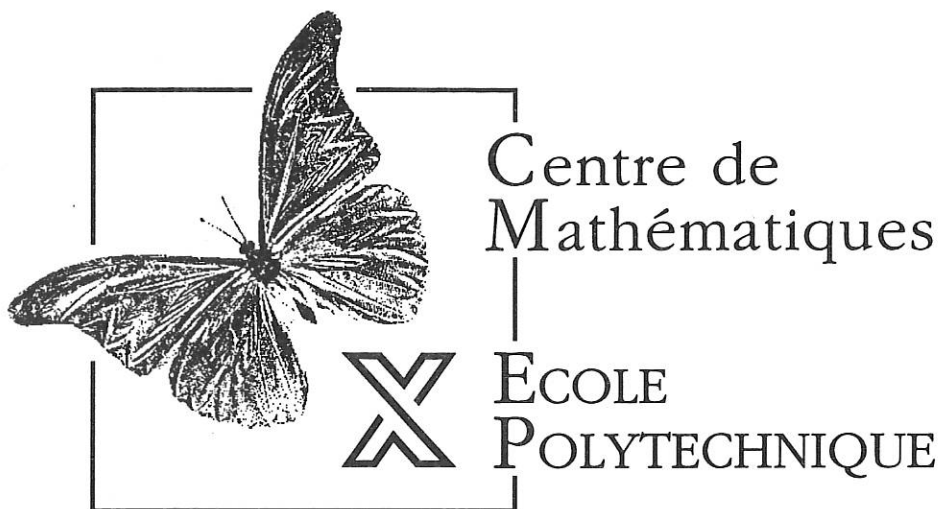
**HAL Id: hal-00943490**

**<https://hal.science/hal-00943490>**

Submitted on 22 Dec 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



U.R.A. 169 du C.N.R.S.

**The topological impact of critical points  
at infinity in a variational problem with  
lack of compactness : the dimension 3**

**Olivier Rey**

**N° 97 - 27**

**décembre 1997**

F-91128 Palaiseau Cedex  
Tél. : (33) 01 69 33 40 88 • Fax : (33) 01 69 33 30 19  
Courrier électr. : [secret@math.polytechnique.fr](mailto:secret@math.polytechnique.fr)  
<http://math.polytechnique.fr>

**Abstract.** We extend to dimension 3 a result previously known in higher dimensions, concerning the topological effect of critical points at infinity on the level sets a functional associated to an elliptic problem with critical nonlinearity.

**Classification AMS.** 35 J 65

**Keywords.** *Nonlinear equations, critical points at infinity, limiting Sobolev exponent, variational problems with lack of compactness.*

# The topological impact of critical points at infinity in a variational problem with lack of compactness : the dimension 3

Olivier Rey

URA 169 CNRS Centre de Mathématiques  
Ecole Polytechnique F - 91128 Palaiseau cedex

December 15, 1997

## Abstract

We extend to dimension 3 a result previously known in higher dimensions, concerning the topological effect of critical points at infinity on the level sets of a functional associated to an elliptic problem with critical nonlinearity.

**Classification AMS.** 35 J 65

**Keywords.** Nonlinear equations, critical points at infinity, limiting Sobolev exponent, variational problems with lack of compactness.

## 1 Introduction and results

Let us consider the nonlinear elliptic problem

$$(P) \begin{cases} -\Delta u = u^p, & u > 0 & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  is a smooth and bounded domain in  $\mathbb{R}^N$ ,  $N \geq 3$ , and  $p = \frac{N+2}{N-2}$  is the critical Sobolev exponent. It has been known, since a work by Pohozaev [15] that  $(P)$  has no solution when  $\Omega$  is starshaped. On the other hand, Kazdan and Warner [10] proved that a solution exists in the special case where  $\Omega$  is an annulus. A major step was accomplished by Bahri and Coron [2] who showed that  $(P)$  has a solution as soon as  $\Omega$  has nontrivial topology, in the

sense that  $H_{2n-1}(\Omega; \mathbb{Q}) \neq 0$  or  $H_n(\Omega; \mathbb{Z}/2\mathbb{Z}) \neq 0$  for some  $n \in \mathbb{N}^*$ . However, contractible domains have been exhibited for which  $(P)$  admits also a solution [8][9], pointing out that the geometry of the domain is of some importance.

A strategy to prove existence or nonexistence of solutions to  $(P)$  is to compute the difference of topology between the level sets of a functional associated to the problem. However, such a functional is noncompact : critical points at infinity occur, that is orbits of the gradient along which the functional remains bounded, its gradient goes to zero, and which do not converge. In view of further results of existence, nonexistence or multiplicity concerning  $(P)$ , it is therefore crucial to know the exact contribution of the critical points at infinity to the relative topology between the level sets. This program was performed in [3], adopting the following strategy.  $(P)$  is approximated by the subcritical problems

$$(P_\varepsilon) \begin{cases} -\Delta u = u^{p-\varepsilon}, & u > 0 & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega \end{cases}$$

where  $\varepsilon$  is a strictly positive small real number. To  $(P_\varepsilon)$  is associated the functional

$$(1) \quad J_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1-\varepsilon} \int_\Omega |u|^{p+1-\varepsilon} \quad u \in H_0^1(\Omega)$$

which is compact, and whose positive critical points are solutions to  $(P_\varepsilon)$ . Such solutions exist and it follows from [12] [20] that, as  $\varepsilon$  goes to zero, either they converge to a solution to  $(P)$ , or they blow up at a finite number of points in  $\Omega$ . Namely,  $(u_\varepsilon)$  being a bounded sequence in  $H_0^1(\Omega)$  of solutions to  $(P_\varepsilon)$ , up to a subsequence we have :

$$(2) \quad u_\varepsilon = u_0 + \sum_{i=1}^k \alpha_i^\varepsilon P\delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + v^\varepsilon$$

with  $u_0$  a solution to  $(P)$  or  $u_0 \equiv 0$ ,  $v^\varepsilon$  goes to zero in  $H_0^1(\Omega)$ ,  $k \in \mathbb{N}$ . Moreover, for  $\lambda \in \mathbb{R}_+^*$ ,  $x \in \mathbb{R}^N$

$$(3) \quad \begin{aligned} \delta_{\lambda,x} : \mathbb{R}^N &\rightarrow \mathbb{R} \\ y &\mapsto \lambda^{\frac{N-2}{2}} (1 + \lambda^2 |x-y|^2)^{-\frac{N-2}{2}} \end{aligned}$$

and  $P\delta_{\lambda,x}$  is defined as the projection of  $\delta_{\lambda,x}$  onto  $H_0^1(\Omega)$ , that is :

$$(4) \quad \Delta P\delta_{\lambda,x} = \Delta\delta_{\lambda,x} \text{ in } \Omega ; P\delta_{\lambda,x} = 0 \text{ on } \partial\Omega .$$

Lastly

$$\begin{aligned} \alpha_i^\varepsilon \in \mathbb{R}, \alpha_i^\varepsilon &\rightarrow \bar{\alpha} = (N(N-2))^{\frac{N-2}{4}} \\ x_i^\varepsilon \in \Omega, x_i^\varepsilon &\rightarrow x_i \in \bar{\Omega} \end{aligned}$$

and

$$(5) \quad \begin{cases} \lambda_i^\varepsilon d(x_i^\varepsilon, \partial\Omega) \rightarrow +\infty \\ \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon \lambda_j^\varepsilon |x_i^\varepsilon - x_j^\varepsilon|^2 \rightarrow +\infty. \end{cases}$$

We note that the functions  $\bar{\alpha}\delta_{\lambda x}$ ,  $\lambda > 0$ ,  $x \in \mathbb{R}^N$ , are the only solutions of the problem

$$-\Delta u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N$$

see [6] - and that  $P\delta_{\lambda x}$ ,  $\lambda > 0$ ,  $x \in \Omega$ , satisfies

$$(6) \quad \begin{cases} -\Delta P\delta_{\lambda x} = \delta_{\lambda x}^p \text{ in } \Omega \\ P\delta_{\lambda x} = 0 \text{ on } \partial\Omega \end{cases}$$

(5) ensures that we have

$$(7) \quad J_\varepsilon(u_\varepsilon) = J_0(u_0) + k \frac{S^{N/2}}{N} + o(1)$$

where  $S = \inf_{u \in H_0^1(\Omega), u \neq 0} |u|_{H_0^1(\Omega)}^2 |u|_{L^{p+1}(\Omega)}^{-2}$  is the Sobolev constant, which depends on  $N$  only. Actually, a result by Schoen [19] says that, under the previous assumptions, we have either  $k = 0$  or  $u_0 \equiv 0$ . If  $k = 0$ ,  $u_\varepsilon$  converges in  $H_0^1(\Omega)$  to a solution of (P); if  $u_0 \equiv 0$ ,  $u_\varepsilon$  blows up as  $\varepsilon$  goes to zero, namely

$$(8) \quad \begin{cases} |\nabla u_\varepsilon|^2, u_\varepsilon^{p+1} \xrightarrow{\varepsilon \rightarrow 0} S^{N/2} \sum_{i=1}^k \delta_{x_i} & (\delta_{x_i} \text{ is the Dirac mass at } x_i) \\ J_\varepsilon(u_\varepsilon) \rightarrow c_k = k \frac{S^{N/2}}{N}. \end{cases}$$

Let us denote by  $G$  and  $H$  respectively the Green's function of the Laplacian with Dirichlet boundary conditions on  $\Omega$ , and its regular part, i.e.

$$(9) \quad \begin{cases} G(x, y) = \frac{1}{|x-y|^{N-2}} - H(x, y) & (x, y) \in \Omega \times \Omega \\ \Delta_x H = 0 \text{ in } \Omega \times \Omega & G = 0 \text{ on } \partial(\Omega \times \Omega). \end{cases}$$

For  $k \in \mathbb{N}^*$  and  $\mathbf{x} = (x_1, \dots, x_k) \in \Omega^k$ , we set  $M(\mathbf{x}) = (m_{ij})_{1 \leq i, j \leq k}$  the matrix defined as

$$(10) \quad m_{ii} = H(x_i, x_i); m_{ij} = -G(x_i, x_j) \quad i \neq j$$

and  $\rho(\mathbf{x})$  denotes the least eigenvalue of  $M$  ( $\rho(\mathbf{x}) = -\infty$  if  $x_i = x_j$  for some  $i \neq j$ ). We define also

$$(11) \quad \begin{aligned} F_{\mathbf{x}} : (\mathbb{R}_+^*)^k &\rightarrow \mathbb{R} \\ \Lambda = (\Lambda_1, \dots, \Lambda_k) &\mapsto \frac{1}{2} \Lambda M(\mathbf{x})^t \Lambda - \sum_{i=1}^k \ell n \Lambda_i. \end{aligned}$$

If  $\rho(\mathbf{x}) > 0$ ,  $F_{\mathbf{x}}$  being strictly convex in  $(\mathbb{R}_+^*)^k$  and infinite on the boundary,  $F_{\mathbf{x}}$  has a unique critical point  $\Lambda(\mathbf{x})$ , which is a minimum. On

$$(12) \quad \rho^+ = \{\mathbf{x} \in \Omega^k / \rho(\mathbf{x}) > 0\}$$

we set

$$(13) \quad \tilde{F}(\mathbf{x}) = F_{\mathbf{x}}(\Lambda(\mathbf{x})) = \frac{k}{2} - \sum_{i=1}^k \ell n \Lambda_i(\mathbf{x})$$

whose differential is given by

$$(14) \quad \tilde{F}'(\mathbf{x}) = \frac{1}{2} \Lambda(\mathbf{x}) M'(\mathbf{x})^t \Lambda(\mathbf{x}) = - \sum_{i=1}^k \frac{\Lambda'_i(\mathbf{x})}{\Lambda_i(\mathbf{x})}.$$

In [3] it is proved :

**Theorem 1** *Assume that  $N \geq 4$ , and  $(u_\varepsilon)$  is a sequence of solutions to  $(P_\varepsilon)$  which blows up at  $k$  points  $x_1, \dots, x_k$  of  $\bar{\Omega}$  as  $\varepsilon$  goes to zero. Then*

- (a)  $\mathbf{x} = (x_1, \dots, x_k) \in \Omega_{d_0}^k$ , with  $d_0 = d_0(\Omega) > 0$  and  $\Omega_{d_0} = \{x \in \Omega / d(x, \partial\Omega) > d_0\}$
- (b)  $\rho(\mathbf{x}) \geq 0$
- (c) either  $\rho(\mathbf{x}) > 0$  and  $\tilde{F}'(\mathbf{x}) = 0$ , or  $\rho(\mathbf{x}) = 0$  and  $\rho'(\mathbf{x}) = 0$ .

*Conversely, if  $\mathbf{x} \in \rho^+$  is a nondegenerate critical point of  $\tilde{F}$ , there exists for  $\varepsilon$  small enough a sequence of solutions to  $(P_\varepsilon)$  which blows up at  $x_1, \dots, x_k$  as  $\varepsilon$  goes to zero.*

*Assuming that zero is a regular value for  $\rho$ , (b) and (c) may be replaced by*

- (b')  $\rho(x) \geq \rho_0(\Omega) > 0$
- (c')  $\tilde{F}'(\mathbf{x}) = 0$ .

Moreover, if  $\rho(\mathbf{x}) > 0$ , the following estimates hold :

$$\frac{1}{(\lambda_i^\varepsilon)^{\frac{N-2}{2}}} \sim \gamma \Lambda_i(\mathbf{x}) \varepsilon^{1/2} ; J_\varepsilon(u_\varepsilon) = c_k + k\gamma_1 \varepsilon |\ln \varepsilon| + k\gamma_2 \varepsilon + 2\gamma_1 \varepsilon \tilde{F}(\mathbf{x}) + o(\varepsilon)$$

$\gamma > 0, \gamma_1 > 0, \gamma_2$  are constants which depend on  $N$  only.

Using these results it is showed, denoting by

$$J_\varepsilon^c = \{u \in H_0^1(\Omega) / J_\varepsilon(u) < c\}$$

the level sets of  $J_\varepsilon$  :

**Theorem 2** Assume that  $N \geq 4$ , and zero is a regular value of  $\rho$ . The contribution to the relative homology

$$H_* (J_\varepsilon^{c_k+\eta}, J_\varepsilon^{c_k-\eta})$$

of the solutions to  $(P_\varepsilon)$  which blow up  $k$  times as  $\varepsilon$  goes to zero,  $0 < \eta < \frac{S^{N/2}}{N}$ , is equal for  $\varepsilon$  small enough to

$$H_* \left( (\Omega^k, \rho^-) \times_{\sigma_k} (D^k, S^{k-1}) \right)$$

with

$$\rho^- = \{\mathbf{x} \in \Omega^k / \rho(\mathbf{x}) \leq 0\} .$$

As the result does not depend on  $\varepsilon$  for  $\varepsilon$  sufficiently small, intercalating the level sets of  $J_0$  between the level sets of  $J_\varepsilon$ , it provides us with the contribution of the critical points at infinity to the relative homology between the level sets of  $J_0$ .

It is to be noticed that the relative topology  $(\Omega^k, \rho^-)$  has been computed in the particular case of thin or expanding annuli-domains [13] [14]. For large enough annuli, this difference turns out to be zero for  $k \geq 2$ , whereas for thin enough annuli, this difference is nonzero, at least for  $k = 2$ . This indicates that some solutions of Yamabe-type equations may transform into critical points at infinity or, conversely, be produced by bifurcation from infinity (an example of such a phenomenon is given in [16]), and this possible link shows that a careful study of the critical points at infinity is necessary to get a good understanding in this kind of problems.

The aim of the paper is to show that the results of [3] are true in dimension 3, namely :



**Theorem 3** *Conclusions of Theorem 1 and 2 hold for  $N = 3$ .*

In fact, the only thing we have to prove is that the statements of Theorem 1 hold in dimension 3, since the statement of Theorem 2 follows exactly in the same way as in [3, Section 5] (for  $N \geq 4$ , one can also use the Morse lemma at infinity established in [4]). Concerning Theorem 1, the strategy is the same as in higher dimensions, but we need a more careful analysis of the properties of the  $v$ -part of  $u$  which occurs in (2).

The next section is devoted to the general setting, which will allow us to prove, in Section 3, that the conclusions of Theorem 1 hold in dimension 3. Section 4 provides us with the estimates required to justify the previous arguments.

**Remark** The extension to dimension 3 of results previously known in higher dimensions relies on improved estimates of some integral quantities. Using the same techniques we can also extend to dimension 4 the conclusion of Theorem 2 in [17], previously known for  $N \geq 5$ , namely :

**Theorem 4** *Assume that  $N \geq 4$ , and  $x_0$  is a nondegenerate critical point of  $\varphi(x) = H(x, x)$ . For  $\varepsilon > 0$  small enough, problem*

$$(P'_\varepsilon) \begin{cases} -\Delta u = u^p + \varepsilon u, & u > 0 & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega \end{cases}$$

*has a solution  $u_\varepsilon$  which blows up at  $x_0$  as  $\varepsilon$  goes to zero, that is  $|\nabla u_\varepsilon|^2 \xrightarrow{\varepsilon \rightarrow 0} S^{N/2} \delta_{x_0}$ .*

In this case, the assumption  $N \geq 4$  is sharp, since in dimension 3  $(P'_\varepsilon)$  has no solution for small  $\varepsilon$  when  $\Omega$  is starshaped [5].

## 2 The setting

From now on, we assume that  $N = 3$ . For  $a > 0$ , we define the subset of  $H_0^1(\Omega)$

$$F_a = \left\{ \alpha \sum_{i=1}^k P\delta_{\lambda_i, x_i} / \lambda d(x_i, \partial\Omega) > \frac{1}{a}, \forall i; \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j|^2 > \frac{1}{a}, \forall i, j, i \neq j \right\}$$

where  $\bar{\alpha} (= 3^{1/4})$  and  $P\delta_{\lambda, x}$  are defined in Section 1. According to (3) (4), we may write

$$(15) \quad P\delta_{\lambda, x} = \delta_{\lambda, x} - \varphi_{\lambda, x}$$

with

$$(16) \quad \Delta\varphi_{\lambda,x} = 0 \text{ in } \Omega; \varphi_{\lambda,x} = \delta_{\lambda,x} \text{ on } \partial\Omega$$

and from the maximum principle we deduce :

$$(17) \quad \varphi_{\lambda,x}(y) = \frac{1}{\lambda^{1/2}}H(x,y) + 0 \left( \frac{1}{\lambda^{3/2}d(x,\partial\Omega)} \right).$$

$(u_\varepsilon)$  being a sequence of solutions to  $(P_\varepsilon)$  which blows up at  $k$  points (not necessarily distinct) of  $\bar{\Omega}$  as  $\varepsilon$  goes to zero, it follows from Section 1 that the distance in  $H_0^1(\Omega)$ -norm between  $u_\varepsilon$  and  $F_a$  goes to zero with  $\varepsilon$ . Since we know from [2, Proposition 7] that, provided that  $dist_{H_0^1(\Omega)}(u, F_a) < a$ , with  $a$  sufficiently small, the problem

$$\text{Minimize } |u - \sum_{i=1}^k \alpha_i P\delta_{\lambda_i, x_i}|_{H_0^1} \text{ with respect to the } \alpha_i, \lambda_i, x_i \text{'s}$$

has a unique solution in the open set

$$|\alpha_i - \bar{\alpha}| < 4a, \lambda_i d(x_i, \partial\Omega) > \frac{1}{4a}, \forall i; \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j|^2 > \frac{1}{4a}, \forall i, j, i \neq j,$$

we can easily prove the existence of a diffeomorphism between a neighbourhood of the eventual solutions to  $(P_\varepsilon)$  which concentrate  $k$  times as  $\varepsilon$  goes to zero and the open set

$$M = \left\{ m = (\alpha, \lambda, \mathbf{x}, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times \Omega^k \times H_0^1(\Omega) / \right.$$

$$\left. |\alpha_i - \bar{\alpha}| < \nu_0, \lambda_i d(x_i, \partial\Omega) > \frac{1}{\nu_0}, \forall i; \right.$$

$$\left. \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j|^2 > \frac{1}{\nu_0}, \forall i, j, i \neq j; v \in E_{\lambda, \mathbf{x}}, |v|_{H_0^1} < \nu_0 \right\}$$

with  $\nu_0 > 0$  some suitable constant, and (here and throughout the sequel,  $P\delta_i$  denotes  $P\delta_{\lambda_i, x_i}$ )

$$(18) \quad E_{\lambda, \mathbf{x}} = \left\{ v \in H_0^1(\Omega) / \langle v, P\delta_i \rangle_{H_0^1} = \langle v, \frac{\partial P\delta_i}{\partial \lambda_i} \rangle_{H_0^1} = \langle v, \frac{\partial P\delta_i}{\partial (x_i)_j} \rangle_{H_0^1} = 0, \right. \\ \left. 1 \leq i \leq k, 1 \leq j \leq 3 \right\}.$$

We know that  $u_\varepsilon$ , solution to  $(P_\varepsilon)$  blowing up at  $k$  points as  $\varepsilon$  goes to zero, may be written as

$$(19) \quad u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P \delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + v^\varepsilon$$

for  $\varepsilon$  small enough, with  $m^\varepsilon = (\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon) \in M$  and  $\alpha_i^\varepsilon \rightarrow \bar{\alpha}$ ,

$$\lambda_i^\varepsilon d(x_i^\varepsilon, \partial\Omega) \rightarrow +\infty, \quad \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon \lambda_j^\varepsilon |x_i^\varepsilon - x_j^\varepsilon|^2 \rightarrow +\infty, \quad v^\varepsilon \rightarrow 0.$$

Moreover, it follows from an argument by Z.C. Han [9] that the concentration points cannot approach the boundary, that is there exists  $d_0 = d_0(\Omega) > 0$  such that for  $\varepsilon$  small enough

$$d(x_i^\varepsilon, \partial\Omega) > d_0 \quad 1 \leq i \leq k.$$

Schoen [19] also showed that there exist  $d'_0(\Omega) > 0$  and  $c_0(\Omega) > 0$  such that for  $\varepsilon$  small enough

$$|x_i^\varepsilon - x_j^\varepsilon| > d'_0, \quad \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} < c_0 \quad 1 \leq i, j \leq k, \quad i \neq j.$$

A proof of the first property follows from Y.Y. Li's work about the scalar curvature problem [11], showing that isolated blow up points are in fact isolated simple blow up points. The second property, concerning the concentration parameters, makes the computations easier, but could be recovered by our own analysis later.

Lastly, multiplying the equation  $-\Delta u_\varepsilon = u_\varepsilon^{p-\varepsilon}$  by  $P \delta_{\lambda_i^\varepsilon, x_i^\varepsilon}$  and integrating on  $\Omega$ , we obtain, using the expansion (19), together with the listed properties of  $\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon$  and integral estimates in [18]

$$C + o(1) = \frac{C}{(\lambda_i^\varepsilon)^{1/2}} + o(1)$$

with  $C = \bar{\alpha} \int_{\mathbb{R}^3} |\nabla \delta_{\lambda, x}|^2 = 3^{-1/4} S^{3/2}$ . Therefore

$$\varepsilon \ln \lambda_i^\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad 1 \leq i \leq k.$$

Collecting these informations, we conclude that there exists a diffeomorphism between a neighbourhood of the eventual solutions to  $(P_\varepsilon)$  with  $k$  peaks and the open set

$$M_\varepsilon = \left\{ m = (\alpha, \lambda, \mathbf{x}, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times \Omega_{d_0}^k \times H_0^1(\Omega) / \right. \\ \left. |\alpha_i - \alpha| < \nu_0, \lambda_i > \frac{1}{\nu_0}, \varepsilon \ell n \lambda_i < \nu_0, 1 \leq i \leq k; \right. \\ \left. \frac{\lambda_i}{\lambda_j} < c_0, |x_i - x_j| > d'_0, 1 \leq i, j \leq k, i \neq j; v \in E_{\lambda, \mathbf{x}}, |v|_{H_0^1} < \nu_0 \right\}$$

$\nu_0, c_0, d_0, d'_0$  being some strictly positive constants suitably chosen. Defining on  $M_\varepsilon$  the functional

$$(20) \quad K_\varepsilon : \quad M_\varepsilon \quad \rightarrow \mathbb{R} \\ m = (\alpha, \lambda, \mathbf{x}, v) \quad \mapsto J_\varepsilon(\sum_{i=1}^k \alpha_i P \delta_{\lambda_i, x_i} + v)$$

we have

**Proposition 1**  *$m = (\alpha, \lambda, \mathbf{x}, v) \in M_\varepsilon$  is a critical point of  $K_\varepsilon$  if and only if  $u = \sum_{i=1}^k \alpha_i P \delta_{\lambda_i, x_i} + v$  is a critical point of  $J_\varepsilon$ . This means that there exists  $(A, B, C, ) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$  such that*

$$(E) \quad \left\{ \begin{array}{l} (E_{\alpha_i}) \quad \frac{\partial K_\varepsilon}{\partial \alpha_i} = 0 \quad 1 \leq i \leq k \\ (E_{\lambda_i}) \quad \frac{\partial K_\varepsilon}{\partial \lambda_i} = B_i \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i^2}, v \rangle_{H_0^1} + \sum_{j=1}^3 C_{ij} \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j}, v \rangle_{H_0^1} \quad 1 \leq i \leq k \\ (E_{(x_i)_j}) \quad \frac{\partial K_\varepsilon}{\partial (x_i)_j} = B_i \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j}, v \rangle_{H_0^1} + \sum_{\ell=1}^3 C_{i\ell} \langle \frac{\partial^2 P \delta_i}{\partial (x_i)_j \partial (x_i)_\ell}, v \rangle_{H_0^1} \quad \begin{matrix} 1 \leq i \leq k \\ 1 \leq j \leq 3 \end{matrix} \\ (E_v) \quad \frac{\partial K_\varepsilon}{\partial v} = \sum_{i=1}^k \left( A_i P \delta_i + B_i \frac{\partial P \delta_i}{\partial \lambda_i} + \sum_{j=1}^3 C_{ij} \frac{\partial P \delta_i}{\partial (x_i)_j} \right) . \end{array} \right.$$

### 3 Proof of Theorem 1 for $N = 3$

#### 3.1 The $v$ -part of $u$

We first look at the last equation of (E), concerning the derivative of  $K_\varepsilon$  with respect to  $v$ . The result is the following :

**Proposition 2** *There exists a smooth map which to any  $(\varepsilon, \alpha, \lambda, \mathbf{x})$  such that  $\varepsilon$  is small enough and  $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$ , associates  $\bar{v} \in E_{\lambda, \mathbf{x}}, |\bar{v}|_{H_0^1} < \nu_0$ , such that  $(E_v)$  is satisfied for some  $(A, B, C) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ . Such a  $\bar{v}$  is*

unique, minimizes  $K_\varepsilon(\alpha, \lambda, \mathbf{x}, v)$  with respect to  $v$  in  $\{v \in E_{\lambda, \mathbf{x}} / |v|_{H_0^1} < \nu_0\}$ , and we have the estimates

$$(21) \quad |\bar{v}|_{H_0^1} = 0\left(\varepsilon + \frac{1}{\lambda}\right)$$

$$(22) \quad \begin{cases} A &= 0(|\beta| + \frac{1}{\lambda} + \varepsilon \ln \lambda + \varepsilon^2) \\ B &= 0(1 + \varepsilon \lambda) \\ C &= 0(\frac{1}{\lambda^3} + \frac{\varepsilon^2}{\lambda}) \end{cases}$$

with

$$(23) \quad \beta = (\beta_1, \dots, \beta_k) = (\bar{\alpha} - \alpha_1, \dots, \bar{\alpha} - \alpha_k).$$

For sake of simplicity,  $\frac{\lambda_i}{\lambda_j}$  being bounded for any  $i, j$ ,  $0(f(\lambda))$  denotes any quantity dominated by  $\sum_{i=1}^k f(\lambda_i)$ .

We sketch the proof of Proposition 2. Expanding  $K_\varepsilon$  with respect to  $v$ , we obtain

$$\begin{aligned} K_\varepsilon(\alpha, \lambda, \mathbf{x}, v) &= K_\varepsilon(\alpha, \lambda, \mathbf{x}, 0) - \int_{\Omega} \left(\sum_{i=1}^k \alpha_i P \delta_i\right)^{5-\varepsilon} v + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \\ &\quad - \frac{5-\varepsilon}{2} \int_{\Omega} \left(\sum_{i=1}^k \alpha_i P \delta_i\right)^4 v^2 + R_{\varepsilon, \alpha, \lambda, \mathbf{x}}(v) \end{aligned}$$

with

$$R_{\varepsilon, \alpha, \lambda, \mathbf{x}}(v) = 0(|v|_{H_0^1}^3), R'_{\varepsilon, \alpha, \lambda, \mathbf{x}}(v) = 0(|v|_{H_0^1}^2), R''_{\varepsilon, \alpha, \lambda, \mathbf{x}}(v) = 0(|v|_{H_0^1})$$

uniformly with respect to  $\varepsilon, \alpha, \lambda, \mathbf{x}$ ,  $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$  and  $\varepsilon$  small enough. Moreover, we know that the quadratic term in  $v$  is coercive, with a modulus of coercivity bounded from below as  $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$  and  $\varepsilon$  is sufficiently small - for a proof of this fact, see [1][17][18]. We claim that

$$(24) \quad \int_{\Omega} \left(\sum_{i=1}^k \alpha_i P \delta_i\right)^{5-\varepsilon} v = 0 \left( \left(\varepsilon + \frac{1}{\lambda}\right) |v|_{H_0^1} \right).$$

Consequently, the implicit function theorem yields the conclusion of Proposition 2, together with estimate (21). Let us prove the claim. Since  $v \in E_{\lambda, \mathbf{x}}$

and  $P\delta_i$  satisfies (6), we may write

$$\begin{aligned} \int_{\Omega} \left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} v &= \int_{\Omega} \left( \left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right) v \\ &= 0 \left( \left| \left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|_{\frac{6}{5}} |v|_{H_0^1} \right) \end{aligned}$$

using Hölder's inequality and Sobolev embedding theorem. Clearly, far from the concentration points  $x_i$ , say outside  $\bigcup_{i=1}^k B(x_i, d)$ , with  $d = \min(d_0, \frac{d_0'}{2})$ ,

$$\delta_i, P\delta_i = 0(\frac{1}{\lambda^{1/2}}), \text{ and } \left| \left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right| = 0(\frac{1}{\lambda^{5/2}}).$$

On  $B_j = B(x_j, d)$ , we write

$$\left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 = \alpha_j^{5-\varepsilon} \left( \delta_j^{5-\varepsilon} - \frac{\delta_j^5}{\lambda_j^{\varepsilon/2}} \right) + 0(\frac{\delta_j^4}{\lambda^{1/2}})$$

since in this subdomain  $P\delta_i, \varphi_j (= \varphi_{\lambda_j, x_j}) = 0(\frac{1}{\lambda^{1/2}}), i \neq j$ , and  $\delta_j \geq \frac{C}{\lambda^{1/2}}$  for some strictly positive constant  $C$ . (We note that,  $\varepsilon \ln \lambda_i$  being small, a quantity as  $\delta_i^\varepsilon$  is close to 1 uniformly in  $\Omega$ , for any  $i$ .) We remark that

$$\begin{aligned} &\left( \delta_j^{5-\varepsilon} - \frac{1}{\lambda_j^{\varepsilon/2}} \right) (x) \\ &= \exp(-\varepsilon \ln \delta_j(x)) - \exp(-\frac{\varepsilon}{2} \ln \lambda_j) \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n \varepsilon^n}{2^n n!} ((\ln \lambda_j - \ln(1 + \lambda_j^2 |x - x_j|^2))^n - (\ln \lambda_j)^n) \\ &= 0 \left( \sum_{n=1}^{+\infty} \frac{\varepsilon^n}{n!} ((\ln \lambda_j)^{n-1} \ln(1 + \lambda_j^2 |x - x_j|^2) + (\ln(1 + \lambda_j^2 |x - x_j|^2))^n) \right) \\ &= 0(\varepsilon \ln(1 + \lambda_j^2 |x - x_j|^2)). \end{aligned}$$

Therefore, on  $B_j$

$$\left| \left( \sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|_{\frac{6}{5}} = 0 \left( \varepsilon^{6/5} (\ln(1 + \lambda_j^2 |x - x_j|^2))^{6/5} \delta_j^6 + \frac{\delta_j^{24/5}}{\lambda^{3/5}} \right)$$

As we have, performing the change of variable  $y = (x - x_j)$  and using spherical coordinates

$$\int_{B_j} (\ln(1 + \lambda_j^2 |x - x_j|^2))^{6/5} \delta_j^6 dx = 4\pi \int_0^{\lambda_j d} (\ln(1 + r^2))^{6/5} \frac{r^2 dr}{(1 + r^2)^3} = 0(1)$$

and

$$\int_{B_j} \delta_j^{24/5} dx = \frac{4\pi}{\lambda_j^{3/5}} \int_0^{\lambda_j d} \frac{r^2 dr}{(1 + r^2)^{12/5}} = 0\left(\frac{1}{\lambda_j^{3/5}}\right)$$

we finally obtain

$$\int_{\Omega} \left| \left( \sum_{i=1}^k \alpha_i P \delta_i \right)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|^{\frac{6}{5}} = 0(\varepsilon^{6/5} + \frac{1}{\lambda^{6/5}})$$

hence the claim.

To complete the proof of Proposition 2, it only remains to show that estimate (22) holds. We proceed as follows : we take the scalar product in  $H_0^1(\Omega)$  of  $(E_v)$  with  $P\delta_i$ ,  $\frac{\partial P\delta_i}{\partial \lambda_i}$ ,  $\frac{\partial P\delta_i}{\partial (x_i)_j}$  respectively,  $1 \leq i \leq k$ ,  $1 \leq j \leq 3$ . On the right hand side, we get a linear system involving the quantities  $A_i, B_i, C_{ij}$ , which is nearly diagonal, invertible, and whose coefficients are given by

$$(25) \quad \begin{cases} \int_{\Omega} \nabla P \delta_i \cdot \nabla P \delta_j = \Gamma_1 \delta_{ij} + 0\left(\frac{1}{\lambda}\right) \\ \int_{\Omega} \nabla \frac{\partial P \delta_i}{\partial \lambda_i} \cdot \nabla \frac{\partial P \delta_j}{\partial \lambda_j} = \frac{\Gamma_2}{\lambda_i^2} \delta_{ij} + 0\left(\frac{1}{\lambda^3}\right) \\ \int_{\Omega} \nabla \frac{\partial P \delta_i}{\partial (x_i)_a} \cdot \nabla \frac{\partial P \delta_j}{\partial (x_j)_b} = \Gamma_3 \lambda_i^2 \delta_{ij} \delta_{ab} + 0\left(\frac{1}{\lambda}\right) \end{cases}$$

$\delta_{ij}, \delta_{ab}$  denoting the Kronecker symbol and  $\Gamma_1, \Gamma_2, \Gamma_3$  being strictly positive constants, and

$$(26) \quad \begin{cases} \int_{\Omega} \nabla P \delta_i \cdot \nabla \frac{\partial P \delta_j}{\partial \lambda_j} = 0\left(\frac{1}{\lambda^2}\right); \\ \int_{\Omega} \nabla P \delta_i \cdot \nabla \frac{\partial P \delta_j}{\partial (x_j)_a} = 0\left(\frac{1}{\lambda}\right); \\ \int_{\Omega} \nabla \frac{\partial P \delta_i}{\partial \lambda_i} \cdot \nabla \frac{\partial P \delta_j}{\partial (x_j)_a} = 0\left(\frac{1}{\lambda^2}\right). \end{cases}$$

These estimates follow easily from (3)(4)(17), and may also be found in [1][17]. On the left hand side we find

$$\left\langle \frac{\partial K_{\varepsilon}}{\partial v}, P \delta_i \right\rangle_{H_0^1} = \frac{\partial K_{\varepsilon}}{\partial \alpha_i}; \left\langle \frac{\partial K_{\varepsilon}}{\partial v}, \frac{\partial P \delta_i}{\partial \lambda_i} \right\rangle_{H_0^1} = \frac{1}{\alpha_i} \frac{\partial K_{\varepsilon}}{\partial \lambda_i}; \left\langle \frac{\partial K_{\varepsilon}}{\partial v}, \frac{\partial P \delta_i}{\partial (x_i)_j} \right\rangle_{H_0^1} = \frac{1}{\alpha_i} \frac{\partial K_{\varepsilon}}{\partial (x_i)_j}$$

and we have :

**Proposition 3** For  $\varepsilon$  small enough and  $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$ , the following estimates hold

$$(27) \quad \frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, \mathbf{x}, \bar{v}) = -K_1 \beta_i + V_{\alpha_i}(\varepsilon, \alpha, \lambda, \mathbf{x})$$

with  $V_{\alpha_i}$  a smooth function which satisfies

$$V_{\alpha_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0(\beta_i^2 + \frac{1}{\lambda} + \varepsilon \ell n \lambda + \varepsilon^2);$$

$$(28) \quad \frac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha, \lambda, \mathbf{x}, \bar{v}) = \frac{1}{\lambda_i} \left( K_2 \varepsilon - K_3 \left( \frac{H(x_i, x_i)}{\lambda_i} - \sum_{\ell \neq i} \frac{G(x_i, x_\ell)}{\lambda_i^{1/2} \lambda_\ell^{1/2}} \right) \right) + V_{\lambda_i}(\varepsilon, \alpha, \lambda, \mathbf{x})$$

with  $V_{\lambda_i}$  a smooth function which satisfies

$$V_{\lambda_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0 \left( \frac{\ell n \lambda}{\lambda^4} + \varepsilon^2 \frac{\ell n \lambda}{\lambda} + |\beta| \left( \frac{\varepsilon}{\lambda} + \frac{1}{\lambda^2} \right) \right);$$

$$(29) \quad \frac{\partial K_\varepsilon}{\partial (x_i)_j}(\alpha, \lambda, \mathbf{x}, \bar{v}) = K_4 \left( \frac{1}{\lambda_i} \frac{\partial H}{\partial a_j}(x_i, x_i) - \sum_{\ell \neq i} \frac{1}{\lambda_i^{1/2} \lambda_j^{1/2}} \frac{\partial G}{\partial a_j}(x_i, x_\ell) \right) + V_{(x_i)_j}(\varepsilon, \alpha, \lambda, \mathbf{x})$$

where  $\frac{\partial}{\partial a_j}$  (resp.  $\frac{\partial}{\partial b_j}$ ) denotes the derivative with respect to the  $j$ -th component of the first (resp. second) variable, and  $V_{(x_i)_j}$  is a smooth function which satisfies

$$V_{(x_i)_j}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0 \left( \frac{1}{\lambda^{3/2}} + \lambda^{1/2} \varepsilon^2 + \frac{|\beta|}{\lambda} \right);$$

$K_1, K_2, K_3, K_4$  are strictly positive constants.

The proof of this proposition is delayed until the next section.

Proof of Proposition 2 completed. From Proposition 3 we deduce that

$$(30) \quad \begin{cases} \frac{\partial K_\varepsilon}{\partial \alpha_i} &= 0(|\beta| + \frac{1}{\lambda} + \varepsilon \ell n \lambda + \varepsilon^2) \\ \frac{\partial K_\varepsilon}{\partial \lambda_i} &= 0 \left( \frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda} \right) \\ \frac{\partial K_\varepsilon}{\partial (x_i)_j} &= 0 \left( \frac{1}{\lambda} + \lambda \varepsilon^2 \right) \end{cases}$$

hence (22), using (25)(26) to invert the linear system involving  $A_i, B_i, C_{ij}$ .

Once  $(E_v)$  is solved, we are left with a finite dimensional system of equations  $(E_{\alpha_i}), (E_{\lambda_i}), (E_{(x_i)_j}), 1 \leq i \leq k, 1 \leq j \leq 3$ , whose left hand side is given



by Proposition 3, and whose right hand side may be estimated using (22), namely

$$\begin{aligned} B_i \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i^2}, \bar{v} \rangle_{H_0^1} + \sum_{j=1}^3 C_{ij} \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j}, \bar{v} \rangle_{H_0^1} &= 0 \left( \left( \frac{|B_i|}{\lambda} + \sum_{j=1}^3 |C_{ij}| \right) |\bar{v}|_{H_0^1} \right) \\ &= 0 \left( \left( \frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda} \right) |\bar{v}|_{H_0^1} \right) \end{aligned}$$

since, as straight forward computations show

$$(31) \quad \left| \frac{\partial^2 P \delta_i}{\partial \lambda_i^2} \right|_{H_0^1} = 0\left(\frac{1}{\lambda^2}\right) \quad \left| \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j} \right|_{H_0^1} = 0(1);$$

in the same way

$$\begin{aligned} B_i \langle \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j}, \bar{v} \rangle_{H_0^1} + \sum_{\ell=1}^3 C_{i\ell} \langle \frac{\partial^2 P \delta_i}{\partial (x_i)_j \partial (x_i)_\ell}, \bar{v} \rangle_{H_0^1} &= 0 \left( \left( \frac{|B_i|}{\lambda^{1/2}} + \sum_{\ell=1}^3 \lambda^2 |C_{i\ell}| \right) |\bar{v}|_{H_0^1} \right) \\ &= 0 \left( \left( \frac{1}{\lambda^{1/2}} + \varepsilon \lambda^{1/2} \right) |\bar{v}|_{H_0^1} \right) \end{aligned}$$

since

$$(32) \quad \left| \frac{\partial^2 P \delta_i}{\partial (x_i)_j \partial (x_i)_\ell} \right|_{H_0^1} = 0(\lambda^2)$$

and

$$(33) \quad \left\langle \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (x_i)_j}, \bar{v} \right\rangle_{H_0^1} = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}\right)$$

(32) follows from an explicit computation, whereas (33) is more subtle and is proved in the next section.

These estimates, together with Proposition 2 and 3, show that with  $v = \bar{v}$ , (E) is equivalent to a new system (E')

$$(E') \begin{cases} \beta_i = \tilde{V}_{\alpha_i}(\varepsilon, \alpha, \lambda, x) \\ K_2 \varepsilon - K_\varepsilon \left( \frac{H(x_i, x_i)}{\lambda_i} - \sum_{\ell \neq i} \frac{G(x_i, x_\ell)}{\lambda_i^{1/2} \lambda_\ell^{1/2}} \right) = \tilde{V}_{\lambda_i}(\varepsilon, \alpha, \lambda, x) \\ \frac{1}{\lambda^{1/2}} \frac{\partial H}{\partial a_j}(x_i, x_i) - \sum_{\ell \neq i} \frac{1}{\lambda_j^{1/2}} \frac{\partial G}{\partial a_j}(x_i, x_\ell) = \tilde{V}_{(x_i)_j}(\varepsilon, \alpha, \lambda, x) \end{cases}$$

$$1 \leq i \leq k \quad 1 \leq j \leq 3$$

where  $\tilde{V}_{\alpha_i}, \tilde{V}_{\lambda_i}, \tilde{V}_{(x_i)_j}$  are smooth functions which satisfy

$$\begin{cases} \tilde{V}_{\alpha_i}(\varepsilon, \alpha, \lambda, x) &= 0(|\beta|^2 + \frac{1}{\lambda} + \varepsilon \ell n \lambda + \varepsilon^2) \\ \tilde{V}_{\lambda_i}(\varepsilon, \alpha, \lambda, x) &= 0\left(\frac{\ell n \lambda}{\lambda^3} + \varepsilon^2 \ell n \lambda + |\beta|(\varepsilon + \frac{1}{\lambda})\right) \\ \tilde{V}_{(x_i)_j}(\varepsilon, \alpha, \lambda, x) &= 0\left(\frac{1}{\lambda} + \varepsilon^2 \lambda + \frac{|\beta|}{\lambda^{1/2}}\right) \end{cases}$$

We are now able to prove Theorem 1 for  $N = 3$ .

### 3.2 Proof of the first part of Theorem 1

The argument is quite similar to what happens for  $n \geq 4$ , we repeat it for convenience of the reader.  $u_\varepsilon$ , satisfying the assumptions of Theorem 1, writes as (19), with  $m^\varepsilon = (\alpha^\varepsilon, \lambda^\varepsilon, \mathbf{x}^\varepsilon, v^\varepsilon)$  a solution of (E). As  $v^\varepsilon$  goes to zero in  $H_0^1(\Omega)$ , it follows from Proposition 2 that  $v^\varepsilon = \bar{v}^\varepsilon$  for  $\varepsilon$  small enough. According to the previous subsection,  $(E_{\alpha_i})$  implies that

$$(34) \quad \beta_i^\varepsilon = 0 \left( \frac{1}{\lambda^\varepsilon} + \varepsilon \ell n \lambda^\varepsilon + \varepsilon^2 \right)$$

with  $\beta_i^\varepsilon = \bar{\alpha} - \alpha_i^\varepsilon$ . From  $(E_{\lambda_i})$  we deduce that

$$K_2 \varepsilon - K_3 \left( \frac{H(x_i^\varepsilon, x_i^\varepsilon)}{\lambda_i^\varepsilon} - \sum_{\ell \neq i} \frac{G(x_i^\varepsilon, x_j^\varepsilon)}{(\lambda_i^\varepsilon)^{1/2} (\lambda_\ell^\varepsilon)^{1/2}} \right) = 0 \left( \frac{1}{(\lambda^\varepsilon)^2} + \frac{\varepsilon \ell n \lambda^\varepsilon}{\lambda^\varepsilon} + \varepsilon^2 \ell n \lambda^\varepsilon \right).$$

Setting

$$(35) \quad \frac{1}{\lambda_i} = \frac{K_2}{K_3} \Lambda_i^2 \varepsilon \quad \Lambda_i > 0$$

we notice that, since

$$\lambda_i^\varepsilon \rightarrow +\infty \quad \varepsilon \ell n \lambda_i^\varepsilon \rightarrow 0 \quad \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} < c_0$$

we have

$$(\Lambda_i^\varepsilon)^2 \varepsilon \rightarrow 0 \quad \varepsilon \ell n \Lambda_i^\varepsilon \rightarrow 0 \quad \frac{\Lambda_i^\varepsilon}{\Lambda_j^\varepsilon} < c_0$$

and  $(E_{\lambda_i})$  provides us with the equation

$$(36) \quad 1 - H(x_i^\varepsilon, x_i^\varepsilon) (\Lambda_i^\varepsilon)^2 + \sum_{\ell \neq i} G(x_i^\varepsilon, x_\ell^\varepsilon) \Lambda_i^\varepsilon \Lambda_\ell^\varepsilon = o(|\Lambda^\varepsilon|^2 + 1).$$

Dividing each of these equations,  $1 \leq i \leq k$ , by  $\Lambda_i^\varepsilon$  respectively, we obtain

$$(37) \quad M(x^\varepsilon)^t \Lambda^\varepsilon + o(|\Lambda^\varepsilon|) = {}^t \left( \frac{1}{\Lambda^\varepsilon} \right) + o\left( \frac{1}{|\Lambda^\varepsilon|} \right)$$

with

$$\Lambda = (\Lambda_1, \dots, \Lambda_k) \quad \frac{1}{\Lambda} = \left( \frac{1}{\Lambda_1}, \dots, \frac{1}{\Lambda_k} \right).$$

The scalar product of (37) with  $r(x^\varepsilon)$ , the unique unit vector with all its components strictly positive associated to the least, simple, eigenvalue of

$M(x^\varepsilon)$  (for a proof of the simplicity of  $\rho$  and that the components of an associated eigenvector have the same sign, see [3, Appendix A]), leads to

$$(38) \quad \rho(\mathbf{x}^\varepsilon) r(\mathbf{x}^\varepsilon) \cdot \Lambda^\varepsilon + o(|\Lambda^\varepsilon|) = r(\mathbf{x}^\varepsilon) \cdot \left( \frac{1}{\Lambda^\varepsilon} \right) + o\left( \frac{1}{|\Lambda^\varepsilon|} \right)$$

From  $(E_{(x_i)_j})$  we deduce also

$$(39) \quad \frac{\partial H}{\partial a_j}(x_i^\varepsilon, x_i^\varepsilon) \Lambda_i^\varepsilon - \sum_{\ell \neq i} \frac{\partial G}{\partial a_j}(x_i^\varepsilon, x_\ell^\varepsilon) \Lambda_\ell^\varepsilon = 0 \left( \varepsilon^{1/2} (|\Lambda^\varepsilon|^2 + \frac{1}{|\Lambda^\varepsilon|^2}) \right)$$

Three cases may occur :

$$i) \Lambda^\varepsilon \rightarrow 0 \quad ii) \Lambda^\varepsilon \rightarrow \bar{\Lambda} \in (\mathbb{R}_+^*)^k \quad iii) \Lambda^\varepsilon \rightarrow +\infty (i.e. \Lambda_i^\varepsilon \rightarrow +\infty, \forall i) .$$

Actually, the first case is impossible, as (38) shows. Indeed, the left hand side would go to zero, since  $\rho$  is bounded from above on  $\Omega_{d_0}^k$ , and the right hand side would go to infinity. Considering the second case, we denote by  $\bar{\mathbf{x}} \in \Omega_{d_0}^k$  the limit (up to a subsequence) of  $(\mathbf{x}^\varepsilon)$ , and (38) yields

$$\rho(\bar{\mathbf{x}}) r(\bar{\mathbf{x}}) \cdot \bar{\Lambda} = r(\bar{\mathbf{x}}) \cdot \left( \frac{1}{\bar{\Lambda}} \right)$$

hence  $\rho(\mathbf{x})$  is strictly positive. Moreover, the limit in (37) provides us with

$$M(\bar{\mathbf{x}}) \bar{\Lambda} = \frac{1}{\bar{\Lambda}} .$$

This equality means that  $\bar{\Lambda}$  is a critical point of  $F_{\bar{\mathbf{x}}}$  in  $(\mathbb{R}_+^*)^k$ , i.e.  $\bar{\Lambda} = \Lambda(\bar{\mathbf{x}})$  according to notations of Section 1, and the limit in (39) yields

$$\frac{\partial M}{\partial x_i}(\bar{\mathbf{x}}) \Lambda(\bar{\mathbf{x}}) = 0$$

which implies, through (14)

$$\frac{\partial \tilde{F}}{\partial x_i}(\bar{\mathbf{x}}) = 0 .$$

Concerning the last case, it follows from (38) that  $\rho(\mathbf{x}^\varepsilon)$  goes to zero, and then  $\rho(\bar{\mathbf{x}}) = 0$ . From (37) we have

$$M(\mathbf{x}^\varepsilon) \Lambda^\varepsilon = o(|\Lambda^\varepsilon|)$$

so that  $\Lambda^\varepsilon$  writes as

$$(40) \quad \Lambda^\varepsilon = \gamma^\varepsilon r(\mathbf{x}^\varepsilon) + \bar{r}^\varepsilon$$

with  $\gamma^\varepsilon \rightarrow +\infty$ ,  ${}^t\bar{r}^\varepsilon \cdot r(\mathbf{x}^\varepsilon) = 0$ , and  $|\bar{r}^\varepsilon| = o(\gamma^\varepsilon)$ . Indeed, since  $\rho(\mathbf{x})$  is simple, the other eigenvalues are bounded from below far from zero as  $\rho(\mathbf{x}^\varepsilon)$  goes to zero, and (40) follows from a decomposition of  $\Lambda^\varepsilon$  in a basis of eigenvectors for  $M(\mathbf{x}^\varepsilon)$ . From (39) we know that

$$\frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon) {}^t\Lambda^\varepsilon = o(|\Lambda^\varepsilon|)$$

that is

$$\gamma^\varepsilon \frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon) {}^t r(\mathbf{x}^\varepsilon) + \frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon) {}^t \bar{r}^\varepsilon = o(\gamma^\varepsilon).$$

The components of the matrix  $\frac{\partial M}{\partial x_i}(\mathbf{x})$  are bounded on any set  $\{\mathbf{x} \in \Omega_{d_0}^k / \rho \geq \rho_0\}$ , for any  $\rho_0 \in \mathbb{R}$ . Therefore

$$(41) \quad \frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon) {}^t r(\mathbf{x}^\varepsilon) = o(1).$$

Derivating the equality  $M(\mathbf{x}) \cdot {}^t r(\mathbf{x}) = \rho(\mathbf{x}) {}^t r(\mathbf{x})$  with respect to  $x_i$ , we find

$$\frac{\partial M}{\partial x_i}(\mathbf{x}) {}^t r(\mathbf{x}) + M(\mathbf{x}) \frac{\partial {}^t r}{\partial x_i}(\mathbf{x}) = \frac{\partial \rho}{\partial x_i}(\mathbf{x}) {}^t r(\mathbf{x}) + \rho(\mathbf{x}) \frac{\partial {}^t r}{\partial x_i}(\mathbf{x}).$$

Taking the scalar product with  $r(\mathbf{x})$ , this yields

$$r(\mathbf{x}) \cdot \frac{\partial M}{\partial x_i}(\mathbf{x}) {}^t r(\mathbf{x}) = \frac{\partial \rho}{\partial x_i}(\mathbf{x})$$

and then (41) implies that

$$\frac{\partial \rho}{\partial x_i}(\bar{\mathbf{x}}) = 0.$$

The first part of Theorem 1 is proved. We note that 0 being assumed to be a regular value for  $\rho$ , case iii) cannot occur. Moreover, there exists  $\rho_0 > 0$  such that  $\rho(\bar{\mathbf{x}}) \geq \rho_0$ . Otherwise there would exist a sequence  $(\mathbf{x}^n)$  in  $\Omega_{d_0}^k$  such that  $\rho(\mathbf{x}^n) > 0$ ,  $\rho(\mathbf{x}^n) \rightarrow 0$  and  $\tilde{F}'(\mathbf{x}^n) = 0$ . Up to a subsequence, we may assume that  $\mathbf{x}^n$  converges to some limit  $\bar{\mathbf{x}}$  in  $\overline{\Omega_{d_0}^k}$ . It follows from the definition of  $\Lambda(\mathbf{x}^n)$  that

$$M(\mathbf{x}^n) {}^t \Lambda(\mathbf{x}^n) = {}^t \left( \frac{1}{\Lambda(\mathbf{x}^n)} \right)$$

The scalar product with  $r(\mathbf{x}^n)$  gives us the equality

$$\rho(\mathbf{x}^n)r(\mathbf{x}^n) \cdot \Lambda(\mathbf{x}^n) = r(\mathbf{x}^n) \cdot \left(\frac{1}{\Lambda}(\mathbf{x}^n)\right)$$

Then  $\rho(\mathbf{x}^n) \rightarrow 0$  implies that  $\Lambda(\mathbf{x}^n) \rightarrow +\infty$ , and the previous argument may be repeated, which shows that

$$\rho(\bar{\mathbf{x}}) = 0 \quad \rho'(\bar{\mathbf{x}}) = 0$$

a contradiction ; hence the announced result. The expansion of  $J_\varepsilon(u^\varepsilon)$  given in Theorem 1 follows from direct computations, using (21) to estimate the  $v$ -part.

### 3.3 The converse part of Theorem 1.

Let  $\bar{\mathbf{x}} \in \Omega^k$  such that  $\rho(\bar{\mathbf{x}}) > 0$  and  $\bar{\mathbf{x}}$  is a nondegenerate critical point of  $\tilde{F}$ . We perform the changes of variables

$$\frac{1}{\lambda_i} = \frac{K_2}{K_3}(\Lambda_i(\bar{\mathbf{x}}) + \zeta_i)^2 \varepsilon \quad x_i = \bar{x}_i + \xi_i$$

$\zeta_i \in \mathbb{R}$  and  $\xi_i \in \mathbb{R}^3$  being assumed to be small. With  $v = \bar{v}(\varepsilon, \alpha, \lambda, \mathbf{x})$ , the previous estimates show that solving (E) is equivalent to finding  $(\beta, \zeta, \xi) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$  such that

$$(42) \quad \beta_i = V_{\beta_i}(\varepsilon, \beta, \zeta, \xi)$$

$$(43) \quad \begin{aligned} & (2H(\bar{x}_i, \bar{x}_i) - \sum_{j \neq i} G(\bar{x}_i, \bar{x}_j) \bar{\Lambda}_j(\bar{\mathbf{x}})) \zeta_i - \sum_{j \neq i} G(\bar{x}_i, \bar{x}_j) \bar{\Lambda}_i(\bar{\mathbf{x}}) \zeta_j \\ & + \left( 2\bar{\Lambda}_i^2(\bar{\mathbf{x}}) \nabla_a H(\bar{x}_i, \bar{x}_i) - \sum_{j \neq i} \bar{\Lambda}_i(\bar{\mathbf{x}}) \bar{\Lambda}_j(\bar{\mathbf{x}}) \nabla_a G(\bar{x}_i, \bar{x}_j) \right) \cdot \xi_i \\ & - \sum_{j \neq i} \bar{\Lambda}_i(\bar{\mathbf{x}}) \bar{\Lambda}_j(\bar{\mathbf{x}}) \nabla_b G(\bar{x}_i, \bar{x}_j) \cdot \xi_j = V_{\zeta_i}(\varepsilon_1, \beta, \zeta, \xi) \end{aligned}$$

$$(44) \quad \begin{aligned} & \nabla_a H(\bar{x}_i, \bar{x}_i) \zeta_i - \sum_{j \neq i} \nabla_a G(\bar{x}_i, \bar{x}_j) \zeta_j + \left( \bar{\Lambda}_i(\bar{\mathbf{x}}) \nabla_{a^2}^2 H(\bar{x}_i, \bar{x}_i) \right. \\ & \left. + \bar{\Lambda}_i(\bar{\mathbf{x}}) \nabla_{ab}^2 H(\bar{x}_i, \bar{x}_i) - \sum_{j \neq i} \bar{\Lambda}_j(\bar{\mathbf{x}}) \nabla_{a^2}^2 G(\bar{x}_i, \bar{x}_j) \right) \cdot \xi_i \\ & - \sum_{j \neq i} \bar{\Lambda}_j(\bar{\mathbf{x}}) \nabla_{ab}^2 G(\bar{x}_i, \bar{x}_j) \cdot \xi_j = V_{\xi_i}(\varepsilon, \beta, \zeta, \xi) \end{aligned}$$

with  $V_{\beta_i}, V_{\zeta_i}, V_{\xi_i}$  smooth functions which satisfy

$$(45) \quad \begin{cases} V_{\beta_i} = 0(\varepsilon|\ell n\varepsilon| + |\beta|^2) \\ V_{\zeta_i} = 0(\varepsilon|\ell n\varepsilon| + |\beta| + |\zeta|^2 + |\xi|^2) \\ V_{\xi_i} = 0(\varepsilon^{1/2} + |\beta| + |\zeta|^2 + |\xi|^2) \end{cases}$$

Equations (42)(43)(44) may be written as

$$(46) \quad \begin{cases} \beta & = V(\varepsilon, \beta, \zeta, \xi) \\ L(\zeta, \xi) & = W(\varepsilon, \beta, \zeta, \xi) \end{cases}$$

$L$  being a fixed linear operator in  $\mathbb{R}^k \times (\mathbb{R}^3)^k$  and  $V, W$  smooth functions which satisfy

$$(47) \quad \begin{cases} V(\varepsilon, \beta, \zeta, \xi) = 0(\varepsilon|\ell n\varepsilon| + |\beta|^2) \\ W(\varepsilon, \beta, \zeta, \xi) = 0(\varepsilon^{1/2} + |\beta| + |\zeta|^2 + |\xi|^2) . \end{cases}$$

Moreover, the determinant of  $L$  is equal, up to a strictly positive number, to the determinant of  $\tilde{F}''(\bar{x})$ , as an easy computation shows.  $\bar{x}$  being assumed to be a nondegenerate critical point of  $\tilde{F}$ ,  $L$  is invertible, and Brouwer's fixed point theorem ensures, provided that  $\varepsilon$  is small enough, the existence of a solution  $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$  to (47), such that

$$|\beta^\varepsilon| = 0(\varepsilon|\ell n\varepsilon|) \quad |\zeta^\varepsilon| = 0(\varepsilon^{1/2}) \quad |\xi^\varepsilon| = 0(\varepsilon^{1/2}) .$$

By construction,  $u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P \delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + \bar{v}(\varepsilon, \alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon)$  with

$$\alpha_i^\varepsilon = \bar{\alpha} - \beta_i^\varepsilon \quad \frac{1}{\lambda_i^\varepsilon} = \frac{K_2}{K_3} (\Lambda_i(\bar{x}) + \zeta_i^\varepsilon)^2 \varepsilon \quad x_i^\varepsilon = \bar{x}_i + \xi_i^\varepsilon$$

is a critical point of  $J_\varepsilon$ , whence

$$-\Delta u_\varepsilon = |u_\varepsilon|^{4-\varepsilon} u_\varepsilon \text{ in } \Omega .$$

Multiplying this equation by  $u_\varepsilon^- = \max(0, -u_\varepsilon)$  and integrating on  $\Omega$ , one obtains

$$(48) \quad \int_{\Omega} |\nabla u_\varepsilon^-|^2 = \int_{\Omega} (u_\varepsilon^-)^{6-\varepsilon} .$$

From Sobolev embedding theorem we derive that

$$(49) \quad \left( \int_{\Omega} (u_\varepsilon^-)^{6-\varepsilon} \right)^{\frac{2}{6-\varepsilon}} \leq C \int_{\Omega} |\nabla u_\varepsilon^-|^2 .$$

(48)(49) show that either  $u_\varepsilon^- \equiv 0$ , or  $|u_\varepsilon^-|_{6-\varepsilon} \geq C^{-\frac{4-\varepsilon}{6-\varepsilon}} \geq C' > 0$ . We know that  $|u_\varepsilon^-|_{6-\varepsilon} \leq |\bar{v}^\varepsilon|_{6-\varepsilon}$  and  $\bar{v}^\varepsilon$  goes to zero in  $H_0^1(\Omega)$ . Therefore,  $u_\varepsilon^- \equiv 0$  for  $\varepsilon$  small enough, and  $u_\varepsilon$  satisfies

$$-\Delta u_\varepsilon = u_\varepsilon^{5-\varepsilon}, u_\varepsilon \geq 0 \text{ in } \Omega, u_\varepsilon = 0 \text{ on } \partial\Omega, u_\varepsilon \neq 0.$$

The strong maximum principle implies that  $u_\varepsilon > 0$  in  $\Omega$ , hence the desired result.

## 4 Proof of proposition 3

We concentrate our attention on formula (29), that is the derivative of the  $K_\varepsilon$  with respect to  $(x_i)_j$ . Formulas (27) and (28), concerning the derivatives of  $K_\varepsilon$  with respect to  $\alpha_i$  and  $\lambda_i$ , may be obtained in the same way, with easier computations which moreover do not differ with the case  $N \geq 4$ . The method that we developp will also allow us to prove estimate (33).

In view of the definition of  $K_\varepsilon$ , we have

$$\begin{aligned} \frac{1}{\alpha_i} \frac{\partial K_\varepsilon}{\partial (x_i)_j}(\alpha, \lambda, \mathbf{x}, \bar{v}) &= \int_\Omega \nabla \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right) \cdot \nabla \frac{\partial P \delta_i}{\partial (x_i)_j} \\ &\quad - \int_\Omega \left| \sum_{\ell=1}^k \alpha_\ell P \delta_\ell + \bar{v} \right|^{4-\varepsilon} \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell + \bar{v} \right) \frac{\partial P \delta_i}{\partial (x_i)_j}. \end{aligned}$$

This may be written as

$$\begin{aligned} (50) \quad \frac{1}{\alpha_i} \frac{\partial K_\varepsilon}{\partial (x_i)_j}(\alpha, \lambda, x, \bar{v}) &= \int_\Omega \nabla \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right) \cdot \nabla \frac{\partial P \delta_i}{\partial (x_i)_j} - \int_\Omega \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{5-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \\ &\quad - (5-\varepsilon) \int_\Omega \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{4-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \bar{v} - \frac{(-5-\varepsilon)(4-\varepsilon)}{2} \int_\Omega \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{3-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \bar{v}^2 \\ &\quad + 0 \left( \int_\Omega \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^2 \left| \frac{\partial P \delta_i}{\partial (x_i)_j} \right| |\bar{v}|^3 + \int_\Omega \left| \frac{\partial P \delta_i}{\partial (x_i)_j} \right| |\bar{v}|^{5-\varepsilon} \right). \end{aligned}$$

The terms where  $\bar{v}$  does not occur may be computed explicitly, using (3)(4)(17), and one finds - see the integral estimates in [1][17] :

$$\begin{aligned} (51) \quad &\int_\Omega \nabla \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right) \cdot \nabla \frac{\partial P \delta_i}{\partial (x_i)_j} - \int_\Omega \left( \sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{5-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \\ &= K_4 \left( \frac{1}{\lambda_i} \frac{\partial H}{\partial a_j}(x_i, x_i) - \sum_{\ell \neq i} \frac{1}{\lambda_i^{1/2} \lambda_\ell^{1/2}} \frac{\partial G}{\partial a_j}(x_i, x_\ell) \right) + 0 \left( \frac{1}{\lambda^2} + \frac{\varepsilon \ell n \lambda}{\lambda} + \frac{|\beta|}{\lambda} \right). \end{aligned}$$

It only remains to control the integrals involving  $\bar{v}$ . In particular, it is of crucial importance to obtain estimates of lower order than  $1/\lambda$  as  $\lambda$  goes to infinity, in order to keep the previous term as a dominating term. The two last integrals in (50) are easy to treat. Namely, using Hölder's inequality and Sobolev embedding theorem, we have

$$\int_{\Omega} \left| \frac{\partial P\delta_i}{\partial(x_i)_j} \right| |\bar{v}|^{5-\varepsilon} = 0 \left[ \left( \int_{\Omega} \left| \frac{\partial P\delta_i}{\partial(x_i)_j} \right|^6 \right)^{1/6} |\bar{v}|_{H_0^1}^{5-\varepsilon} \right].$$

We note that, according to (3)

$$(52) \quad \frac{\partial \delta_{\lambda,x}}{\partial x_j}(y) = \frac{\lambda^{5/2}(y-x)_j}{(1+\lambda^2|x-y|^2)^{3/2}} \quad y \in \mathbb{R}^3$$

and, according to (4)(15)(17) and the maximum principle

$$(53) \quad \frac{\partial \varphi_{\lambda,x}}{\partial x_j}(y) = 0 \left( \frac{1}{\lambda^{1/2}} \right) \text{ uniformly in } \Omega$$

$x$  being assumed to remain far from the boundary of  $\Omega$  ( $d(x, \partial\Omega) > d_0$ ). As a consequence, we have

$$(54) \quad \frac{\partial P\delta_i}{\partial x_i} = 0(\lambda_i \delta_i) \text{ uniformly in } \Omega.$$

$\int_{\mathbb{R}^N} \delta_{\lambda,x}^6(y) dy$  being a constant, depending on  $N$  only, it follows that

$$(55) \quad \int_{\Omega} \left| \frac{\partial P\delta_i}{\partial(x_i)_j} \right| |\bar{v}|^{5-\varepsilon} = 0(\lambda |\bar{v}|_{H_0^1}^{5-\varepsilon}).$$

In the same way, writing  $(\sum_{\ell=1}^k \alpha_{\ell} P\delta_{\ell})^2 \left| \frac{\partial P\delta_j}{\partial(x_i)_j} \right| = 0(\lambda_i \sum_{\ell=1}^k \delta_{\ell}^3)$ , we get

$$(56) \quad \int_{\Omega} \left( \sum_{\ell=1}^k \alpha_{\ell} P\delta_{\ell} \right)^2 \left| \frac{\partial P\delta_i}{\partial(x_i)_j} \right| \bar{v}^3 = 0(\lambda |\bar{v}|_{H_0^1}^3).$$

Let us now consider the linear and the quadratic terms in  $\bar{v}$ . Outside  $\bigcup_{\ell=1}^k B_{\ell}$ , with  $B_{\ell} = B(x_{\ell}, d)$ ,  $P\delta_{\ell} = 0(\frac{1}{\lambda^{1/2}})$  and  $\frac{\partial P\delta_i}{\partial(x_i)_j} = 0(\frac{1}{\lambda^{1/2}})$ . Therefore

$$\int_{\left(\bigcup_{\ell=1}^k B_{\ell}\right)^c} \left( \sum_{\ell=1}^k \alpha_{\ell} P\delta_{\ell} \right)^{4-\varepsilon} \frac{\partial P\delta_i}{\partial(x_i)_j} \bar{v} = 0\left(\frac{1}{\lambda^{5/2}} |\bar{v}|_{H_0^1}\right)$$



$$\int_{\left(\bigcup_{\ell=1}^k B_\ell\right)^c} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{3-\varepsilon} \frac{\partial P \delta_i}{\partial(x_i)_j} \bar{v}^2 = 0\left(\frac{1}{\lambda^2} |\bar{v}|_{H_0^1}^2\right)$$

using Hölder's inequality and Sobolev embedding theorem. On  $B_q$ ,  $q \neq i$ ,

$$\sum_{\ell=1}^k \alpha_\ell P \delta_\ell = 0(\delta_q) \text{ and } \frac{\partial P \delta_i}{\partial(x_i)_j} = 0\left(\frac{1}{\lambda^{1/2}}\right), \text{ so that}$$

$$\int_{B_q} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{4-\varepsilon} \frac{\partial P \delta_i}{\partial(x_i)_j} \bar{v} = 0\left(\frac{1}{\lambda^{1/2}} \left(\int_{B_q} \delta_q^{24/5}\right)^{5/6} |\bar{v}|_{H_0^1}\right)$$

$$\int_{B_q} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{3-\varepsilon} \frac{\partial P \delta_i}{\partial(x_i)_j} \bar{v}^2 = 0\left(\frac{1}{\lambda^{1/2}} \left(\int_{B_q} \delta_q^{9/2}\right)^{2/3} |\bar{v}|_{H_0^1}^2\right)$$

still using Hölder's inequality and Sobolev embedding theorem. A simple computation leads to

$$(57) \quad \left(\int_{B_q} \delta_q^{24/5}\right)^{5/6} = 0\left(\frac{1}{\lambda^{1/2}}\right) \quad \left(\int_{B_q} \delta_q^{9/2}\right)^{2/3} = 0\left(\frac{1}{\lambda^{1/2}}\right)$$

whence

$$(58) \quad \begin{cases} \int_{B_i^c} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{4-\varepsilon} \frac{\partial P \delta_i}{\partial(x_i)_j} \bar{v} = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda}\right) \\ \int_{B_i^c} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{3-\varepsilon} \frac{\partial P \delta_i}{\partial(x_i)_j} \bar{v} = 0\left(\frac{|\bar{v}|_{H_0^1}^2}{\lambda}\right). \end{cases}$$

We turn now to the last and most delicate part, the integrals on  $B_i$ . We first note that

$$(59) \quad \begin{cases} \left| \int_{B_q} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{4-\varepsilon} \frac{\partial \varphi_i}{\partial(x_i)_j} \bar{v} \right| \leq \frac{C}{\lambda^{1/2}} \int_{B_q} \delta_i^4 |\bar{v}| = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda}\right) \\ \left| \int_{B_q} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{3-\varepsilon} \frac{\partial \varphi_i}{\partial(x_i)_j} \bar{v}^2 \right| \leq \frac{C}{\lambda^{1/2}} \int_{B_q} \delta_i^3 |\bar{v}|^2 = 0\left(\frac{|\bar{v}|_{H_0^1}^2}{\lambda}\right) \end{cases}$$

proceeding as previously. In order to estimate the terms involving  $\frac{\partial \delta_i}{\partial(x_i)_j}$ , we write on  $B_i$

$$(60) \quad \sum_{\ell=1}^k \alpha_\ell P \delta_\ell = \alpha_i \delta_i + \left(\sum_{\ell \neq i} \alpha_\ell P \delta_\ell - \alpha_i \varphi_i\right)(x_i) + 0\left(\frac{|x - x_i|}{\lambda^{1/2}}\right)$$

and

$$(61) \quad \begin{aligned} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell\right)^{4-\varepsilon} &= \alpha_i^{4-\varepsilon} \delta_i^{4-\varepsilon} + (4-\varepsilon) \alpha_i^{3-\varepsilon} \left(\sum_{\ell \neq i} \alpha_\ell P \delta_\ell - \alpha_i \varphi_i\right)(x_i) \delta_i^{3-\varepsilon} \\ &\quad + 0\left(\frac{\delta_i^3 |x - x_i|}{\lambda^{1/2}} + \frac{\delta_i^2}{\lambda}\right). \end{aligned}$$

Concerning the last quantities in (61), we have

$$\frac{1}{\lambda} \int_{B_i} \delta_i^2 \left| \frac{\partial \delta_i}{\partial x_i} \right| |\bar{v}| \leq C \int_{B_i} \delta_i^3 |\bar{v}| = 0 \left[ \left( \int_{B_i} \delta_i^{18/5} \right)^{5/6} |\bar{v}|_{H_0^1} \right]$$

and

$$\begin{aligned} \frac{1}{\lambda^{1/2}} \int_{B_i} \delta_i^3 |x - x_i| \left| \frac{\partial \delta_i}{\partial x_i} \right| |\bar{v}| &\leq C \lambda^{1/2} \int_{B_i} \delta_i^4 |x - x_i| |\bar{v}| \\ &= 0 \left[ \lambda^{1/2} \left( \int_{B_i} \delta_i^{24/5} |x - x_i|^{6/5} \right)^{5/6} |\bar{v}|_{H_0^1} \right]. \end{aligned}$$

Straightforward computation yield

$$(62) \quad \left( \int_{B_i} \delta_i^{18/5} \right)^{5/6} = 0 \left( \frac{1}{\lambda} \right) \quad \left( \int_{B_i} \delta_i^{24/5} |x - x_i|^{6/5} \right)^{5/6} = 0 \left( \frac{1}{\lambda^{3/2}} \right)$$

so that the contribution of the last term in (61), multiplied by  $\frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}$ , to the integral on  $B_i$ , is dominated by  $\frac{|\bar{v}|_{H_0^1}}{\lambda}$ . Let us now compute the contribution of the first term. We recall that

$$-\Delta P \delta_{\lambda, x} = 3\delta_{\lambda, x}^5 \text{ in } \mathbb{R}^3$$

so that

$$-\Delta \frac{\partial P \delta_{\lambda, x}}{\partial x_j} = 15 \delta_{\lambda, x}^4 \frac{\partial \delta_{\lambda, x}}{\partial x_j}.$$

As  $\bar{v} \in E_{\lambda, x}$ , we know that

$$\int_{\Omega} \nabla \frac{\partial P \delta_i}{\partial (x_i)_j} \cdot \nabla \bar{v} = - \int_{\Omega} \Delta \frac{\partial P \delta_i}{\partial (x_i)_j} \cdot \bar{v} = 15 \int_{\Omega} \delta_i^4 \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} = 0$$

and

$$\int_{\Omega} \delta_i^{4-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} = \int_{\Omega} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}.$$

Therefore

$$(63) \quad \begin{aligned} \int_{\Omega} \left( \sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{4-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \bar{v} &= \alpha_i^{4-\varepsilon} \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} \\ &+ (4-\varepsilon) \alpha_i^{3-\varepsilon} \left( \sum_{\ell \neq i} \alpha_{\ell} P \delta_{\ell} - \alpha_i \varphi_i \right) (x_i) \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} + 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda} \right) \end{aligned}$$

Concerning the quadratic term in  $\bar{v}$ , we write on  $B_i$

$$\left( \sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} = \alpha_i^{3-\varepsilon} \delta_i^{3-\varepsilon} + 0 \left( \frac{\delta_i^2}{\lambda^{1/2}} \right).$$

As

$$\frac{1}{\lambda^{1/2}} \int_{B_i} \delta_i^2 \left| \frac{\partial \varphi_i}{\partial x_i} \right| \bar{v}^2 \leq C \frac{1}{\lambda} \left( \int_{B_i} \delta_i^3 \right)^{2/3} |\bar{v}|_{H_0^1}^2 = 0 \left( \frac{(\ell n \lambda)^{2/3}}{\lambda^2} |\bar{v}|_{H_0^1}^2 \right)$$

$$\frac{1}{\lambda^{1/2}} \int_{B_i} \delta_i^2 \left| \frac{\partial \varphi_i}{\partial x_i} \right| \bar{v}^2 \leq C \frac{1}{\lambda} \left( \int_{B_i} \delta_i^{9/2} \right)^{2/3} |\bar{v}|_{H_0^1}^2 = 0 \left( |\bar{v}|_{H_0^1}^2 \right)$$

and

$$\int_{B_i} \delta_i^{3-\varepsilon} \left| \frac{\partial \varphi}{\partial x_i} \right| \bar{v}^2 \leq \frac{C}{\lambda^{1/2}} \left( \int_{B_i} \delta_i^{9/2} \right)^{2/3} |\bar{v}|_{H_0^1}^2 = 0 \left( \frac{|\bar{v}|_{H_0^1}^2}{\lambda} \right)$$

we have

$$(64) \quad \int_{\Omega} \left( \sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} \frac{\partial P \delta_i}{\partial (x_i)_j} \bar{v}^2 = \alpha_i^{3-\varepsilon} \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}^2 + 0 \left( |\bar{v}|_{H_0^1}^2 \right)$$

Using, as previously, Hölder's inequality and Sobolev embedding theorem, and the fact that  $\frac{\partial \delta_i}{\partial (x_i)_j} = 0(\lambda \delta_i)$ ,  $\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} = 0(\varepsilon \ell n (1 + \lambda_i^2 |x - x_i|^2))$ , we find, in view of (63) (64)

$$\left| \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} \right| \leq C \varepsilon \lambda_i \left( \int_{B_i} \delta_i^6 (\ell n (1 + \lambda_i^2 |x - x_i|^2))^{6/5} \right)^{5/6} |\bar{v}|_{H_0^1}$$

$$= 0(\varepsilon \lambda |\bar{v}|_{H_0^1})$$

$$\left| \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} \right| \leq C \lambda_i \left( \int_{B_i} \delta_i^{24/5} \right)^{5/6} |\bar{v}|_{H_0^1} = 0(\lambda^{1/2} |\bar{v}|_{H_0^1})$$

$$\left| \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}^2 \right| \leq C \lambda_i \left( \int_{B_i} \delta_i^6 \right)^{2/3} |\bar{v}|_{H_0^1}^2 = 0(\lambda |\bar{v}|_{H_0^1}^2)$$

In view of (21), we obtain for the  $\bar{v}$ -part in the expansion of  $\frac{\partial K_{\varepsilon}}{\partial (x_i)_j}$  a quantity dominated by  $\frac{1}{\lambda} + \lambda \varepsilon^2$ . This is sufficient to get (30), and therefore estimate (22) in Proposition 2. However, we need sharper estimates in (63) and (64) to prove (29).

For these purposes, we remark that  $\delta_i$  is even and  $\frac{\partial \delta_i}{\partial (x_i)_j}$  is odd with respect to the variable  $(x - x_i)_j$ . Splitting  $\bar{v}$  in an even and an odd part with respect to this variable in a neighbourhood of  $x_i$ , we are able to obtain a better estimate than (21) concerning the odd part. As the even part has no contribution to the integrals, because of the oddness of  $\frac{\partial \delta_i}{\partial (x_i)_j}$ , the method will provide us with a suitable control of the quantities involving  $\bar{v}$  and  $\bar{v}^2$ .

Let us make this precise. Firstly, we set

$$(65) \quad \bar{v} = \sum_{i=1}^k v_i + w$$

with  $v_i$  the projection of  $\bar{v}$  onto  $H_0^1(B_i)$ , that is

$$(66) \quad \Delta v_i = \Delta \bar{v} \text{ in } B_i ; v_i = 0 \text{ on } \partial B_i$$

$v_i$  being continued by 0 in  $\Omega \setminus B_i$ . Note that  $w \in H_0^1(\Omega)$  is harmonic in  $B_i$ , and is orthogonal to  $v_i$ , that is

$$(67) \quad \Delta w = 0 \text{ in } B_i ; \int_{\Omega} \nabla w \cdot \nabla v_i = 0 \quad \forall i, 1 \leq i \leq k .$$

As a consequence, we have

$$(68) \quad \int_{\Omega} |\nabla \bar{v}|^2 = \sum_{i=1}^k \int_{\Omega} |\nabla v_i|^2 + \int_{\Omega} |\nabla w|^2 .$$

We split  $v_i$  in an even part  $v^e$  and an odd part  $v^o$  with respect to  $(x - x_i)_j$ . On  $B_i$ ,  $\bar{v} = v^e + v^o + w$ , whence, in view of (63) and (64)

$$\begin{aligned} \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} &= \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} (v^o + w) \\ &= \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} w + 0(\varepsilon \lambda |v^o|_{H_0^1}) \end{aligned}$$

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} = \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (v^o + w) = \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} w + 0(\lambda^{1/2} |v^o|_{H_0^1})$$

$$\begin{aligned} \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}^2 &= \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2v^e v^o + 2(\bar{v} - w)w + w^2) \\ &= \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2\bar{v} - w)w + 0(\lambda |\bar{v}|_{H_0^1} |v^o|_{H_0^1}) \end{aligned}$$

We claim that

$$(69) \quad |v^o|_{H_0^1} = 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}} \right).$$

estimate which will be sufficient to conclude. Concerning the integrals where  $w$  occurs, we use the fact that  $w$  is harmonic in  $B_i$ . Let  $\psi$  the solution of

$$\Delta \psi = \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \text{ in } B_i, \quad \psi = 0 \text{ on } \partial B_i.$$

Thus we have

$$(70) \quad \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} w = \int_{B_i} \Delta \psi \cdot w = \int_{\partial B_i} \frac{\partial \psi}{\partial n} w.$$

Let  $G_i$  the Green's function for the Laplacian on  $B_i$ , that is

$$G_i(x, y) = \frac{1}{|x - y|} - \frac{d}{|x| |y - \frac{d^2 x}{|x|^2}|} \quad (x, y) \in B_i^2.$$

$\psi$  is given by

$$\psi(y) = \int_{B_i} G_i(x, y) \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} dx \quad y \in B_i$$

and its normal derivative by

$$\frac{\partial \psi}{\partial n}(y) = \int_{B_i} \frac{\partial G_i}{\partial n_y}(x, y) \left( \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} \right) (x) dx \quad y \in \partial B_i$$

with

$$\frac{\partial G_i}{\partial n_y}(x, y) = \frac{x \cdot y - d^2}{d|x - y|^3} + d^2 \frac{|x|^2 - x \cdot y}{|x|^3 |y - \frac{d^2 x}{|x|^2}|} = 0 \left( \frac{1}{|x - y|^2} \right).$$

For  $x \in B_i \setminus B(y, \frac{d}{2})$

$$\frac{\partial G_i}{\partial n_y}(x, y) = 0(1) \quad \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j}(x) = 0(\varepsilon \lambda \delta_i^5 \ell n(1 + \lambda_i^2 |x - x_i|^2))$$

and for  $x \in B_i \cap B(y, \frac{d}{2})$

$$\frac{\partial G_i}{\partial n_y}(x, y) = 0 \left( \frac{1}{|x - y|^2} \right) \quad \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_j}{\partial (x_i)_j}(x) = 0 \left( \frac{\varepsilon \ell n \lambda}{\lambda^{3/2}} \right).$$

Therefore

$$\begin{aligned}\frac{\partial \psi}{\partial n}(y) &= 0 \left( \varepsilon \lambda \int_{B_i} \delta_i^5 \ell n(1 + \lambda_i^2 |x - x_i|^2) dx + \frac{\varepsilon \ell n \lambda}{\lambda^{5/2}} \int_{|x| \leq \frac{d}{2}} \frac{dx}{|x|^2} \right) \\ &= 0(\varepsilon \lambda^{1/2})\end{aligned}$$

and

$$\int_{\partial B_i} \frac{\partial \psi}{\partial n} w = 0(\varepsilon \lambda^{1/2} \int_{\partial B_i} |w|)$$

so that, from (70) we deduce

$$(71) \quad \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} w = 0(\varepsilon \lambda^{1/2} |\bar{v}|_{H_0^1}).$$

Concerning the second integral in which  $w$  occurs, we set  $\psi'$  as

$$\Delta \psi' = \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \text{ in } B_i ; \psi' = 0 \text{ on } \partial B_i$$

and we may write

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} w = \int_{\partial B_i} \frac{\partial \psi'}{\partial n} w.$$

Proceeding as previously, we have, for  $y \in \partial B_i$

$$\begin{aligned}\frac{\partial \psi'}{\partial n}(y) &= 0 \left( \lambda \int_{B_i} \delta_i^4 + \frac{1}{\lambda^2} \int_{|x| \leq \frac{d}{2}} \frac{dx}{|x|^2} \right) \\ &= 0(1)\end{aligned}$$

and

$$(72) \quad \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} w = 0(|\bar{v}|_{H_0^1}).$$

Lastly, we define  $\psi''$  as

$$\Delta \psi'' = \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2\bar{v} - w) \text{ in } B_i ; \psi'' = 0 \text{ on } \partial B_i$$

and we write

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2\bar{v} - w) w = \int_{\partial B_i} \frac{\partial \psi''}{\partial n} w.$$

Note that, as  $2\bar{v} - w \in W^{1,2}(B_i)$ ,  $\psi'' \in W^{3,2}(B_i)$ , and  $\nabla\psi''|_{\partial B_i} \in L^\infty(\partial B_i)$ . Actually, the integral definition of  $\psi''$  yields

$$\begin{aligned} \frac{\partial\psi''}{\partial n}(y) &= 0 \left( \lambda \int_{B_i} \delta_i^4 |2\bar{v} - w| + \frac{1}{\lambda^2} \int_{B(y, \frac{d}{2})} |(2\bar{v} - w)(x)| \frac{dx}{|x - y|^2} \right) \\ &= 0 \left( \lambda |\bar{v}|_{H_0^1} \left( \int_{B_i} \delta_i^{24/5} \right)^{5/6} + \frac{1}{\lambda^2} |\bar{v}|_{H_0^1} \right) \\ &= 0(\lambda^{1/2} |\bar{v}|_{H_0^1}) \end{aligned}$$

uniformly for  $y \in \partial B_i$ . Therefore

$$(73) \quad \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial\delta_i}{\partial(x_i)_j} (2\bar{v} - w)w = 0(\lambda^{1/2} |\bar{v}|_{H_0^1}^2) .$$

(71) (72) (73) and claim (69), together with (21), show that

$$\begin{aligned} \int_{B_i} \delta_i^4 \left( \delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial\delta_i}{\partial(x_i)_j} \bar{v} &= 0 \left( \frac{\varepsilon}{\lambda^{1/2}} + \lambda^{1/2} \varepsilon^2 \right) \\ \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial\delta_i}{\partial(x_i)_j} \bar{v} &= 0 \left( \frac{1}{\lambda} + \varepsilon \right) \\ \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial\delta_i}{\partial(x_i)_j} \bar{v}^2 &= 0 \left( \frac{1}{\lambda^{3/2}} + \lambda^{1/2} \varepsilon \right) . \end{aligned}$$

Then, from (50) (51) (55) (56) (58) (63) (64) and (21) we deduce (29), that is the desired estimate of  $\frac{\partial K_\varepsilon}{\partial(x_i)_j}$ .

We are also able to derive (33) from the above estimates. Namely, we have

$$\begin{aligned} \left\langle \frac{\partial^2 P\delta_i}{\partial\lambda_i \partial(x_i)_j}, \bar{v} \right\rangle_{H_0^1} &= \int_{\Omega} \nabla \frac{\partial^2 P\delta_i}{\partial\lambda_i \partial(x_i)_j} \cdot \nabla \bar{v} \\ &= \int_{\Omega} \left( 60\delta_i^3 \frac{\partial\delta_i}{\partial\lambda_i} \frac{\partial\delta_i}{\partial(x_i)_j} + 15\delta_i^4 \frac{\partial^2\delta_i}{\partial\lambda_i \partial(x_i)_j} \right) (v^e + v^o + w) . \end{aligned}$$

The contribution of the subdomain  $\Omega \setminus B_i$  to the integral is very small. Indeed, outside  $B_i$

$$\delta_i^3 \frac{\partial\delta_i}{\partial\lambda_i} \frac{\partial\delta_i}{\partial(x_i)_j} = 0 \left( \frac{1}{\lambda^{7/2}} \right) \quad \delta_i^4 \frac{\partial^2\delta_i}{\partial\lambda_i \partial(x_i)_j} = 0 \left( \frac{1}{\lambda^{7/2}} \right) .$$

On  $B_i$ , the contribution of  $v^e$  is zero because of evenness with respect to the first variable and, as  $\frac{\partial \delta_i}{\partial \lambda_i} = 0(\frac{\delta_i}{\lambda_i})$ ,  $\frac{\partial \delta_i}{\partial (x_i)_j} = 0(\lambda_i \delta_i)$ ,  $\frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} = 0(\delta_i)$

$$\left| \int_{B_i} \left( 60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_i^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \right) v^o \right| \leq C |v^o|_{H_0^1} = 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}} \right)$$

according to (69). Lastly

$$\int_{B_i} \left( 60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_i^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \right) w = \int_{B_i} \Delta \psi''' \cdot w = \int_{\partial B_i} \frac{\partial \psi'''}{\partial n} w$$

with

$$\Delta \psi''' = 60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_i^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \text{ in } B_i ; \psi''' = 0 \text{ on } \partial B_i .$$

For  $y \in \partial B_i$ ,  $\frac{\partial \psi'''}{\partial n}$  satisfies

$$\left| \frac{\partial \psi'''}{\partial n}(y) \right| \leq C \int_{B_i} \left| \frac{\partial G}{\partial n_y}(x, y) \right| \delta_i^5(x) dx = 0 \left( \frac{1}{\lambda^{1/2}} \right) .$$

As a consequence

$$\int_{\partial B_i} \frac{\partial \psi'''}{\partial n} w = 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}} \right)$$

and (33) follows.

### Proof of Claim (69)

The estimate of  $|v^o|_{H_0^1}$  will make the proof of Proposition 3 complete. For sake of simplicity, we may assume that  $i = j = 1$  and, up to a translation, that  $x_1 = 0$ . We write

$$(74) \quad v^o = \tilde{v}^o + aP\delta_1 + b \frac{\partial P\delta_1}{\partial \lambda_1} + \sum_{\ell=1}^3 c_\ell \frac{\partial P\delta_1}{\partial (x_1)_\ell}$$

with

$$\langle P\delta_1, \tilde{v}^o \rangle_{H_0^1} = \left\langle \frac{\partial P\delta_1}{\partial \lambda_1}, \tilde{v}^o \right\rangle_{H_0^1} = \left\langle \frac{\partial P\delta_1}{\partial (x_i)_\ell}, \tilde{v}^o \right\rangle_{H_0^1} = 0 \quad 1 \leq \ell \leq 3 .$$

Taking the scalar product in  $H_0^1(\Omega)$  of (74) with  $P\delta_1$ ,  $\frac{\partial P\delta_1}{\partial \lambda_1}$ ,  $\frac{\partial P\delta_1}{\partial (x_i)_\ell}$ ,  $1 \leq \ell \leq 3$ , provides us with an invertible linear system in  $a, b, c_\ell$ , whose coefficients are given by (25) (26). On the left hand side, we find

$$(75) \quad \int_{B_1} \nabla P\delta_1 \cdot \nabla v^o = 0$$



since

$$\int_{B_1} \nabla \delta_1 \cdot \nabla v^o = 0$$

because of evenness of  $\delta_1$  and oddness of  $v^o$  with respect to the first variable, and

$$\int_{B_1} \nabla \varphi_1 \cdot \nabla v^o = 0$$

because of harmonicity of  $\varphi_1$  and nullity of  $v^o$  on  $\partial B_1$ . In the same way

$$(76) \quad \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial \lambda_1} \cdot \nabla v^o = \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_\ell} \cdot \nabla v^o = 0 \quad \ell = 2, 3.$$

Lastly, we have

$$\begin{aligned} \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla v^o &= \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla (\bar{v} - v^e - w) \\ &= - \int_{\Omega \setminus B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla \bar{v} - \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla w \end{aligned}$$

since  $\bar{v} \in E_{\lambda, \mathbf{x}}$  and  $v^e$  is even with respect to the first variable, zero on  $\partial B_1$ , and  $\frac{\partial \varphi}{\partial x_1}$  is harmonic in  $B_1$ . On one hand

$$\int_{\Omega \setminus B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla \bar{v} \leq \frac{C}{\lambda^{1/2}} \int_{\Omega} |\nabla \bar{v}| = 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}} \right).$$

On the other hand, let  $\psi^{(4)}$  be such that

$$\Delta \psi^{(4)} = \Delta \frac{\partial P \delta_1}{\partial (x_1)_1} = -15\delta_1^4 \frac{\partial \delta_i}{\partial (x_1)_1} \text{ in } B_1; \psi^{(4)} = 0 \text{ on } \partial B_1.$$

Writing

$$\psi^{(4)} = \frac{\partial P \delta_1}{\partial (x_1)_1} + \theta$$

we have

$$\begin{aligned} \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla w &= \int_{B_1} \nabla (\psi^{(4)} - \theta) \cdot \nabla w \\ &= - \int_{\partial B_1} \frac{\partial \theta}{\partial n} \cdot w \end{aligned}$$

since  $w$  and  $\theta$  are harmonic in  $B_1$ , and  $\psi^{(4)}$  is zero on  $\partial B_1$ . Using, as previously, an integral representation for  $\psi^{(4)}$ , we obtain for  $y \in \partial B_1$

$$\begin{aligned} \frac{\partial \psi^{(4)}}{\partial n}(y) &= -15 \int_{B_1} \frac{\partial G_1}{\partial n_y}(x, y) \left( \delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1} \right) (x) dx \\ &= -15 \int_{B(0, \frac{d}{2})} \left( \frac{\partial G_1}{\partial n_y}(0, y) + o(|x|) \right) \left( \delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1} \right) (x) dx + o\left(\frac{1}{\lambda^{5/2}}\right) \end{aligned}$$

since  $\nabla_x \frac{\partial G_1}{\partial n_y}(x, y)$  is bounded in  $B(0, \frac{d}{2}) \times \partial B_1$ ,  $\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1} = o\left(\frac{1}{\lambda^{5/2}}\right)$  in  $B_1 \setminus B(0, \frac{d}{2})$  and  $\frac{\partial G_1}{\partial n_y}(x, y) = o\left(\frac{1}{|x-y|^2}\right)$  in  $B_1 \times \partial B_1$ .  $\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1}$  being odd with respect to the first variable, and

$$\int_{B(0, \frac{d}{2})} |x| \delta_1^4 \left| \frac{\partial \delta_1}{\partial (x_1)_1} \right| dx = o\left(\frac{1}{\lambda^{1/2}}\right)$$

we find

$$\frac{\partial \psi^{(4)}}{\partial n}(y) = o\left(\frac{1}{\lambda^{1/2}}\right)$$

so that, noticing that  $\frac{\partial}{\partial n} \left( \frac{\partial P \delta_1}{\partial (x_1)_1} \right) = o\left(\frac{1}{\lambda^{1/2}}\right)$  on  $\partial B_1$

$$\frac{\partial \theta}{\partial n} = o\left(\frac{1}{\lambda^{1/2}}\right) \text{ on } \partial B_1 .$$

It follows that

$$\int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla w = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}\right)$$

and

$$(77) \quad \int_{B_1} \nabla \frac{\partial P \delta_1}{\partial (x_1)_1} \cdot \nabla v^o = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}\right) .$$

Inverting the linear system involving  $a, b, c_\ell$ , whose coefficients are given by (25) (26) and whose left hand side is given by (75) (76) (77), the following estimates are obtained :

$$(78) \quad a = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{7/2}}\right) \quad b = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}}\right) \quad c_1 = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}}\right) \quad c_\ell = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{7/2}}\right) \quad \ell = 2, 3 .$$

Using again (25) (26), this implies through (74)

$$(79) \quad |v^o - \tilde{v}^o|_{H_0^1} = o\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{3/2}}\right) \quad |v^o|_{H_0^1}^2 = |\tilde{v}^o|_{H_0^1}^2 + o\left(\frac{|\bar{v}|_{H_0^1}^2}{\lambda^3}\right) .$$

We turn now to the last step, which consists in estimating  $\tilde{v}^o$  in  $H_0^1(\Omega)$ . The scalar product of  $(E_v)$  with  $v^o$  yields the equality

$$(80) \quad \int_{\Omega} \nabla \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) \cdot \nabla v^o - \int_{\Omega} \left| \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right|^{4-\varepsilon} \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) v^o$$

$$= \int_{\Omega} \nabla \sum_{i=1}^k \left( A_i P \delta_i + B_i \frac{\partial P \delta_i}{\partial \lambda_i} + \sum_{j=1}^3 C_{ij} \frac{\partial P \delta_i}{\partial (x_i)_j} \right) \cdot \nabla \left( a P \delta_1 + b \frac{\partial P \delta_1}{\partial \lambda_1} + \sum_{j=1}^3 c_j \frac{\partial P \delta_1}{\partial (x_1)_j} \right)$$

whence

$$(81) \quad \int_{\Omega} \nabla \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) \cdot \nabla v^o - \int_{\Omega} \left| \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right|^{4-\varepsilon} \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) v^o$$

$$= 0 \left( |\bar{v}|_{H_0^1} \left( \frac{1}{\lambda^{5/2}} + \frac{\varepsilon^2}{\lambda^{1/2}} \right) \right)$$

using (22) (25) (26) and (78). Concerning the first integral, we know that

$$\int_{\Omega} \nabla \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) \cdot \nabla v^o = -3 \sum_{i=2}^k \alpha_i \int_{B_1} \delta_i^5 v^o + \int_{B_1} |\nabla v^o|^2$$

since  $-\Delta P \delta_i = 3\delta_i^5$  in  $\Omega$ ,  $v^o$  is zero in  $\Omega \setminus B_1$ ,  $\bar{v} = v^e + v^o + w$  in  $B_1$  with  $v^e$  even and  $v^o$  odd with respect to the first variable, and  $w$  harmonic in  $B_1$ . Therefore, as  $\delta_i^5 = 0(\frac{1}{\lambda^{5/2}})$  in  $B_1$  for  $2 \leq i \leq k$ , we find, taking account of (79) :

$$(82) \quad \int_{\Omega} \nabla \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) \cdot \nabla v^o = \int_{B_1} |\nabla \tilde{v}^o|^2 + 0 \left( \frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}} \right)$$

Let us consider the second integral, which may be restricted to  $B_1$ , since  $v^o$  is zero in  $B_1 \setminus \Omega$ . We expand

$$\int_{B_1} \left| \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right|^{4-\varepsilon} \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) v^o$$

$$= \alpha_1^{5-\varepsilon} \int_{B_1} P \delta_1^{5-\varepsilon} v^o + (5-\varepsilon) \alpha_1^{4-\varepsilon} \int_{B_1} P \delta_1^{4-\varepsilon} \left( \sum_{i=2}^k \alpha_i P \delta_i + \bar{v} \right) v^o$$

$$+ 0 \left[ \int_{B_1} \left( \delta_1^3 \left( \sum_{i=2}^k \delta_i^2 + |\bar{v}|^2 \right) + \sum_{i=2}^k \delta_i^5 + |\bar{v}|^{5-\varepsilon} \right) |v^o| \right].$$

Estimating the last term is easy, namely

$$\begin{aligned}
& \int_{B_1} \left( \delta_1^3 \left( \sum_{i=2}^k \delta_i^2 + |\bar{v}|^2 \right) + \sum_{i=2}^k \delta_i^5 + |\bar{v}|^{5-\varepsilon} \right) |v^o| \\
&= 0 \left[ \frac{1}{\lambda} \left( \int_{B_1} \delta_1^{18/5} \right)^{5/6} |v^o|_{H_0^1} + |\bar{v}|_{H_0^1}^2 |v^o| + \frac{1}{\lambda^{5/2}} |v^o|_{H_0^1} + |\bar{v}|_{H_0^1}^{5-\varepsilon} |v^o|_{H_0^1} \right] \\
&= 0 \left[ |v^o|_{H_0^1} \left( \frac{1}{\lambda^2} + |\bar{v}|_{H_0^1}^2 \right) \right].
\end{aligned}$$

Concerning the remaining terms, we write

$$\begin{aligned}
\int_{B_1} P \delta_1^{5-\varepsilon} v^o &= \int_{B_1} \delta_1^{5-\varepsilon} v^o - (5-\varepsilon) \int_{B_1} \delta_1^{4-\varepsilon} \left( \varphi_1(0) + 0\left(\frac{|x|}{\lambda^{1/2}}\right) \right) v^o \\
&\quad + 0 \left( \frac{1}{\lambda} \int_{B_1} \delta_1^3 |v^o| \right).
\end{aligned}$$

Using evenness of  $\delta_1$  and oddness of  $v^o$  with respect to the first variable, and noticing that

$$\left( \int_{B_1} \delta_1^{24/5} |x|^{6/5} \right)^{5/6} = 0 \left( \frac{1}{\lambda^{3/2}} \right) \quad \left( \int_{B_1} \delta_i^{18/5} \right)^{5/6} = 0 \left( \frac{1}{\lambda} \right)$$

we obtain

$$(83) \quad \int_{B_1} P \delta_1^{5-\varepsilon} v^o = 0 \left( \frac{|v^o|_{H_0^1}}{\lambda^2} \right).$$

In the same way, we have

$$\begin{aligned}
& \int_{B_1} P \delta_1^{4-\varepsilon} \left( \sum_{i=2}^k \alpha_i P \delta_i \right) v^o \\
&= \int_{B_1} \left( \delta_1^{4-\varepsilon} + 0\left(\frac{\delta_1^3}{\lambda^{1/2}}\right) \right) \left( \sum_{i=2}^k \alpha_i P \delta_i(x_1) + 0\left(\frac{|x|}{\lambda^{1/2}}\right) \right) v^o \\
&= 0 \left( \frac{1}{\lambda^{1/2}} \int_{B_1} \delta_1^4 |x| |v^o| + \frac{1}{\lambda} \int_{B_1} \delta_1^3 |v^o| \right)
\end{aligned}$$

whence

$$(84) \quad \int_{B_1} P\delta_1^{4-\varepsilon} \left( \sum_{i=2}^k \alpha_i P\delta_i \right) v^o = 0 \left( \frac{|v^o|_{H_0^1}}{\lambda^{3/2}} \right).$$

The last term to consider writes as

$$(5 - \varepsilon)\alpha_1^{4-\varepsilon} \int_{B_1} P\delta_1^{4-\varepsilon} \bar{v}v^o = (15 + 0(\varepsilon)) \int_{B_1} P\delta_1^{4-\varepsilon} (v^e + v^o + w)v^o.$$

On one hand

$$\begin{aligned} & \int_{B_1} P\delta_1^{4-\varepsilon} (v^e + v^o)v^o \\ &= \int_{B_1} \left[ \delta_1^{4-\varepsilon} - (4 - \varepsilon)\delta_1^{3-\varepsilon} (\varphi_1(0) + 0(\frac{|x|}{\lambda^{1/2}})) + 0(\frac{\delta_1^2}{\lambda}) \right] (v^e + v^o)v^o \\ &= \int_{B_1} \delta_1^{4-\varepsilon} (v^o)^2 + 0 \left( \frac{1}{\lambda^{1/2}} \int_{B_1} \delta_1^3 (v^o)^2 + \frac{1}{\lambda^{1/2}} \int_{B_1} \delta_1^3 |x| |v_1| |v^o| + \frac{1}{\lambda} \int_{B_1} \delta_1^2 |v_1| |v^o| \right) \\ &= \int_{B_1} \delta_1^4 (v^o)^2 + 0 \left( \varepsilon \ell n \lambda |v^o|_{H_0^1}^2 + \frac{1}{\lambda^{1/2}} \left( \int_{B_1} \delta_1^{9/2} \right)^{2/3} |v^o|_{H_0^1}^2 \right. \\ & \quad \left. + \frac{1}{\lambda^{1/2}} \left( \int_{B_1} \delta_1^{9/2} |x|^{3/2} \right)^{2/3} |\bar{v}|_{H_0^1} |v^o|_{H_0^1} + \frac{1}{\lambda} \left( \int_{B_1} \delta_1^3 \right)^{2/3} |\bar{v}|_{H_0^1} |v^o|_{H_0^1} \right) \\ &= \int_{B_1} \delta_1^4 (v^o)^2 + o(|v^o|_{H_0^1}^2) + 0 \left( \frac{(\ell n \lambda)^{2/3}}{\lambda^2} |\bar{v}|_{H_0^1} |v^o|_{H_0^1} \right) \end{aligned}$$

because of (57) and

$$\left( \int_{B_1} \delta_1^{9/2} |x|^{3/2} \right)^{2/3} = 0 \left( \frac{(\ell n \lambda)^{2/3}}{\lambda^2} \right) \quad \left( \int_{B_1} \delta_1^3 \right)^{2/3} = 0 \left( \frac{(\ell n \lambda)^{2/3}}{\lambda} \right).$$

On the other hand

$$\int_{B_1} P\delta_1^{4-\varepsilon} wv^o = \int_{B_1} \Delta \psi^{(5)} \cdot w = \int_{\partial B_1} \frac{\partial \psi^{(5)}}{\partial n} w$$

with  $\psi^{(5)}$  defined as

$$\Delta \psi^{(5)} = P\delta_1^{4-\varepsilon} v^o \text{ in } B_1; \psi^{(5)} = 0 \text{ on } \partial B_1.$$

The normal derivative of  $\psi^{(5)}$  at  $y \in \partial B_1$  is given by

$$\begin{aligned}
\frac{\partial \psi^{(5)}}{\partial n}(y) &= \int_{B_1} \frac{\partial G_1}{\partial n_y}(x, y) P \delta_1^{4-\varepsilon}(x) v^o(x) dx \\
&= 0 \left[ \int_{\substack{B_1 \\ |x-y| \geq \frac{d}{2}}} \delta_1^4 |v^o| dx + \frac{1}{\lambda^2} \int_{\substack{B_1 \\ |x-y| \geq \frac{d}{2}}} |v^o| \frac{dx}{|x-y|^2} \right] \\
&= 0 \left[ \left( \int_{B_1} \delta_1^{24/5} \right)^{5/6} |v^o|_{H_0^1} + \frac{|v^o|_{H_0^1}}{\lambda^2} \right] \\
&= 0 \left( \frac{|v^o|_{H_0^1}}{\lambda^{1/2}} \right)
\end{aligned}$$

because of (57). Consequently

$$\int_{B_1} P \delta_1^{4-\varepsilon} w v^o = 0 \left( \frac{|v^o|_{H_0^1}}{\lambda^{1/2}} \int_{\partial B_1} |w| \right)$$

and so

$$\int_{B_1} P \delta_1^{4-\varepsilon} w v^o = 0 \left( \frac{|\bar{v}|_{H_0^1} |v^o|_{H_0^1}}{\lambda^{1/2}} \right).$$

This yields finally, taking account of (82)(83)(84)

$$\begin{aligned}
&\int_{\Omega} \nabla \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) \cdot \nabla v^o - \int_{\Omega} \left| \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right|^{4-\varepsilon} \left( \sum_{i=1}^k \alpha_i P \delta_i + \bar{v} \right) v^o \\
&= \int_{B_1} |\nabla v^o|^2 - 15 \int_{B_1} \delta_1^4 (v^o)^2 + o \left( |v^o|_{H_0^1}^2 \right) + 0 \left( \frac{|\bar{v}|_{H_0^1} |v^o|_{H_0^1}}{\lambda^{1/2}} \right) \\
&= \int_{\Omega} |\nabla \tilde{v}^o|^2 - 15 \int_{\Omega} \delta_1^4 (\tilde{v}^o)^2 + o \left( |\tilde{v}^o|_{H_0^1}^2 \right) + 0 \left( \frac{|\bar{v}|_{H_0^1} |\tilde{v}^o|_{H_0^1}}{\lambda^{1/2}} + \frac{|\bar{v}|_{H_0^1}^2}{\lambda^2} \right)
\end{aligned}$$

because of (79). Comparing with (81) and (79), and the quadratic form

$$v \mapsto \int_{\Omega} |\nabla v|^2 - 15 \int_{\Omega} \delta_1^4 v^2$$

being coercive on the subset  $\left[ \text{Span}(P \delta_1, \frac{\partial P \delta_1}{\partial \lambda_1}, \frac{\partial P \delta_1}{\partial (x_1)_j}, 1 \leq j \leq 3) \right]_{H_0^1}^{\perp}$ , (69) follows.

## References

- [1] A. Bahri, *Critical points at infinity in some variational problems*, Pitman Research Notes in Math. Series 182, Longman (1989).
- [2] A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent : the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 255-294.
- [3] A. Bahri, Y. Li, O. Rey, *On a variational problem with lack of compactness : the topological effect of the critical points at infinity*, Calculus of Variations and Partial Diff. Equ. **3** (1995), 67-94.
- [4] M. Ben Ayed, H. Chtioui, M. Hammani, *A Morse lemma at infinity for Yamabe-type problems on domains*, to appear.
- [5] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437-477.
- [6] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271-297.
- [7] E.N. Dancer, *A note on an equation with critical exponent*, Bull. London Math. Soc. **20** (1988), 600-602.
- [8] W.Y. Ding, *Positive solutions of  $\Delta u + u^{(n+2)/(n-2)} = 0$  on contractible domains*, J. Part. Diff. Equ. **2** (1989), 83-88.
- [9] Z.C. Han, *Asymptotic approach to singular solutions for non linear elliptic equations involving critical Sobolev exponent*, Ann. Inst. Poincaré, Anal. non linéaire **8** (1991), 159-174.
- [10] J. Kazdan, F. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567-597.
- [11] Y.Y. Li, *Prescribing scalar curvature on  $S^n$  and related problems, Part I*, J. Diff. Equ. **120** (1995), 319-410.
- [12] P.L. Lions, *The concentration compactness principle in the calculus of variations, The limit case*, Rev. Mat. Iberoamericana **1.1**, **1.2** (1985), 145-201 and 45-121.

- [13] M. Ould Ahmedou, K. Ould El Mehdi, *Computation of the difference of topology at infinity for Yamabe-type problems on thin annuli-domains, Part I*, to appear in Duke Math. J.
- [14] M. Ould Ahmedou, K. Ould El Mehdi, *On an elliptic problem with critical nonlinearity in expanding annuli*, to appear.
- [15] S. Pohozaev, *Eigenfunctions of the equation  $\Delta u + \lambda f(u) = 0$* , Soviet Math. Dokl. **6**, (1965) 1408-1411.
- [16] O. Rey, *Bifurcation from infinity in a nonlinear elliptic equation involving the limiting Sobolev exponent*, Duke Math. J. **60** (1990), 815-861.
- [17] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1-52.
- [18] O. Rey, *Blow-up points of solutions to elliptic equations with limiting nonlinearity*, Diff. and Int. Equ. **4** (1991), 1155-1167.
- [19] R. Schoen, *Courses at Stanford University (1988) and New-York University (1989)*, unpublished.
- [20] M. Struwe, *A global compactness result for elliptic boundary value problem involving limiting nonlinearities*, Math. Z. **187** (1984), 511-517.