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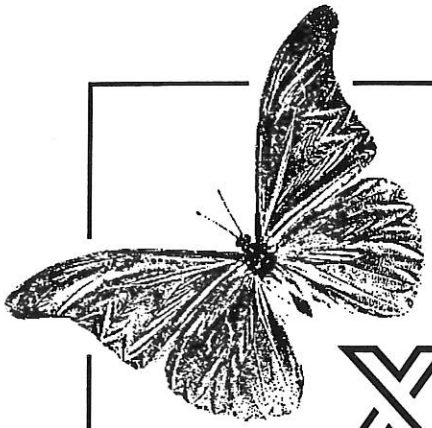

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Boundary effect for an elliptic Neumann problem with critical nonlinearity

Olivier REY

N° 1127 **février 1996**

F–91128 Palaiseau Cedex
Tél. : ((33)) (1) 69 33 40 88 • Fax : ((33)) (1) 69 33 30 19
Internet : secret@orphee.polytechnique.fr

RÉSUMÉ. — Nous étudions les problèmes elliptiques avec non-linéarité critique et conditions au bord de Neumann de la forme (P_μ) : $-\Delta u + \mu u = u^{\frac{n+2}{n-2}}, u > 0$ sur Ω; $\frac{\partial u}{\partial \nu} = 0$ sur $\partial\Omega$ - où Ω est un domaine borné et régulier de $\mathbf{R}^n, n \geq 3$, et μ est un paramètre strictement positif. Nous montrons, pour $n \geq 7$, que si u est une solution de (P_μ) d'énergie petite, u se concentre quand μ tend vers l'infini en un point du bord où la courbure moyenne H est positive, et critique quand elle est strictement positive. Réciproquement, nous montrons, pour $n \geq 5$, que si $a > 0$ est une valeur critique de H induisant une différence de topologie entre les ensembles de niveau de cette fonction, il existe pour μ assez grand une solution de (P_μ) qui se concentre en un point y du bord tel que $H(y) = a$ et $H'(y) = 0$. Enfin, si $n \geq 6$ et $y^1, \cdots, y^k$ sont k points critiques distincts et non-dégénérés de H, il existe pour μ assez grand une solution de (P_μ) qui se concentre en chacun des points y^i, $1 \leq i \leq k$.

ABSTRACT. — We are interested in elliptic problems with critical nonlinearity and Neumann boundary conditions, namely (P_μ) : $-\Delta u + \mu u = u^{\frac{n+2}{n-2}}, u > 0$ in Ω; $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ - where Ω is a smooth bounded domain in $\mathbf{R}^n, n \geq 3$, and μ is a strictly positive parameter. We show, for $n \geq 7$, and u a small energy solution of (P_μ), that u concentrates as μ goes to infinity at a point of the boundary such that the mean curvature H is positive, and critical if it is strictly positive. Conversely, we show, for $n \geq 5$, and $a > 0$ a critical value for H inducing a difference of topology between the level sets of H, that there exists for μ large enough a solution of (P_μ) which concentrates at a point y of the boundary such that $H(y) = a$ and $H'(y) = 0$. Lastly, if $n \geq 6$ and $y^1, \cdots, y^k$ are k distinct critical points of H, there exists for μ large enough a solution of (P_μ) which concentrates at each of the points y^i, $1 \leq i \leq k$.

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MOTS-CLÉS. — *problèmes variationnels non-compacts, problèmes elliptiques avec non-linéarité critique, conditions au bord de Neumann, courbure moyenne.*

KEYWORDS. — *noncompact variational problems, elliptic problems with critical nonlinearity, Neumann boundary conditions, mean curvature.*

1 - Introduction and results

This paper is devoted to the study of elliptic problems with critical nonlinearity and Neumann boundary conditions, namely

$$(1.1) \qquad \begin{cases} -\Delta u + \mu u &= u^p & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega \end{cases}$$

where μ is a strictly positive constant, Ω is a smooth and bounded domain in $\mathbb{R}^n, n \geq 3$, and $p = \frac{n+2}{n-2}$.

Such kind of problems appears in the modelization of activator-inhibitor systems [16], as chemotactic behaviour of cells [12]. If $n = 1, 2$, or $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, standard variational methods apply. In particular, Lin, Ni and Takagi [14], Li and Ni [13] proved that (1.1) does not have any nonconstant solution for small μ, whereas it admits at least one for μ sufficiently large. It was conjectured that this situation was actually general : in contrast with the similar Dirichlet problem, in which the exponent p plays a crucial role, μ would have been here the most important parameter to decide of existence or nonexistence of nontrivial solutions. However, by proving that for $n = 4, 5, 6, p = \frac{n+2}{n-2}$ and Ω a ball, the problem did admit a nontrivial solution for any μ strictly positive, Adimurthi and Yadava [2] - see also Budd, Knaap and Peletier [10] - showed that the feature in the critical case is different, independently of the arguments which have anyhow to be modifed with respect to the subcritical case.

Adimurthi and Mancini [1] - see also Wang [20] - established that for any Ω in $\mathbb{R}^n, n \geq 3$, and $p = \frac{n+2}{n-2}$, (1.1) admits a solution for μ large enough. They use the same method as Brezis and Nirenberg in [9], which consists in proving, by the choice of suitable test functions, that the infinimum of the functional associated to the problem is below the first level at which a lack of compactness occurs. Any minimizing sequence thus converges to a nontrivial solution. The considered test functions are solutions to the equation

$$(1.2) \qquad -\Delta u = u^{\frac{n+2}{n-2}}, \ u > 0 \text{ in } \mathbb{R}^n$$

which, with the additional assumptions $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n)$, may all be written as

$$(1.3) \qquad U_{\lambda,y}(x) = \frac{\lambda^{\frac{n-2}{2}}}{(1 + \lambda^2|x-y|^2)^{\frac{n-2}{2}}} \qquad \lambda > 0, \ y \in \mathbb{R}^n$$

up to the multiplicative constant

$$\bar{\alpha} = (n(n-2))^{\frac{n-2}{4}} . \tag{1.4}$$

Setting

$$I_\mu(u) = \frac{\int_\Omega |\nabla u|^2 + \mu \int_\Omega u^2}{(\int_\Omega |u|^{p+1})^{\frac{2}{p+1}}} \qquad u \in H^1(\Omega), p = \frac{n+2}{n-2} \tag{1.5}$$

a functional whose positive critical points, up to a multiplicative constant, are solutions to (1.1), and denoting

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^2}{(\int_\Omega |u|^{p+1})^{\frac{2}{p+1}}} \qquad p = \frac{n+2}{n-2} \tag{1.6}$$

the best Sobolev constant, it follows from a standard application of the concentration - compactness principle [15] that I_μ is compact below the level $S/2^{2/n}$. On the other hand, expanding $I_\mu(U_{\lambda,y})$ as $y \in \partial\Omega, H(y) > 0$, where H denotes the mean curvature of $\partial\Omega$, and λ going to infinity, one obtains [1, Lemma 2.2]

$$\inf_{u \in H_0^1(\Omega)} I_\mu(u) < \frac{S}{2^{2/n}} \tag{1.7}$$

for μ large enough, and the existence of a solution to (1.1) follows.

Note that in striking contrast with what happens in the Dirichlet case, where the maximum of concentrated solutions cannot approach the boundary, Adimurthi, Pacella and Yadava showed in [4] that for μ large enough, a solution u_μ of (1.1), which satisfies $I_\mu(u_\mu) < \frac{S}{2^{2/n}}$, attains its maximum at only one point $y_\mu \in \partial\Omega$. Up to a subsequence, (y_μ) converges to some $y^0 \in \partial\Omega$ as μ goes to infinity, and u_μ satisfies

$$|\nabla u_\mu|^2 \underset{\mu \to +\infty}{\rightharpoonup} \frac{S^{n/2}}{2} \delta_{y^0} \tag{1.8}$$

in the sense of measures in Ω.

Furthermore, they proved that if $n \geq 7$ and u_μ minimizes I_μ, (y_μ) is a maximizing sequence for the mean curvature H of the boundary $\partial\Omega$ of Ω. Dropping the assumption that u_μ is a minimizing sequence, it is proved in [5] that the accumulation points of (y_μ) are more generally critical with respect to H, and precise L^∞-estimates are obtained.

Some partial converse results are also available. Firsteval, we know from [4] that for $n \geq 7$ and any $y^0 \in \partial\Omega$ such that $H(y^0) > 0$ is a strict local maximum of H on $\partial\Omega$, there exists, for μ large enough, a solution u_μ of (1.1), which satisfies

$I_\mu(u_\mu) < \frac{S}{2^{2/n}}$ and concentrates at y^0 as μ goes to infinity. In [3], Adimurthi, Mancini and Yadava showed that the same result holds assuming that $n \geq 6$ and $y^0 \in \partial\Omega$ is a nondegenerate critical point of H, with $H(y^0) > 0$.

The aim of this paper is, adapting to the present context the methods that we previously developped to analyze similar Dirichlet problems - see, for instance [17] [18] [8] - to give a general setting for the study of problems as (1.1), whatever the boundary conditions may be. The interesting point is that, although the results are qualitatively different, the same kind of arguments allows to conclude. On one hand, we find again the above results, with some extensions and improvements ; on the other hand, we look for solutions which concentrate at several points at the same time, and we prove that, in contrast with what happened in the Dirichlet case, the interactions between the different concentrated peaks remain neglictable with respect to the boundary effects induced by the positive mean curvature. Therefore, such solutions behave as a superposition of single concentrated solutions.

Namely, we state :

Theorem 1.1 —

a) Let $n \geq 7$ and u_μ be a solution of (1.1) such that $I_\mu(u_\mu) < \frac{S}{2^{2/n}}$. Let $y^0 \in \partial\Omega$ be the point at which, up to a subsequence, u_μ concentrates, as μ goes to infinity (in the sense $|\nabla u_\mu|^2 \underset{\mu \to +\infty}{\rightharpoonup} \frac{S^{n/2}}{2} \delta_{y^0}$, or y^0 is an accumulation point of the sequence (y_μ) such that u_μ attains its maximum at y_μ). Then

$$H(y^0) \geq 0$$

and

$$\text{if } H(y^0) > 0, \ H'(y^0) = 0 \ .$$

b) Let $n \geq 6$ and $y^0 \in \partial\Omega$ be a nondegenerate critical point of H, $H(y^0) > 0$. For μ large enough, (1.1) admits a solution u_μ which concentrates at y^0 (in the previous sense) as μ goes to infinity.

In fact, the nondegeneracy assumption way be weakened in the following way. Denoting

$$H^a = \{y \in \partial\Omega / H(y) < a\} \tag{1.9}$$

the level sets of H, we have

Theorem 1.2 — *Let $n \geq 5$, and assume that $a > 0$ is a critical value of H such that the relative topology $(H^{a+\delta}, H^{a-\delta})$ is nontrivial for any $\delta > 0$ sufficiently small.*

There exists a solution u_μ of (1.1), which concentrates, as μ goes to infinity, at a point $y^0 \in \partial\Omega$ such that $H(y^0) = a$ (for $n \geq 6$, we prove that $H'(y^0) = 0$).

Solutions which concentrate at one point may be glued together, and we obtain

Theorem 1.3 — *Let $n \geq 6$, and $y^1, \cdots, y^k$ k distinct points of $\partial\Omega$, each one a nondegenerate critical point of H such that $H(y^i) > 0$. There exists, for μ large enough, a solution u_μ of (1.1) which concentrates at $y^1, \cdots, y^k$ as μ goes to infinity (in the sense $|\nabla u_\mu|^2 \underset{\mu \to +\infty}{\rightharpoonup} \frac{S^{n/2}}{2} \sum_{i=1}^{k} \delta_{y^i}$).*

Of course, we could also prove that u_μ being a solution of (1.1) concentrating at k distinct points y^i of $\partial\Omega$ as μ goes to infinity, these points have to satisfy $H(y^i) > 0, H'(y^i) = 0$ (assuming, as in Theorem 1.1, $n \geq 7$).

Theorem 1.3 allows to state multiplicity results, in connection with the number of critical points of H on

$$H^+ = \{y \in \partial\Omega / H(y) > 0\} \tag{1.10}$$

Theorem 1.4 — *Let $n \geq 6$. Assume that H admits k nondegenerate critical points on H^+. For μ large enough, (1.1) admits at least $2^k - 1$ solutions.*

The nondegeneracy condition which occurs in Theorem 1.3 could be weakened, as previously, leading to multiplicity results in relation with the relative topology between the level sets of H. For instance, (1.1) should admit, for μ large enough, at least $2^{\text{cat}(H^+)} - 1$ solutions, where cat (H^+) is the Ljusternik-Schnirelman category of H^+. Note that all our results are available assuming that $n \geq 5, 6$ or 7, the low dimension spaces being the most delicate cases. However, the same should hold for any $n \geq 3$.

The proof of the theorems is based upon a parametrization of $H^1(\Omega)$ in a neighbourhood of the potential concentrated solutions u that we look for. This parametrization is performed in the next section.

In this neighbourhood, u splits into a small part v which is showed to be inessential, and a concentrated part which depends on a finite number of parameters, namely the points and the speeds of concentration. It is proved in Section 3 that u being a solution to (1.1) is equivalent to the fulfilment of a finite dimensional system, whose study will give us the announced results in Section 4.

The proofs require some technical results which, for the convenience of the reader, may be found in appendix. A serious amont of computations is also delayed until

the end of the paper, so that the main arguments appear more clearly. However, and unfortunately, these computations are as heavy as they are impossible to avoid to make the demonstration rigorous.

2 - The framework

Here and throughout the sequel, we assume that $n \geq 3$, and $p = \frac{n+2}{n-2}$.

We proceed to a suitable parametrization of the variational problem that we are interested in. Let us introduce the following notations.

For $k \in \mathbb{N}^*, \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}_+^k, \lambda = (\lambda_1, \cdots, \lambda_k) \in (\mathbb{R}_+^*)^k, y = (y^1, \cdots, y^k) \in (\partial\Omega)^k$, we set

$$\varphi(\alpha, \lambda, y) = \varphi_{\alpha,\lambda,y} = \sum_{i=1}^{k} \alpha_i U_i \tag{2.1}$$

where $U_i = U_{\lambda_i, y^i}$ and U_{λ_i, y^i} is defined by (1.3). For $\epsilon > 0, B_\epsilon$ denotes the subset of $\{(\alpha, \lambda, y) \in \mathbb{R}_+^k \times (\mathbb{R}_+^*)^k \times (\partial\Omega)^k\}$ such that

$$\begin{cases} \dfrac{\bar{\alpha}}{2} < \alpha_i < 2\bar{\alpha} \ , \text{ with } \bar{\alpha} = (n(n-2))^{\frac{n-2}{4}} \\ \lambda_i > \dfrac{1}{\epsilon} \qquad 1 \leq i \leq k \\ |y^i - y^j|^2 > \epsilon \qquad 1 \leq i < j \leq k \end{cases} \tag{2.2}$$

We define

$$\begin{aligned} V_\epsilon = \{ & u \in H^1(\Omega) / \exists (\lambda, y) \in (\mathbb{R}_+^*)^k \times (\partial\Omega)^k \text{ s.t.} \\ & (\bar{\alpha}, \lambda, y) \in B_\epsilon \text{ and } |\nabla(u - \varphi(\bar{\alpha}, \lambda, y))|_2 < \epsilon \} \end{aligned} \tag{2.3}$$

with the notation $\bar{\alpha} = (\bar{\alpha}, \cdots, \bar{\alpha})$.

We have the following proposition :

Proposition 2.1 — *Let $k \in \mathbb{N}^*$. There exists $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon \leq \epsilon_0$, and any $u \in V_\epsilon$, the problem*

$$\underset{(\alpha,\lambda,y) \in B_{4\epsilon}}{\text{Minimize}} \ |\nabla(u - \varphi(\alpha, \lambda, y))|_2$$

has a unique solution (up to permutations of the indices).

This statement is a direct adaptation to our case of [7, Proposition 7]. Its proof is given in Appendix A.

Once the unicity of the solution of the minimization problem is ensured, it is not difficult to see that it actually induces a diffeomorphism between V_{ϵ_0} and an open subset M' of the manifold

$$\begin{aligned} M =&\{(\alpha,\lambda,y,v)\in\mathbb{R}^k\times(\mathbb{R}_+^*)^k\times(\partial\Omega)^k\times H^1(\Omega)\\ &(\alpha,\lambda,y)\in B_{\epsilon_0}, v\in E_{\lambda,y}, |\nabla v|_2<\epsilon_0\} \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} E_{\lambda,y} =&\Bigg\{v\in H^1(\Omega)/\ \forall\ (i,j),\ 1\le i\le k\ ,\ 1\le j\le n\\ &\int_\Omega \nabla v.\nabla U_i=\int_\Omega \nabla v.\nabla\frac{\partial U_i}{\partial\lambda_i}=\int_\Omega \nabla v.\nabla\frac{\partial U_i}{\partial y_j^i}=0\Bigg\}. \end{aligned} \tag{2.5}$$

M' clearly contains some open subset N

$$\begin{aligned} N =&\{(\alpha,\lambda,y,v)\in\mathbb{R}^k\times(\mathbb{R}_+^*)^k\times(\partial\Omega)^k\times H^1(\Omega)\ \text{s.t.}\\ &|\alpha_i-\bar\alpha|<\eta_0, \lambda_i>\frac{1}{\eta_0},\forall\ i\ ;\ |y^i-y^j|>d_0,\forall\ i,j,i\neq j\ ;\\ &v\in E_{\lambda,y}, |\nabla v|_2<\eta_0\} \end{aligned} \tag{2.6}$$

for some constants $\eta_0>0, d_0>0$ (we note that, as d, d_0 may be chosen as small as we want, provided that $\eta_0>0$ is then taken sufficiently small).

Let us now consider the variational problem. The 0-homogeneity of I_μ with respect to u, which induces a simplified analysis in case of a single point of concentration, is not an advantage in the case of several points of concentration that we are interested in. Rather than I_μ, we consider the functional

$$J_\mu(u)=\frac{1}{2}\int_\Omega|\nabla u|^2+\frac{\mu}{2}\int_\Omega u^2-\frac{1}{p+1}\int_\Omega|u|^{p+1}\qquad u\in H^1(\Omega) \tag{2.7}$$

whose positive critical points are solutions to (1.1). We define on N the corresponding functional

$$\begin{aligned} &K_\mu\ :\ N\to\mathbb{R}\\ &(\alpha,\lambda,y,v)\mapsto K_\mu\left(\sum_{i=1}^k\alpha_iU_i+v\right). \end{aligned} \tag{2.8}$$

(α, λ, y, v) is a critical point of K_μ on N if and only if $u = \sum_{i=1}^{k} \alpha_i U_i + v$ is a critical point of J_μ on $H^1(\Omega)$. We set, for $1 \leq i \leq k$ and $1 \leq j \leq n-1$

$$\left\{ \begin{array}{l} \phi_i(\alpha, \lambda, y, v) = \displaystyle\int_\Omega \nabla v . \nabla U_i \\ \psi_i(\alpha, \lambda, y, v) = \displaystyle\int_\Omega \nabla v . \nabla \frac{\partial U_i}{\partial \lambda_i} \\ \xi_{i,j}(\alpha, \lambda, y, v) = \displaystyle\int_\Omega \nabla v . \nabla \frac{\partial U_i}{\partial \tau_j^i} \end{array} \right. \tag{2.9}$$

where the $\tau_j^{i\prime}s, 1 \leq j \leq n-1$, build an orthonormal system of coordinates on the tangent space to $\partial\Omega$ at y^i. Considering $K_\mu, \phi_i, \psi_i, \xi_{i,j}$ as functionals defined on an open subset of $\mathbb{R}_+^k \times (\mathbb{R}_+^*)^k \times (\partial\Omega)^k \times H^1(\Omega)$, (α, λ, y, v) will be a critical point of K_μ restricted to N if and only if there exists $A = (A_i)_{1 \leq i \leq k} \in \mathbb{R}^k, B = (B_i)_{1 \leq i \leq k} \in \mathbb{R}^k, C = (C_{i,j})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n-1}} \in \mathbb{R}^{(n-1)k}$ such that

$$K'_\mu = \sum_{i=1}^{k} (A_i \phi'_i + B_i \psi'_i + \sum_{j=1}^{n-1} C_{i,j} \xi'_{i,j}) \, . \tag{2.10}$$

The next section is devoted to a careful analysis of this equation.

3 - Analysis of the system

We are going to solve equation (2.10) step by step, beginning with the variable v.

In order to make the statements simpler, we assume that

$$\frac{\lambda_i}{\lambda_j} \leq \kappa_1 \qquad 1 \leq i, j \leq k \tag{3.1}$$

where κ_1 is a strictly positive constant, to be determined later. This assumption is not restrictive, since it will turn out that (3.1) always holds for the solutions that we look for. For sake of simplicity, we shall also denote, in the following, by $0(f(\lambda))$ a quantity which is dominated by $\sum_{i=1}^{k} f(\lambda_i)$ as the $\lambda_i^{\prime}s$ go to infinity.

Lastly, for v, w in $H^1(\Omega)$, we set

$$\left\{ \begin{array}{c} < v, w > = \displaystyle\int_\Omega \nabla v . \nabla w + \mu \int_\Omega vw \\ ||v||^2 = < v, v > \end{array} \right. \tag{3.2}$$

3.1 The v-part of u.

Here is the main statement of this subsection.

Proposition 3.1 — *There exists $\mu_0 > 0, \delta_0 > 0$, and a smooth map which to any $\mu > \mu_0$ and (α, λ, y) - such that $(\alpha, \lambda, y, 0) \in N$, (3.1) holds, and, for $1 \leq i \leq k$*

$$\text{(3.3)} \qquad \begin{cases} \inf\left(\frac{\mu}{\lambda_i^{\frac{3}{2}}}, \frac{\mu}{\lambda_i^{\frac{7}{4}}} + \frac{\mu^{\frac{1}{2}}}{\lambda_i^{\frac{5}{4}}}\right) < \delta_0 \text{ if } n = 5 \\ \frac{\mu(\mathrm{Log}\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} < \delta_0 \text{ if } n = 6 \\ \frac{\mu}{\lambda_i^2} < \delta_0 \text{ if } n \geq 7 \end{cases}$$

- associates $\bar{v} \in E_{\lambda,y}$ such that, at the point $(\alpha, \lambda, y, \bar{v})$

$$\text{(3.4)} \qquad \frac{\partial K_\mu}{\partial v} = \sum_{i=1}^{k} \left(A_i \frac{\partial \phi_i}{\partial v} + B_i \frac{\partial \psi_i}{\partial v} + \sum_{j=1}^{n-1} C_{i,j} \frac{\partial \xi_{i,j}}{\partial v} \right)$$

where $A_i, B_i, C_{i,j}$ are some real numbers depending on α, λ, y, μ.

Moreover, there exists $\nu_0 > 0$, independent of α, λ, y, μ such that such a $\bar{v}$ is unique in $\{v \in E_{\lambda,y} / |v|_{H^1} < \nu_0\}$, and we have the estimate

$$\text{(3.5)} \qquad \begin{aligned} ||\bar{v}|| = 0 \Bigg[& \left(\frac{1}{\lambda} + \inf \left(\frac{\mu}{\lambda^{3/2}}, \frac{\mu}{\lambda^{7/4}} + \frac{\mu^{1/2}}{\lambda^{5/4}} \right) \right) \text{ if } n = 5 \ ; \\ & \left(\frac{1}{\lambda} + \mu \frac{(\mathrm{Log}\ \lambda)^{2/3}}{\lambda^2} \right) \text{ if } n = 6 \ ; \left(\frac{1}{\lambda} + \frac{\mu}{\lambda^2} \right) \text{ if } n \geq 7 \Bigg] \end{aligned}$$

Proof of the proposition.

We expand $K_\mu(\alpha, \lambda, y, v)$ with respect to v in a neighbourhood of 0. According to the definition of K_μ, we find

$$\text{(3.6)} \qquad K_\mu(\alpha, \lambda, y, v) = K_\mu(\alpha, \lambda, y, 0) + L_{\mu,\alpha,\lambda,y}(v) + Q_{\mu,\alpha,\lambda,y}(v) + R_{\mu,\alpha,\lambda,y}(v)$$

with

$$\text{(3.7)} \qquad L_{\mu,\alpha,\lambda,y}(v) = \mu \int_\Omega \varphi_{\alpha,\lambda,y} v - \int_\Omega \varphi_{\alpha,\lambda,y}^p v$$

$$Q_{\mu,\alpha,\lambda,y}(v) = \frac{1}{2}\left(\int_\Omega |\nabla v|^2 + \mu \int_\Omega v^2 - p\int_\Omega \varphi_{\alpha,\lambda,y}^{p-1} v^2\right) \tag{3.8}$$

and

$$R_{\mu,\alpha,\lambda,y}(v) = 0\ (||v||^{\min(3,p+1)}). \tag{3.9}$$

Moreover, the derivatives of $R_{\mu,\alpha,\lambda,y}$ satisfy

$$R'_{\mu,\alpha,\lambda,y}(v) = 0(||v||^{\min(2,p)})\ ;\ R''_{\mu,\alpha,\lambda,y}(v) = 0(||v||^{\min(1,p-1)}) \tag{3.10}$$

uniformly with respect to μ, α, λ, y verifying the assumptions of the proposition.

At this stage, we need the following crucial result, about the uniform coercivity of the quadratic form $Q_{\mu,\alpha,\lambda,y}$.

Lemma 3.1 — *There exists $\rho > 0$ such that for any μ and (α, λ, y) verifying the assumptions of Proposition 3.1 (with μ_0 large enough, δ_0 small enough)*

$$Q_{\mu,\alpha,\lambda,y}(v) \geq \rho||v||^2 \quad \forall v \in E_{\lambda,y}\ .$$

The proof of this lemma is given in Appendix B. (In fact, for $n = 5, 6$, we do not need (3.3) to be satisfied, but only $\frac{\mu}{\lambda_i^2} < \delta_0$, $1 \leq i \leq k$, with δ_0 small enough.)

We claim that $L_{\mu,\alpha,\lambda,y}$, which is a continuous linear form on $E_{\lambda,y}$ equipped with the scalar product (3.2), is going to zero as μ goes to infinity, uniformly with respect to α, λ, y satisfying the assumptions of the proposition. Writing

$$\begin{cases} L_{\mu,\alpha,\lambda,y}(v) = < \ell_{\mu,\alpha,\lambda,y}, v > \\ Q_{\mu,\alpha,\lambda,y}(v) = \dfrac{1}{2} < T_{\mu,\alpha,\lambda,y} v, v > \end{cases} \tag{3.11}$$

with $\ell_{\mu,\alpha,\lambda,y} \in E_{\lambda,y}$ and $T_{\mu,\alpha,\lambda,y} \in \mathcal{L}(E_{\lambda,y})$ a continuous, symmetric and coercive operator whose modulus of coercivity is bounded from below, the derivative of K_μ with respect to v on $E_{\lambda,y}$ may be written as

$$\ell_{\mu,\alpha,\lambda,y} + T_{\mu,\alpha,\lambda,y} v + 0\ (||v||^{\min(2,p)}) \tag{3.12}$$

and the second derivative is equal to

$$T_{\mu,\alpha,\lambda,y} + 0\ (||v||^{\min(1,p-1)})\ . \tag{3.13}$$

Therefore, the implicit functions theorem provides us with a smooth map which to any μ and (α, λ, y) as in the statement of Proposition 3.1, with μ_0 large enough, associates $\bar{v} \in E_{\lambda,y}$ such that (3.4) is satisfied. Moreover, we have the estimate

$$||\bar{v}|| = 0\ (||\ell_{\mu,\alpha,\lambda,y}||) \tag{3.14}$$

and $\bar{v}$ is the unique solution of (3.4) in a suitable neighbourhood of zero in $E_{\lambda,y}$.

It only remains to prove the claim which, through (3.14), will also provide us with estimate (3.5). Using Hölder's inequality and Sobolev embedding theorem, we have, for any $v \in E_{\lambda,y}$

$$\mu \int_\Omega \varphi_{\alpha,\lambda,y} v = 0\ \left[\mu \left(\int_\Omega |v|^{p+1} \right)^{\frac{1}{p+1}} \sum_i \left(\int_\Omega U_i^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \right] .$$

From the computations in Appendix D - see (D.36) - and (3.1), we deduce that

$$\begin{aligned} &\mu \int_\Omega \varphi_{\alpha,\lambda,y} v \\ &= 0\ \left[||v|| \left(\frac{\mu}{\lambda^{3/2}} \text{ if } n = 5\ ;\ \frac{\mu (\mathrm{Log}\lambda)^{2/3}}{\lambda^2} \text{ if } n = 6\ ;\ \frac{\mu}{\lambda^2} \text{ if } n \geq 7 \right) \right] . \end{aligned} \tag{3.15}$$

The estimate for $n = 5$ ($p = \frac{7}{3}$) may be improved in the following way. We write

$$\begin{aligned} \mu \int_\Omega \varphi_{\alpha,\lambda,y} v &= 0\ \left[\mu \sum_i \int_{\Omega \cap B(y^i, \frac{1}{\lambda_i^{1/2}})} U_i v + \mu \sum_i \int_{\Omega \setminus B(y^i, \frac{1}{\lambda_i^{1/2}})} U_i v \right] \\ &= 0\ \left[\mu \left(\int_\Omega |v|^{10/3} \right)^{\frac{3}{10}} \sum_i \left(\int_{\Omega \cap B(y^i, \frac{1}{\lambda_i^{1/2}})} U_i^{\frac{10}{7}} \right)^{7/10} \right. \\ &\quad \left. + \mu \left(\int_\Omega v^2 \right)^{1/2} \sum_i \left(\int_{\Omega \setminus B(y^i, \frac{1}{\lambda_i^{1/2}})} U_i^2 \right)^{1/2} \right] \\ &= 0 \left(\frac{\mu ||v||}{\lambda^{7/4}} + \frac{\mu |v|_2}{\lambda^{5/4}} \right) \end{aligned}$$

according to (D.24,36). Noticing that, in view of (3.2), $\mu |v|_2 \leq \mu^{1/2} ||v||$, we obtain

$$\mu \int_\Omega \varphi_{\alpha,\lambda,y} v = 0\ \left[\left(\frac{\mu}{\lambda^{7/4}} + \frac{\mu^{1/2}}{\lambda^{5/4}} \right) ||v|| \right] \text{ for } n = 5\ . \tag{3.16}$$

In order to estimate the second term in (3.7), we expand

$$\varphi^p_{\alpha,\lambda,y} = \sum_i \alpha_i^p U_i^p + 0\left(\sum_{\substack{i,j \\ i\neq j}} U_i^{p-1} U_j\right). \tag{3.17}$$

On one hand, denoting by B_ℓ the ball of center y^ℓ and radius $d_0/2$, we have, in view of (1.3)

$$\begin{aligned}
&\int_\Omega U_i^{p-1} U_j v \\
&= \int_{B_i} U_i^{p-1} U_j v + \int_{B_j} U_i^{p-1} U_j v + \int_{\Omega\setminus B_i\cup B_j} U_i^{p-1} U_j v \\
&= 0\left(\frac{1}{\lambda_j^{\frac{n-2}{2}}}\int_{B_i} U_i^{p-1}|v| + \frac{1}{\lambda_i^2}\int_\Omega U_j|v| + \frac{1}{\lambda_i^2\lambda_j^{\frac{n-2}{2}}}\int_{\Omega\setminus B_i\cup B_j}|v|\right) \\
&= 0\left[|v|_{H^1}\left(\frac{1}{\lambda_j^{\frac{n-2}{2}}}\left(\int_{B_i} U_i^{\frac{(p-1)(p+1)}{p}}\right)^{\frac{p}{p+1}} + \frac{1}{\lambda_i^2}\left(\int_{B_j} U_j^{\frac{(p+1)}{p}}\right)^{\frac{p}{p+1}} + \frac{1}{\lambda_i^2\lambda_j^{\frac{n-2}{2}}}\right)\right]
\end{aligned}$$

using Hölder's inequalities and Sobolev embedding theorem. Estimates (D.36,38) and (3.1) yield

$$\int_\Omega U_i^{p-1} U_j v = 0\left[|v|_{H^1}\left(\frac{1}{\lambda^3} \text{ if } n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^4} \text{ if } n=6\ ;\ \frac{1}{\lambda^4} \text{ if } n\geq 7\right)\right]. \tag{3.18}$$

On the other hand

$$\begin{aligned}
\int_\Omega U_i^p v &= -\frac{1}{n(n-2)}\int_\Omega \Delta U_i \cdot v \qquad \text{because of (1.2-4)} \\
&= -\frac{1}{n(n-2)}\int_{\partial\Omega}\frac{\partial U_i}{\partial v}\cdot v \qquad \text{integrating by parts, since } v\in E_{\lambda,y}.
\end{aligned}$$

Using the embedding of $H^1(\Omega)$ into $L^{\frac{2(n-1)}{n-2}}(\partial\Omega)$ and Hölder's inequality, we obtain

$$\int_\Omega U_i^p v = 0\left[|v|_{H^1}\left(\int_{\partial\Omega}\left|\frac{\partial U_i}{\partial\nu}\right|^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}}\right].$$

There for, estimate (D.49) shows that

$$\int_\Omega U_i^p v = 0\left(\frac{|v|_{H^1}}{\lambda}\right). \tag{3.19}$$

From (3.15-19), we deduce that

$$||\ell_{\mu,\alpha,\lambda,y}|| = 0\left[\left(\frac{1}{\lambda} + \inf\left(\frac{\mu}{\lambda^{3/2}}, \frac{\mu}{\lambda^{7/4}} + \frac{\mu^{1/2}}{\lambda^{5/4}}\right)\right) \text{ if } n = 5\ ;\right.$$

$$\left.\left(\frac{\mu}{\lambda} + \mu\frac{(\text{Log }\lambda)^{2/3}}{\lambda^2}\right) \text{ if } n = 6\ ;\ \left(\frac{1}{\lambda} + \frac{\mu}{\lambda^2}\right) \text{ if } n \geq 7\right]$$

The claim is proved and (3.14) makes the proof of Proposition 3.1 complete.

Before ending this section, let us show how the coefficients $A_i, B_i, C_{i,j}$ which occur in (3.4) may be estimated. According to the definition of $K_\mu, \phi_i, \psi_i, \xi_{i,j}$, (3.4) means that for any $w \in H^1(\Omega)$

(3.20)

$$\int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla w + \mu\int_\Omega (\varphi_{\alpha,\lambda,y} + \bar{v})w - \int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v})w$$

$$= \sum_{i=1}^{k}\left(A_i\int_\Omega \nabla U_i.\nabla w + B_i\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla w + \sum_{j=1}^{n-1} C_{i,j}\int_\Omega \nabla\frac{\partial U_i}{\partial\tau_j^i}.\nabla w\right)$$

Taking successively $w = U_i, \frac{\partial U_i}{\partial\lambda_i}, \frac{\partial U_i}{\partial\tau_j^i}, 1 \leq j \leq n-1$, we obtain $k(n+1)$ equations which build a linear system for the $k(n+1)$ coeffcicients $A_i, B_i, C_{i,j}$. This system is nearly diagonal, since the functions $U_i, \frac{\partial U_i}{\partial\lambda_i}, \frac{\partial U_i}{\partial\tau_j^i}$ are nearly orthogonal for the bilinear form $(v,w) \mapsto \int_\Omega \nabla v.\nabla w$ - see estimates (D.26). As a consequence, the system is invertible, and computing the terms on the left hand side, which are equal to

$$\frac{\partial K_\mu}{\partial v}.U_i\ ,\ \frac{\partial K_\mu}{\partial v}.\frac{\partial U_i}{\partial\lambda_i}\ ,\ \frac{\partial K_\mu}{\partial v}.\frac{\partial U_i}{\partial\tau_j^i} \qquad \begin{matrix} 1 \leq i \leq k \\ 1 \leq j \leq n-1 \end{matrix}$$

respectively, we obtain an estimate of A_i, B_i, C_{ij}. The result is given in Appendix C, Lemma C.2.

3 - Optimization with respect to α and λ

Once the functional K_μ has been minimized with respect to v, we set

$$\widetilde{K}_\mu(\alpha,\lambda,y) = K_\mu(\alpha,\lambda,y,\bar{v}) \tag{3.21}$$

which is defined, for $\mu > \mu_0$, on

$$N_\mu = \{(\alpha,\lambda,y) \in \mathbb{R}^k \times (\mathbb{R}_+^k)^k \times (\partial\Omega)^k / (\alpha,\lambda,y,0) \in N, (3.1) \text{ and } (3.3) \text{ hold}\}\ .$$

In order to determine the critical points of $\widetilde{K}_\mu$ on N_μ, we consider the derivatives of $\widetilde{K}_\mu$ with respect to α and λ. Our aim in this subsection is to find, for given $y \in (\partial\Omega)^k$ - such that $|y^i - y^j| > d_0$, $1 \leq i, j \leq k$ - $(\alpha, \lambda) \in \mathbb{R}_+^k \times (\mathbb{R}_+^*)^k$ such that $(\alpha, \lambda, y) \in N_\mu$, and

$$\text{(3.22)} \qquad \begin{cases} \dfrac{\partial \widetilde{K}_\mu}{\partial \alpha_i} = \dfrac{\partial K_\mu}{\partial \alpha_i} + \dfrac{\partial K_\mu}{\partial v} \cdot \dfrac{\partial \bar{v}}{\partial \alpha_i} = 0 \\ \dfrac{\partial \widetilde{K}_\mu}{\partial \lambda_i} = \dfrac{\partial K_\mu}{\partial \lambda_i} + \dfrac{\partial K_\mu}{\partial v} \cdot \dfrac{\partial \bar{v}}{\partial \lambda_i} = 0 \end{cases} \qquad 1 \leq i \leq k \ .$$

Note that, in this subsection, we assume that

$$\text{(3.23)} \qquad \mu < \kappa_2 \lambda_i \qquad 1 \leq i \leq k$$

where $\kappa_2 > 0$ is a constant to be determined later. This assumption, which makes the computations and the statements clearer, will appear to be unrestrictive for our purposes. Then, (3.5) turns out to be

$$\text{(3.24)} \qquad ||\bar{v}|| = 0\left(\frac{1}{\lambda^{3/4}} \text{ if } n = 5 \ ; \ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda} \text{ if } n = 6 \ ; \ \frac{1}{\lambda} \text{ if } n \geq 7\right).$$

From the definition of K_μ and (3.24), computations show - see Lemma C.1 in appendix

$$\text{(3.25)} \qquad \begin{cases} \dfrac{\partial K_\mu}{\partial \alpha_i}(\alpha, \lambda, y, \bar{v}) = -C_6 \eta_i - C_7 \dfrac{H(y^i)}{\lambda_i} \\ \quad + 0\left[\eta_i^2 + \left(\dfrac{1}{\lambda^{3/2}} \text{ if } n = 5 \ ; \ \dfrac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^2} \text{ if } n = 6 \ ; \ \dfrac{1}{\lambda^2} \text{ if } n \geq 7\right)\right] \\ \dfrac{\partial K_\mu}{\partial \lambda_i}(\alpha, \lambda, y, \bar{v}) = C_8 \dfrac{H(y^i)}{\lambda_i^2} - C_9 \dfrac{\mu}{\lambda_i^3} \\ \quad + 0\left[\dfrac{|\eta_i|}{\lambda_i^2} + \left(\dfrac{1}{\lambda^{5/2}} \text{ if } n = 5 \ ; \ \dfrac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^3} \text{ if } n = 6 \ ; \ \dfrac{1}{\lambda^3} \text{ if } n \geq 7\right)\right] \end{cases}$$

with

$$\eta_i = \frac{\alpha_i}{\bar{\alpha}} - 1$$

and C_6, C_7, C_8, C_9 are strictly positive constants which depends on n only. Let us now estimate the remaining quantities, $\frac{\partial K_\mu}{\partial v} \cdot \frac{\partial \bar{v}}{\partial \alpha_i}$ and $\frac{\partial K_\mu}{\partial v} \cdot \frac{\partial \bar{v}}{\partial \lambda_i}$. According to (2.9)

and (3.4), we have

$$
(3.26)\qquad \begin{cases} \dfrac{\partial K_\mu}{\partial v}\cdot\dfrac{\partial \bar v}{\partial \alpha_i} = \displaystyle\sum_{j=1}^{k}\Big(A_j\int_\Omega \nabla U_j.\nabla\frac{\partial \bar v}{\partial \alpha_i} + B_j\int_\Omega \nabla\frac{\partial U_j}{\partial \lambda_j}.\nabla\frac{\partial \bar v}{\partial \alpha_i} \\ \qquad\qquad + \displaystyle\sum_{\ell=1}^{n-1} C_{j,\ell}\int_\Omega \nabla\frac{\partial U_j}{\partial \tau_\ell^j}.\nabla\frac{\partial \bar v}{\partial \alpha_i}\Big) \\ \dfrac{\partial K_\mu}{\partial v}.\dfrac{\partial \bar v}{\partial \lambda_i} = \displaystyle\sum_{j=1}^{k}\Big(A_j\int_\Omega \nabla U_j.\nabla\frac{\partial \bar v}{\partial \lambda_i} + B_j\int_\Omega \nabla\frac{\partial U_j}{\partial \lambda_j}.\nabla\frac{\partial \bar v}{\partial \lambda_i} \\ \qquad\qquad + \displaystyle\sum_{\ell=1}^{n-1} C_{j,\ell}\int_\Omega \nabla\frac{\partial U_j}{\partial \tau_\ell^j}.\nabla\frac{\partial \bar v}{\partial \lambda_i}\Big). \end{cases}
$$

Since $\bar v \in E_{\lambda,y}$

$$
(3.27)\qquad \int_\Omega \nabla U_j.\nabla \bar v = \int_\Omega \nabla\frac{\partial U_j}{\partial \lambda_j}.\nabla \bar v = \int_\Omega \nabla\frac{\partial U_j}{\partial \tau_\ell^j}.\nabla \bar v = 0 \qquad \begin{matrix} 1\le j\le k \\ 1\le \ell\le n-1\ . \end{matrix}
$$

Differentiating these equalities with respect to α_i, we see that they are as well satisfied with $\frac{\partial \bar v}{\partial \alpha_i}$ instead of $\bar v$. This means that $\frac{\partial \bar v}{\partial \alpha_i} \in E_{\lambda,y}$, and it follows from (3.26) that

$$
(3.28)\qquad \frac{\partial K_\mu}{\partial v}.\frac{\partial \bar v}{\partial \alpha_i} = 0\ .
$$

Differentiating (3.27) with respect to λ_i, one obtains

$$
\int_\Omega \nabla U_j.\nabla\frac{\partial \bar v}{\partial \lambda_i} = -\int_\Omega \nabla\frac{\partial U_j}{\partial \lambda_i}.\nabla \bar v = 0
$$

$$
\int_\Omega \nabla\frac{\partial U_j}{\partial \lambda_j}.\nabla\frac{\partial \bar v}{\partial \lambda_i} = -\int_\Omega \nabla\frac{\partial^2 U_j}{\partial \lambda_i\partial \lambda_j}.\nabla \bar v = \begin{cases} 0 \text{ if } i\neq j \\ 0\Big(\dfrac{||v||}{\lambda_i^2}\Big) \text{ if } i=j \end{cases}
$$

$$
\int_\Omega \bar v\frac{\partial U_j}{\partial \tau_\ell^j}.\nabla\frac{\partial v}{\partial \lambda_i} = -\int_\Omega \nabla\frac{\partial^2 U_j}{\partial \lambda_i\partial \tau_\ell^j}.\nabla \bar v = \begin{cases} 0 \text{ if } i\neq j \\ 0(||\bar v||) \text{ if } i=j \end{cases}
$$

since

$$
\int_\Omega \left|\nabla\frac{\partial^2 U_i}{\partial \lambda_i^2}\right|^2 = 0\Big(\frac{1}{\lambda_i^4}\Big) \text{ and } \int_\Omega \left|\nabla\frac{\partial^2 U_i}{\partial \lambda_i\partial y_\ell^i}\right|^2 = 0\ (1)\ ,
$$

as a simple computation shows - see (D.30). Therefore, (3.26), the estimate of $B_i, C_{i,\ell}$ given by Lemma C.2, and (3.24), imply that

$$
(3.29)\qquad \frac{\partial K_\mu}{\partial v}.\frac{\partial \bar v}{\partial \lambda_i} = 0\left(\frac{1}{\lambda^{11/4}} \text{ if } n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^3} \text{ if } n=6\ ;\ \frac{1}{\lambda^3} \text{ if } n\ge 7\right)\ .
$$

Finally, from (3.26-29), we deduce

$$(3.30)\quad \begin{cases} \dfrac{\partial \widetilde{K}_\mu}{\partial \alpha_i} = -C_6\eta_i - C_7\dfrac{H(y^i)}{\lambda_i} \\ \qquad\qquad + 0\left[\eta_i^2 + \left(\dfrac{1}{\lambda^{3/2}} \text{ if } n=5\ ;\ \dfrac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^2} \text{ if } n=6\ ;\ \dfrac{1}{\lambda^2} \text{ if } n\geq 7\right)\right] \\ \dfrac{\partial \widetilde{K}_\mu}{\partial \lambda_i} = C_8\dfrac{H(y^i)}{\lambda_i^2} - C_9\dfrac{\mu}{\lambda_i^3} \\ \qquad\qquad + 0\left[\dfrac{|\eta_i|}{\lambda_i^2} + \left(\dfrac{1}{\lambda^{5/2}} \text{ if } n=5\ ;\ \dfrac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^3} \text{ if } n=6\ ;\ \dfrac{1}{\lambda^3} \text{ if } n\geq 7\right)\right]. \end{cases}$$

Let us assume now that

$$(3.31)\qquad H(y^i) > c > 0 \qquad 1\leq i\leq k\ .$$

With the change of variables

$$(3.32)\qquad \zeta_i = \frac{C_9}{C_8}\ \frac{\mu}{H(y^i)}\ \frac{1}{\lambda_i} - 1\ ;\ \xi_i = -C_6\mu\eta_i - C_7\mu H(y^i)\frac{1}{\lambda_i}$$

(3.30) writes

$$(3.33)\quad \begin{cases} \dfrac{\partial \widetilde{K}_\mu}{\partial \alpha_i} = \dfrac{\xi_i}{\mu} \\ \qquad\qquad + 0\left(\dfrac{1}{\mu^{3/2}} \text{ if } n=5\ ;\ \dfrac{(\mathrm{Log}\mu)^{4/3}}{\mu^2} \text{ if } n=6\ ;\ \dfrac{1}{\mu^2} \text{ if } n\geq 7\right) \\ \dfrac{\partial \widetilde{K}_\mu}{\partial \lambda_i} = -\dfrac{C_8^3}{C_9^2}\ \dfrac{H^3(y^i)}{\mu^2}\ \zeta_i(1+\zeta_i)^2 \\ \qquad\qquad + 0\left(\dfrac{1}{\mu^{5/2}} \text{ if } n=5\ ;\ \dfrac{(\mathrm{Log}\mu)^{\frac{4}{3}}}{\mu^3} \text{ if } n=6\ ;\ \dfrac{1}{\mu^3} \text{ if } n\geq 7\right). \end{cases}$$

Then, $\frac{\partial \widetilde{K}_\mu}{\partial \alpha_i} = \frac{\partial \widetilde{K}_\mu}{\partial \lambda_i} = 0,\ 1\leq i\leq k$, turns out to be equivalent to

$$(3.34)\qquad \begin{cases} \xi_i = V_i(\mu, y, \xi, \zeta) \\ \zeta_i = W_i(\mu, y, \xi, \zeta) \end{cases} \qquad 1\leq i\leq k$$

where V_i, W_i are smooth functions which satisfy

$$V_i, W_i = 0\left(\frac{1}{\mu^{1/2}} \text{ if } n=5\ ;\ \frac{(\mathrm{Log}\mu)^{\frac{4}{3}}}{\mu} \text{ if } n=6\ ;\ \frac{1}{\mu} \text{ if } n\geq 7\right)$$

in a neighbourhood of $\xi = \zeta = 0$. Therefore, Brouwer's fixed point theorem ensures the existence, for μ large enough, of $\xi^{\mu,y}, \zeta^{\mu,y}$, such that (3.34) is satisfied. This means that for any $y \in (\partial\Omega)^k$ such that $|y^i - y^j| > d_0$ for $i \neq j$ and (3.31) holds, and any μ large enough (μ bounded from below in function of c), there exist $\alpha^{\mu,y}$ and $\lambda^{\mu,y}$ such that

$$\text{(3.35)} \qquad \begin{cases} \dfrac{\partial \widetilde{K}_\mu}{\partial \alpha_i}\,(\alpha^{\mu,y}, \lambda^{\mu,y}, y, \bar{v}(\alpha^{\mu,y}, \lambda^{\mu,y}, y)) = 0 \\ \dfrac{\partial \widetilde{K}_\mu}{\partial \lambda_i}\,(\alpha^{\mu,y}, \lambda^{\mu,y}, y, \bar{v}(\alpha^{\mu,y}, \lambda^{\mu,y}, y)) = 0 \end{cases} \qquad 1 \leq i \leq k.$$

Moreover, (3.30) implies

$$\text{(3.36)} \qquad \begin{cases} \alpha_i^{\mu,y} = \bar{\alpha} - \bar{\alpha}\dfrac{C_7 C_8}{C_6 C_9}\,\dfrac{H^2(y^i)}{\mu} \\ \qquad + 0\left(\dfrac{1}{\mu^{3/2}} \text{ if } n = 5\ ;\ \dfrac{(\mathrm{Log}\mu)^{4/3}}{\mu^2} \text{ if } n = 6\ ;\ \dfrac{1}{\mu^2} \text{ if } n \geq 7\right) \\ \lambda_i^{\mu,y} = \dfrac{C_9}{C_8}\,\dfrac{\mu}{H(y^i)} \\ \qquad + 0(\mu^{1/2} \text{ if } n = 5\ ;\ (\mathrm{Log}\mu)^{4/3} \text{ if } n = 6\ ;\ 1 \text{ if } n \geq 7)\ . \end{cases}$$

We verify that the assumptions of Proposition 3.1 are satisfied by $(\mu, \alpha^{\mu,y}, \lambda^{\mu,y}, y)$, and

$$\frac{\lambda_i^{\mu,y}}{\lambda_j^{\mu,y}} \leq \kappa_1\ ,\ \mu < \kappa_2 \lambda_i^{\mu,y} \qquad 1 \leq i \leq k,\ 1 \leq j \leq k,\ i \neq j$$

provided that (3.31) holds and μ is large enough, with

$$\text{(3.38)} \qquad \kappa_1 = \frac{2 \max\limits_{y \in \partial\Omega} H(y)}{c}\ ;\ \kappa_2 = \frac{2C_8 \max\limits_{y \in \partial\Omega} H(y)}{C_9}.$$

We note that we have also, for μ large enough

$$\text{(3.39)} \qquad \kappa_3 \lambda_i^{\mu,y} < \mu \quad 1 \leq i \leq k, \quad \text{with} \quad \kappa_3 = \frac{C_8 c}{2C_9}.$$

Further computations, giving estimates of the second derivatives of $\widetilde{K}_\mu$ with respect to the $\alpha_i' s$ and $\lambda_i' s$ - see, for instance, [19, Appendix C] - show that $(\alpha^{\mu,y}, \lambda^{\mu,y})$ such the assumptions of Proposition 3.1 and (3.35) hold is actually unique, and is a nondegenerate critical point of the map

$$(\alpha, \lambda) \mapsto \widetilde{K}_\mu(\alpha, \lambda, y, \bar{v}(\alpha, \lambda, y))\ .$$

As a consequence, implicit function theorem provides us with a smooth map which to any $y \in (\partial\Omega)^k$, such that $|y^i - y^j| > d_0$ for $i \neq j$ and $H(y^i) > c$ for any i, associates $(\alpha^{\mu,y}, \lambda^{\mu,y})$ such that (3.35) is satisfied, together with the estimates (3.36). (We recall that $d_0 > 0$ and $c > 0$ may be chosen arbitrarily small, provided that μ is then assumed to be large enough.)

3.3 A function which depends on y only.

On

$$P_{c,d_0} = \{y \in (\partial\Omega)^k / |y^i - y^j| > d_0, 1 \leq i \leq j \leq k \ ; \ H(y^i) > c, 1 \leq i \leq k\} \tag{3.40}$$

we set

$$\widetilde{\widetilde{K}}_\mu(y) = \widetilde{K}_\mu(\alpha^{\mu,y}, \lambda^{\mu,y}, y) = \widetilde{K}_\mu(\alpha^{\mu,y}, \lambda^{\mu,y}, y, \bar{v}(\alpha^{\mu,y}, \lambda^{\mu,y}, y)) \ . \tag{3.41}$$

According to (3.36,24), we have

$$||\bar{v}(\alpha^{\mu,y}, \lambda^{\mu,y}, y)|| = 0\left(\frac{1}{\mu^{3/4}} \text{ if } n = 5 \ ; \ \frac{(\mathrm{Log}\mu)^{\frac{2}{3}}}{\mu} \text{ if } n = 6 \ ; \ \frac{1}{\mu} \text{ if } n \geq 7\right) \tag{3.42}$$

and from subsection 3.1 we derive

$$\begin{aligned} \widetilde{K}_\mu(\alpha^{\mu,y}, \lambda^{\mu,y}, y) &= K_\mu(\alpha^{\mu,y}, \lambda^{\mu,y}, y, 0) \\ &+ 0\left(\frac{1}{\mu^{3/2}} \text{ if } n = 5 \ ; \ \frac{(\mathrm{Log}\mu)^{\frac{4}{3}}}{\mu^2} \text{ if } n = 6 \ ; \ \frac{1}{\mu^2} \text{ if } n \geq 7\right) . \end{aligned} \tag{3.43}$$

The definition of K_μ yields

$$K_\mu(\alpha, \lambda, y, 0) = \frac{1}{2}\left(\int_\Omega (|\nabla\varphi_{\alpha,\lambda,y}|^2 + \mu\varphi^2_{\alpha,\lambda,y})\right) - \frac{1}{p+1}\int_\Omega \varphi^{p+1}_{\alpha,\lambda,y} \ .$$

Therefore

$$\begin{aligned} K_\mu(\alpha, \lambda, y, 0) = \frac{1}{2}\Bigg[&\sum_{i=1}^k \alpha_i^2\left(\int_\Omega |\nabla U_i|^2 + \mu\int_\Omega U_i^2\right) \\ &+ \sum_{\substack{1\leq i,j\leq k \\ i\neq j}} \alpha_i\alpha_j\left(\int_\Omega \nabla U_i.\nabla U_j + \mu\int_\Omega U_iU_j\right)\Bigg] \\ &- \frac{1}{p+1}\left[\sum_{i=1}^k \alpha_i^{p+1}\int_\Omega U_i^{p+1} + 0\left(\sum_{\substack{1\leq i,j\leq k \\ i\neq j}} \int_\Omega U_i^pU_j\right)\right] \end{aligned} \tag{3.44}$$

and from (D.6,17,21,25,31,42) and (3.36,41,43), we deduce

$$\begin{aligned}\tilde{\tilde{K}}_\mu(y) = &-\frac{(n-1)(n-3)}{4}kS^{n/2} + \frac{C_{10}}{\mu}\sum_{i=1}^{k} H^2(y^i) \\ &+ 0\left(\frac{1}{\mu^{3/2}} \text{ if } n=5\ ;\ \frac{(\text{Log}\mu)^{\frac{4}{3}}}{\mu^2} \text{ if } n=6\ ;\ \frac{1}{\mu^2} \text{ if } n\geq 7\right)\end{aligned} \tag{3.45}$$

with

$$C_{10} = \frac{C_8}{2C_9}\left((n(n-2)-1)\frac{C_7}{C_6}S^{n/2} + \bar{\alpha}^2((n-2)^2C_4 - C_1 + 2\frac{C_5C_8}{C_9})\right) > 0\ .$$

According to the definition of $\tilde{\tilde{K}}_\mu$, we have also

$$\frac{\partial\,\tilde{\tilde{K}}_\mu}{\partial\tau_j^i} = \frac{\partial K_\mu}{\partial\tau_j^i} + \frac{\partial K_\mu}{\partial v}.\frac{\partial\bar{v}}{\partial\tau_j^i} \qquad 1\leq i\leq k\ ,\ 1\leq j\leq n-1 \tag{3.46}$$

at the point $(\alpha^{\mu,y}, \lambda^{\mu,y}, y, \bar{v}(\alpha^{\mu,y}, \lambda^{\mu,y}, y))$. Similarly to (3.26), we have

$$\begin{aligned}\frac{\partial K_\mu}{\partial v}.\frac{\partial\bar{v}}{\partial\tau_j^i} = \sum_{\ell=1}^{k}\Bigg(&A_\ell\int_\Omega \nabla U_j.\nabla\frac{\partial\bar{v}}{\partial\tau_j^i} \\ &+ B_\ell\int_\Omega \nabla\frac{\partial U_\ell}{\partial\lambda_\ell}.\nabla\frac{\partial\bar{v}}{\partial\tau_j^i} + \sum_{m=1}^{n-1} C_{\ell,m}\int_\Omega \nabla\frac{\partial U_\ell}{\partial\tau_m^\ell}.\nabla\frac{\partial\bar{v}}{\partial\tau_j^i}\Bigg)\end{aligned} \tag{3.47}$$

and, differentiating (3.27) with respect to y_j^i, we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla U_\ell.\nabla\frac{\partial\bar{v}}{\partial y_j^i} = -\int_\Omega \nabla\frac{\partial U_\ell}{\partial y_j^i}.\nabla\bar{v} = 0 \\ \displaystyle\int_\Omega \nabla\frac{\partial U_\ell}{\partial\lambda_\ell}.\nabla\frac{\partial\bar{v}}{\partial y_j^i} = -\int_\Omega \nabla\frac{\partial^2 U_\ell}{\partial y_j^i\partial\lambda_\ell}.\nabla\bar{v} = \begin{cases} 0 \text{ if } i\neq\ell \\ 0(||\bar{v}||) \text{ if } i=\ell \end{cases} \\ \displaystyle\int_\Omega \nabla\frac{\partial U_\ell}{\partial y_m^\ell}.\nabla\frac{\partial\bar{v}}{\partial y_j^i} = -\int_\Omega \nabla\frac{\partial^2 U_\ell}{\partial y_j^i\partial y_m^\ell}.\nabla\bar{v} = \begin{cases} 0 \text{ if } i\neq\ell \\ 0(\lambda_i^2||\bar{v}||) \text{ if } i=\ell \end{cases} \end{cases} \tag{3.48}$$

since $\int_\Omega |\nabla\frac{\partial^2 U_i}{\partial y_j^i\partial\lambda_i}|^2 = 0(1)$ and $\int_\Omega |\nabla\frac{\partial^2 U_i}{\partial y_j^i\partial y_m^i}|^2 = 0(\lambda_i^2)$ - see (D.30). On the other hand, taking successively $w = U_i, \frac{\partial U_i}{\lambda y_i}, \frac{\partial U_i}{\partial\tau_j^i}, 1\leq i\leq k\ ,\ 1\leq j\leq n-1$, in (3.20), we obtain a linear system

$$\frac{\partial K_\mu}{\partial\alpha_i} = \sum_{\ell=1}^{k}\left(A_\ell\int_\Omega \nabla U_\ell.\nabla U_i + B_\ell\int_\Omega \nabla\frac{\partial U_\ell}{\partial\lambda_\ell}.\nabla U_i + \sum_{m=1}^{n-1} C_{\ell,m}\int_\Omega \nabla\frac{\partial U_\ell}{\partial\tau_m^\ell}.\nabla U_i\right)$$

$$\frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \lambda_i}=\sum_{\ell=1}^{k}\left(A_\ell\int_\Omega \nabla U_\ell.\nabla\frac{\partial U_i}{\partial\lambda_i}+B_\ell\int_\Omega\nabla\frac{\partial U_\ell}{\partial\lambda_\ell}.\nabla\frac{\partial U_i}{\partial\lambda_i}+\sum_{m=1}^{n-1}C_{\ell,m}\int_\Omega\nabla\frac{\partial U_\ell}{\partial\tau_m^\ell}.\nabla\frac{\partial U_i}{\partial\lambda_i}\right)$$

$$\frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \tau_j^i}=\sum_{\ell=1}^{k}\left(A_\ell\int_\Omega \nabla U_\ell.\nabla\frac{\partial U_i}{\partial\tau_j^i}+B_\ell\int_\Omega\nabla\frac{\partial U_\ell}{\partial\lambda_\ell}.\nabla\frac{\partial U_i}{\partial\tau_j^i}+\sum_{m=1}^{n-1}C_{\ell,m}\int_\Omega\nabla\frac{\partial U_\ell}{\partial\tau_m^\ell}.\nabla\frac{\partial U_i}{\partial\tau_j^i}\right).$$

At the considered point, we have - see (3.22,28-29)

$$\frac{\partial K_\mu}{\partial\alpha_i}=0\ ;\ \frac{\partial K_\mu}{\partial\lambda_i}=0\left(\frac{1}{\lambda^{1/4}}\text{ if }n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^3}\text{ if }n=6\ ;\ \frac{1}{\lambda^3}\text{ if }n\geq 7\right)$$

with $\lambda=\lambda^{\mu,y}$ and, according to Lemma C.1 in appendix - see (C.5,7)

$$\frac{\partial K_\mu}{\partial\tau_j^i}=0\left(\frac{1}{\lambda^{1/2}}\text{ if }n=5\ ;\ \frac{1}{\lambda}\text{ if }n\geq 6\right).$$

Therefore, using (D.27), we deduce from this invertible linear system that

$$(3.49)\qquad\begin{cases}A_\ell=0\left(\frac{1}{\lambda^{3/2}}\text{ if }n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^2}\text{ if }n=6\ ;\ \frac{1}{\lambda^2}\text{ if }n\geq 7\right)\\ B_\ell=0\left(\frac{1}{\lambda^{1/2}}\text{ if }n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda}\text{ if }n=6\ ;\ \frac{1}{\lambda}\text{ if }n\geq 7\right)\\ C_{\ell,m}=0\left(\frac{1}{\lambda^{5/2}}\text{ if }n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^3}\text{ if }n=6\ ;\ \frac{1}{\lambda^3}\text{ if }n\geq 7\right).\end{cases}$$

Finally, (3.47-49,24) yield

$$\frac{\partial K_\mu}{\partial v}.\frac{\partial\bar v}{\partial\tau_j^i}=0\left(\frac{1}{\lambda^{5/4}}\text{ if }n=5\ ;\ \frac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^2}\text{ if }n=6\ ;\ \frac{1}{\lambda^2}\text{ if }n\geq 7\right).$$

This, together with (3.46) and (C.7) - see Lemma C.1 in appendix - gives, for $n\geq 6$

$$\frac{\partial\,\tilde{\tilde K}_\mu(y)}{\partial\tau_j^i}=-\frac{C_8}{\lambda_i}\frac{\partial H}{\partial\tau_j^i}(y^i)+0\left(\frac{(\mathrm{Log}\lambda)^{\frac{4}{3}}}{\lambda^{7/6}}\text{ if }n=6\ ;\ \frac{1}{\lambda^{\inf(\frac{7}{6},\frac{n}{n-2})}}\text{ if }n\geq 7\right)$$

with $\lambda=\lambda^{\mu,y}$, so that, using (3.36)

$$(3.50)\qquad\begin{aligned}\frac{\partial\,\tilde{\tilde K}_\mu}{\partial\tau_j^i}(y)&=-\frac{C_8^2}{C_9}\frac{H(y^i)}{\mu}\frac{\partial H}{\partial\tau_j^i}(y^i)\\&\quad+0\left(\frac{(\mathrm{Log}\mu)^{\frac{4}{3}}}{\mu^{7/6}}\text{ if }n=6\ ;\ \frac{1}{\mu^{\inf(\frac{7}{6},\frac{n}{n-2})}}\text{ if }n\geq 7\right).\end{aligned}$$

We are now able to prove the theorems.

4 - Proof of the theorems

4.1. Proof of Theorem 1.1 a)

Let $n \geq 7$ and u_μ be a solution of (1.1) such that

$$I_\mu(u_\mu) = \frac{\int_\Omega |\nabla u_\mu|^2 + \mu \int_\Omega u_\mu^2}{(\int_\Omega |u_\mu|^{p+1})^{\frac{2}{p+1}}} < \frac{S}{2^{2/n}} \tag{4.1}$$

Multiplying (1.1) by u_μ and integrating on Ω, we get

$$\int_\Omega |\nabla u_\mu|^2 + \mu \int_\Omega u_\mu^2 = \int_\Omega u_\mu^{p+1} \tag{4.2}$$

As a consequence, there exists $x \in \Omega$ such that $u_\mu(x) \geq \mu^{\frac{1}{p-1}}$, and $\max_{\bar{\Omega}} u_\mu$ goes to infinity as μ goes to infinity. It follows from a classical blow-up analysis - see [11], or [4, Lemma 2.2] - that $\max_{\bar{\Omega}} u_\mu$ is attained at a unique point $y_\mu \in \partial\Omega$ in $\bar{\Omega}$, and

$$|\nabla(u_\mu - \bar{\alpha} U_{\lambda_\mu, y_\mu})|_2 \to 0 \quad \text{as} \quad \mu \to +\infty \tag{4.3}$$

with

$$\lambda_\mu = \left(\frac{u_\mu(y_\mu)}{\bar{\alpha}}\right)^{\frac{2}{n-2}} \geq \frac{\mu^{1/2}}{\bar{\alpha}^{\frac{2}{n-2}}} \tag{4.4}$$

Therefore, Proposition 2.1 ensures that, for μ large enough, there is a unique way to write

$$u_\mu = \alpha_\mu U_{\lambda'_\mu, y'_\mu} + v_\mu \quad (\alpha_\mu, \lambda'_\mu, y'_\mu, v_\mu) \in N \tag{4.5}$$

where N is defined in (2.6). Since $(\alpha_\mu, \lambda'_\mu, y'_\mu)$ minimizes $|\nabla(u_\mu - \alpha U_{\lambda,y})|_2$ with respect to (α, λ, y) in a set containing $(\bar{\alpha}, \lambda_\mu, y_\mu)$, (4.3) implies that

$$|\nabla v_\mu|_2 \to 0 \quad \text{as} \quad \mu \to +\infty \,. \tag{4.6}$$

Then, it follows from Lemma A.1 in appendix that

$$\alpha_\mu \to \bar{\alpha}, \frac{\lambda_\mu}{\lambda'_\mu} \to 1 \,, \; \lambda_\mu \lambda'_\mu |y_\mu - y'_\mu|^2 \to 0 \quad \text{as} \quad \mu \to +\infty \,. \tag{4.7}$$

From (4.1,2) we deduce

$$\int_\Omega |\nabla u_\mu|^2 + \mu \int_\Omega u_\mu^2 < \frac{S^{n/2}}{2}. \tag{4.8}$$

In view of (4.5), estimate (D.6) yields

$$\int_\Omega |\nabla u_\mu|^2 = \alpha_\mu^2 \frac{S^{n/2}}{2(n(n-2))^{\frac{n-2}{2}}} - \alpha_\mu^2 C_1 \frac{H(y'_\mu)}{\lambda'_\mu} + 0(\frac{1}{\lambda'^2_\mu} + |\nabla v_\mu|_2^2) \tag{4.9}$$

so that, using (4.4,6-7)

$$\int_\Omega |\nabla u_\mu|^2 = \frac{S^{n/2}}{2} + o(1) \quad \text{as} \quad \mu \to +\infty \, .$$

As a consequence

$$\mu \int_\Omega u_\mu^2 = o(1) \quad \text{as} \quad \mu \to +\infty \tag{4.10}$$

and, since, according to (D.21) and (4.4,7)

$$\mu \int_\Omega U^2_{\lambda'_\mu, y'_\mu} = \mu \frac{C_5}{\lambda'^2_\mu} + o(\frac{\mu}{\lambda'^3_\mu}) = 0(1) \quad \text{as} \quad \mu \to +\infty \tag{4.11}$$

(4.6,10) imply that

$$\mu \int_\Omega v_\mu^2 = 0(1) \qquad |v_\mu|_{H^1} = o(1) \quad \text{as} \quad \mu \to +\infty$$

Proceeding as in Subsection 3.1, and using (4.11), we obtain

$$\mu \int_\Omega u_\mu^2 = \mu \alpha_\mu^2 \frac{C_5}{\lambda'^2_\mu} + \mu \int_\Omega v_\mu^2 + 0\left(\frac{\mu}{\lambda'^2_\mu}|v_\mu|_{H^1} + \frac{\mu}{\lambda'^3_\mu}\right) \, . \tag{4.12}$$

Following again the arguments of Subsection 3.1, and using (D.17), we have also

$$\int_\Omega u_\mu^{p+1} = \alpha_\mu^{p+1} \frac{S^{n/2}}{2(n(n-2))^{\frac{n}{2}}} - \alpha_\mu^{p+1} C_4 \frac{H(y'_\mu)}{\lambda'_\mu} + 0\left(|v_\mu|^2_{H^1} + \frac{1}{\lambda'^2_\mu}\right). \tag{4.13}$$

From (4.9,12-13), we deduce

$$\begin{aligned} I_\mu(u_\mu) = & \frac{S}{2^{2/n}} + \frac{2^{\frac{n-2}{n}}(n(n-2))^{\frac{n-2}{2}}}{S^{\frac{n-2}{2}}} \left(-C_{11}\frac{H(y'_\mu)}{\lambda'_\mu} + C_5 \frac{\mu}{\lambda'^2_\mu} + \frac{\mu}{\alpha_\mu^2}\int_\Omega v_\mu^2\right) \\ & + 0\left(\frac{1}{\lambda'^2_\mu} + \frac{\mu}{\lambda'^3_\mu} + \frac{\mu}{\lambda_{\mu'^2}}|v_\mu|^2_{H^1} + |v_\mu|^2_{H^1}\right) \end{aligned} \tag{4.14}$$

with

$$C_{11} = C_1 - (n-2)^2 C_4 = \frac{(n-2)^2 \sigma_{n-2}}{2} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-3}{2})}{\Gamma(n)} > 0$$

$$C_5 = \frac{\sigma_{n-1}}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)} > 0.$$

Therefore, as

$$\frac{1}{\lambda_\mu'^2} + \frac{\mu}{\lambda_\mu'^3} + \frac{\mu}{\lambda_\mu'^2}|v_\mu|_{H^1}^2 + |v_\mu|_{H^1}^2 = o(1 + \frac{\mu}{\lambda_\mu'^2})$$

(4.1) shows that necessarily

$$\frac{\mu}{\lambda_\mu'^2} = o(1) \tag{4.15}$$

as μ goes to infinity. It follows from (4.15) and Proposition 3.1 that, actually

$$v_\mu = \bar{v}(\alpha_\mu, \lambda_\mu', y_\mu')$$

for μ large enough. (3.5) yields

$$\|v_\mu\|^2 = \int_\Omega |\nabla v_\mu|^2 + \mu \int_\Omega v_\mu^2 = 0\left(\frac{1}{\lambda_\mu'^2} + \frac{\mu^2}{\lambda_\mu'^4}\right).$$

Hence, coming back to (4.14), we find

$$I_\mu(u_\mu) = \frac{S}{2^{2/n}} + \frac{2^{\frac{n-2}{2}}(n(n-2))^{\frac{n-2}{2}}}{S^{\frac{n-2}{2}}}\left(-C_{11}\frac{H(y_\mu')}{\lambda_\mu'} + C_5\frac{\mu}{\lambda_\mu'^2}\right) + 0\left(\frac{1}{\lambda_\mu'^2} + \frac{\mu^2}{\lambda_\mu'^4}\right) \tag{4.16}$$

Then, (4.1,15) show that, necessarily

$$H(y_\mu') > 0 \tag{4.17}$$

whence, if y^o is an accumulation point of (y_μ)

$$H(y^o) \geq 0 \tag{4.18}$$

since (4.7) shows that y^o is also an accumulation point of (y_μ'). If $H(y^o) > 0$, we can assume that for μ large enough, $H(y_\mu') > c > 0$. (4.1,15-16) ensure that, for μ large enough, we have

$$\mu < \frac{2C_{11} \max_{y \in \partial\Omega} H(y)}{C_5} \lambda_\mu' = \kappa_2 \lambda_\mu' .$$

Thus, it follows from Subsection 3.2 that

$$(\alpha_\mu, \lambda_\mu') = (\alpha^{\mu, y_\mu'}, \lambda^{\mu, y_\mu'})$$

and, u_μ being a solution to (1.1), y_μ' has to be a critical point of $\tilde{\tilde{K}}_\mu$. Then, it follows from (3.50) that $H'(y^o) = 0$.

4.2 Proof of Theorem 1.2

(3.45) shows that, for $n \geq 5$ and $k = 1$, up to additive and multiplicative constants, $\tilde{\tilde{K}}_\mu(y)$ is close to $H^2(y)$ in L^∞-norm on

$$H^c = \{y \in \partial\Omega / H(y) > c\}$$

for any fixed $c > 0$, provided that μ is large enough. Indeed, defining on H^c

$$L_\mu(y) = \frac{\mu}{C_{10}} \left(\tilde{\tilde{K}}_\mu(y) + \frac{(n-1)(n-3)}{4} S^{n/2} \right)$$

we have

$$L_\mu(y) = H^2(y) + 0\left(\frac{1}{\mu^{1/2}} \text{ if } n = 5; \frac{(\text{Log}\mu)^{4/3}}{\mu} \text{ if } n = 6; \frac{1}{\mu} \text{ if } n \geq 7 \right). \tag{4.19}$$

We assume that $a > 0$ is a critical value for H such that, for any $\delta > 0$ sufficiently small, the relative topology $(H^{a+\delta}, H^{a-\delta})$ is nontrivial. By excision, the relative topology $(H^{a+\delta} \cap H^c, H^{a-\delta} \cap H^c)$ is nontrivial, where $c > 0$ is chosen such that $c < a$ (and then, $c < a - \delta$, for δ sufficiently small). (4.19) shows that for μ large enough,

$$L_\mu^{(a-\delta)^2 - c_\mu} \subset H^{a-\delta} \cap H^c \subset H^{a+\delta} \cap H^c \subset L_\mu^{(a+\delta)^2 + c_\mu}$$

with c_μ going to zero as μ goes to infinity. It follows that L_μ has a critical point between the levels $a^2 - c_\mu$ and $a^2 + c_\mu$. This means that, for μ large enough, (1.1) has a solution $u_\mu = \alpha_\mu U_{\lambda_\mu, y_\mu} + v_\mu$ which concentrates as μ goes to infinity, up to a subsequence, at a point $y^o = \lim_{\mu \to +\infty} y_\mu \in \partial\Omega$, such that $H(y^o) = a$. y_μ being a critical point of $\tilde{\tilde{K}}_\mu$, it follows from (3.50) that, if $n \geq 6$, $H'(y^o) = 0$.

Remarks.

1 - If a is an isolated critical value for H, by intercalating the level sets of L_μ between the level sets of H^2, we see that in fact (1.1) admits at least as many distinct solutions u_μ of the previous type as the relative topology $(H^{a+\delta}, H^{a-\delta})$ has generators.

2 - What we find is actually a critical point of K_μ, which provides us with a solution $u_\mu = \alpha_\mu U_{\lambda_\mu, y_\mu} + v_\mu$ of

$$\begin{cases} -\Delta u_\mu + \mu u_\mu = |u_\mu|^{p-1} u_\mu & \text{in} \quad \Omega \\ \dfrac{\partial u_\mu}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.20}$$

with $y_\mu \in \partial\Omega$, and $\alpha_\mu \to \alpha$, $\lambda_\mu \to +\infty$, $\|v_\mu\| \to 0$ as $\mu \to +\infty$. Let us show that in fact $u_\mu > 0$ in Ω. First it is clear, since $\|v_\mu\|$ goes to zero, that $u_\mu \not\equiv 0$ for

μ large enough. Secondly, multiplying the equation in (4.20) by $u_\mu^- = \max(0, -u_\mu^-)$ and integrating over Ω, we obtain

$$\int_\Omega |\nabla u_\mu^-|^2 + \mu \int_\Omega (u_\mu^-)^2 = \int_\Omega (u_\mu^-)^{p+1}$$

As, from the continuous embedding of $L^{p+1}(\Omega)$ in $H^1(\Omega)$, we have also

$$\left(\int_\Omega (u_\mu^-)^{p+1}\right)^{\frac{2}{p+1}} \leq C_\Omega \left(\int_\Omega |\nabla u_\mu^-|^2 + \mu \int_\Omega (u_\mu^-)^2\right)$$

this implies

$$\textit{either } \int_\Omega (u_\mu^-)^{p+1} \geq \frac{1}{C_\Omega^{n/2}} \ , \ \textit{or } u_\mu^- \equiv 0 \ .$$

On the other hand

$$0 \leq u_\mu^- = \max(0, -\alpha_\mu U_{\lambda_\mu, y_\mu} - v_\mu) \leq |v_\mu|$$

and $|v_\mu|_{p+1} \to 0$ as $\mu \to +\infty$. We conclude that for μ large enough, $u_\mu^- \equiv 0$, and $u_\mu \geq 0$. Since u_μ satisfies

$$-\Delta u_\mu + \mu u_\mu \geq 0 \quad \text{in} \quad \Omega \ ; \ u_\mu \geq 0 \quad \text{in} \quad \Omega \ ; \ u_\mu \not\equiv 0$$

the strong maximum principle ensures that $u_\mu > 0$ in Ω, and then u_μ is a solution to (1.1).

4.3 Proof of Theorems 1.1 b), 1.3, 1.4

Since Theorems 1.1 b) and 1.4 are straight forward consequences of Theorem 1.3, it is sufficient to prove this result.

Let $n \geq 6$, and $\bar{y}^1, \cdots, \bar{y}^k$ k distinct points of $\partial\Omega$, each one a nondegenerate critical point of H such that $H(\bar{y}^i) > 0$. We set

$$c = \frac{1}{2} \min_{1 \leq i \leq k} H(\bar{y}^i) \ ; \ d_0 = \frac{1}{2} \min_{1 \leq i < j \leq k} |\bar{y}^i - \bar{y}^j|$$

and we consider the function $\tilde{\tilde{K}}_\mu$ defined on P_{c,d_0} as in (3.40-41). In order to show Theorem 1.3, it is sufficient to prove that for μ large enough, $\tilde{\tilde{K}}_\mu$ admits in P_{c,d_0} a critical point $y_\mu = (y_\mu^i)_{1 \leq i \leq k}$ such that for any i, $1 \leq i \leq k$, $y_\mu^i \to \bar{y}^i$ as $\mu \to +\infty$. (This provides us with a solution $u_\mu = \sum_{i=1}^k \alpha_\mu^i U_{\lambda_\mu^i, y_\mu^i} + v_\mu$ of (4.20) and, according to the argument of the above remark, $u_\mu > 0$, and is then a solution to (1.1) which concentrates at $\bar{y}^1, \cdots, \bar{y}^k$ as $\mu \to +\infty$.) For any i, $1 \leq i \leq k$, we set, for y^i in a neighbourhood of $\bar{y}^i$

$$\xi^i = y^i - \bar{y}^i$$

and we denote by $(\xi_j^i)_{1\le j\le n-1}$ the coordinates of ξ^i obtained by projecting y^i on the tangent space to $\partial\Omega$ at $\bar{y}^i$. Since $\bar{y}^i$ is assumed to be a nondegenerate critical point of H, it follows from (3.50) that

$$\frac{\partial \widetilde{\widetilde{K}}_\mu}{\partial \tau_j^i}(\bar{y}^i) = 0 \qquad 1 \le i \le k \qquad 1 \le j \le n-1$$

is equivalent to

$$\xi_j^i = V_\mu^{i,j}(\xi) \qquad 1 \le i \le k \qquad 1 \le j \le n-1 \tag{4.21}$$

where $V_\mu^{i,j}$ is a smooth function such that

$$V_\mu^{i,j}(\xi) = 0\left[|\xi|^2 + \left(\frac{(\mathrm{Log}\ \mu)^{4/3}}{\mu^{1/6}} \quad \text{if} \quad n=6; \frac{1}{\mu^{\inf(\frac{1}{6},\frac{2}{n-2})}} \quad \text{if} \quad n \ge 7\right)\right]$$

Thus, it follows from Brouwer's fixed point theorem that, for μ large enough, there exists $\xi \in (\mathbf{R}^{n-1})^k$ such that (4.21) is satisfied, with

$$\xi_j^i = 0\left(\frac{(\mathrm{Log}\ \mu)^{4/3}}{\mu^{1/6}} \quad \text{if} \quad n=6; \frac{1}{\mu^{\inf(\frac{1}{6},\frac{2}{n-2})}} \quad \text{if} \quad n \ge 7\right)$$

Hence the result.

APPENDIX

A. A minimization result

In this appendix, we give a proof of Proposition 3.1, adapted from [7]. We recall that, for $k \in \mathbf{N}^*$, $\varepsilon > 0$

$$B_\varepsilon = \left\{ (\alpha, \lambda, y) \in \mathbf{R}^k \times (\mathbf{R}_+^*)^k \times (\partial\Omega)^k / \frac{\bar{\alpha}}{2} < \alpha_i < 2\bar{\alpha} \ , \lambda_i > \frac{1}{\varepsilon} \ , 1 \leq i \leq k \ ; \right.$$

$$\left. |y^i - y^j|^2 > \varepsilon \ , 1 \leq i < j \leq k \right\}$$

$V_\varepsilon = \{u \in H^1(\Omega) / \exists (\lambda, y) \in (\mathbf{R}_+^*)^k \times (\partial\Omega)^k / (\bar{\alpha}, \lambda, y) \in B_\varepsilon \ , |\nabla(u - \varphi(\bar{\alpha}, \lambda, y))|_2 < \varepsilon\}$

with

$$\varphi(\alpha, \lambda, y) = \sum_{i=1}^{k} \alpha_i U_{\lambda_i, y^i}$$

and

$$\bar{\alpha} = (\bar{\alpha}, \cdots, \bar{\alpha})$$

We have :

Lemma A1.—

Let (ε_m) be a sequence in $\mathbf{R}_+^*$ such that ε_m goes to zero as m goes to infinity, and $(\alpha^m, \lambda^m, y_m) \in B_{\varepsilon_m}, (\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m) \in B_{\varepsilon_m}$ such that

$$(A.1) \qquad |\nabla(\varphi(\alpha^m, \lambda^m, y_m) - \varphi(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m))|_2 \underset{m \to +\infty}{\longrightarrow} 0$$

Then (up to permutations on the indices of $(\tilde{\alpha}_i^m, \tilde{\lambda}_i^m, \tilde{y}_m^i)_{1 \leq i \leq k}$)

$$(A.2) \qquad \alpha_i^m - \tilde{\alpha}_i^m \to 0 \qquad \frac{\lambda_i^m}{\tilde{\lambda}_i^m} \to 1 \qquad \lambda_i^m \tilde{\lambda}_i^m |y_m^i - \tilde{y}_m^i|^2 \to 0$$

as $m \to +\infty$, $1 \leq i \leq k$.

Proof of Lemma A1

Integral estimates show - see Appendix D and [6] - the uniform convergences

$$(A.3) \qquad |\nabla U_{\lambda, y}|_2 \to \frac{S^{n/2}}{2(n(n-2))^{\frac{n-2}{2}}} \quad \text{as} \quad \lambda \to +\infty$$

$$(A.4) \qquad \int_\Omega \nabla U_{\lambda,y}.\nabla U_{\lambda',y'} \to 0 \quad \text{as} \quad \lambda,\lambda',\frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda}+\lambda\lambda'|y-y'|^2 \to +\infty \ .$$

Thus, assumptions of Lemma A1 imply the existence of $C>0$ such that

$$\forall m \in \mathbf{N},\ \forall i, 1 \le i \le k,\ \exists j, 1 \le j \le k$$

such that

$$(A.5) \qquad \frac{\lambda_i^m}{\tilde\lambda_j^m}+\frac{\tilde\lambda_j^m}{\lambda_i^m}+\lambda_i^m\tilde\lambda_j^m|y_m^i-\tilde y_m^j|^2 \le C \ .$$

Such a j is unique. Indeed, assume that (A.5) holds for another index $\ell \neq j$. We would have

$$\begin{aligned}\tilde\lambda_j^m\tilde\lambda_\ell^m|\tilde y_m^j-\tilde y_m^\ell|^2 &\le 2\tilde\lambda_j^m\tilde\lambda_\ell^m(|\tilde y_m^j-y_m^i|^2+|y_m^i-\tilde y_m^\ell|^2)\\ &\le 2\frac{\tilde\lambda_\ell^m}{\lambda_i^m}\lambda_i^m\tilde\lambda_j^m|y_m^i-\tilde y_m^j|^2+2\frac{\tilde\lambda_j^m}{\lambda_i^m}\lambda_i^m\tilde\lambda_\ell^m|y_m^i-\tilde y_m^j|^2\\ &\le 4C^2\end{aligned}$$

a contradiction for m large enough, since $(\tilde\alpha^m,\tilde\lambda_m) \in B_{\varepsilon_m}$ means

$$\tilde\lambda_j^m\tilde\lambda_\ell^m|\tilde y_m^j-\tilde y_m^\ell|^2 > \frac{1}{\varepsilon_m}$$

and ε_m goes to zero as m goes to infinity.

Up to a permutation of the indices, we may assume that $j=i$. With this choice, we note that for $j \neq i$

$$|y_m^i-\tilde y_m^j| \ge |\tilde y_m^j-\tilde y_m^i|-|y_m^i-\tilde y_m^i| > \varepsilon_m^{1/2}-C^{1/2}\varepsilon_m > \frac{\varepsilon_m^{1/2}}{2}$$

for m large enough, so that

$$\lambda_i^m\tilde\lambda_j^m|y_m^i-\tilde y_m^j|^2 > \frac{1}{4\varepsilon_m}$$

and, according to (A.4)

$$\int_\Omega \nabla U_{\lambda_i^m,y_i^m}.\nabla U_{\tilde\lambda_j^m,\tilde y_j^m} \to 0 \quad \text{as} \quad m \to +\infty \ .$$

Consequently

$$|\nabla(\sum_{i=1}^k(\alpha_i^m U_{\lambda_i^m,y_m^i}-\tilde\alpha_i^m U_{\tilde\lambda_i^m,\tilde y_m^i}))|_2^2 = \sum_{i=1}^k|\nabla(\alpha_i^m U_{\lambda_i^m,y_m^i}-\tilde\alpha_i^m U_{\tilde\lambda_i^m,\tilde y_m^i})|_2^2+o(1) \ .$$

(A.1,3) then imply, for any i, $1 \leq i \leq k$, that

$$\alpha_i^m - \tilde{\alpha}_i^m \to 0 \quad \text{as} \quad m \to +\infty$$

and

$$(A.6) \qquad |\nabla(U_{\lambda_i^m, y_m^i} - U_{\tilde{\lambda}_i^m, \tilde{y}_m^i})|_2^2 \to 0 \quad \text{as} \quad m \to +\infty \, .$$

From (A.3) we have

$$|\nabla(U_{\lambda_i^m, y_m^i} - U_{\tilde{\lambda}_i^m, \tilde{y}_m^i})|_2^2 = S^{n/2} - 2 \int_\Omega \nabla U_{\lambda_i^m, y_m^i} . \nabla U_{\tilde{\lambda}_i^m, \tilde{y}_m^i} + o(1)$$

and from (A.4,6) we see that the quantity

$$\frac{\lambda_i^m}{\tilde{\lambda}_i^m} + \frac{\tilde{\lambda}_i^m}{\lambda_i^m} + \lambda_i^m \tilde{\lambda}_i^m |y_i^m - \tilde{y}_i^m|^2$$

is bounded. Then, up to a subsequence, we may assume that

$$\frac{\tilde{\lambda}_i^m}{\lambda_i^m} \to \delta \in \mathbf{R}_+^*, \quad \lambda_i^m (\tilde{y}_m^i - y_m^i) \to a \in \mathbf{R}^n.$$

Thus, with a change of variable in the integral

$$\begin{aligned} |\nabla(U_{\lambda_i^m, y_m^i} - U_{\tilde{\lambda}_i^m, \tilde{y}_m^i})|_2^2 &= \int_{\lambda_i^m(\Omega - y_m^i)} |\nabla U_{1,o} - \nabla U_{\frac{\tilde{\lambda}_i^m}{\lambda_i^m}, \lambda_i^m (\tilde{y}_m^i - y_m^i)}|^2 \\ &= \frac{1}{2} \int_{\mathbf{R}^n} |\nabla U_{1,o} - \nabla U_{\delta,a}|^2 + o(1) \end{aligned}$$

using Lebesgue convergence theorem. It follows from (A.6) that $\delta = 1$ and $a = 0$, and the proof of Lemma A.1 is complete.

Lemma A.2.—

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon, 0 < \varepsilon < \varepsilon_0$, and $u \in V_\varepsilon$,

$$\inf_{(\alpha, \lambda, y) \in B_{4\varepsilon}} |\nabla(u - \varphi(\alpha, \lambda, y))|_2$$

is achieved in $B_{2\varepsilon}$ and is not achieved in $B_{4\varepsilon} \setminus B_{2\varepsilon}$

Proof of Lemma A.2.

Let us show first that the infimum is achieved in $\overline{B}_{4\varepsilon}$.

Let $(\alpha^m, \lambda^m, y_m)$ be a minimizing sequence in $B_{4\varepsilon}$. Up to a subsequence, we may assume that, for any i, $1 \leq i \leq k$

$$\begin{aligned} &\alpha_i^m \to \alpha_i^0 \qquad \frac{\bar{\alpha}}{2} \leq \alpha_i^0 \leq 2\bar{\alpha} \\ &\lambda_i^m \to \lambda_i^0 \qquad \frac{1}{\varepsilon} \leq \lambda_i^0 \leq +\infty \\ &y_m^i \to y_0^i \in \partial\Omega \end{aligned}$$

with $|y_0^i - y_0^j|^2 \geq \varepsilon, 1 \leq i < j \leq k$. The only thing to prove is that λ_i^o is finite. Since $u \in V_\varepsilon$, there exists (λ, y) such that $(\alpha, \lambda, y) \in B_\varepsilon$ and $v \in H^1(\Omega), \|v\| < \varepsilon$, such that

$$u = \varphi(\bar{\alpha}, \lambda, y) + v \tag{A.7}$$

and we have

$$|\nabla(\varphi(\bar{\alpha}, \lambda, y) + v - \varphi(\alpha^m, \lambda^m, y_m)|_2 < \varepsilon$$

Therefore

$$|\nabla(\varphi(\bar{\alpha}, \lambda, y) - \varphi(\alpha^m, \lambda^m, y_m))|_2 < 2\varepsilon\ . \tag{A.8}$$

Proceeding as in the proof of the previous lemma, we deduce from this inequality that for ε small enough, up to a permutation of the indices, $\alpha_i^m, y_m^i, \frac{\lambda_i^m}{\lambda_i}$ have to be close to $\alpha_i, y^i, 1$ respectively. This implies that λ_i^o is finite, and the infimum is achieved in $\overline{B}_{4\varepsilon}$.

Suppose now, arguing by contradiction, that there exists a sequence u_m in $H^1(\Omega)$ such that $u_m \in V_{\varepsilon_m}$ with ε_m going to zero as m goes to infinity, and such that for any m the infimum of $|\nabla(u - \varphi(\alpha, \lambda, y))|_2, (\alpha, \lambda, y) \in B_{4\varepsilon_m}$, is achieved in $\overline{B}_{4\varepsilon_m} \setminus B_{2\varepsilon_m}$. Writing u as in (A.7) we obtain, similarly to (A.8), a sequence (λ^m, y_m) such that $(\bar{\alpha}, \lambda^m, y_m) \in B_{\varepsilon_m}$, $(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m) \in \overline{B}_{4\varepsilon_m} \setminus B_{2\varepsilon_m}$, and

$$|\nabla(\varphi(\bar{\alpha}, \lambda^m, y_m) - \varphi(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m)|_2 = o(1)$$

Lemma A.1 applies, which gives, up to a permutation of the indices of $(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m)$

$$\frac{\lambda_i^m}{\tilde{\lambda}_i^m} = 1 + o(1) \qquad 1 \leq i \leq k\ . \tag{A.9}$$

$(\bar{\alpha}, \lambda^m, y_m) \in B_{\varepsilon_m}$ implies that $\lambda_i^m > \frac{1}{\varepsilon_m}$, and $(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}) \in \overline{B}_{4\varepsilon_m} \setminus B_{2\varepsilon_m}$ that $\tilde{\lambda}_i^m \leq \frac{1}{2\varepsilon_m}$, a contradiction with (A.9). Hence the lemma.

Proof of Proposition 2.1

We argue again by contradiction. If the proposition is false, Lemma A.2 ensures the existence of a sequence (ε_m) in $\mathbf{R}_+^*$ going to zero, a sequence of functions u_m in V_{ε_m}, and sequences $(\alpha^m, \lambda^m, y_m)$ and $(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m)$ in $B_{2\varepsilon_m}$ such that, for any m

$$(\alpha^m, \lambda^m, y_m) \neq (\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m) \tag{A.10}$$

and

$$\inf_{(\alpha,\lambda,y)\in B_{4\varepsilon_m}} |\nabla(u-\varphi(\alpha,\lambda,y)|_2 = |\nabla(u-\varphi(\alpha^m,\lambda^m,y_m)|_2 = |\nabla(u-\varphi(\tilde{\alpha}^m,\tilde{\lambda}^m,\tilde{y}_m))|_2$$

As a consequence, for any i, $1 \leq i \leq k$, and any m

$$\begin{aligned} v_m &= u_m - \varphi(\alpha^m, \lambda^m, y_m) \in E_{\lambda^m, y_m} \qquad |\nabla v_m|_2 < \varepsilon_m \\ \tilde{v}_m &= u_m - \varphi(\tilde{\alpha}^m, \tilde{\lambda}^m, \tilde{y}_m) \in E_{\tilde{\lambda}^m, \tilde{y}_m} \qquad |\nabla \tilde{v}_m|_2 < \varepsilon_m \ . \end{aligned} \tag{A.11}$$

Lemma A.1 implies, up to a permutation of the indices of $(\tilde{\alpha}_m, \tilde{\lambda}^m, \tilde{y}_m)$

$$\alpha_i^m - \tilde{\alpha}_i^m = o(1) \qquad \frac{\lambda_i^m}{\tilde{\lambda}_i^m} = 1 + o(1) \qquad \lambda_i^m \tilde{\lambda}_i^m |y_m^i - \tilde{y}_m^i|^2 = o(1)$$

for any i, $1 \leq i \leq k$. Setting

$$a_i^m = \alpha_i^m - \tilde{\alpha}_i^m \qquad b_i^m = \frac{\tilde{\lambda}_i^m}{\lambda_i^m} - 1 \qquad c_i^m = \tilde{\lambda}_i^m (y_m^i - \tilde{y}_m^i)$$

we have

$$a_i^m = o(1) \qquad b_i^m = o(1) \qquad c_i^m = o(1)$$

and our aim is to prove that $a_i^m = b_i^m = 0, c_i^m = 0$. For sake of simplicity, we omit in the following computations the index m, U_i denotes $U_{\lambda_i^m, y_m^i}$ and $\widetilde{U}_i$ denotes $U_{\tilde{\lambda}_i^m, \tilde{y}_m^i}$. We are going to show, successively, that

$$a_i = o(\Sigma_j(|a_j| + |b_j| + |c_j|)) \tag{A.12}$$

$$b_i = o(\Sigma_j(|a_j| + |b_j| + |c_j|)) \tag{A.13}$$

$$c_i = o(\Sigma_j(|a_j| + |b_j| + |c_j|)) \tag{A.14}$$

for any $i, 1 \leq i \leq k$, whence, for m large enough, $a_i = b_i = c_i = 0$, a contradiction with (A.10) - from which Proposition 2.1 follows. To make the proof complete, let us show, for example, (A.12). We have

$$\int_\Omega \nabla(\varphi(\alpha,\lambda,y) - \varphi(\tilde{\alpha},\tilde{\lambda},\tilde{y})).\nabla U_i = \int_\Omega \nabla(\tilde{v}-v).\nabla U_i = \int_\Omega \nabla\tilde{v}.\nabla(U_i - \tilde{U}_i) \tag{A.15}$$

because of (A.11). It is easy to see that, for any j, $1 \leq j \leq k$

$$(A.16) \qquad |\nabla U_j - \nabla \widetilde{U}_j| = 0\left((|b_j| + |c_j|)|\nabla U_j| + |c_j| \frac{\lambda_j^{\frac{n}{2}}}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} \right) .$$

Indeed

$$\begin{aligned} \lambda_j^2|x-y^j|^2 &= \tilde{\lambda}_j^2 \frac{\lambda_j^2}{\tilde{\lambda}_j^2} \left(|x-\tilde{y}^j|^2 + |\tilde{y}^j - y^{j2}| + 2(x-\tilde{y}^j).(\tilde{y}^j - y^j) \right) \\ &= (1+0(|b_j|))(\tilde{\lambda}_j|x-\tilde{y}^j|^2 + c_j^2 + 0(|c_j|\tilde{\lambda}_j|x-\tilde{y}^j|)) \end{aligned}$$

so that

$$\frac{1+\lambda_j^2|x-y^j|^2}{1+\tilde{\lambda}_j|x-\tilde{y}^j|^2} = 1 + 0(|b_j| + |c_j|)$$

uniformly with respect to $x \in \Omega$. Therefore

$$\begin{aligned} |\nabla U_j - \nabla \widetilde{U}_j| &= (n-2)\left| \lambda_j^{\frac{n+2}{2}} \frac{y^j - x}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} - \tilde{\lambda}_j^{\frac{n+2}{2}} \frac{\tilde{y}^j - x}{(1+\tilde{\lambda}_j^2|x-\tilde{y}^j|^2)^{\frac{n}{2}}} \right| \\ &= \frac{(n-2)\lambda_j^{\frac{n+2}{2}}}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} \left| y^j - x - (\frac{\tilde{\lambda}_j}{\lambda_j})^{\frac{n+2}{2}} \left(\frac{1+\lambda_j^2|x-y^j|^2}{1+\tilde{\lambda}_j^2|x-\tilde{y}^j|^2} \right)^{\frac{n}{2}} (\tilde{y}^j - x) \right| \\ &= \frac{(n-2)\lambda_j^{\frac{n+2}{2}}}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} \left(0(|b_j| + |c_j|)|y^j - x| + 0(\frac{|c_j|}{\lambda_j}) \right) \\ &= 0(|b_j| + |c_j|)|\nabla U_j| + 0\left(|c_j| \frac{\lambda_j^{\frac{n}{2}}}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} \right) \end{aligned}$$

hence (A.16). From this we deduce, for $i \neq j$

$$\begin{aligned} \int_\Omega \nabla(U_j - \widetilde{U}_j).\nabla U_i &= 0\left[(|b_j| + |c_j|) \int_\Omega \left(|\nabla U_j| + |c_j| \frac{\lambda_j^{\frac{n}{2}}}{(1+\lambda_j^2|x-y^j|^2)^{\frac{n}{2}}} \right) |\nabla U_i| \right] \\ &= o(|b_j| + |c_j|) \end{aligned}$$

since, as simple computations show - see the estimates in Appendix D - the integral goes to zero as m goes to infinity. Thus

$$(A.17) \qquad \int_\Omega \nabla(\alpha_j U_j - \tilde{\alpha}_j \widetilde{U}_j).\nabla U_i = o(|a_j| + |b_j| + |c_j|) \quad \text{for} \quad i \neq j .$$

As we have also, according to (A.11,16) and straightforward computations

$$\int_\Omega \nabla \tilde{v}.\nabla(U_i - \tilde{U}_i) = o\left[(\int_\Omega |\nabla U_i - \nabla \widetilde{U}_i|^2)^{1/2} \right] = o(|b_i| + |c_i|)$$

(A.15) yields

$$(A.18) \qquad \int_\Omega \nabla(\alpha_i U_i - \tilde{a}_i \widetilde{U}_i).\nabla U_i = o(\Sigma_j(|a_j| + |b_j| + |c_j|))$$

On the other hand, we claim that

$$(A.19) \qquad \int_\Omega \nabla(U_i - \widetilde{U}_i).\nabla U_i = o(|b_i| + |c_i|) \ .$$

Indeed

$$\begin{aligned}
&\frac{1}{(n-2)^2}\int_\Omega (\nabla U_i - \nabla \widetilde{U}_i).\nabla U_i \\
&= \int_\Omega \left(\frac{\lambda_i^{\frac{n+2}{2}}(y^i - x)}{(1+\lambda_i^2|y^i - x|^2)^{n/2}} - \frac{\tilde{\lambda}_i^{\frac{n+2}{2}}(\tilde{y}^i - x)}{(1+\tilde{\lambda}_i^2|\tilde{y}^i - x|^2)^{n/2}} \right) \cdot \frac{\lambda_i^{\frac{n+2}{2}}(y^i - x)}{(1+\lambda_i^2|y^i - x|^2)^{n/2}} dx \\
&= \int_{\lambda_i(y^i - \Omega)} \left(\frac{x}{(1+|x|^2)^{\frac{n}{2}}} - \frac{(1+b_i)^{\frac{n}{2}}((1+b_i)x - c_i)}{(1+|(1+b_i)x - c_i|^2)^{\frac{n}{2}}} \right) \cdot \frac{x}{(1+|x|^2)^{n/2}} dx \\
&= \int_{\lambda_i(y^i - \Omega)} \left[x - \left((1 + \frac{n+2}{2} b_i)x - c_i + n\frac{c_i.x - b_i|x|^2}{1+|x|^2} x \right) + 0(b_i^2 + |c_i|^2) \right] \cdot \frac{x}{(1+|x|^2)^n} dx \\
&= \int_{\lambda_i(y^i - \Omega)} \left(b_i(\frac{n+2}{2}|x|^2 - \frac{n|x|^4}{|+|x|^2}) + c_i.x(1 - \frac{n|x|^2}{1+|x|^2}) + 0(b_i^2 + |c_i|^2) \right) \frac{dx}{(1+|x|^2)^n} \ .
\end{aligned}$$

$\lambda_i(y^i - \Omega)$ goes to a half space P of $\mathbf{R}^n$ such that $0 \in \partial P$. With a good choice of the coordinates, we may assume that $P = \{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x_n > 0\}$. Because of oddness

$$\int_P x_j (1 - \frac{n|x|^2}{1+|x|^2}) \frac{dx}{(1+|x|^2)^n} = 0 \qquad 1 \le j \le n-1$$

and, integrating by parts with respect to x_n, we have

$$\int_p x_n \frac{n|x|^2}{1+|x|^2} \frac{dx}{(1+|x|^2)^n} = \int_P \frac{d}{dx_n} \left(\frac{1}{(1+|x|^2)^n} \right) \left(-\frac{|x|^2}{2} \right) dx = \int_P \frac{x_n}{(1+|x|^2)^n} dx \ .$$

Still integrating by parts, we have also

$$\begin{aligned}
\int_P \frac{n|x|^4}{1+|x|^2} \frac{dx}{(1+|x|^2)^n} &= \frac{n}{2}\sigma_{n-1} \int_0^{+\infty} \frac{r^{n+3} dr}{(1+r^2)^{n+1}} \\
&= \frac{\sigma_{n-1}}{2} \int_0^{+\infty} \frac{d}{dr} \left(\frac{1}{(1+r^2)^n} \right) \left(-\frac{r^{n+2}}{2} \right) dr \\
&= \frac{n+2}{4} \sigma_{n-1} \int_0^{+\infty} \frac{r^{n+1}}{(1+r^2)^n} dr \\
&= \frac{n+2}{2} \int_P \frac{|x|^2}{(1+|x|^2)^n} dx
\end{aligned}$$

Finally, we find

$$\int_\Omega (\nabla U_i - \nabla \widetilde{U}_i).\nabla U_i = o(b_i + |c_i|)$$

and (A.19) is proved. Coming back to (A.18), we obtain

$$(\alpha_i - \tilde{\alpha}_i) \int_\Omega |\nabla U_i|^2 = o(\Sigma_j(|a_j| + |b_j| + |c_j|))$$

hence (A.12), using (D.6).

(A.13) and (A.14) may be proven using the same arguments, just changing in (A.15) $U_i, \widetilde{U}_i$ by $\frac{\partial U_i}{\partial \lambda_i}, \frac{\partial \widetilde{U}}{\partial \tilde{\lambda}_i}$ and $\frac{\partial U_i}{\partial y_i}, \frac{\partial U_i}{\partial \tilde{y}_i}$ respectively. The computations are quite similar, and we omit them.

B - THE COERCIVITY OF $Q_{\mu,\alpha,\lambda,y}$

On

$$E_{\lambda,y} = \left\{ v \in H^1(\Omega) / \int_\Omega \nabla v.\nabla U_i = \int_\Omega \nabla v.\nabla \frac{\partial U_i}{\partial \lambda_i} = \int_\Omega \nabla v.\nabla \frac{\partial U_i}{\partial y_j^i} = 0 \right. \tag{2.4}$$
$$\left. 1 \le i \le k \ ; \ 1 \le j \le n \right\}$$

we consider the quadratic form

$$Q_{\mu,\alpha,\lambda,y}(v) = \frac{1}{2}\left(\int_\Omega |\nabla v|^2 + \mu \int_\Omega v^2 - p \int_\Omega \varphi_{\alpha,\lambda,y}^{p-1} v^2 \right) .$$

We want to prove that $Q_{\mu,\alpha,\lambda,y}$ is coercive, with a modulus of coercivity independent of $(\mu, \alpha, \lambda, y) \in \mathbf{R}_+^* \times \mathbf{R}^k \times (\mathbf{R}_+^*)^k \times (\partial\Omega)^k$ for

$$\begin{aligned} \mu_0 < \mu < \delta_0 \lambda_i^2 \qquad & 1 \le i \le k \\ |\alpha_i - \bar{\alpha}| < \eta_0 \qquad & 1 \le i \le k \\ \lambda_i > \frac{1}{\eta_0} \qquad & 1 \le i \le k \\ |y^i - y^j| > d_0 \qquad & 1 \le i < j \le k \end{aligned} \tag{B.1}$$

where $d_0 > 0$ is a given constant, and η_0, μ_0, δ_0 are strictly positive constants which have to be chosen independently of each other. We remark that

$$\varphi_{\alpha,\lambda,y}^{p-1} = \sum_{i=1}^k \alpha_i^{p-1} U_i^{p-1} + 0(\sum_{\substack{i,j \\ i \neq j}} U_i^{p-2} \inf(U_i, U_j))$$

and, using Hölder's inequality and Sobolev embedding theorem

$$\int_\Omega U_i^{p-2} \inf(U_i, U_j) v^2 = 0 \left[\|v\| \left(\int_\Omega \inf(U_i^{p+1}, U_i^{\frac{(p-2)(p+1)}{p-1}} U_j^{\frac{p+1}{p-1}}) \right)^{\frac{p-1}{p+1}} \right] = o(\|v\|)$$

as $\lambda_i, \lambda_j \to +\infty$, as it follows from (D.47). Therefore, choosing η_0 sufficiently small, and noticing that $p\bar{\alpha}^{p-1} = n(n+2)$, we have actually to study the coercivity on $E_{\lambda,y}$ of the quadratic form

$$(B.2) \qquad \widetilde{Q}_{\mu,\lambda,y}(v) = \frac{1}{2} \left(\int_\Omega |\nabla v|^2 + \mu \int_\Omega v^2 - n(n+2) \sum_{i=1}^k \int_\Omega U_i^{p-1} v^2 \right)$$

In the context of Dirichlet boundary conditions, a similar result has been proved by Bahri [6]. It has been adapted in [4] to our case, for $k = 1$. What we need here is a result which holds for arbitrary k in $\mathbf{N}^*$. For $1 \leq i \leq k$, we define

$$\Omega_i = \Omega \cap B(y^i, \frac{d_0}{2}) \ .$$

We remark that, if $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$. For $v \in H^1(\Omega)$, we consider

$$v_i = v.1_{\Omega_i}$$

where 1_{Ω_i} is the characteristic function of Ω_i. We set also

$$E_i^- = \mathrm{Vect}(U_i; \frac{\partial U_i}{\partial \lambda_i} \ ; \ \frac{\partial U_i}{\partial y_j^i}, 1 \leq j \leq n)$$

$$E_i^+ = \{w \in H^1(\Omega_i) / \int_{\Omega_i} \nabla U_i . \nabla w = \int_{\Omega_i} \nabla \frac{\partial U_i}{\partial \lambda_i} . \nabla w = \int_{\Omega_i} \nabla \frac{\partial U_i}{\partial y_j^i} . \nabla w = 0, 1 \leq j \leq n\} \ .$$

$H^1(\Omega_i) = E_i^- \oplus E_i^+$, so that there is a unique way to write

$$v_i = v_i^+ + v_i^- \quad v_i^+ \in E_i^+, v_i^- \in E_i^-$$

On one hand, according to [4, Lemma 3.4], we know that there exists $\rho_i > 0$ such that, for any $v \in H^1(\Omega)$

$$(B.3) \qquad \frac{1}{2} \left(\int_{\Omega_i} |\nabla v_i^+|^2 + \mu \int_{\Omega_i} v_i^{+2} - n(n+2) \int_{\Omega_i} U_i^{p-1} v_i^{+2} \right) \geq \rho_i \left(\int_{\Omega_i} |\nabla v_i^+|^2 + \mu \int_{\Omega_i} v_i^{+2} \right)$$

provided that μ and λ_i are large enough, and μ/λ_i^2 is small enough. On the other hand, let us estimate v_i^-, assuming that $v \in E_{\lambda,y}$. We write

$$(B.4) \qquad v_i^- = a_i U_i + b_i \frac{\partial U_i}{\partial \lambda_i} + \sum_{j=1}^{n} c_j^i \frac{\partial U_i}{\partial y_j^i}$$

and, as usual, we are going to evaluate a_i, b_i, c_j^i, multiplying the gradient of (B.4) by $\nabla U_i, \nabla \frac{\partial U_i}{\partial \lambda_i}, \nabla \frac{\partial U_i}{\partial y_j^i}$ successively, and integrating on Ω_i. We obtain a nearly diagonal and invertible linear system whose left hand side is given by

$$\int_{\Omega_i} \nabla v_i^- . \nabla U_i = \int_{\Omega_i} \nabla v . \nabla U_i = -\int_{\Omega \setminus \Omega_i} \nabla v . \nabla U_i = 0\left(\frac{1}{\lambda_i^{\frac{n-2}{2}}} \int_{\Omega} |\nabla v|\right) = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\right)$$

and, in the same way

$$\int_{\Omega_i} \nabla v_i^- . \nabla \frac{\partial U_i}{\partial \lambda_i} = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n}{2}}}\right) \qquad \int_{\Omega} \nabla v_i^- . \nabla \frac{\partial U_i}{\partial y_j^i} = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\right)$$

Then, using (D.27) to solve the linear system, we find

$$a_i = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\right) \; ; \; b_i = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n-4}{2}}}\right) \; ; c_j^i = 0\left(\frac{\|v\|}{\lambda_i^{\frac{n+2}{2}}}\right)$$

and, from (B.4) and (D.26) we deduce

$$(B.5) \qquad \|v_i^-\|^2 = 0\left(\frac{1}{\lambda_i^{n-2}}(1 + \frac{\mu}{\lambda_i^2})\|v\|^2\right)$$

Combining (B.3) and (B.5) we get, for any $v \in E_{\lambda,y}$

$$(B.6) \qquad \begin{aligned} &\frac{1}{2}\left(\int_{\Omega_i} |\nabla v|^2 + \mu \int_{\Omega_i} v^2 - n(n+2) \int_{\Omega_i} U_i^{p-1} v^2\right) \\ &\geq \frac{\rho_i}{2}\left(\int_{\Omega_i} |\nabla v|^2 + \mu \int_{\Omega_i} v^2\right) + 0\left(\frac{\|v\|^2}{\lambda_i^{n-2}}\right) \end{aligned}$$

provided that μ and λ_i are large enough, and μ/λ_i^2 is small enough. We write

$$\begin{aligned} \widetilde{Q}_{\mu,\lambda,y}(v) &= \frac{1}{2}\Big(\int_{\Omega} |\nabla v|^2 + \mu \int_{\Omega} v^2 - n(n+2) \sum_{i=1}^{k} \int_{\Omega} U_i^{p-1} v^2\Big) \\ &= \frac{1}{2}\|v\|^2 - \frac{1}{2}\sum_{i=1}^{k}\left(\int_{\Omega_i} |\nabla v|^2 + \mu \int_{\Omega_i} v^2\right) + \frac{1}{2}\sum_{i=1}^{k}\left(\int_{\Omega_i} |\nabla v|^2 + \mu \int_{\Omega_i} v^2\right. \\ &\left. - n(n+2) \int_{\Omega_i} U_i^{p-1} v^2\right) - \frac{n(n+2)}{2} \sum_{i=1}^{k} \int_{\Omega \setminus \Omega_i} U_i^{p-1} v^2. \end{aligned}$$

As we have

$$\int_{\Omega\setminus\Omega_i} U_i^{p-1}v^2 = 0\left(\frac{1}{\lambda_i^2}\int_\Omega v^2\right) = 0\left(\frac{\|v\|^2}{\mu\lambda_i^2}\right)$$

it follows from (B.6) that, for μ, λ_i large enough, $\frac{\mu}{\lambda_i^2}$ small enough, $1 \le i \le k$

$$\widetilde{Q}_{\mu,\lambda,y}(v) \ge \frac{1}{2}\left(\int_{\Omega\setminus\cup_{i=1}^k\Omega_i}|\nabla v|^2 + \mu\int_{\Omega\setminus\cup_{i=1}^k\Omega_i} v^2\right) + \frac{1}{2}\sum_{i=1}^k\frac{\rho_i}{2}\left(\int_{\Omega_i}|\nabla v|^2 + \mu\int_{\Omega_i} v^2\right)$$
$$+ 0\left(\sum_{i=1}^k\left(\frac{1}{\lambda_i^{n-2}} + \frac{1}{\mu\lambda_i^2}\right)\|v\|^2\right)$$

Consequently, setting $\rho = \frac{1}{2}\min(\frac{1}{2}; \frac{\rho_i}{4}, 1 \le i \le k)$, and choosing in (B.1) μ_0 large enough, η_0 and δ_0 small enough, we obtain

$$\widetilde{Q}_{\mu,\lambda,y}(v) \ge \rho\|v\|^2 \quad \forall v \in E_{\lambda,y}$$

This concludes the proof of Lemma 3.1.

C- THE DERIVATIVES OF K_μ WITH RESPECT TO α, λ, y

In this appendix, we prove

Lemma C.1.—

Let μ, α, λ, y satisfy the assumptions of Proposition 3.1, be such that

$$(C.1) \qquad \mu < \kappa_2\lambda_i \qquad 1 \le i \le k$$

with κ_2 a strictly positive constant. We have

$$(C.2) \qquad \frac{\partial K_\mu}{\partial\alpha_i}(\alpha,\lambda,y,\bar{v}) = -C_6\eta_i - C_7\frac{H(y^i)}{\lambda_i}$$
$$+ 0\left[\eta_i^2 + \left(\frac{1}{\lambda^{3/2}}\text{ if } n=5; \frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^2}\text{if} n=6\ ; \frac{1}{\lambda^2}\text{if } n\ge 7\right)\right]$$

with

$$(C.3) \qquad \alpha_i = \bar{\alpha}(1+\eta_i)$$

and

$$C_6 = \frac{2n}{(n(n-2))^{\frac{n+2}{4}}}S^{n/2} > 0$$
$$C_7 = \bar{\alpha}(C_1 - \bar{\alpha}^{p-1}C_4)$$
$$= (n(n-2))^{\frac{n-2}{4}}(n-1)(n-2)\sigma_{n-2}\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-3}{2})}{4\Gamma(n)} > 0\ .$$

$$(C.4)\qquad \begin{aligned}\frac{\partial K_\mu}{\partial \lambda_i}(\alpha,\lambda,y,\bar v) &= C_8\frac{H(y^i)}{\lambda_i^2} - C_9\frac{\mu}{\lambda_i^3}\\ &+ 0\left[\frac{|\eta_i|}{\lambda_i^2} + \left(\frac{1}{\lambda^{5/2}} \text{ if } n=5; \frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^3} \text{if} n=6\ ; \frac{1}{\lambda^3} \text{ if } n\geq 7\right)\right]\end{aligned}$$

with

$$C_8 = \frac{\bar\alpha^2}{2}(C_1-(n-2)^2C_4) = (n(n-2))^{\frac{n-2}{2}}(n-2)^2\sigma_{n-2}\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-3}{2})}{4\Gamma(n)} > 0$$

$$C_9 = \frac{\bar\alpha^2}{2}C_5 = (n(n-2))^{\frac{n-2}{2}}\sigma_{n-1}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{8\Gamma(n-2)} > 0\ .$$

Lastly,

$$(C.5)\qquad \frac{\partial K_\mu}{\partial \tau_j^i}(\alpha,\lambda,y,\bar v) = 0\left(\frac{1}{\lambda^{1/2}} \text{ if } n=5; \frac{(\mathrm{Log}\lambda)^{1/2}}{\lambda} \text{ if } n=6\ ; \frac{1}{\lambda} \text{ if } n\geq 7\right).$$

Moreover, if $n \geq 6$, *and*

$$(C.6)\qquad \kappa_3\lambda_i < \mu \qquad 1\leq i\leq k$$

with κ_3 *a strictly positive constant,*

$$(C.7)\qquad \begin{aligned}\frac{\partial K_\mu}{\partial \tau_j^i}(\alpha,\lambda,y,\bar v) &= -\frac{C_8}{\lambda_i}\frac{\partial H}{\partial \tau_j^i}(y^i)\\ &+ 0\left(\frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^{7/6}} \text{ if } n=6\ ; \frac{1}{\lambda^{\inf(\frac{7}{6},\frac{n}{n-2})}} \text{ if } n\geq 7\right).\end{aligned}$$

As a byproduct, we obtain also

Lemma C.2.—

Let $A_i, B_i, C_{i,j}$, $1\leq i\leq k$, $1\leq j\leq n-1$ *the coefficients which occur in (3.4). With the same assumptions on* μ,α,λ,y *as above, we have*

$$\begin{aligned}A_i &= 0(|\eta| + \frac{1}{\lambda})\\ B_i &= 0(1)\\ C_{i,j} &= 0\left(\frac{1}{\lambda^{5/2}} \text{ if } n=5\ ; \frac{1}{\lambda^3} \text{ if } n\geq 6\right).\end{aligned}$$

Proof of Lemma C.2

It is a direct consequence of Lemma C.1. Indeed, considering the linear system in $A_i, B_i, C_{i,j}$ that we obtain taking $w = U_i, \frac{\partial U_i}{\partial \lambda_i}, \frac{\partial U_i}{\partial \tau_j^i}$ $1 \leq i \leq k, 1 \leq j \leq n-1$, in (3.20), it follows from the definition of K_μ that the $k(n+1)$ terms on the left hand side are equal to

$$\frac{\partial K_\mu}{\partial v}.U_i = \frac{\partial K_\mu}{\partial \alpha_i}$$

$$\frac{\partial K_\mu}{\partial v}.\frac{\partial U_i}{\partial \lambda_i} = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \lambda_i}$$

$$\frac{\partial K_\mu}{\partial v}.\frac{\partial U_i}{\partial \tau_j^i} = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \tau_j^i}.$$

Using estimates (D.27) to invert the system, estimates of $A_i, B_i, C_{i,j}$ follow from (C.2,4-5,7). We also remark that from (C.2,4-5) only we derive

$$(C.8) \qquad \begin{cases} A_i = 0(|\eta| + \frac{1}{\lambda}) \\ B_i = 0(1) \\ C_{i,j} = 0\left(\frac{1}{\lambda^{5/2}} \text{ if } n=5\ ; \frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^3} \text{ if } n=6\ ; \frac{1}{\lambda^3} \text{ if } n \geq 6\right). \end{cases}$$

Proof of Lemma C.1

According to the definition of K_μ, we have

$$(C.9) \qquad \begin{aligned} \frac{\partial K_\mu}{\partial \alpha_i}(\alpha, \lambda, y, \bar{v}) = &\int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla U_i + \mu \int_\Omega (\varphi_{\alpha,\lambda,y} + \bar{v})U_i \\ &- \int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v})U_i \end{aligned}$$

and similar expressions for $\frac{\partial K_\mu}{\partial \lambda_i}$ and $\frac{\partial K_\mu}{\partial y_j^i}$, replacing U_i by $\alpha_i \frac{\partial U_i}{\partial \lambda_i}$ and $\alpha_i \frac{\partial U_i}{\partial y_j^i}$ respectively. Thus

$$\begin{aligned} \frac{\partial K_\mu}{\partial \alpha_i} = &\alpha_i \int_\Omega |\nabla U_i|^2 + \sum_{\substack{j=1 \\ j\neq i}}^k \alpha_j \int_\Omega \nabla U_i.\nabla U_j + \mu\alpha_i \int_\Omega U_i^2 + \mu \sum_{\substack{j=1 \\ j\neq 1}}^k \alpha_j \int_\Omega U_i U_j \\ &- \alpha_i^p \int_\Omega U_i^{p+1} + \mu \int_\Omega U_i \bar{v} - p\alpha_i^{p-1} \int_\Omega U_i^p \bar{v} \\ &+ 0\left(\sum_{\substack{j=1 \\ j\neq i}}^k \int_\Omega U_i^p U_j + \sum_{\substack{j=1 \\ j\neq i}}^k \int_\Omega U_j^p U_j\right) + 0\left(\int_\Omega U_i^{p-1}\bar{v}^2\right) + 0\left(\int_\Omega U_i|\bar{v}|^p \text{ if } p > 2\right) \end{aligned}$$

since

$$
\begin{aligned}
&|\varphi_{\alpha,\lambda,y}+\bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y}+\bar{v}) \\
&= |\alpha_i U_i + \sum_{\substack{j \\ j\neq i}} \alpha_j U_j + \bar{v}|^{p-1}(\alpha_i U_i + \sum_{\substack{j \\ j\neq i}} \alpha_j U_j + \bar{v}) \\
&= \alpha_i^p U_i^p + p\alpha_i U_i^{p-1p-1}(\sum_{\substack{j \\ j\neq i}} \alpha_j U_j + \bar{v}) \\
&+ 0\left(U_i^{p-2}(\sum_{\substack{j \\ j\neq i}} U_j^2 + \bar{v}^2)\right) + 0\left((\sum_{\substack{j \\ j\neq i}} U_j^p + |\bar{v}|^p) \text{ if } p>2\right).
\end{aligned}
$$

Proceeding as in the proof of Proposition 3.1, we obtain

$$
\begin{aligned}
&\mu\int_\Omega U_i\bar{v} - p\alpha_i^{p-1}\int_\Omega U_i^p\bar{v} \\
&\qquad = 0\left[\|\bar{v}\|\left(\frac{1}{\lambda^{3/4}} \text{ if } n=5\ ; \frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda} \text{ if } n=6\ ; \frac{1}{\lambda} \text{ if } n\geq 7\right)\right]
\end{aligned}
$$

and Hölder's inequality, Sobolev embedding theorem and estimate (D.17) give

$$
\int_\Omega U_i^{p-1}\bar{v}^2 = 0(\|\bar{v}\|^2) \qquad \int_\Omega U_i|\bar{v}|^p = 0(\|\bar{v}\|^p)\ .
$$

Therefore, (3.5) and (C.1) yield

$$
\begin{aligned}
&\mu\int_\Omega U_i\bar{v} - p\alpha_i^{p-1}\int_\Omega U_i^p\bar{v} + 0\left(\int_\Omega U_i^{p-1}\bar{v}^2\right) + 0\left(\int_\Omega U_i|\bar{v}|^p \text{ if } p>2\right) \\
&\qquad = 0\left(\frac{1}{\lambda^{3/2}} \text{ if } n=5\ ; \frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^2} \text{ if } n=6\ ; \frac{1}{\lambda^2} \text{ if } n\geq 7\right).
\end{aligned}
$$

Thus, using estimates (D.6,17,21,26,31,42) and (C.1), we obtain (C.2).

(C.4) is obtained exactly in the same way, noticing that $\frac{\partial U_i}{\partial\lambda_i} = 0(\frac{U_i}{\lambda_i})$ and using the estimates of Appendix D. The computations are quite similar and we omit them.

It only remains to estimate $\frac{\partial K_\mu}{\partial\tau_j^i}$. Without loss of generality, we may assume, up to a translation and a rotation of the orthonormal referential of $\mathbf{R}^n$, that

$$
(C.11) \qquad y^i = 0 \qquad (\tau_j^i)_{1\leq j\leq n-1} = (y_j^i)_{1\leq j\leq n-1}\ ,
$$

Because of (C.10), we have

(*C*.12)

$$\frac{\partial K_\mu}{\partial y_j^i} = \alpha_i^2 \int_\Omega \nabla U_i . \nabla \frac{\partial U_i}{\partial y_j^i} + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{k} \alpha_i \alpha_j \int_\Omega \nabla U_\ell . \nabla \frac{\partial U_i}{\partial y_j^i} - \alpha_i^p \int_\Omega U_i^p \frac{\partial U_i}{\partial y_j^i}$$

$$+ \mu \alpha_i^2 \int_\Omega U_i \frac{\partial U_i}{\partial y_j^i} + \mu \sum_{\substack{\ell=1 \\ \ell \neq i}}^{k} \alpha_i \alpha_\ell \int_\Omega U_\ell \frac{\partial U_i}{\partial y_j^i} + \mu \alpha_i \int_\Omega \frac{\partial U_i}{\partial y_j^i} \bar{v}$$

$$- p \alpha_i^p \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j \int_\Omega U_i^{p-1} \frac{\partial U_i}{\partial y_j^i} U_j - p \alpha_i^p \int_\Omega U_i^{p-1} \frac{\partial U_i}{\partial y_j^i} \bar{v}$$

$$+ 0 \left(\int_\Omega U_i^{p-2} |\frac{\partial U_i}{\partial y_j^i}| (\sum_{\substack{j=1 \\ j \neq i}}^{k} U_j^2 + \bar{v}^2) \right) + 0 \left(\int_\Omega |\frac{\partial U_i}{\partial y_j^i}| (\sum_{\substack{j=1 \\ j \neq i}}^{k} U_j^p + |\bar{v}|^p) \text{ if } p > 2 \right).$$

Proceeding again as in the proof of Proposition 3.1, we deduce from estimates (D.41,50) and (C.1)

$$\mu \alpha_i \int_\Omega \frac{\partial U_i}{\partial y_j^i} \bar{v} - p \alpha_i^p \int_\Omega U_i^{p-1} \frac{\partial U_i}{\partial y_j^i} \bar{v} = 0(\|\bar{v}\|).$$

Noticing that $|\frac{\partial U_i}{\partial y_j^i}| = 0(\lambda_i U_i)$, Hölder's inequality and (D.17) yield

$$\int_\Omega U_i^{p-2} |\frac{\partial U_i}{\partial y_j^i}| \bar{v}^2 = 0(\lambda_i \|\bar{v}\|^2) \; ; \int_\Omega |\frac{\partial U_i}{\partial y_j^i}| |\bar{v}|^p = 0(\lambda_i \|\bar{v}\|^p).$$

Therefore, from estimates (D.8,19,23,26,31,43-45), (3.5) and (C.1), we deduce (C.5).

Proof of (C.7)

All the equalities of Lemma C.1 are proved, except (C.7), which is the most delicate to obtain. To improve our estimate of $\frac{\partial K_\mu}{\partial y_j^i}$, we inspire ourself of an idea of [3].

We remark that with our choice of coordinates - see (C.10) -, Ω is nearly symmetric with respect to the x_n-xaxis in a neighbourhood of $y^i = 0$. Therefore, there exists φ, a diffeomorphism close to identity in a neighbourhood of 0, such that $\varphi(\Omega)$ is symmetric with respect to the n-th axis in this neighborhood. $\frac{\partial U_i}{\partial y_j^i}, 1 \leq j \leq n-1$, being odd with respect to x_j, $\widetilde{\frac{\partial U_i}{\partial y_j^i}} = \frac{\partial U_i}{\partial y_j^i} \circ \varphi^{-1}$ is nearly odd with respect to j-th coordinate. At the same time, we split $\tilde{v} = v \circ \varphi^{-1}$ into an even and an odd part with respect to the j-th coordinate, and we are able to show that the odd part is

smaller than the even one. Therefore, through this transformation and this splitting, we may get improved estimates of integrals where both $\frac{\partial U_i}{\partial y_j^i}$ and $\bar{v}$ occur.

Let us now prove this in details, for $i = 1$, $j = 1$ (according to our choice of origin and basis in $\mathbf{R}^n$, $y^1 = 0$ and $\mathbf{R}^{n-1} \times \{0\}$ is the tangent space to $\partial\Omega$ at 0). We take $R > 0$, small enough, such that

$$(C.12) \qquad \Omega \cap B(0,R) = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x \in B(0,R), x_n > f(x')\}$$

with

$$(C.13) \qquad f(x') = \sum_{i=1}^{n-1} f_i x_i^2 + \sum_{1 \le i \le j \le \ell \le n-1} g_{ij\ell} x_i x_j x_\ell + 0(|x'|^4).$$

We set

$$(C.14) \qquad \begin{cases} \varphi : \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}^{n-1} \times \mathbf{R} \\ \qquad (x', x_n) \mapsto z = (z', z_n) = (x', x_n - f(x')). \end{cases}$$

From these definitions, we see that there exists V a neighbourhood of 0 in $\mathbf{R}^n$ and $R' > 0$ such that φ induces a diffeomorphism between V and $B(0,R')$. As a consequence, φ induces a diffeomorphism between $W = V \cap \Omega$ and $\varphi(W) = B^+ = B^+(0,R')$, with

$$B^+(0,R') = \{z = (z', z_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / z \in B(0,R'), z_n > 0\}.$$

In the following, for u a function defined on W, we denote by $\tilde{u}$ the corresponding function defined on B^+ such that

$$u(x) = \tilde{u}(z) \quad \text{with} \quad z = \varphi(x) \ .$$

Let $\zeta \in C^\infty(\mathbf{R}^+, \mathbf{R})$ a smooth function such that

$$(C.15) \qquad 0 \le \xi \le 1 \ ; \ \zeta(t) = 1 \text{ if } 0 \le t \le \frac{1}{2} \ ; \ \zeta(t) = 0 \text{ if } t \ge 1 \ .$$

We set, for $T > 0$, and $z \in \mathbf{R}^n$

$$(C.16) \qquad \tilde{\zeta}_T(z) = \zeta(T|z|)$$

and we notice that if T is large enough, $\text{supp}(\tilde{\zeta}_T) \subset B(0,R')$. For obvious reasons, we denote by ζ_T the function defined on $\mathbf{R}^n$ as

$$(C.17) \qquad \zeta_T(x) = \tilde{\zeta}_T(z) \text{ if } x \in V \ ; \ \zeta_T(x) = 0 \text{ if } x \in \mathbf{R}^n \setminus V \ .$$

For T large enough, ζ_T satisfies

$$(C.18) \qquad \zeta_T(x) = 1 \quad \text{for} \quad |x| \leq \frac{1}{4T} \; ; \; \zeta_T(x) = 0 \quad \text{for} \quad |x| \geq \frac{2}{T}.$$

In the following, we assume

$$(C.19) \qquad T = \lambda_1^q \qquad 0 < q \leq \frac{1}{3} \; .$$

We set

$$(C.20) \qquad \bar{v} = v_1 + v_2 + v_3$$

with

$$(C.21) \qquad \begin{cases} v_1(x) = \tilde{v}_1(z) = \dfrac{1}{2}\tilde{\zeta}_T(z)(\tilde{\bar{v}}(z) + \tilde{\bar{v}}(\bar{z})) \\ v_2(x) = \tilde{v}_2(z) = \dfrac{1}{2}\tilde{\zeta}_T(z)(\tilde{\bar{v}}(z) - \tilde{\bar{v}}(\bar{z})) \\ v_3(x) = (1 - \zeta_T(x))\bar{v}(x) \end{cases}$$

and

$$\bar{z} = (-z_1, z_2, \cdots, z_n) \; .$$

v_1 and v_2 are assumed to be zero out of V. The crucial point is to get a good estimate of v_2, which represents the "odd part" of v with respect to x_1. But before estimating v_2 itself, we need a convenient estimate of v_3 in a fixed neighbourhood of 0. In order to perform the next computations, we remark that

$$(C.22) \qquad |\varphi'(x)| = |\frac{dz}{dx}| = 1 \quad \text{on} \quad \mathbf{R}^n$$

$$(C.23) \qquad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial z_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial z_n}, \quad 0 \leq i \leq n-1 \qquad \frac{\partial}{\partial x_n} = \frac{\partial}{\partial z_n}$$

and

$$(C.24) \qquad |x|^2 = |z|^2 + 2\sum_{i=1}^{n-1} f_i z_i^2 z_n + 0(|z|^4) \; .$$

Therefore, for any $\delta \in \mathbf{R}$

$$(C.25) \qquad \begin{aligned} (1 + \lambda^2|x|^2)^\delta = (1 + \lambda^2|z|^2)^\delta + 2\delta\lambda^2(1 + \lambda^2|z|^2)^{\delta-1}(\sum_{i=1}^{n-1} f_i z_i^2)z_n \\ + 0\left(\lambda^2(1 + \lambda^2|z|^2)^{\delta-1}|z|^4\right) . \end{aligned}$$

Estimate of v_3

More precisely, what we need is an estimate of v_3 in W. For this purpose, we set

$$(C.26) \qquad v_4 = \zeta_S v_3 = \zeta_S(1-\zeta_T)\bar{v}$$

where $S > 0$ is a fixed number which is chosen such that

$$(C.27) \qquad \begin{cases} \zeta_S(x) = 1 \quad \forall x \in W \quad (\text{i.e. } \tilde{\zeta}_S(z) = 1 \ \forall z \in B^+, \quad \text{that is} \quad S > \dfrac{1}{R'}) \\ \zeta_S(x) = 0 \quad \forall x \in W', \quad \text{a neighbourhood of} \quad \{y^2, \cdots, y^k\} \quad \text{in} \quad \Omega\,. \end{cases}$$

In order to estimate v_4, we write

$$(C.28) \qquad v_4 = aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^{n} c_j \frac{\partial U_1}{\partial y_j^1} + v_4'$$

with

$$(C.29) \qquad \int_\Omega \nabla U_1 . \nabla v_4' = \int_\Omega \nabla \frac{\partial U_1}{\partial \lambda_1} . \nabla v_4' = \int_\Omega \nabla \frac{\partial U_1}{\partial y_j^1} . \nabla v_4' = 0 \quad 1 \le j \le n\,.$$

Multiplying the gradient of (C.28) by $\nabla U_1, \nabla \frac{\partial U_1}{\partial \lambda_1}, \nabla \frac{\partial U_1}{\partial y_j^1}$ respectively and integrating on Ω, we obtain an invertible linear system in a, b, c_j which provides us with an estimate of these coefficients. On the left hand side, we have quantities such that, in view of (C.26)

$$\begin{aligned} \int_\Omega \nabla U_1 . \nabla v_4 &= 0\left(\int_{\substack{x\in\Omega \\ \frac{1}{4T} \le |x|}} |\nabla U_1|(|\nabla \bar{v}| + T|\bar{v}|)\right) \\ &= 0\left((1+\frac{T}{\mu^{1/2}})\|\bar{v}\|(\int_{\frac{1}{4T}\le|x|} |\nabla U_1|^2)^{1/2}\right). \end{aligned}$$

The integrals $\int_\Omega \nabla \frac{\partial U_1}{\partial \lambda_i} . \nabla v_4$ and $\int_\Omega \nabla \frac{\partial U_1}{\partial y_j^1} . \nabla v_4$ may be treated in the same way, and, using (C.6,19) and estimates (D. 29) we get

$$(C.30) \qquad \begin{cases} \displaystyle\int_\Omega \nabla U_1 . \nabla v_4 = 0(\frac{T^{\frac{n-2}{2}}}{\lambda_1^{\frac{n-2}{2}}}\|\bar{v}\|) \\ \displaystyle\int_\Omega \nabla \frac{\partial U_1}{\partial \lambda_1} . \nabla v_4 = 0(\frac{T^{\frac{n-2}{2}}}{\lambda_1^{\frac{n}{2}}}\|\bar{v}\|) \\ \displaystyle\int_\Omega \nabla \frac{\partial U_1}{\partial y_j^1} . \nabla v_4 = 0(\frac{T^{\frac{n}{2}}}{\lambda_1^{\frac{n-2}{2}}}\|\bar{v}\|) \qquad 1 \le j \le n\,. \end{cases}$$

Therefore, through (D.27), the linear system yields

$$(C.31) \qquad a = 0\left(\frac{T^{\frac{n-2}{2}}}{\lambda_1^{\frac{n-2}{2}}}\|\bar{v}\|\right); b = 0\left(\frac{T^{\frac{n-2}{2}}}{\lambda_1^{\frac{n-4}{2}}}\|\bar{v}\|\right); c_j = 0\left(\frac{T^{\frac{n}{2}}}{\lambda_1^{\frac{n+2}{2}}}\|\bar{v}\|\right)$$

In order to estimate v_4', we take $w = v_4'$ in (3.20). On the right hand side we find

$$\mathcal{R} = \sum_{i=1}^{k}\left(A_i\int_\Omega \nabla U_i.\nabla v_4' + B_i\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla v_4' + \sum_{j=1}^{n-1} C_{ij}\int_\Omega \nabla\frac{\partial U_i}{\tau_j^i}.\nabla v_4'\right) .$$

From (C.28, 29) we deduce

$$\begin{aligned}\mathcal{R} = \sum_{i=2}^{k}\Bigg(& A_i\int_\Omega \nabla U_i.\nabla(v_4 - aU_1 - b\frac{\partial U_i}{\partial\lambda_1} - \sum_{j=1}^{n} c_j\frac{\partial U_1}{\partial y_j^1})\\ &+ B_i\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla(v_4 - aU_1 - b\frac{\partial U_1}{\partial\lambda_1} - \sum_{j=1}^{n} c_j\frac{\partial U_1}{\partial y_j^1})\\ &+ \sum_{\ell=1}^{n-1} C_{i,\ell}\int_\Omega \nabla\frac{\partial U}{\partial\tau_\ell^i}.\nabla(v_4 - aU_1 - b\frac{\partial U_1}{\partial\lambda_i} - \sum_{j=1}^{n} c_j\frac{\partial U_1}{\partial y_j^1})\Bigg) .\end{aligned}$$

We note that, for $2 \le i \le k$, since ζ_S is identically zero in a neighbourhood of y^i

$$\int_\Omega \nabla U_i.\nabla v_4 = \int_\Omega \nabla U_i.\nabla(\zeta_S(1-\zeta_T)\bar{v}) = 0(\frac{1}{\lambda_i^{\frac{n-2}{2}}}\int_\Omega(|\nabla\bar{v}| + T|\bar{v}|)$$

and similar expressions for $\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla v_4$ and $\int_\Omega \nabla\frac{\partial U_i}{\partial\tau_j^i}.\nabla v_4$. In view of (C.19) and (C.6), we find

$$(C.32) \qquad \begin{cases} \displaystyle\int_\Omega \nabla U_i.\nabla v_4 = 0(\frac{\|\bar{v}\|}{\lambda_i^{\frac{n-2}{2}}}) \\ \displaystyle\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla v_4 = 0(\frac{\|\bar{v}\|}{\lambda_i^{\frac{n}{2}}}) \\ \displaystyle\int_\Omega \nabla\frac{\partial U_i}{\partial\tau_j^i}.\nabla v_4 = 0(\frac{\|\bar{v}\|}{\lambda_i^{\frac{n-2}{2}}}) \\ 2 \le i \le k,\ 1 \le j \le n-1 . \end{cases}$$

This, together with (C.8,19,31) and (D.26), leads to

$$(C.33) \qquad \mathcal{R} = 0\left[(\frac{|\eta|}{\lambda^{\frac{n-2}{2}}} + \frac{1}{\lambda^{\frac{n}{2}}})\|\bar{v}\|\right]$$

The left hand side is equal to
(C.34)

$$\mathcal{L} = \int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla v_4' + \mu \int_\Omega (\varphi_{\alpha,\lambda,y} + \bar{v}) v_4' - \int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v}) v_4' \ .$$

Let us compute the first integral

$$\begin{aligned}
&\int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla v_4' \\
&= \sum_{i=1}^k \alpha_i \int_\Omega \nabla U_i.\nabla v_4' + \int_\Omega \nabla \bar{v}.\nabla v_4' \\
&= \sum_{i=2}^k \alpha_i \int_\Omega U_i.\nabla(v_4 - aU_1 - b'\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1}) \\
&\quad + \int_\Omega \nabla(v_1 + v_2 + (1-\zeta_S)v_3 + v_4').\nabla v_4' \\
&= \sum_{i=2}^k \alpha_i \left(\int_\Omega \nabla U_i.\nabla v_4 - a\int_\Omega \nabla U_i.\nabla U_1 - b\int_\Omega \nabla U_i.\nabla \frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \int_\Omega \nabla U_i.\nabla \frac{\partial U_1}{\partial y_j^1} \right) \\
&\quad + \int_\Omega |\nabla v_4'|^2 + \int_\Omega \nabla(v_1 + v_2 + (1-\zeta_S)v_3).\nabla v_4 \\
&\quad - \int_\Omega \nabla(\bar{v} - v_4).\nabla(aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j'}) \\
&= \int_\Omega |\nabla v_4'|^2 + \int_\Omega \nabla(v_1 + v_2 + (1-\zeta_S)v_3)\nabla(\zeta_S v_3) + 0(\frac{\|\bar{v}\|}{\lambda^{\frac{n-2}{2}}})
\end{aligned}$$

using (C.30-32) and (D.26). Moreover, we have

$$\begin{aligned}
&\int_\Omega \nabla(v_1 + v_2 + (1-\zeta_S)v_3)\nabla(\zeta_S v_3) \\
&= \int_\Omega \nabla(v_1 + v_2).\nabla v_3 + \int_\Omega \nabla((1-\zeta_S)v_3).\nabla(\zeta_S v_3) \\
&= \int_\Omega \nabla(\zeta_T \bar{v}).\nabla((1-\zeta_T)\bar{v}) + \int_\Omega \nabla((1-\zeta_S)v_3)\nabla(\zeta_S v_3) \\
&= \int_\Omega \zeta_T(1-\zeta_T)|\nabla \bar{v}|^2 + \int_\Omega (1-2\zeta_T)\nabla\zeta_T.\bar{v}\nabla\bar{v} - \int_\Omega |\nabla\zeta_T|^2|\bar{v}|^2 \\
&+ \int_\Omega \zeta_S(1-\zeta_S)|\nabla v_3|^2 + \int_\Omega (1-2\zeta_S)\nabla\zeta_S.v_3\nabla v_3 - \int_\Omega |\nabla\zeta_S|^2 v_3^2 \\
&\geq 0\left(T\int_\Omega |\bar{v}||\nabla\bar{v}| + T^2\int_\Omega |\bar{v}|^2 + \int_\Omega |v_3||\nabla v_3| + \int_\Omega v_3^2 \right) \\
&\geq 0\left(\frac{T}{\lambda^{1/2}}\|\bar{v}\|^2 \right)
\end{aligned}$$

using Schwarz inequality, (C.19) and (C.6). Therefore

$$(C.35) \qquad \int_\Omega \nabla(\varphi_{\alpha,\lambda,y}+\bar v).\nabla v_4' \geq \int_\Omega |\nabla v_4'|^2 + 0(\frac{T}{\lambda^{1/2}}\|\bar v\|^2 + \frac{\|\bar v\|}{\lambda^{\frac{n-2}{2}}}) .$$

Let us consider now the second integral in (C.34). According to (C.28)

$$\begin{aligned}
&\mu\int_\Omega(\varphi_{\alpha,\lambda,y}+\bar v)v_4' \\
&= \mu\sum_{i=1}^k \alpha_i \int_\Omega U_i(v_4 - aU_1 - b\frac{\partial U_1}{\partial\lambda_1} - \sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1}) + \mu\int_\Omega v_4'^2 \\
&+ \mu\int_\Omega (v_1+v_2+(1-\zeta_S)v_3 + aU_1 + b\frac{\partial U_1}{\partial\lambda_1} + \sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1}) \\
&\qquad .(\zeta_S v_3 - aU_1 - b\frac{\partial U_1}{\partial\lambda_1} - \sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1}).
\end{aligned}$$

We have

$$\begin{aligned}
\int_\Omega U_1 v_4 &= \int_\Omega \zeta_S(1-\zeta_T)U_1\bar v \\
&= 0\left(\int_{\frac{1}{2T}\leq|x|} U_1|\bar v|\right) \\
&= 0\left((\int_\Omega \bar v^2)^{1/2}(\int_{\frac{1}{4T}\leq|x|} U_1^2)^{1/2}\right)
\end{aligned}$$

hence

$$(C.36) \qquad \int_\Omega U_1 v_4 = 0(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n-1}{2}}}\|\bar v\|)$$

because of (D.25) and (C.6), and, for $2\leq i\leq k$

$$\int_\Omega U_i v_4 = 0\left(\frac{1}{\lambda_i^{\frac{n-2}{2}}}(\int_\Omega|\bar v|^2)^{1/2}\right) = 0(\frac{1}{\lambda^{\frac{n-1}{2}}}\|\bar v\|)$$

because of (C.6). Therefore, using (C.1,31) and (D.21-23, 31)

$$\mu\sum_{i=1}^n \alpha_i\int_\Omega U_i(v_4 - aU_1 - b\frac{\partial U_1}{\partial\lambda_1} - \sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1}) = 0(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n-3}{2}}}) .$$

Lastly

$$\mu\int_\Omega (v_1+v_2+(1-\zeta_S)v_3+aU_1+b\frac{\partial U_1}{\partial\lambda_1}+\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j'})(\zeta_S v_3-aU_1-b\frac{\partial U_1}{\partial\lambda_1}-\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1})$$

$$=\mu\int_\Omega(\zeta_T+(1-\zeta_S)(1-\zeta_T))\zeta_S(1-\zeta_T)\bar v^2-\mu\int_\Omega(aU_1+b\frac{\partial U_1}{\partial\lambda_1}+\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1})^2$$

$$-\mu\int_\Omega(aU_1+b\frac{\partial U_1}{\partial\lambda_1}+\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1})(v_1+v_2+(1-2\zeta_S)v_3)$$

$$\geq 0\left[\mu\int_\Omega\left(a^2U_1^2+b^2(\frac{\partial U_1}{\partial\lambda_1})^2+\sum_{j=1}^n c_j^2(\frac{\partial U_1}{\partial y_j'})^2\right)+\mu\int_\Omega\left|aU_1+b\frac{\partial U_1}{\partial\lambda_1}+\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j'}\right||\bar v|\right]$$

$$\geq 0\left(\frac{T^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\|\bar v\|^2\right)$$

because of (C.1,6,19,31) (D.31) and, still using (C.6) (D.31)

$$\int_\Omega U_1|\bar v|=0\left[(\int_\Omega\bar v^2)^{1/2}(\int_\Omega U_1^2)^{1/2}\right]=0\left(\frac{1}{\lambda^{3/2}}\|\bar v\|\right)$$

$$\int_\Omega|\frac{\partial U_1}{\partial\lambda_1}||\bar v|=0\left(\frac{1}{\lambda^{5/2}}\|\bar v\|\right)$$

$$\int_\Omega|\frac{\partial U_1}{\partial y_j^1}||\bar v|=0\left(\frac{1}{\lambda^{1/2}}\|\bar v\|\right)\ .$$

Thus

$$(C.37)\qquad \mu\int_\Omega(\varphi_{\alpha,\lambda,y}+\bar v)v_4'\geq\mu\int_\Omega v_4'^2+0\left(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n-2}{2}}}\|\bar v\|\right)\ .$$

The third integral in (C.34) writes (note that for $n\geq 6$, $p\leq 2$)

$$\int_\Omega|\varphi_{\alpha,\lambda,y}+\bar v|^{p-1}(\varphi_{\alpha,\lambda,y}+\bar v)v_4'$$

$$=\int_\Omega\left((\alpha_1U_1)^p+p(\alpha_1U_1)^{p-1}(\sum_{j=2}^k\alpha_jU_j+\bar v)+0(\sum_{j=2}^k U_j^p+|\bar v|^p)\right)v_4'$$

$$=\int_\Omega\left(p\alpha_1^{p-1}U_1^{p-1}\sum_{j=2}^k\alpha_jU_j+0(\sum_{j=2}^n U_j^p+|\bar v|^p)\right)(v_4-aU_1-b\frac{\partial U_1}{\partial\lambda_1}-\sum_{j=1}^n c_j\frac{\partial U_1}{\partial y_j^1})$$

$$+\alpha_1^p\int_\Omega U_1^pv_4'+p\alpha_1^{p-1}\int_\Omega U_1^{p-1}v_4'^2+p\alpha_1^{p-1}\int_\Omega U_1^{p-1}(\bar v-v_4')v_4'.$$

We have, for $2 \leq j \leq k$, using Hölder's inequality, Sobolev embedding theorem and (D.38)

$$\int_\Omega U_1^{p-1} U_j v_4 = 0\left[\frac{\|\bar{v}\|}{\lambda_j^{\frac{n-2}{2}}}\left(\int_\Omega U_1^{\frac{(p-1)(p+1)}{p}}\right)^{\frac{p}{p+1}}\right]$$

$$= 0\left[\|\bar{v}\|\left(\frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^4} \text{ if } n=6\ ; \frac{1}{\lambda^{\frac{n+2}{2}}} \text{ if } n \geq 7\right)\right]$$

$$\int_\Omega U_j^p |v_4| = 0\left(\frac{1}{\lambda_j^{\frac{n+2}{2}}}\int_\Omega |\bar{v}|\right) = 0\left(\frac{1}{\lambda^{\frac{n+3}{2}}}\|\bar{v}\|\right)$$

because of (C.6). Since we have also, using Hölder's inequality, (D.17), and noticing that $\frac{\partial U_i}{\partial \lambda_i} = 0(\frac{U_i}{\lambda_i})$, $\frac{\partial U_i}{\partial y_j^i} = 0(\lambda_i U_i)$

$$(C.38) \quad \int_\Omega U_1 |\bar{v}|^p = 0(\|\bar{v}\|^p)\ ; \int_\Omega |\frac{\partial U_1}{\partial \lambda_1}||\bar{v}|^p = 0(\frac{\|\bar{v}\|^p}{\lambda_1})\ ; \int_\Omega |\frac{\partial U_1}{\partial y_j^1}||\bar{v}|^p = 0(\lambda_1 \|\bar{v}\|^p)$$

(D.42), (C.19,31) and (3.24) lead to

(C.39)

$$\int_\Omega \left(p\alpha_1^{p-1} U_1^{p-1} \sum_{j=2}^k \alpha_j U_j + 0(\sum_{j=2}^k U_j^p + |\bar{v}|^p)\right)(v_4 - aU_1 - b\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1})$$

$$= 0\left(\|\bar{v}\|(\frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^2} \text{ if } n=6\ ; \frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n>6)\right).$$

We have also

$$\int_\Omega U_1^p v_4' = -\frac{1}{n(n-2)}\int_\Omega \Delta U_1 . v_4'$$

$$= -\frac{1}{n(n-2)}\int_{\partial\Omega} \frac{\partial U_1}{\partial \nu} v_4' \quad \text{integrating by parts}$$

$$= -\frac{1}{n(n-2)}\int_{\partial\Omega} \frac{\partial U_1}{\partial \nu}(v_4 - aU_1 - b\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1})$$

$$= 0\left(\|\bar{v}\|(\int_{\substack{\partial\Omega \\ \frac{1}{4T} \leq |x|}} |\frac{\partial U_1}{\partial \nu}|^{\frac{2(n-1)}{n}})^{\frac{n}{2(n-1)}} + \int_{\partial\Omega} |\frac{\partial U_1}{\partial \nu}||aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j}|\right).$$

Using (C.19,31) and (D.51,56), we get

$$(C.40) \qquad \int_\Omega U_1^p v_4' = 0(\frac{T^{\frac{n-4}{2}}}{\lambda_1^{\frac{n-2}{2}}}\|\bar{v}\|)\ .$$

Lastly

$$\begin{aligned}
&\int_\Omega U_1^{p-1}(\bar v - v_4')v_4' \\
&= \int_\Omega U_1^{p-1}(\bar v - v_4 + aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1})(v_4 - aU_1 - b\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1}) \\
&= \int_\Omega U_1^{p-1}\left((1-(1-\zeta_T)\zeta_S)\bar v + aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1}\right) \\
&\qquad\qquad \cdot\left((1-\zeta_T)\zeta_S \bar v - aU_1 - b\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1}\right) \\
&= 0\left[\int_\Omega U_1^{p-1}\left(|\bar v|\left|aU_1 + b\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j \frac{\partial U_1}{\partial y_j^1}\right| + a^2U_1^2 + b^2(\frac{\partial U_1}{\partial \lambda_1})^2 + \sum_{j=1}^n c_j^2(\frac{\partial U_1}{\partial y_j^1})^2\right)\right. \\
&\qquad\qquad \left. + \int_{\substack{\Omega \\ |x|\geq \frac{1}{4T}}} U_1^{p-1}\bar v^2\right].
\end{aligned}$$

Using (D.2,17), and the fact that, in view of (C.6)

$$\int_{\substack{\Omega \\ |x|\geq \frac{1}{4T}}} U_1^{p-1}\bar v^2 = 0\left(\frac{T^4}{\lambda^2}\int_\Omega |\bar v|^2\right) = 0\left(\frac{T^4}{\lambda^3}\|\bar v\|^2\right)$$

we get, because of Hölder's and Sobolev inequalities, (C.19) and (C.31)

$$\begin{aligned}
&\int_\Omega U_1^{p-1}(\bar v - v_4')v_4' \\
&= 0\left[\|\bar v\|\left(|a| + \frac{|b|}{\lambda_1} + \sum_{j=1}^n \lambda_1|c_j|\right) + a^2 + \frac{b^2}{\lambda_1^2} + \lambda_1^2\sum_{j=1}^n c_j^2 + \frac{T^4}{\lambda^3}\|\bar v\|^2\right] \\
&= 0\left((\frac{T^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} + \frac{T^4}{\lambda^3})\|\bar v\|^2\right).
\end{aligned}$$

This, together with (C.19, 39-40) and (3.24), yields

$$\begin{aligned}
(C.41)\qquad &\int_\Omega |\varphi_{\alpha,\lambda,y} + \bar v|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar v)v_4' = p\alpha_1^{p-1}\int_\Omega U_1^{p-1}v_4'^2 \\
&\qquad + 0\left[\|\bar v\|\left(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n-2}{2}}} + (\frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^2} \text{ if } n=6\ ; \frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n\geq 7)\right)\right]
\end{aligned}$$

and, from (C.33-35,37,40), (3.24) and Lemma 3.1 we deduce

$$\|v_4'\|^2 = 0\left[\|\bar{v}\|\left(T\frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^{3/2}} \text{ if } n=6\ ; \frac{T}{\lambda^{3/2}}+\frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n\geq 7\right)\right].$$

Then, (C.1,19,28,31) and (D.26,31) allow to conclude

$$(C.42) \qquad \|v_4\|^2 = 0\left[\|\bar{v}\|\left(T\frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^{3/2}} \text{ if } n=6\ ; \frac{T}{\lambda^{3/2}}+\frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n\geq 7\right)\right].$$

Estimate of v_2

We are now able to evaluate $\|v_2\|$. We proceed as previously, writing

$$(C.43) \qquad v_2 = a'U_1 + b'\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^{n} c_j'\frac{\partial U_1}{\partial y_j^1} + v_2'$$

with

$$(C.44) \qquad \int_\Omega \nabla U_1.\nabla v_2' = \int_\Omega \nabla\frac{\partial U_1}{\partial \lambda_1}.\nabla v_2' = \int_\Omega \nabla\frac{\partial U_1}{\partial y_j^1}.\nabla v_2' = 0 \quad 1\leq j\leq n$$

We multiply the gradient of (C.43) by $\nabla U_1, \nabla\frac{\partial U_1}{\partial\lambda_1}, \nabla\frac{\partial U_1}{\partial y_j^1}, 1\leq j\leq n$, respectively, and integrate over Ω, in order to obtain a linear system in the coefficients a', b', c_j', whose left hand terms are $\int_\Omega \nabla U_1.\nabla v_2, \int_\Omega \nabla\frac{\partial U_1}{\partial\lambda_1}.\nabla v_2, \int_\Omega \nabla\frac{\partial U_1}{\partial y_j^1}.\nabla v_2$ respectively. Let us estimate these quantities. Using (C.14,21-23,25), we have

$$\begin{aligned}
&\int_\Omega \nabla U_1.\nabla v_2 dx \\
&= \sum_{i=1}^{n}\int_\Omega -\frac{(n-2)\lambda_1^{\frac{n+2}{2}}x_i}{(1+\lambda_i^2|x|^2)^{n/2}}\frac{\partial v_2}{\partial x_i}dx \\
&= -\frac{(n-2)}{2}\lambda_1^{\frac{n+2}{2}}\left(\sum_{i=1}^{n-1}\int_{B^+}\left[\frac{z_i}{(1+\lambda_1^2|z|^2)^{n/2}} - \frac{n\lambda_1^2(\sum_{j=1}^{n-1}f_jz_j^2z_n)z_i}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}}\right.\right. \\
&+0\left(\frac{\lambda_1^2|z|^5}{(1+\lambda^2|z|^2)^{\frac{n+2}{2}}}\right)\Bigg]\cdot\left[T\frac{z_i}{|z|}\xi'(T|z|)(\tilde{\bar{v}}(z)-\tilde{\bar{v}}(\bar{z}))+\tilde{\zeta}_T(z)(\frac{\partial\tilde{\bar{v}}}{\partial z_i}-\frac{\partial\tilde{\bar{v}}(\bar{z})}{\partial z_i})\right. \\
&\left.-(2f_iz_i+0(|z|^2))(T\frac{z_n}{|z|}\xi'(T|z|)(\tilde{\bar{v}}(z)-\tilde{\bar{v}}(\bar{z}))+\tilde{\zeta}_T(z)(\frac{\partial\tilde{\bar{v}}}{\partial z_n}-\frac{\partial\tilde{\bar{v}}(\bar{z})}{\partial z_n}))\right]dz \\
&+\int_{B^+}\left[\frac{z_n}{(1+\lambda_1^2|z|^2)^{n/2}}-\frac{n\lambda_1^2(\sum_{j=1}^{n-1}f_jz_j^2)z_n^2}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}}+0\left(\frac{\lambda_1^2|z|^5}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}}\right)\right] \\
&\left.\cdot\left[T\frac{z_n}{|z|}\xi'(T|z|)(\tilde{\bar{v}}(z)-\tilde{\bar{v}}(\bar{z}))+\tilde{\zeta}_T(z)(\frac{\partial\tilde{\bar{v}}}{\partial z_n}-\frac{\partial\tilde{\bar{v}}(\bar{z})}{\partial z_n})\right]dz\right).
\end{aligned}$$

Performing the change of variable $\bar{z} \mapsto z$ for the terms in which $\widetilde{\bar{v}}(\bar{z})$ or $\frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_i}$ occur, and using the symmetry of B^+ with respect to $z_1 = 0$, we get

$$\int_\Omega \nabla U_1 . \nabla v_2 dx = 0 \left(\lambda_1^{\frac{n+2}{2}} \int_{B^+} \frac{|z|^3}{(1+\lambda_1^2|z|^2)^{n/2}} (T|\widetilde{\bar{v}}(z)| + |\nabla \widetilde{\bar{v}}(z)|) dz \right).$$

Noticing that $\int_{B^+} \widetilde{\bar{v}}^2(z) dz = \int_W \bar{v}^2 dx$ and $\int_{B^+} |\nabla \widetilde{\bar{v}}(z)|^2 dz = 0(\int_W |\nabla \bar{v}(x)|^2 dx)$, this yields

$$\int_\Omega \nabla U_1 . \nabla v_2 dx = 0 \left[\lambda_1^{\frac{n+2}{2}} \left(\int_{B^+} \frac{|z|^6}{(1+\lambda_1^2|z|^2)^n} \right)^{1/2} (1 + \frac{T}{\mu^{1/2}}) \|\bar{v}\| \right].$$

Using (C.6,19) and (D.58) we get finally

$$(C.45) \qquad \int_\Omega \nabla U_1 . \nabla v_2 dx = 0 \left[\left(\frac{(\mathrm{Log}\lambda_1)^{1/2}}{\lambda_1^2} \text{ if } n = 6 \; ; \frac{1}{\lambda_1^2} \text{ if } n > 6 \right) \|\bar{v}\| \right].$$

In the same way, we have

$$(C.46) \qquad \int_\Omega \nabla \frac{\partial U_1}{\partial \lambda_1} . \nabla v_2 dx = 0 \left[\left(\frac{(\mathrm{Log}\lambda)^{1/2}}{\lambda_1^3} \text{ if } n = 6 \; ; \frac{1}{\lambda_1^3} \text{ if } n > 6 \right) \|\bar{v}\| \right].$$

Similarly, we compute, for $2 \leq i \leq n-1$

$$\begin{aligned}
&\int_\Omega \nabla \frac{\partial U_1}{\partial y_i^1} . \nabla v_2 dx \\
&= \sum_{j=1}^n \int_\Omega \nabla \frac{\partial^2 U_1}{\partial y_i^1 \partial x_j} \frac{\partial v_2}{\partial x_j} dx \\
&= (n-2)\lambda_1^{\frac{n+2}{2}} \sum_{j=1}^n \int_\Omega \left(\frac{\delta_{ij}}{(1+\lambda_1^2|x|^2)^{\frac{n}{2}}} - \frac{n\lambda_1^2 x_i x_j}{(1+\lambda_1^2|x|^{2\frac{n+2}{2}})} \right) \frac{\partial v_1}{\partial x_j} dx \\
&= (n-2)\lambda_1^{\frac{n+2}{2}} \left(\sum_{j=1}^n \int_{B^+} \left[\frac{\delta_{ij}}{(1+\lambda_1^2|z|^2)^{\frac{n}{2}}} - \frac{n\lambda_1^2 (\sum_{\ell=1}^{n-1} f_\ell z_\ell^2) z_n \delta_{ij}}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} \right.\right. \\
&\left. - \frac{n\lambda_1^2 z_i z_j}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} + \frac{n(n+2)\lambda_1^4 (\sum_{\ell=1}^{n-1} f_\ell z_\ell^2) z_i z_j z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} + 0\left(\frac{\lambda_1^2|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} \right) \right] \\
&\cdot \left[T\frac{z_j}{|z|}\xi'(T|z|)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_j} - \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_j}) \right. \\
&- (2f_j z_j + 0(|z|^2)) \left(T\frac{z_n}{|z|}\xi'(T|z|)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) + \tilde{\xi}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_n} - \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_n}) \right) \Big] dz \\
&+ \int_{B^+} \left[\frac{-n\lambda_1^2 z_i z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} + \frac{n(n+2)\lambda_1^4 (\sum_{\ell=1}^{n-1} f_\ell z_\ell^2) z_i z_n^2}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} + 0\left(\frac{\lambda_1^4|z|^6}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} \right) \right] \\
&\cdot \left. \left[T\frac{z_n}{|z|}\xi'(T|z|)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_n} - \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_n}) \right] dz \right\} \Big) .
\end{aligned}$$

The dominating terms have no contribution to the integral since they are odd with respect to z_1, and we get

$$\int_\Omega \nabla \frac{\partial U_1}{\partial y_i^1}.\nabla v_2 dx = 0\left(\lambda_1^{\frac{n+2}{2}} \int_{B+} \frac{|z|^2}{(1+\lambda_1^2|z|^2)^{n/2}}(T|\,\widetilde{\bar{v}}(z)| + |\nabla\,\widetilde{\bar{v}}(z)|)dz\right)$$
$$= \left(\lambda_1^{\frac{n+2}{2}}\left(\int_{B+} \frac{|z|^4}{(1+\lambda_1^2|z|^2)^n}\right)^{1/2}(1+\frac{T}{\mu^{1/2}})\|\bar{v}\|\right).$$

Thus, using (C.6,19) and (D.59), we obtain

$$(C.47) \qquad \int_\Omega \nabla \frac{\partial U_1}{\partial y_i^1}.\nabla v_2 dx = 0\left(\frac{\|\bar{v}\|}{\lambda_1}\right) \qquad 2 \le i \le n-1\ .$$

It remains to estimate $\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_2 dx$. We remark that, since $\bar{v} \in E_{\lambda,y}$

$$(C.48) \qquad \int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_2 = -\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_1 - \int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_3\ .$$

On one hand

$$\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_3 = 0\left[\|\bar{v}\|\left(\int_{\substack{\Omega \\ |x|\ge\frac{1}{4}}} |\nabla \frac{\partial U_1}{\partial y_1^1}|^2\right)^{1/2}\right].$$

Indeed, according to (C.21)

$$\int_\Omega |\nabla v_3|^2 = 0\left(\int_\Omega |\nabla \bar{v}|^2 + T^2\int_\Omega \bar{v}^2\right) = 0(\|\bar{v}\|^2)$$

because of (C.6,19). (D.29) then yields

$$(C.49) \qquad \int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_3 = 0\left(\frac{T^{\frac{n}{2}}}{\lambda_1^{\frac{n-2}{2}}}\|\bar{v}\|\right).$$

On the other hand, using (C.14,21-23,25)

$$
\begin{aligned}
&\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_1 \\
&= \sum_{j=1}^n \int_\Omega \frac{\partial^2 U_1}{\partial y_1^1 \partial x_j}\frac{\partial v_1}{\partial x_j}dx \\
&= (n-2)\lambda_1^{\frac{n+2}{2}} \sum_{j=1}^n \int_\Omega \left(\frac{\delta_{ij}}{(1+\lambda_1^2|x|^2)^{\frac{n}{2}}} - \frac{n\lambda_1^2 x_1 x_j}{(1+\lambda_2^2|x|^2)^{n+2}} \right) \frac{\partial v_1}{\partial x_i}dx \\
&= (n-2)\lambda_1^{\frac{n+2}{2}} \left(\sum_{j=1}^{n-1} \int_{B^+} \left[\frac{\delta_{1j}}{(1+\lambda_1^2|z|^2)^{\frac{n}{2}}} - \frac{n\lambda_1^2(\sum_{\ell=1}^{n-1} f_\ell z_\ell^2)z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}}\delta_{1j} \right.\right. \\
&- \frac{n\lambda_1^2 z_1 z_j}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} + \frac{n(n+2)\lambda_1^4(\sum_{\ell=1}^{n-1} f_\ell z_\ell^2)z_1 z_j z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} + 0\left(\frac{\lambda_1^2|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} \right) \Bigg] \\
&\cdot \left[T\frac{z_j}{|z|}\xi'(T|z|)(\widetilde{\widetilde{v}}(z)+\widetilde{\widetilde{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\widetilde{v}}}{\partial z_j} + \frac{\partial \widetilde{\widetilde{v}}(\bar{z})}{\partial z_j}) \right. \\
&\left. -(2f_j z_j + 0(|z|^2)) \left(T\frac{z_n}{|z|}\xi'(T|z|)(\widetilde{\widetilde{v}}(z)+\widetilde{\widetilde{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\widetilde{v}}}{\partial z_n} + \frac{\partial \widetilde{\widetilde{v}}(\bar{z})}{\partial z_n}) \right) \right] dz \\
&+ \int_{B^+} \left[-\frac{n\lambda_1^2 z_1 z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} + n(n+2)\lambda_1^4 \frac{(\sum_{\ell=1}^{n-1} f_\ell z_\ell^2)z_1 z_n^2}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} + 0\left(\frac{\lambda_1^4|z|^6}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} \right) \right] \\
&\left[T\frac{z_n}{|z|}\xi'(T|z|)(\widetilde{\widetilde{v}}(z)+\widetilde{\widetilde{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\widetilde{v}}}{\partial z_n} + \frac{\partial \widetilde{\widetilde{v}}(\bar{z})}{\partial z_n}) \right] dz \Bigg) \\
&= 0\left(\lambda_1^{\frac{n+2}{2}} \int_{B^+} \frac{|z|^2}{(1+\lambda_1^2|z|^2)^{\frac{n}{2}}} (T|\widetilde{\widetilde{v}}(z)| + |\nabla \widetilde{\widetilde{v}}(z)|)dz \right).
\end{aligned}
$$

As previously, for the estimate of $\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_2 dx$, we obtain, using Schwarz inequality, (C.6,19) and (D.59)

$$
\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_1 = 0\left(\frac{\|\bar{v}\|}{\lambda_1} \right). \tag{C.50}
$$

Combining (C.47-49) and (C.19), we get

$$
\int_\Omega \nabla \frac{\partial U_1}{\partial y_1^1}.\nabla v_2 = 0\left(\frac{\|\bar{v}\|}{\lambda_1} \right). \tag{C.51}
$$

From (C.43-44,49) and (D.27) we deduce

$$(C.52)\qquad \begin{cases} a' = 0\left[\left(\dfrac{(\mathrm{Log}\lambda)^{1/2}}{\lambda^2} \text{ if } n=6; \dfrac{1}{\lambda^2} \text{ if } n>6\right)\|\bar{v}\|\right] \\ b' = 0\left[\left(\dfrac{(\mathrm{Log}\lambda)^{1/2}}{\lambda} \text{ if } n=6; \dfrac{1}{\lambda} \text{ if } n>6\right)\|\bar{v}\|\right] \\ c'_j = 0\left[\dfrac{\|\bar{v}\|}{\lambda^3}\right] . \end{cases}$$

It only remains to estimate v'_2, taking $w = v'_2$ in (3.20). On the right hand side, we find

$$\begin{aligned} \mathcal{R}' &= \sum_{i=1}^{k}\left(A_i\int_\Omega \nabla U_i.\nabla v'_2 + B_i\int_\Omega \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla v'_2 + \sum_{j=1}^{n-1} C_{i,j}\int_\Omega \nabla\frac{\partial U_i}{\partial\tau_j^i}.\nabla v'_2\right) \\ &= \sum_{i=2}^{k}\left(\int_\Omega (A_i\nabla U_i + B_i\nabla\frac{\partial U_i}{\partial\lambda_i} + \sum_{j=1}^{n-1} C_{i,j}\nabla\frac{\partial U_i}{\partial\tau_j^i}).\nabla(v_2 - a'U_1 - b'\frac{\partial U_1}{\partial\lambda_1} - \sum_{j=1}^{\ell} c'_\ell\frac{\partial U_1}{\partial y_\ell^1})\right) \end{aligned}$$

because of (C.42-43). It is easy to see, from the definition of v_2 in (C.21), that, for $2 \le i \le n$

$$\begin{aligned} \int_\Omega \nabla U_i.\nabla v_2 &= 0\left(\frac{1}{\lambda_i^{\frac{n-2}{2}}}\int_\Omega |\nabla v_2|\right) \\ &= 0\left(\frac{1}{\lambda_i^{\frac{n-2}{2}}}\int_{|x|\le\frac{2}{T}} (|\nabla\bar{v}| + T|\bar{v}|)\right) \\ &= 0\left[\frac{1}{\lambda_i^{\frac{n-2}{2}}}\frac{1}{T^{\frac{n}{2}}}\left((\int_\Omega |\nabla\bar{v}|^2)^{1/2} + T(\int_\Omega \bar{v}^2)^{1/2}\right)\right] \end{aligned}$$

using Schwarz inequality. Finally, taking account of (C.6,19), we find

$$(C.53)\qquad \begin{cases} \displaystyle\int_\Omega \nabla U_i.\nabla v_2 = 0(\frac{1}{\lambda^{\frac{n-2}{2}}T^{\frac{n}{2}}}\|\bar{v}\|) \\ \displaystyle\int_\Omega \frac{\partial U_i}{\partial\lambda_i}.\nabla v_2 = 0(\frac{1}{\lambda^{\frac{n}{2}}T^{\frac{n}{2}}}\|\bar{v}\|) \qquad 2 \le i \le k\ ,\ 1 \le j \le n-1\ . \\ \displaystyle\int_\Omega \frac{\partial U_i}{\partial\tau_j^i}.\nabla v_2 = 0(\frac{1}{\lambda^{\frac{n-2}{2}}T^{\frac{n}{2}}}\|\bar{v}\|) \end{cases}$$

This together with (C.8,19,52) and (D.26) leads to

$$\mathcal{R}' = 0\left[(|\eta| + \frac{1}{\lambda})\frac{1}{\lambda^{\frac{n-2}{2}}T^{\frac{n}{2}}}\|\bar{v}\|\right] .$$

The left hand side is equal to
(C.55)
$$\mathcal{L} = \int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla v_2' + \mu \int_\Omega (\varphi_{\alpha,\lambda,y} + \bar{v}) v_2' - \int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v}) v_2' \ .$$

We begin by computing the first integral.

$$\begin{aligned}
&\int_\Omega \nabla(\varphi_{\alpha,\lambda,y} + \bar{v}).\nabla v_2' \\
&= \sum_{i=1}^k \alpha_i \int_\Omega \nabla U_i.\nabla v_2' + \int_\Omega \nabla \bar{v}.\nabla v_2' \\
&= \sum_{i=2}^k \alpha_i \int_\Omega \nabla U_i.\nabla(v_2 - a'U_1 - b'\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) + \int_\Omega \nabla(v_1 + v_2' + v_3).\nabla v_2' \\
&= \sum_{i=2}^k \alpha_i \left(\int_\Omega \nabla U_i.\nabla v_2 - a\int_\Omega \nabla U_i.\nabla U_1 - b'\int_\Omega \nabla U_i.\nabla \frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j' \int_\Omega \nabla U_i.\nabla \frac{\partial U_1}{\partial y_j^1} \right) \\
&+ \int_\Omega |\nabla v_2'|^2 + \int_\Omega \nabla(v_1 + v_3).\nabla v_2 - \int_\Omega \nabla(\bar{v} - v_2).\nabla(a'U_1 + b'\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) \\
&= \int_\Omega |\nabla v_2'|^2 + \int_\Omega \nabla(v_1 + v_3).\nabla v_2 + 0\left(\|\bar{v}\|(\frac{1}{\lambda^{\frac{n-2}{2}} T^{\frac{n}{2}}} + \frac{1}{\lambda^5}) \right)
\end{aligned}$$

using (C.19,43-47,51-53) and (D.26). Moreover

$$\begin{aligned}
&\int_\Omega \nabla v_1.\nabla v_2 \\
&= \sum_{i=1}^n \int_\Omega \frac{\partial v_1}{\partial x_i}\frac{\partial v_2}{\partial x_i} dx \\
&= \sum_{i=1}^{n-1} \int_{B^+} \left[T\frac{z_i}{|z|}\xi'(T|z|)(\widetilde{\bar{v}}(z) + \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_i} + \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_i}) \right. \\
&\left. -(2f_i z_i + 0(|z|^2))(T\frac{z_n}{|z|}\xi'(T|z|))(\widetilde{\bar{v}}(z) + \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_n} + \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_n}) \right] \\
&\cdot \left[T\frac{z_i}{|z|}\xi'(T|z|)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_i} - \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_i}) \right. \\
&\left. -(2f_i z_i + 0(|z|^2))(T\frac{z_n}{|z|}\xi'(T|z|))(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) + \tilde{\zeta}_T(z)(\frac{\partial \widetilde{\bar{v}}}{\partial z_n} - \frac{\partial \widetilde{\bar{v}}(\bar{z})}{\partial z_n}) \right] dz
\end{aligned}$$

$$+\int_{B^+}\left[T\frac{z_n}{|z|}\xi'(T|z|)(\tilde{\bar{v}}(z)+\tilde{\bar{v}}(\bar{z}))+\tilde{\zeta}_T(z)(\frac{\partial\tilde{\bar{v}}}{\partial z_n}+\frac{\partial\tilde{\bar{v}}(\bar{z})}{\partial z_n})\right]$$

$$\cdot\left[T\frac{z_n}{|z|}\xi'(T|z|)(\tilde{\bar{v}}(z)-\tilde{\bar{v}}(\bar{z}))+\tilde{\zeta}_T(z)(\frac{\partial\tilde{\bar{v}}}{\partial z_n}-\frac{\partial\tilde{\bar{v}}(\bar{z})}{\partial z_n})\right]dz$$

$$=0\left(\int_{|z|\leq\frac{1}{T}}|z|^2(|\nabla\tilde{\bar{v}}|^2+T^2|\tilde{\bar{v}}|^2)\right)\text{ using oddness properties with respect to } z_1$$

$$=0\left(\frac{1}{T^2}(\int_\Omega|\nabla\bar{v}|^2+T^2\bar{v}^2)\right)\text{ coming back to the variable } x \text{ through (C.14)}$$

$$=0\left(\frac{1}{T^2}\|\bar{v}\|^2\right)$$

because of (C.1,19). We have also, for any $\sigma>0$

$$|\int_\Omega\nabla v_3.\nabla v_2|=|\int_\Omega\nabla v_4.\nabla v_2|\leq(\int_\Omega|\nabla v_4|^2)^{1/2}(\int_\Omega|\nabla v_2|^2)^{1/2}\leq\sigma\int_\Omega|\nabla v_2|^2+\frac{1}{4\sigma}\|v_4\|^2\,.$$

Thus, using (C.19,42) and (3.24), we get

(*C*.56)

$$\int_\Omega\nabla(\varphi_{\alpha,\lambda,y}+\bar{v}).\nabla v_2'=(1+0(\sigma))\int_\Omega|\nabla v_2'|^2$$

$$+0\left[\|\bar{v}\|\left((\frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda T^2}+\frac{T(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\sigma\lambda^{3/2}})\text{ if } n=6\,;(\frac{1}{\lambda T^2}+\frac{1}{\sigma}(\frac{T}{\lambda^{3/2}}+\frac{1}{\lambda^{\frac{n+2}{n-2}}}))\text{ if } n\geq 7\right)\right].$$

We compute now the second integral in (C.55). According to (C.43), we have

(*C*.57)

$$\mu\int_\Omega(\varphi_{\alpha,\lambda,y}+\bar{v})v_2'$$

$$=\mu\sum_{i=1}^n\alpha_i\int_\Omega U_i(v_2-a'U_1-b'\frac{\partial U_1}{\partial\lambda_1}-\sum_{j=1}^n c_j'\frac{\partial U_1}{\partial y_j^1})+\mu\int_\Omega v_2'^2$$

$$+\mu\int_\Omega(v_1+v_3+a'U_1+b'\frac{\partial U_1}{\partial\lambda_1}+\sum_{j=1}^n c_j'\frac{\partial U_1}{\partial y_j^1})(v_2-a'U_1-b'\frac{\partial U_1}{\partial\lambda_1}-\sum_{j=1}^n c_j'\frac{\partial U_1}{\partial y_j^1})\,.$$

Using (C.21-22,25) and oddness properties with respect to z_1, we may write

$$\int_\Omega U_1 v_2 = \int_{B^+} \lambda_1^{\frac{n-2}{2}} \left[\frac{1}{(1+\lambda_1^2|z|^2)^{\frac{n-2}{2}}} - \frac{(n-2)\lambda_1^2(\sum_{i=1}^{n-1} f_i z_i^2) z_n}{(1+\lambda_1^2|z|^2)^{n/2}} + 0\left(\frac{\lambda_1^2|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n}{2}}}\right) \right]$$
$$\cdot \frac{1}{2}\tilde{\zeta}_T(z)(\tilde{\bar{v}}(z) - \tilde{\bar{v}}(\bar{z}))dz$$
$$= 0\left(\lambda_1^{\frac{n+2}{2}} \int_{|z|\leq \frac{1}{T}} \frac{|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n}{2}}} |\tilde{\bar{v}}(z)| dz\right)$$
$$= 0\left[\lambda_1^{\frac{n+2}{2}} \left(\int_{|z|\leq \frac{1}{T}} \frac{|z|^8 dz}{(1+\lambda_1^2|z|^2)^n}\right)^{1/2} \left(\int_{B^+} \tilde{\bar{v}}^2(z) dz\right)^{1/2}\right]$$
$$= 0\left[\frac{\lambda_1^{\frac{n+2}{2}}}{\mu^{1/2}} \|\bar{v}\| \left(\frac{T^{\frac{n-8}{2}}}{\lambda_1^n} \text{ if } n \leq 7 \; ; \frac{(\mathrm{Log}\lambda_1)^{1/2}}{\lambda_1^8} \text{ if } n = 8 \; ; \frac{1}{\lambda_1^{\frac{n+8}{2}}} \text{ if } n \geq 9\right)\right]$$

using (D.60). Therefore, (C.6) yields

$$(C.58) \qquad \mu \int_\Omega U_1 v_2 = 0\left[\left(\frac{T^{\frac{8-n}{2}}}{\lambda^{\frac{n-3}{2}}} \text{ if } n = 6,7 \; ; \frac{(\mathrm{Log}\lambda)^{1/2}}{\lambda^{5/2}} \text{ if } n = 8 \; ; \frac{1}{\lambda^{5/2}}\right) \|\bar{v}\|\right] .$$

For $2 \leq i \leq n$, we have also

$$\mu \int_\Omega U_i v_2 = 0\left(\frac{\mu}{\lambda^{\frac{n-2}{2}}} \int_\Omega |\bar{v}\|\right) = 0\left(\frac{\mu^{1/2}}{\lambda^{\frac{n-2}{2}}} \|\bar{v}\|\right)$$

so that, taking account of (C.1)

$$(C.59) \qquad \mu \int_\Omega U_i v_2 = 0\left(\frac{\|\bar{v}\|}{\lambda^{\frac{n-3}{2}}}\right) \qquad 2 \leq i \leq n .$$

Because of oddness with respect to z_1 - see (C.21) - we know that

$$\int_\Omega v_1 v_2 = \int_{B^+} \frac{1}{4}\tilde{\zeta}_T^2(z)(\tilde{\bar{v}}^2(z) - \tilde{\bar{v}}^2(\bar{z}))dz = 0 .$$

We write

$$(C.60) \qquad \mu \int_\Omega v_2 v_3 = \mu \int_\Omega v_2 v_4 = \mu \int_\Omega (v_2' + a' U_1 + b' \frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j'}) v_4 .$$

On one hand

$$(C.61) \qquad |\mu \int_\Omega v_2' v_4| \leq \mu^{1/2} (\int_\Omega v_2'^2)^{1/2} \mu^{1/2} (\int_\Omega v_4^2)^{1/2} \leq \sigma\mu \int_\Omega v_2'^2 + \frac{1}{4\sigma} \|v_4\|^2 .$$

On the other hand

$$(C.62) \qquad \int_\Omega U_1 v_4 = 0(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n}{2}}}\|\bar{v}\|) \quad \text{as given by (C.36)}$$

and similar computations show that

$$(C.63) \qquad \int_\Omega \frac{\partial U_1}{\partial \lambda_1} v_4 = 0(\frac{T^{\frac{n-4}{2}}}{\lambda^{\frac{n+2}{2}}}\|\bar{v}\|) \qquad \int_\Omega \frac{\partial U_1}{\partial y_j^1} v_4 = 0(\frac{T^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\|\bar{v}\|) \ .$$

Note that the same estimates obviously hold for v_3 instead of v_4. Lastly, using (C.21-22), (C.1) and (D.31)

$$\begin{aligned}
&\mu \int_\Omega (v_2 - v_1)(a' U_1 + b' \frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j'}) \\
&\qquad = -\mu \int_{B^+} \tilde{\zeta}_T(z)\, \widetilde{\bar{v}}(\bar{z})(a' \tilde{U}_1 + b' \frac{\widetilde{\partial U_1}}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\widetilde{\partial U_1}}{\partial y_j'}) dz \\
&\qquad = 0\left(\lambda^{1/2}\|\bar{v}\|(\frac{|a'|}{\lambda} + \frac{|b'|}{\lambda^2} + \sum_{j=1}^n |c_j'|)\right) .
\end{aligned}$$

Collecting this last estimate with (C.58-63,52) (3.24) and (D.21-23,31), we deduce from (C.57)

$$\begin{aligned}
(C.64) \qquad &\mu \int_\Omega (\varphi_{\alpha,\lambda,y} + \bar{v}) v_2' = \mu(1 + 0(\sigma)) \int_\Omega v_2'^2 \\
&\qquad + 0\left[\frac{1}{\sigma}\left(\frac{T(\mathrm{Log}\lambda)^{2/3}}{\lambda^{3/2}} \text{ if } n = 6\ ; \frac{T}{\lambda^{3/2}} + \frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n \geq 7\right)\|\bar{v}\|\right] .
\end{aligned}$$

It only remains to estimate the third integral in (C.55), i.e.

$$\begin{aligned}
&\int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v}) v_2' \\
&= \int_\Omega \left((\alpha_1 U_1)^p + p(\alpha_1 U_1)^{p-1}(\sum_{j=2}^k \alpha_j U_j + \bar{v}) + 0(\sum_{j=2}^k U_{j=2}^p U_j^p + |\bar{v}|^p)\right) v_2' \\
&= \int_\Omega \left(\alpha_1^p U_1^p + p(\alpha_1 U_1)^{p-1}(\sum_{j=2}^n \alpha_j U_j) + 0(\sum_{j=2}^n U_j^p + |\bar{v}|^p)\right) \\
&\qquad\qquad .(v_2 - a' U_1 - b' \frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) \\
&+ p\alpha_1^{p-1} \int_\Omega U_1^{p-1} v_2'^2 + p\alpha_1^{p-1} \int_\Omega U_1^{p-1}(\bar{v} - v_2') v_2' .
\end{aligned}$$

We have, for $2 \leq j \leq k$, using Hölder's inequality

$$\int_\Omega U_1^{p-1} U_j v_2 = 0 \left[\frac{1}{\lambda_j^{\frac{n-2}{2}}} \left(\int_\Omega |v_2|^{p+1} \right)^{\frac{1}{p+1}} \left(\int_\Omega U_1^{\frac{(p-1)(p+1)}{p}} \right)^{\frac{p}{p+1}} \right]$$

and, from (C.21)

$$\begin{aligned} \int_\Omega |v_2|^{p+1} &= \int_{B^+} \frac{1}{2^{p+1}} \tilde{\zeta}_T^{p+1}(z) |\,\widetilde{\overline{v}}(z) - \widetilde{\overline{v}}(\bar{z})|^{p+1} dz \\ &= 0 \left(\int_{B^+} |\,\widetilde{\overline{v}}(z)|^{p+1} dz \right) \\ &= 0 \left(\int_W |\bar{v}|^{p+1} \right) . \end{aligned}$$

Therefore, Sobolev embedding theorem and (D.38) yield

$$\int_\Omega U_1^{p-1} U_j v_2 = 0 \left[\|\bar{v}\| \left(\frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^4} \text{ if } n = 6 \; ; \frac{1}{\lambda^{\frac{n+2}{2}}} \text{ if } n \geq 7 \right) \right] .$$

Because of (C.21) and (C.6), we have also, for $2 \leq j \leq k$

$$\int_\Omega U_j^p |v_2| = 0 \left(\frac{1}{\lambda_j^{\frac{n+2}{2}}} \int_\Omega |v_2| \right) = 0 \left(\frac{1}{\lambda_j^{\frac{n+2}{2}}} \int_{\substack{x \in \Omega \\ |x| \leq \frac{2}{T}}} |\bar{v}| \right) = 0 \left(\frac{\|\bar{v}\|}{\lambda_j^{\frac{n+3}{2}} T^{\frac{n}{2}}} \right)$$

and, through a change of variables $x \to z$, it follows easily from (C.21) that

$$\int_\Omega |\bar{v}|^p v_2 = 0(\|\bar{v}\|^{p+1}) .$$

Thus, (C.19, 38, 52) (3.24) and (D.42-43) lead to

(*C*.65)

$$\int_\Omega \left(p(\alpha_1 U_1)^{p-1} (\sum_{j=2}^k \alpha_j U_j) + 0(\sum_{j=2}^k U_j^p + |\bar{v}|^p) \right) (v_2 - a' U_1 - b' \frac{\partial U_1}{\partial \lambda} - \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1})$$

$$= 0 \left[\|\bar{v}\| \left(\frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^2} \text{ if } n = 6 \; ; \frac{1}{\lambda^{\frac{n+2}{n-2}}} \text{ if } n \geq 7 \right) \right]$$

noticing that $\frac{\partial U_1}{\partial \lambda_1} = 0(\frac{U_1}{\lambda_1})$ and $\frac{\partial U_1}{\partial y_j^1} = 0(\lambda_1 U_1)$. Then, we compute

$$\begin{aligned} &\int_\Omega U_1^p (a' U_1 + b' \frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) \\ &= 0(|a'| + \frac{|b'|}{\lambda_1^2} + \sum_{j=1}^n \frac{|c_j'|}{\lambda_1}) \qquad \text{according to (D.17-19)} \\ &= 0 \left[\left(\frac{(\mathrm{Log}\lambda_1)^{1/2}}{\lambda_1^2} \text{ if } n = 6 \; ; \frac{1}{\lambda_1^2} \text{ if } n \geq 7 \right) \|\bar{v}\| \right] \qquad \text{according to (C.52)} \end{aligned}$$

and, in view of (C.21-22,25)

$$\begin{aligned}
&\int_\Omega U_1^p v_2 \\
&= \int_{B^+} \lambda_1^{\frac{n+2}{2}} \left(\frac{1}{(1+\lambda_1^2|z|^2)^{\frac{n+2}{2}}} - \frac{(n+2)\lambda_1^2(\sum_{i=1}^{n-1} f_i^2)z_n}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} + 0(\frac{\lambda_1^2|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}}) \right) \\
&\qquad \cdot \frac{1}{2}\tilde{\zeta}_T(z)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z}))dz \\
&= 0\left(\lambda_1^{\frac{n+6}{2}} \int_{B^+} \frac{|z|^4}{(1+\lambda_1^2|z|^2)^{\frac{n+4}{2}}} |\widetilde{\bar{v}}(z)|dz \right) \\
&= 0\left[\lambda_1^{\frac{n+6}{2}} \left(\int_{B^+} \frac{|z|^8 dz}{(1+\lambda_1^2|z|^2)^{n+4}} \right)^{1/2} \left(\int_{B^+} \widetilde{\bar{v}}^2(z)dz \right)^{1/2} \right]
\end{aligned}$$

where we used oddness properties with respect to z_1. Therefore, (C.6) and (D.61) imply

$$\int_\Omega U_1^p v_2 = 0(\frac{\|\bar{v}\|}{\lambda_1^{3/2}}) .$$

Thus, we obtain

$$(C.66) \qquad \int_\Omega \alpha_1^p U_1^p (v_2 - a'U_1 - b'\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) = 0\left(\frac{\|\bar{v}\|}{\lambda^{\frac{3}{2}}}\right) .$$

Lastly

$$\begin{aligned}
&\int_\Omega U_1^{p-1}(\bar{v} - v_2')v_2' \\
&= \int_\Omega U_1^{p-1}(\bar{v} - v_2 + a'U_1 + b'\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1})(v_2 - a'U_1 - b'\frac{\partial U_1}{\partial \lambda_1} - \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) \\
&= \int_\Omega U_1^{p-1} \left((v_1 + v_3)v_2 + (2v_2 - \bar{v})(a'U_1 + b'\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}) \right. \\
&\qquad \left. -(a'U_1 + b'\frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U}{\partial y_j^1})^2 \right) .
\end{aligned}$$

On one hand, using (C.21) and (C.6)

$$\int_\Omega U_1^{p-1} v_3 v_2 = 0\left(\int_{\substack{\Omega \\ |x| \geq \frac{1}{4T}}} U_1^{p-1}\bar{v}^2 \right) = 0\left(\frac{T^4}{\lambda^2} \int_\Omega \bar{v}^2 \right) = 0\left(\frac{T^4}{\lambda^3} \|\bar{v}\|^2 \right) .$$

On the other hand, using (C.21-22,25)

$$\begin{aligned}
&\int_\Omega U_1^{p-1} v_1 v_2 \\
&= \int_{B^+} \lambda_1^2 \left(\frac{1}{(1+\lambda_1^2|z|^2)^2} - \frac{4\lambda_1^2(\sum_{i=1}^n f_i z_i^2) z_n}{(1+\lambda_1^2|z|^2)^3} + 0\left(\frac{\lambda_1^2|z|^4}{(1+\lambda_1^2|z|^2)^3}\right)\right) \\
&\qquad \cdot \frac{1}{4}\tilde{\zeta}_T^2(z)(\widetilde{\bar{v}}(z) - \widetilde{\bar{v}}(\bar{z})) \\
&= 0\left(\lambda_1^4 \int_{\substack{z\in B^+ \\ |z|\le \frac{1}{T}}} \frac{|z|^4}{(1+\lambda_1^2|z|^2)^3} |\widetilde{\bar{v}}(z)| dz\right) \\
&= 0\left(\lambda_1^4 \left(\int_{|z|\le\frac{1}{T}} \frac{|z|^8}{(1+\lambda_1^2|z|^2)^6}\right)^{\frac{1}{2}} \left(\int_{B^+} \widetilde{\bar{v}}^2(z) dz\right)^{\frac{1}{2}}\right) \\
&= 0\left(\frac{\|\bar{v}\|}{\lambda_1^{\frac{5}{2}} T^{\frac{n-4}{2}}}\right)
\end{aligned}$$

where we used again oddness properties with respect to z_1, (C.6) and (D.65). It is easy to see, using (C.21), (D.17), and noticing that $\frac{\partial U_1}{\partial \lambda_1} = 0(\frac{U_1}{\lambda_1})$, $\frac{\partial U_1}{y_j^1} = 0(\lambda_1 u_1)$, that

$$\begin{aligned}
&\int_\Omega U_1^{p-1}(2v_2 - \bar{v})\left(a' U_1 + b' \frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}\right) \\
&= 0\left[\|\bar{v}\|\left(|a'| + \frac{|b'|}{\lambda_1} + \lambda_1 \sum_{j=1}^n |c_j'|\right)\right] \\
&= 0\left[\left(\frac{(\mathrm{Log}\lambda)^{1/2}}{\lambda^2} \text{ if } n=6\ ; \frac{1}{\lambda^2} \text{ if } n\ge 7\right)\|\bar{v}\|^2\right]
\end{aligned}$$

because of (C.52). It follows also from (D.17) and the previous remark that

$$\begin{aligned}
&\int_\Omega U_1^{p-1}\left(a' U_1 + b' \frac{\partial U_1}{\partial \lambda_1} + \sum_{j=1}^n c_j' \frac{\partial U_1}{\partial y_j^1}\right)^2 \\
&= 0\left(a'^2 + \frac{b'^2}{\lambda_1^2} + \lambda_1^2 \sum_{j=1}^n c_j'^2\right) \\
&= 0\left[\left(\frac{\mathrm{Log}\ \lambda}{\lambda^4} \text{ if } n=6\ ; \frac{1}{\lambda^4} \text{ if } n\ge 7\right)\|\bar{v}\|^2\right] \quad \text{using again (C.52).}
\end{aligned}$$

From (C 65-66) and these last estimates, taking account of (C.19) and (3.24), we conclude

$$(C.67)\quad \int_\Omega |\varphi_{\alpha,\lambda,y} + \bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y} + \bar{v}) v_2' = p\alpha_1^{p-1} \int_\Omega U_1^{p-1} v_2'^2 + 0\left(\frac{\|\bar{v}\|}{\lambda^{\inf(\frac{3}{2}, \frac{n+2}{n-2})}}\right).$$

Then, collecting (C.54-56,64,67), α_1 being close to $\bar{\alpha}$ and σ being chosen small enough, Lemma 3.1 applies and we obtain

(*C*.68)

$$\|v_2'\|^2 = 0\left[\left(\left(\frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda T^2}+\frac{T(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^{3/2}}\right) \text{ if } n=6\ ; \left(\frac{1}{\lambda T^2}+\frac{T}{\lambda^{3/2}}+\frac{1}{\lambda^{\frac{n+2}{n-2}}}\right) \text{ if } n\geq 7\right)\|\bar{v}\|\right].$$

Then, (C.43,52) and (D.26) allow to conclude

(*C*.69)

$$\|v_2\|^2 = 0\left[\left(\left(\frac{(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda T^2}+\frac{T(\mathrm{Log}\lambda)^{\frac{2}{3}}}{\lambda^{\frac{3}{2}}}\right) \text{ if } n=6\ ; \left(\frac{1}{\lambda T^2}+\frac{T}{\lambda^{\frac{3}{2}}}+\frac{1}{\lambda^{\frac{n+2}{n-2}}}\right) \text{ if } n\geq 7\right)\|\bar{v}\|\right].$$

In order to make $\|v_2\|$ as small as possible, the best choice for q in (C.19) is $q=\frac{1}{6}$. In this case (C.42,69) (3.24) lead to

(*C*.70)

$$\begin{cases} \|v_4\| = 0\left(\dfrac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^{4/3}} \text{ if} n=6\ ; \dfrac{1}{\lambda^{\inf(\frac{4}{3},\frac{n}{n-2})}} \text{ if } n\geq 7\right) \\ \|v_2\| = 0\left(\dfrac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^{7/6}} \text{ if} n=6\ ; \dfrac{1}{\lambda^{\inf(\frac{7}{6},\frac{n}{n-2})}} \text{ if } n\geq 7\right) \end{cases}.$$

Estimate of $\frac{\partial K_\mu}{\partial y_1^1}$

We are now able to prove (C.7) - which means, with our assumptions, obtaining the suitable estimate for $\frac{\partial K_\mu}{\partial y_1^1}$. Since we can write, for $n\geq 6$ ($p\leq 2$) and σ, $2-p<\sigma<1$

$$\begin{aligned} &|\varphi_{\alpha,\lambda,y}+\bar{v}|^{p-1}(\varphi_{\alpha,\lambda,y}+\bar{v}) \\ &= \alpha_1^p U_1^p + p\alpha_1^{p-1}U_1^{p-1}(\sum_{j=2}^k \alpha_j U_j+\bar{v}) + \frac{p(p-1)}{2}\alpha_1^{p-2}U_1^{p-2}(\sum_{j=2}^k \alpha_j U_j+\bar{v})^2 \\ &\qquad + 0\left(U_1^{-\sigma}|\sum_{j=2}^k \alpha_j U_j+\bar{v}|^{p+\sigma}\right) \\ &= \alpha_1^p U_1^p + p\alpha_1^{p-1}U_1^{p-1}\bar{v} + \frac{p(p-1)}{2}\alpha_1^{p-2}U_1^{p-2}\bar{v}^2 \\ &\qquad + 0\left(U_1^{p-1}\sum_{j=2}^k U_j + U_1^{p-2}\sum_{j=2}^k U_j^2 + U_1^{-\sigma}\sum_{j=2}^k U_j^{p+\sigma} + U_1^{-\sigma}|\bar{v}|^{p+\sigma}\right) \end{aligned}$$

we have

$$\frac{\partial K_\mu}{\partial y_1^1} = \alpha_1^2 \int_\Omega \nabla U_1 . \nabla \frac{\partial U_1}{\partial y_1^1} + \sum_{j=2}^k \alpha_1 \alpha_j \int_\Omega \nabla U_j . \nabla \frac{\partial U_1}{\partial y_1^1} + \mu \alpha_1^2 \int_\Omega U_1 \frac{\partial U_1}{\partial y_1^1}$$

$$+ \mu \sum_{j=2}^k \alpha_1 \alpha_j \int_\Omega U_j \frac{\partial U_1}{\partial y_1^1} - \alpha_1^p \int_\Omega U_1^p \frac{\partial U_1}{\partial y_1^1}$$

$$- p\alpha_1^{p-1} \int_\Omega U_1^{p-1} \frac{\partial U_1}{\partial y_1^1} \bar{v} - \frac{p(p-1)}{2} \alpha_1^{p-2} \int_\Omega U_1^{p-2} \frac{\partial U_1}{\partial y_1^1} \bar{v}^2$$

$$= 0 \left[\int_\Omega \left(U_1^{p-1} \sum_{j=2}^k U_j + U_1^{p-2} \sum_{j=2}^k U_j^2 + U_1^{-\sigma} \sum_{j=2}^k U_j^{p+\sigma} + U_1^{-\sigma} |\bar{v}|^{p+\sigma} \right) |\frac{\partial U_1}{\partial y_1^1}| \right] .$$

Estimates of Appendix D - see (D.8,19,23,26,31,43-44,46) - give

$$\alpha_1^2 \int_\Omega \nabla U_1 . \nabla \frac{\partial U_1}{\partial y_1^1} + \sum_{j=2}^k \alpha_1 \alpha_j \int_\Omega \nabla U_j . \nabla \frac{\partial U_1}{\partial y_1^1} + \mu \alpha_1^2 \int_\Omega U_1 \frac{\partial U_1}{\partial y_1^1} + \mu \sum_{j=2}^k \alpha_1 \alpha_j \int_\Omega U_j \frac{\partial U_1}{\partial y_1^1}$$

$$- \alpha_1^p \int_\Omega U_1^p \frac{\partial U_1}{\partial y_1^1} + 0 \left[\int_\Omega \left(U_1^{p-1} \sum_{j=2}^k U_j + U_1^{p-2} \sum_{j=2}^k U_j^2 + U_1^{-\sigma} \sum_{j=2}^k U_j^{p+\sigma} \right) |\frac{\partial U_1}{\partial y_1^1}| \right]$$

$$= -\frac{C_8}{\lambda_1} \frac{\partial H}{\partial y_1^1}(y) + 0 \left(\frac{1}{\lambda^{\inf(2,(n-2)(1-\sigma))}} \right)$$

and, noticing that $|\frac{\partial U_1}{\partial y_1^1}| = 0(\lambda_1 U_1)$, Hölder's inequality, (D.17) and (3.24) yield

$$\int_\Omega U_1^{-\sigma} |\bar{v}|^{p+\sigma} |\frac{\partial U}{\partial y_1^1}| = 0(\lambda_1 \|\bar{v}\|^{p+\sigma})$$

$$= 0 \left(\frac{(\mathrm{Log}\lambda)^{\frac{2}{3}(\sigma+2)}}{\lambda^{1+\sigma}} \text{ if } n = 6 \; ; \frac{1}{\lambda^{\frac{4}{n-2}+\sigma}} \text{ if} n \geq 7 \right) .$$

Choosing $\sigma = \frac{n(n-4)}{(n-1)(n-2)}$, we get

$$(C.71) \qquad \frac{\partial K_\mu}{\partial y_1^1} = -\frac{C_8}{\lambda_1} \frac{\partial H}{\partial y_1^1}(y) + 0 \left(\frac{(\mathrm{Log}\lambda)^{\frac{26}{5}}}{\lambda^{8/5}} \text{ if } n = 6 \; ; \frac{1}{\lambda^{\frac{n+2}{n-1}}} \text{ if } n \geq 7 \right)$$

$$+ \left| \int_\Omega U_1^{p-1} \frac{\partial U_1}{\partial y_1^1} \bar{v} \right| + \left| \int_\Omega U_1^{p-2} \frac{\partial U_1}{\partial y_1^1} \bar{v}^2 \right|$$

and it only remains to estimate the two last integrals. We have

$$\int_\Omega U_1^{p-1} \frac{\partial U_1}{\partial y_1^1} \bar{v} = -\frac{1}{n(n+2)} \int_\Omega \Delta \frac{\partial U_1}{\partial y_1^1} . \bar{v}$$

$$= -\frac{1}{n(n+2)} \int_{\partial\Omega} \frac{\partial^2 U_1}{\partial \nu \partial y_1^1} . \bar{v} \qquad \text{integrating by parts.}$$

We note that

$$\frac{\partial^2 U_1}{\partial x_i \partial y_1^1} = (n-2)\lambda_1^{\frac{n+2}{2}} \left(\frac{\delta_{1i}}{(1+\lambda_1^2|x|^2)^{\frac{n}{2}}} - \frac{n\lambda_1^2 x_1 x_i}{(1+\lambda_1^2|x|^2)^{\frac{n+2}{2}}} \right) .$$

Therefore

$$\frac{\partial^2 U_1}{\partial \nu \partial y_1^1} = 0\left(\frac{\lambda_1^{\frac{n+2}{2}}}{(1+\lambda_1^2|x|^2)^{\frac{n}{2}}} \right) \tag{C.72}$$

and, in $\partial\Omega \cap \overline{W}$ - since, according to (C.12,13)

$$\nu(x) = \frac{1}{(1+|\nabla f|^2)^{1/2}} \left(2f_1 x_1 + 0(|x'|^2), \cdots, 2f_{n-1}x_{n-1} + 0(|x'|^2), -1\right)$$

we have, noticing that on $\partial\Omega$, $x_n = f(x') = \sum_{i=1}^{n-1} f_i x_i^2 + 0(|x'|^3)$

$$\frac{\partial^2 U_1}{\partial \nu \partial y_1^1} = \frac{1}{(1+|\nabla f|^2)^{1/2}} \frac{(n-2)\lambda_1^{\frac{n+2}{2}}}{(1+\lambda_1^2|x|^2)^{n/2}} \left(2f_1 x_1 - \frac{n\lambda_1^2 x_1}{1+\lambda_1^2|x|^2} \sum_{i=1}^{n-1} f_i x_i^2 + 0(|x'|^2) \right) .$$

Whence, since on $\partial\Omega \cap \overline{W}$, because of (C.13-14), $|x|^2 = |x'|^2 + 0(|x'|^4)$

$$\frac{\partial^2 U_1}{\partial \nu \partial y_1^1} = \frac{1}{(1+|\nabla f|^2)^{1/2}} \frac{(n-2)\lambda_1^{\frac{n+2}{2}}}{(1+\lambda_1^2|x'|^2)^{n/2}} \left(2f_1 x_1 - \frac{n\lambda_1^2 x_1}{1+\lambda_1^2|x'|^2} \sum_{i=1}^{n-1} f_i x_i^2 + 0(|x'|^2) \right) . \tag{C.73}$$

Then, in view of (C.20-21), we compute

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial^2 U_1}{\partial \nu \partial y_1^1} v_3 &= 0\left[\left(\int_{\substack{x\in\partial\Omega \\ |x|\geq \frac{1}{4T}}} \left| \frac{\partial^2 U}{\partial \nu \partial y_1^1} \right|^{\frac{2(n-1)}{n}} \right)^{\frac{n}{2(n-1)}} \|\bar{v}\| \right] \\ &= 0\left(\frac{T^{\frac{n}{2}}}{\lambda_1^{\frac{n}{2}}} \|\bar{v}\| \right) \qquad \text{using (D.57)} . \end{aligned}$$

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial^2 U_1}{\partial \nu \partial y_1^1} v_2 &= 0\left(\int_{\substack{x'\in\mathbf{R}^{n-1} \\ |x'|\leq \frac{2}{T}}} \frac{\lambda_1^{\frac{n+2}{2}}|x'|}{(1+\lambda_1^2|x'|^2)^{\frac{n}{2}}} v_2(x', f(x'))dx' \right) \\ &= 0\left[\lambda_1^{\frac{n+2}{2}} \left(\int_{\mathbf{R}^{n-1}} \frac{|x'|^{\frac{2(n-1)}{n}} dx'}{(1+\lambda_1^2|x'|^2)^{n-1}} \right)^{\frac{n}{2(n-1)}} \|v_2\| \right] \\ &= 0(\|v_2\|) \qquad \text{using (D.63)} . \end{aligned}$$

Lastly, (C.14,21-22,25,73) lead to

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial^2 U_1}{\partial\nu\partial y_1^1} v_1 \\
&= \int_{z'\in\mathbf{R}^{n-1}} \frac{(n-2)\lambda_1^{\frac{n+2}{2}}}{(1+\lambda_1^2|z'|^2)^{\frac{n}{2}}} \left(2f_1 z_1 - \frac{n\lambda_1^2 z_1}{1+\lambda_1^2|z|^2} \sum_{i=1}^{n-1} f_i z_1^2 + 0(|z'|^2) \right) .\tilde{\zeta}_T(z)(\widetilde{\bar{v}}(z',0)dz \\
&= 0\left[\int_{|z'|\le\frac{1}{T}} \frac{\lambda_1^{\frac{n+2}{2}}|z'|^2}{(1+\lambda_1^2|z|'^2)^{\frac{n}{2}}} |\widetilde{\bar{v}}(z',0)|dz' \right] \quad \text{using oddness properties with respect to} z_1 \\
&= 0\left[\lambda_1^{\frac{n+2}{2}} \left(\int_{\mathbf{R}^{n-1}} \frac{|z|^{\frac{4(n-1)}{n}} dz'}{(1+\lambda_1^2|z'|^2)^{n-1}} \right)^{\frac{n}{2(n-1)}} \|\bar{v}\| \right] \\
&= 0\left(\frac{\|\bar{v}\|}{\lambda_1} \right) \qquad \text{using (D.64) .}
\end{aligned}$$

From (C.20, 70), (3.24), $T = \lambda_1^{1/6}$, and the previous estimates we conclude

$$(C.74) \qquad \int_\Omega U_1^{p-1} \frac{\partial U_1}{\partial y_1^1} \bar{v} = 0\left(\frac{(\mathrm{Log}\lambda)^{2/3}}{\lambda^{7/6}} \text{ if } n=6 \; ; \frac{1}{\lambda^{\inf(\frac{7}{6},\frac{n}{n-2})}} \text{ if } n\ge 7 \right) .$$

In the same way we compute, for $1 \le i \le 3$

$$\begin{aligned}
\int_\Omega U_1^{p-2} \frac{\partial U_1}{\partial y_1^1} v_i v_3 &= 0\left[\left(\int_{\substack{\Omega \\ |x|\ge\frac{1}{4T}}} |U_1^{p-2} \frac{\partial U_1}{\partial y_1^1}|^{\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}} \|v_i\|\|v_3\| \right] \\
&= 0\left(\frac{T^3}{\lambda^2} \|\bar{v}\|^2 \right) \qquad \text{using (D.39)}
\end{aligned}$$

and, for $1 \le i \le 2$

$$\begin{aligned}
\int_\Omega U_1^{p-2} \frac{\partial U_1}{\partial y_1^1} v_i v_2 &= 0\left(\lambda_1 \int_\Omega U_1^{p-1} |v_i| \, |v_2| \right) \\
&= 0(\lambda_1 \|v_i\| \, \|v_2\|) \\
&= 0(\lambda_1 \|\bar{v}\| \, \|v_2\|)
\end{aligned}$$

Lastly, using (C.21-22,25), we find

$$\int_\Omega U_1^{p-2}\frac{\partial U_1}{\partial y_1^1}v_1^2$$

$$= \int_\Omega \frac{(n-2)\lambda_1^4 x_1}{(1+\lambda_1^2|x|^2)^3}v_1^2 dx$$

$$= (n-2)\lambda_1^4 \int_{B+} \frac{z_1}{(1+\lambda_1^2|z|^2)^3}\left(1-\frac{2\lambda_1^2(\sum_{i=1}^{n-1} f_i z_i^2)z_n}{3(1+\lambda_1^2|z|^2)}+0(\frac{\lambda_1^2|z|^4}{1+\lambda_1^2|z|^2})\right)$$

$$\cdot\frac{1}{4}\tilde{\zeta}_1^2(z)(\tilde{\bar{v}}(z)+\tilde{\bar{v}}(\bar{z}))^2 dz$$

$$= 0\left(\lambda_1^6\int_{B+}\frac{|z|^5}{(1+\lambda_1^2|z|^2)^4}\tilde{\bar{v}}^2(z)dz\right) \quad \text{because of oddness properties with respect to } z_1$$

$$= 0\left(\lambda_1^6(\int_{\mathbf{R}^n}\frac{|z|^{\frac{5n}{2}}dz}{(1+\lambda_1^2|z|^2)^{2n}})^{\frac{2}{n}}\|\bar{v}\|^z\right)$$

$$= 0\left(\frac{\|\bar{v}\|^2}{\lambda_1}\right) \quad \text{using (D.62).}$$

Therefore, taking account of (3.24) and (C.70), we obtain

$$(C.75)\qquad \int_\Omega U_1^{p-2}\frac{\partial U_1}{\partial y_1'}\bar{v}^2 = 0\left(\frac{(\mathrm{Log}\lambda)^{4/3}}{\lambda^{7/6}} \text{ if } n=6;\ \frac{1}{\lambda^{\inf(\frac{7}{6},\frac{n}{n-2})}} \text{ if } n\geq 7\right).$$

Finally, (C.71, 74-75) show that (C.7) is satisfied.

D. ESTIMATES

In this last appendix, we collect the estimates of the different integral quantities which occur in the paper. In order to be as clear, short, but complete as possible, we detail the computations each time that they are necessary to convince the reader of the result, and omit them each time that they are similar to previous ones or may be easily performed.

We recall that, for $y\in\partial\Omega$, $\lambda>0$, $x\in\mathbf{R}^n$

$$(D.1)\qquad \begin{cases} U_{\lambda,y}(x) = \dfrac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-y|^2)^{\frac{n-2}{2}}} \\ \dfrac{\partial U_{\lambda,y}}{\partial\lambda}(x) = \dfrac{n-2}{2}\lambda^{\frac{n-4}{2}}\dfrac{1-\lambda^2|x-y|^2}{(1+\lambda^2|x-y|^2)^{\frac{n}{2}}} \\ \dfrac{\partial U_{\lambda,y}}{\partial y_j}(x) = -\dfrac{\partial U_{\lambda,y}}{\partial x_j}(x) = (n-2)\lambda^{\frac{n+2}{2}}\dfrac{(x-y)_j}{(1+\lambda^2|x-y|^2)^{\frac{n}{2}}} \end{cases}$$

We notice that

$$\frac{\partial U_{\lambda,y}}{\partial \lambda} = 0(\frac{U_{\lambda,y}}{\lambda}) \qquad \frac{\partial U_{\lambda,y}}{\partial y_j} = 0(\lambda U_{\lambda,y})$$

From now on, we assume that $n \geq 5$.

For $y \in \partial\Omega$, $H(y)$ denotes the mean curvature of the boundary at y. If we assume that, up to a translation and a rotation, $y = 0$ and that, for $R > 0$ small enough,

$$(D.2) \qquad \Omega \cap B(0,R) = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x \in B(0,R), x_n > f(x')\}$$

with

$$(D.3) \qquad f(x') = \sum_{i=1}^{n-1} f_i x_i^2 + \sum_{1 \leq i \leq j \leq \ell \leq n-1} g_{ij\ell} x_i x_j x_\ell + 0(|x'|^4)$$

we have

$$(D.4) \qquad H(0) = \frac{2}{n-1} \sum_{i=1}^{n-1} f_i \; ; \; H'(0) = \frac{2}{n-1} \left(\sum_{j=1}^{i-1} g_{jji} + \sum_{j=i+1}^{n-1} g_{ijj} + 3g_{iii} \right) .$$

We set

$$(D.5) \qquad \sigma_{n-1} = mes(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

where S^{n-1} is the unit sphere of $\mathbf{R}^n$.

We begin with the three crucial estimates

$$(D.6) \qquad \int_\Omega |\nabla U_{\lambda,y}|^2 = \frac{S^{n/2}}{2(n(n-2))^{\frac{n-2}{2}}} - C_1 \frac{H(y)}{\lambda} + 0(\frac{1}{\lambda^2})$$

with

$$C_1 = \frac{(n-2)^2 \sigma_{n-2}}{4} \frac{\Gamma(\frac{n+3}{2})\Gamma(\frac{n-3}{2})}{\Gamma(n)}$$

$$(D.7) \qquad \int_\Omega \nabla U_{\lambda,y} . \nabla \frac{\partial U_{\lambda,y}}{\partial \lambda} = \frac{C_1}{2} \frac{H(y)}{\lambda^2} + 0(\frac{1}{\lambda^3})$$

$$(D.8) \qquad \int_\Omega \nabla U_{\lambda,y} . \nabla \frac{\partial U_{\lambda,y}}{\partial \tau_i} = -\frac{C_1}{2\lambda} \frac{\partial H}{\partial \tau_i}(y) + 0 \left(\frac{\mathrm{Log}\lambda}{\lambda^3} \text{ if } n = 5 \; ; \frac{1}{\lambda^3} \text{ if } n \geq 6 \right) .$$

Proof.

(D.6) is established in [1], and the computations to show (D.7) are quite similar. We do not reproduce them, since they can be easily deduced from the proof of (D.8), which is the most delicate integral to estimate. We note that, according to (D.1)

$$(D.9) \qquad \frac{\partial^2 U_{\lambda,y}}{\partial y_i \partial x_j} = (n-2)\lambda^{\frac{n+2}{2}} \left(\frac{\delta_{ij}}{(1+\lambda^2|x-y|^2)^{\frac{n}{2}}} - \frac{n\lambda^2(x-y)_i(x-y)_j}{(1+\lambda^2|x-y|^2)^{\frac{n+2}{2}}} \right) .$$

In particular

$$\frac{\partial^2 U_{\lambda,y}}{\partial y_i \partial x_j} = 0 \left(\frac{\lambda^{\frac{n+2}{2}}}{(1+\lambda^2|x-y|^2)^{\frac{n}{2}}} \right) .$$

Without loss of generality, we may assume that $y = 0$, $\tau_i = y_i$ for any $i, 1 \leq i \leq n-1$, and (D.2-3) hold. In the following computations, U denotes the function $U_{\lambda,y}$. (D.1,9) yield

$$\int_{\Omega \setminus B_R} \nabla U . \nabla \frac{\partial U}{\partial y_i} = 0 \left(\lambda^{n+2} \int_{\mathbf{R}^n \setminus B_R} \frac{|x| dx}{(+\lambda^2|x|^2)^n} \right) = 0 \left(\lambda \int_{\lambda R}^{+\infty} \frac{r^n dr}{(1+r^2)^n} \right)$$

using the spherical coordinates and a change of scale. Therefore

$$(D.10) \qquad \int_{\Omega \setminus B_R} \nabla U . \nabla \frac{\partial U}{\partial y_i} = 0(\frac{1}{\lambda^{n-2}}) .$$

On the other hand, setting

$$B_R^+ = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x \in B(0,R), x_n > 0\}$$

it follows from oddness properties that

$$\int_{B_R^+} \nabla U . \nabla \frac{\partial U}{\partial y_i} = 0 .$$

As a consequence

$$(D.11) \qquad \int_{\Omega} \nabla U . \nabla \frac{\partial U}{\partial y_i} = - \int_{\omega'} \nabla U . \nabla \frac{\partial U}{\partial y_i} + \int_{\omega''} \nabla U . \nabla \frac{\partial U}{\partial y_i} + 0(\frac{1}{\lambda^{n-2}})$$

with

$$(D.12) \qquad \begin{cases} \omega' = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x \in B(0,R), f(x') > 0 \text{ and } 0 < x_n < f(x')\} \\ \omega'' = \{x \in (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} / x \in B(0,R), f(x') < 0 \text{ and } f(x') < x_n < 0\} . \end{cases}$$

Let us estimate the integral on ω'. For $a > 0$, we define

$$(D.13) \qquad L_a = \{x \in \mathbf{R}^n / |x_i| < a \ , 1 \leq i \leq n\} .$$

We choose a small enough, so that $L_a \subset B_R$. Proceeding as previously, it is clear that

$$\int_{\omega' \cap L_a^c} \nabla U . \nabla \frac{\partial U}{\partial y_i} = 0(\frac{1}{\lambda^{n-2}}) .$$

Therefore, from (D.1,9) we deduce

$$\int_{\omega'} \nabla U . \nabla \frac{\partial U}{\partial y_i} = \int_{\omega' \cap L_a} \nabla U . \nabla \frac{\partial U}{\partial y_i} + 0(\frac{1}{\lambda^{n-2}})$$

hence

$$\begin{aligned} \int_{\omega'} \nabla U . \nabla \frac{\partial U}{\partial y_i} &= -(n-2)^2 \lambda^{n+2} \int_{\omega' \cap La} \frac{x_i dx}{(1+\lambda^2|x|^2)^n} \\ &+ n(n-2)^2 \lambda^{n+4} \int_{\omega' \cap La} \frac{|x|^2 x_i dx}{(1+\lambda^2|x|^2)^{n+1}} + 0(\frac{1}{\lambda^{n-2}}) \end{aligned}$$

or

$$\begin{aligned} (D.14) \qquad \int_{\omega'} \nabla U . \nabla \frac{\partial U}{\partial y_i} &= -(n-1)(n-2)^2 \lambda^{n+2} \int_{\omega' \cap La} \frac{x_i dx}{(1+\lambda^2|x|^2)^n} \\ &- n(n-2)^2 \lambda^{n+2} \int_{\omega' \cap La} \frac{x_i dx}{(1+\lambda^2|x|^2)^{n+1}} + 0(\frac{1}{\lambda^{n-2}}) . \end{aligned}$$

The same expansion holds with ω'' instead of ω'. Setting

$$(D.15) \qquad \Delta_a = \{x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1} / |x_i| < a \ , 1 \leq i \leq n-1\}$$

we have

$$\begin{aligned} \int_{\omega' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n} &= \int_{\substack{\Delta_a \\ f(x')>0}} \left(\int_0^{f(x')} \frac{dx_n}{(1+\lambda^2(|x'|^2+x_n^2))^n} \right) dx' \\ &= \frac{1}{\lambda} \int_{\substack{\Delta_a \\ f(x')>0}} \frac{x_i}{(1+\lambda^2|x'|^2)^{n-\frac{1}{2}}} \left(\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}} \frac{dy}{(1+y^2)^n} \right) dx' \end{aligned}$$

with $y = \frac{\lambda x_n}{(1+\lambda^2|x'|^2)^{1/2}}$. We compute

$$\begin{aligned} \int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}} \frac{dy}{(1+y^2)^n} &= \frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}} + 0\left[\left(\frac{\lambda |f(x')|}{(1+\lambda^2|x'|^2)^{1/2}} \right)^3 \right] \\ &= \frac{\lambda}{(1+\lambda^2|x'|^2)^{1/2}} (\sum_{j=1}^{n-1} f_j x_j^2 + \sum_{1 \leq j \leq \ell \leq m \leq n-1} g_{j\ell m} x_j x_\ell x_m) \\ &+ 0\left(\frac{\lambda |x'|^4}{(1+\lambda^2|x'|^2)^{1/2}} \right) . \end{aligned}$$

It follows that

$$\int_{\omega' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n}$$
$$= \sum_{j=1}^{n-1} f_j \int_{\substack{\Delta_a \\ f(x')>0}} \frac{x_j^2 x_i dx'}{(1+\lambda^2|x'|^2)^n} + \sum_{1 \le j \le \ell \le m \le n-1} g_{j\ell m} \int_{\substack{\Delta_a \\ f(x')>0}} \frac{x_i x_j x_\ell x_m dx'}{(1+\lambda^2|x'|^2)^n}$$
$$+ 0\left(\int_{\substack{\Delta_a \\ f(x')>0}} \frac{|x'|^5 dx'}{(1+\lambda^2|x'|^2)^n} \right) .$$

We note that the same expression still holds for ω'' intead of ω', with opposite sign and integrals taken on $\{x' \in \Delta_a / f(x') < 0\}$ intead of $\{x' \in \Delta_a / f(x') > 0\}$. Therefore

$$\int_{\omega' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n} - \int_{\omega'' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n}$$
$$= \sum_{j=1}^{n-1} f_j \int_{\Delta_a} \frac{x_j^2 x_i dx'}{(1+\lambda^2|x'|^2)^n} + \sum_{1 \le j \le \ell \le m \le n-1} g_{j\ell m} \int_{\Delta_a} \frac{x_i x_j x_\ell x_m dx'}{(1+\lambda^2|x'|^2)^n}$$
$$+ 0\left(\int_{\Delta_a} \frac{|x'|^5 dx'}{(1+\lambda^2|x'|^2)^n} \right) .$$

It follows from (D.15), oddness properties and

$$\int_{\Delta_a} \frac{|x'|^5 dx'}{(1+\lambda^2|x|^2)^n} = 0\left(\frac{1}{\lambda^{n+4}} \int_0^{\lambda\sqrt{n-1}a} \frac{r^{n+3} dr}{(1+r^2)^n} \right) = 0(\frac{1}{\lambda^{n+4}})$$

that

$$\int_{\omega' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n} - \int_{\omega'' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^n}$$
$$= (\sum_{j=1}^{i-1} g_{jji} + \sum_{j=i+1}^{n-1} g_{ijj}) \int_{\Delta_a} \frac{x_1^2 x_2^2 dx'}{(1+\lambda^2|x'|^2)^n} + g_{iii} \int_{\Delta_a} \frac{x_1^4 dx'}{(1+\lambda^2|x'|^2)^n} + 0(\frac{1}{\lambda^{n+4}}) .$$

In the same way, we get

$$\int_{\omega' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^{n+1}} - \int_{\omega'' \cap L_a} \frac{x_i dx}{(1+\lambda^2|x|^2)^{n+1}}$$
$$= (\sum_{j=1}^{i-1} g_{jji} + \sum_{j=i+1}^{n-1} g_{ijj}) \int_{\Delta_a} \frac{x_1^2 x_2^2 dx'}{(1+\lambda^2|x'|^2)^{n+1}} + g_{iii} \int_{\Delta_a} \frac{x_1^4 dx'}{(1+\lambda^2|x'|^2)^{n+1}} + 0(\frac{1}{\lambda^{n+4}})$$

so that, in view of (D.14), with a rescaling in the integrals

(*D*.16)

$$
\begin{aligned}
&-\int_{\omega'} \nabla U.\nabla \frac{\partial U}{\partial y_i} + \int_{\omega''} \nabla U.\nabla \frac{\partial U}{\partial y_i} \\
&= -\frac{1}{\lambda}\left[(\sum_{j=1}^{i-1} g_{jji} + \sum_{j=i+1}^{n-1} g_{ijj}) \left((n-1)(n-2)^2 \int_{\lambda\Delta_a} \frac{x_1^2 x_2^2 dx'}{(1+|x'|^2)^n} \right.\right. \\
&\left. -n(n-2)^2 \int_{\lambda\Delta_a} \frac{x_1^2 x_2^2 dx'}{(1+|x'|^2)^{n+1}} \right) + g_{iii} \left((n-1)(n-2)^2 \int_{\lambda\Delta_a} \frac{x_1^4 dx'}{(1+|x'|^2)^n} \right. \\
&\left.\left. -n(n-2)^2 \int_{\lambda\Delta_a} \frac{x_1^4 dx'}{(1+|x'|^2)^{n+1}} \right) \right] + 0(\frac{1}{\lambda^2}) \, .
\end{aligned}
$$

Integrating by parts, we have, for $\sigma = n$ or $\sigma = n+1$,

$$
\begin{aligned}
\int_{\lambda\Delta_a} \frac{x_1^2 x_2^2 dx'}{(1+|x'|^2)^\sigma} &= \frac{1}{2(\sigma-1)} \int_{\lambda\Delta_a} \frac{x_1^2 dx'}{(1+|x'|^2)^{\sigma-1}} \\
&= \frac{1}{2(n-1)(\sigma-1)} \int_{\lambda\Delta_a} \frac{|x'|^2 dx'}{(1+|x'|^2)^{\sigma-1}} \\
&= \frac{\sigma_{n-2}}{2(n-1)(\sigma-1)} \int_0^{+\infty} \frac{r^n dr}{(1+r^2)^{\sigma-1}} + 0(\frac{1}{\lambda^{2\sigma-n-3}})
\end{aligned}
$$

and

$$
\int_{\lambda\Delta_a} \frac{x_1^4 dx'}{(1+|x'|^2)^\sigma} = \frac{3}{2(\sigma-1)} \int_{\lambda\Delta_a} \frac{x_1^2 dx'}{(1+|x'|^2)^{\sigma-1}}.
$$

Then, (D.11, 16) yield

$$
\begin{aligned}
&\int_\Omega \nabla U.\nabla \frac{\partial U}{\partial y_i} \\
&= -\frac{(n-2)^2 \sigma_{n-2}}{2(n-1)\lambda} (\sum_{j=1}^{i=1} g_{jji} + \sum_{j=i+1}^{n-1} g_{ijj} + 3g_{iii}) \left(\int_0^{+\infty} \frac{r^n dr}{(1+r^2)^{n-1}} - \int_0^{+\infty} \frac{r^n dr}{(1+r^2)^n} \right) \\
&+ 0(\frac{1}{\lambda^2}) \, .
\end{aligned}
$$

Therefore, using (D.4)

$$
\int_\Omega \nabla U.\nabla \frac{\partial U}{\partial y_i} = -\frac{(n-2)^2 \sigma_{n-2}}{4\lambda} \frac{\partial H}{\partial y_i}(0) \int_0^{+\infty} \frac{r^{n+2} dr}{(1+r^2)^n} + 0(\frac{1}{\lambda^2}) \, .
$$

Since, as standard computations show

$$
\int_0^{+\infty} \frac{r^{n+2} dr}{(1+r^2)^n} = \frac{\Gamma(\frac{n+3}{2})\Gamma(\frac{n-3}{2})}{2\Gamma(n)}
$$

(D.8) is proved.

Similarly to (D.6-8), we have

$$(D.17) \qquad \int_\Omega U_{\lambda,y}^{p+1} = \frac{S^{n/2}}{2(n(n-2))^{n/2}} - C_4 \frac{H(y)}{\lambda} + 0(\frac{1}{\lambda^2})$$

with

$$C_4 = \frac{\sigma_{n-2}}{4} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n)}$$

$$(D.18) \qquad \int_\Omega U_{\lambda,y}^{p} \frac{\partial U_{\lambda,y}}{\partial \lambda} = \frac{n-2}{2n} C_4 \frac{H(y)}{\lambda^2} + 0(\frac{1}{\lambda^3})$$

$$(D.19) \qquad \int_\Omega U_{\lambda,y}^{p} \frac{\partial U_{\lambda,y}}{\partial \tau_i} = -\frac{n-2}{2n} \frac{C_4}{\lambda} \frac{\partial H}{\partial \tau_i}(y) + 0(\frac{1}{\lambda^2}) \ .$$

(D.17-19) are obtained in the same way as (D.6-8), and we omit the computations. In addition to these estimates, we remark that

$$\int_{\substack{\Omega \\ |x-y| \geq \rho}} U_{\lambda,y}^{p+1} = 0 \left(\int_{\lambda\rho}^{+\infty} \frac{r^{n-1} dr}{(1+r^2)^n} \right)$$

so that, for $\frac{\lambda}{T}$ going to infinity

$$(D.20) \qquad \int_{\substack{\Omega \\ |x-y| \geq \frac{1}{4T}}} U_{\lambda,y}^{p+1} = 0(\frac{T^n}{\lambda^n}) \ .$$

Concerning quadratic integral estimates of $U_{\lambda,y}$, we have

$$(D.21) \qquad \int_\Omega U_{\lambda,y}^2 = \frac{C_5}{\lambda^2} + 0 \left(\frac{\mathrm{Log}\lambda}{\lambda^3} \text{ if } n = 5 \ ; \frac{1}{\lambda^3} \text{ if } n \geq 6 \right)$$

with

$$C_5 = \frac{\sigma_{n-1}}{4} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)}$$

$$(D.22) \qquad \int_\Omega U_{\lambda,y} \frac{\partial U_{\lambda,y}}{\partial \lambda} = -\frac{C_5}{2\lambda^3} + 0 \left(\frac{\mathrm{Log}\lambda}{\lambda^4} \text{ if } n = 5 \ ; \frac{1}{\lambda^4} \text{ if } n \geq 6 \right)$$

$$(D.23) \qquad \int_\Omega U_{\lambda,y}\frac{\partial U_{\lambda,y}}{\partial \tau_i} = 0\left(\frac{\mathrm{Log}\lambda}{\lambda^3} \text{ if} n = 5\ ; \frac{1}{\lambda^3} \text{ if } n \geq 6\right)\ .$$

Proof. With the previous notations, we write

$$\int_\Omega U^2 = \int_{\Omega\setminus B_R} U^2 + \int_{B_R^+} U^2 - \int_{\omega'} U^2 + \int_{\omega''} U^2.$$

We compute

$$\int_{\Omega\setminus B_R} U^2 = 0\left(\int_{|x|\geq R}\frac{\lambda^{n-2}dx}{(1+\lambda^2|x|^2)^{n-2}}\right) = 0\left(\frac{1}{\lambda^2}\int_{\lambda R}^{+\infty}\frac{r^{n-1}dr}{(1+r^2)^{n-2}}\right) = 0(\frac{1}{\lambda^{n-2}})$$

$$\int_{B_R^+} U^2 = \frac{1}{2}\int_{B_R} U^2 = \frac{\sigma_{n-1}}{2}\int_0^{\lambda R}\frac{r^{n-1}dr}{(1+r^2)^{n-2}} = \frac{\sigma_{n-1}}{4}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{\Gamma(n-2)} + 0(\frac{1}{\lambda^{n-2}})$$

since

$$\int_0^{+\infty}\frac{r^{n-1}dr}{(1+r^2)^{n-2}} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-4}{2})}{2\Gamma(n-2)}\ .$$

Lastly

$$\int_{\omega'} U^2 = \int_{\omega'\cap L_a} U^2 + \int_{\omega'\cap L_a^c} U^2 = \int_{\omega'\cap L_a} U^2 + 0(\frac{1}{\lambda^{n-2}})$$

and

$$\begin{aligned}\int_{\omega'\cap L_a} U^2 &= \lambda^{n-2}\int_{\substack{\Delta_a \\ f(x')>0}}\left(\int_0^{f(x')}\frac{dx_n}{(1+\lambda^2|x'|^2+\lambda^2x_n^2)^{n-2}}\right)dx' \\ &= 0\left(\lambda^{n-2}\int_{\substack{\Delta_a \\ f(x')>0}}\frac{|x'|^2dx'}{(1+\lambda^2|x'|^2)^{n-2}}\right) \quad \text{using (D.3).}\end{aligned}$$

Proceeding in the same way to integrate U^2 on ω'', we obtain

$$\begin{aligned}-\int_{\omega'} U^2 + \int_{\omega''} U^2 &= 0\left(\lambda^{n-2}\int_{\Delta_a}\frac{|x'|^2dx'}{(1+\lambda^2|x'|^2)^{n-2}}\right) \\ &= 0\left(\frac{1}{\lambda^3}\int_0^{\lambda a}\frac{r^n dr}{(1+r^2)^{n-2}}\right) \\ &= 0\left(\frac{\mathrm{Log}\lambda}{\lambda^3} \text{ if } n = 5\ ; \frac{1}{\lambda^3} \text{ if } n \geq 6\right)\end{aligned}$$

whence (D.21). The computations which lead to (D.22) are quite similar. Concerning (D.23), assuming that $\tau_i = y_i$, $1 \leq i \leq n-1$, we write again

$$\int_\Omega U\frac{\partial U}{\partial y_i} = \int_{\Omega\setminus B_R} U\frac{\partial U}{\partial y_i} + \int_{B_R^+} U\frac{\partial U}{\partial y_i} - \int_{\omega'} U\frac{\partial U}{\partial y_i} + \int_{\omega''} U\frac{\partial U}{\partial y_i}$$

and we compute, using (D.1)

$$\int_{\Omega\setminus B_R} U\frac{\partial U}{\partial y_i} = 0\left(\int_{|x|\geq R}\frac{\lambda^n|x|dx}{(1+\lambda^2|x|^2)^{n-1}}\right) = 0\left(\frac{1}{\lambda}\int_{\lambda R}^{+\infty}\frac{r^n dr}{(1+r^2)^{n-1}}\right) = 0(\frac{1}{\lambda^{n-2}})$$

$\int_{B_R^+} U\frac{\partial U}{\partial y_i} = 0$ because of oddness with respect of y_i

$$\begin{aligned}\int_{\omega'} U\frac{\partial U}{\partial y_i} &= \int_{\omega'\cap L_a} U\frac{\partial U}{\partial y_i} + 0(\frac{1}{\lambda^{n-2}})\\ &= \lambda^n\int_{\substack{\Delta_a\\ f(x')>0}}\left(\int_0^{f(x')}\frac{dx_n}{(1+\lambda^2|x'|^2+\lambda^2x_n^2)^{n-1}}\right)x_i dx'\\ &= \lambda^{n-1}\int_{\substack{\Delta_a\\ f(x')>0}}\left(\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}}\frac{dy}{(1+y^2)^{n-1}}\right)\frac{x_i dx'}{(1+\lambda^2|x'|^2)^{n-\frac{3}{2}}}.\end{aligned}$$

(D.3) yields

$$\begin{aligned}\int_0^{\frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}}}\frac{dy}{(1+y^2)^{n-1}} &= \frac{\lambda f(x')}{(1+\lambda^2|x'|^2)^{1/2}} + 0\left(\frac{\lambda^3|f(x')|^3}{(1+\lambda^2|x'|^2)^{3/2}}\right)\\ &= \frac{\lambda}{(1+\lambda^2|x'|^2)^{1/2}}\sum_{j=1}^{n-1}f_j x_j^2 + 0\left(\frac{\lambda|x'|^3}{(1+\lambda^2|x'|^2)^{1/2}}\right).\end{aligned}$$

Therefore

$$\int_{\omega'} U\frac{\partial U}{\partial y_i} = \lambda^n\sum_{j=1}^{n-1}f_j\int_{\substack{\Delta_a\\ f(x')>0}}\frac{x_j^2x_i dx'}{(1+\lambda^2|x'|^2)^{n-1}} + 0\left(\lambda^n\int_{\substack{\Delta_a\\ f(x')>0}}\frac{|x'|^4dx'}{(1+\lambda^2|x'|^2)^{n-1}}\right).$$

Treating in the same way the integral on ω'', we obtain

$$\begin{aligned}-\int_{\omega'} U\frac{\partial U}{\partial y_i} + \int_{\omega''} U\frac{\partial U}{\partial y_i} &= -\lambda^n\sum_{j=1}^{n-1}f_j\int_{\Delta a}\frac{x_j^2x_i dx'}{(1+\lambda^2|x'|^2)^{n-1}}\\ &\quad + 0\left(\lambda^n\int_{\Delta_a}\frac{|x'|^4dx'}{(1+\lambda^2|x'|^2)^{n-1}}\right).\end{aligned}$$

The first integral vanishes, by oddness of the integrated function, and

$$\begin{aligned}\lambda^n\int_{\Delta_a}\frac{|x'|^4dx'}{(1+\lambda^2|x|^2)^{n-1}} &= 0\left(\frac{1}{\lambda^3}\int_0^{\lambda\sqrt{n-1}a}\frac{r^{n+2}dr}{(1+r^2)^{n-1}}\right)\\ &= 0\left(\frac{\mathrm{Log}\lambda}{\lambda^3}\text{ if } n=5\ ;\ \frac{1}{\lambda^3}\text{ if } n\geq 6\right).\end{aligned}$$

(D.23) follows.

Remark.

We have

$$(D.24) \qquad \int_{\Omega \setminus B(y, \frac{1}{\lambda^{1/2}})} U_{\lambda,y}^2 = 0(\frac{1}{\lambda^{5/2}}) \quad \text{if } n = 5 \ .$$

Indeed, we write

$$\int_{\Omega \setminus B(y, \frac{1}{\lambda^{1/2}})} U_{\lambda,y}^2 = 0 \left(\lambda^{n-2} \int_{B(0,2S) \setminus B(0, \frac{1}{\lambda^{1/2}})} \frac{dx}{(1+\lambda^2|x|^2)^{n-2}} \right)$$

hence, for $n = 5$

$$\int_{\Omega \setminus B(y, \frac{1}{\lambda^{1/2}})} U_{\lambda,y}^2 = 0 \left(\frac{1}{\lambda^2} \int_{\lambda^{1/2}}^{2\lambda S} \frac{r^4 dr}{(1+r^2)^3} \right)$$

and the announced result. In addition, we remark that

$$\int_{\substack{\Omega \\ |x-y| \geq \rho}} U_{\lambda,y}^2 = 0 \left(\frac{1}{\lambda^2} \int_{\lambda\rho}^{+\infty} \frac{r^{n-1} dr}{(1+r^2)^{n-2}} \right)$$

so that, for $\frac{\lambda}{T}$ going to infinity, we have

$$(D.25) \qquad \int_{\substack{\Omega \\ |x-y| \geq \frac{1}{4T}}} U_{\lambda,y}^2 = 0 \left(\frac{T^{n-4}}{\lambda^{n-2}} \right) \ .$$

We write now the following table, which is often used throughout the paper :

(D.26)

$$\begin{cases} \int_\Omega |\nabla U_i|^2 = \dfrac{S^{n/2}}{2(n(n-2))^{\frac{n-2}{2}}} + 0(\dfrac{1}{\lambda}) \\ \int_\Omega |\nabla \dfrac{\partial U_i}{\partial \lambda_i}|^2 = \dfrac{C_2}{\lambda_i^2} + 0(\dfrac{1}{\lambda^3}) \qquad C_2 = C_2(n) > 0 \\ \int_\Omega |\nabla \dfrac{\partial U_i}{\partial y_j^i}|^2 = C_3 \lambda_i^2 + 0(\lambda_i) \qquad C_3 = C_3(n) > 0 \\ \int_\Omega \nabla U_i . \nabla \dfrac{\partial U_i}{\partial \lambda_i} = 0(\dfrac{1}{\lambda^2}) \; ; \int_\Omega \nabla U_i . \nabla \dfrac{\partial U_i}{\partial y_j^i} = 0(\dfrac{1}{\lambda}) \\ \int_\Omega \nabla \dfrac{\partial U_i}{\partial \lambda_i} . \nabla \dfrac{\partial U_i}{\partial y_j^i} = 0(\dfrac{1}{\lambda^2}) \; ; \int_\Omega \nabla \dfrac{\partial U_i}{\partial y_j^i} . \nabla \dfrac{\partial U_i}{\partial y_\ell^i} = 0(\dfrac{1}{\lambda}) \qquad j \neq \ell \\ \text{and, for } i \neq j \\ \int_\Omega \nabla U_i . \nabla U_j = 0(\dfrac{1}{\lambda^{n-2}}) \; ; \int_\Omega \nabla U_i . \nabla \dfrac{\partial U_j}{\partial \lambda_j} = 0(\dfrac{1}{\lambda^{n-1}}) \; ; \int_\Omega \nabla U_i . \nabla \dfrac{\partial U_j}{\partial y_i^j} = 0(\dfrac{1}{\lambda^{n-2}}) \\ \int_\Omega \nabla \dfrac{\partial U_i}{\partial \lambda_i} . \nabla \dfrac{\partial U_j}{\partial \lambda_j} = 0(\dfrac{1}{\lambda^n}); \int_\Omega \nabla \dfrac{\partial U_i}{\partial \lambda_i} . \nabla \dfrac{\partial U_j}{\partial y_\ell^j} = 0(\dfrac{1}{\lambda^{n-1}}); \int_\Omega \nabla \dfrac{\partial U_i}{\partial y_\ell^i} . \nabla \dfrac{\partial U_j}{\partial y_m^j} = 0(\dfrac{1}{\lambda^{n-2}}) \end{cases}$$

with, as previously, $U_i = U_{\lambda_i, y^i}$ and $0(f(\lambda))$ denotes a quantity dominated by $\sum_{i=1}^k f(\lambda_i)$. Setting, for $1 \leq i \leq k$, $3 \leq \ell \leq n+1$

$$\varphi_{(n+1)(i-1)+1} = U_i \qquad \varphi_{(n+1)(i-1)+2} = \frac{\partial U_i}{\partial \lambda_i} \qquad \varphi_{(n+1)(i-1)+\ell} = \frac{\partial U_i}{\partial \tau_\ell^i}$$

and, for $1 \leq a, b \leq (n+1)k$

$$m_{a,b} = \int_\Omega \nabla \varphi_a . \nabla \varphi_b$$

it follows from (D.26) that the matrix $M = (m_{a,b})$ is invertible with

$$\det M = C \prod_{i=1}^k \lambda_i^{2(n-2)} + 0(\lambda^{2k(n-2)-2}) \qquad C = \left(\frac{S^{n/2} C_2 C_3^{n-1}}{2(n(n-2))^{\frac{n-2}{2}}} \right)^k .$$

The coefficients of $M^{-1} = (m'_{a,b})$ may be easily estimated through (D.26). Writing

$$M^{-1} = (A_{ij})_{1 \leq i,j \leq k}$$

where the A_{ij}'s are $(n+1)\times(n+1)$ matrices, we have

$$(D.27)\quad \begin{cases} A_{ii} = \begin{bmatrix} 0(1) & 0(1) & 0(\frac{1}{\lambda^2}) & \cdots & 0(\frac{1}{\lambda^3}) \\ 0(1) & 0(\lambda^2) & 0(\frac{1}{\lambda^2}) & \cdots & 0(\frac{1}{\lambda^2}) \\ 0(\frac{1}{\lambda^3}) & 0(\frac{1}{\lambda^2}) & 0(\frac{1}{\lambda^2}) & & \\ \vdots & \vdots & & \ddots & 0(\frac{1}{\lambda^5}) \\ & & & 0(\frac{1}{\lambda^5}) & \ddots \\ 0(\frac{1}{\lambda^3}) & 0(\frac{1}{\lambda^2}) & & & 0(\frac{1}{\lambda^2}) \end{bmatrix} \\ A_{ij} = \begin{bmatrix} 0(\frac{1}{\lambda^{n-2}}) & 0(\frac{1}{\lambda^{n-3}}) & 0(\frac{1}{\lambda^n}) & \cdots & 0(\frac{1}{\lambda^n}) \\ 0(\frac{1}{\lambda^{n-3}}) & 0(\frac{1}{\lambda^{n-4}}) & 0(\frac{1}{\lambda^{n-1}}) & \cdots & \frac{0(1}{\lambda^{n-1})} \\ 0(\frac{1}{\lambda^n}) & 0(\frac{1}{\lambda^{n-1}}) & & & \\ \vdots & \vdots & & 0(\frac{1}{\lambda^{n+2}}) & \\ 0(\frac{1}{\lambda^n}) & 0(\frac{1}{\lambda^{n-1}}) & & & \end{bmatrix} \quad i \neq j \end{cases}$$

Proof of (D.26)

The estimates of the integrals over Ω of $|\nabla U_i|^2, \nabla U_i.\nabla\frac{\partial U_i}{\partial\lambda_i}, \nabla U_i.\nabla\frac{\partial U_i}{\partial y_j^i}$ follow from (D.6-8). The integrals of $|\nabla\frac{\partial U_i}{\partial\lambda_i}|^2$, $|\nabla\frac{\partial U_i}{\partial y_j^i}|^2, \nabla\frac{\partial U_i}{\partial\lambda_i}.\nabla\frac{\partial U_i}{\partial y_j^i}$ and $\nabla\frac{\partial U_i}{\partial y_j^i}.\nabla\frac{\partial U_i}{\partial y_\ell^i}$ are computed in the same way, very easily since, in contrast with (D.6-8), we are only interested here in rough estimates.

Concerning integrals where both U_i and U_j occur, $i \neq j$, let us estimate the first one. We recall that y^i and y^j are assumed to remain far from each other, i.e. $|y^i - y^i| > d_0$. We write

$$\int_\Omega \nabla U_i.\nabla U_j = \int_{B_i} \nabla U_i.\nabla U_j + \int_{B_j} \nabla U_i.\nabla U_j + \int_{\Omega\setminus B_i\cup B_j} \nabla U_i.\nabla U_j$$

where B_i and B_j are balls centered respectively at y^i and y^j, with radius $\frac{d_0}{2}$. It follows from (D.1) that

$$|\nabla U_i| = 0(\frac{1}{\lambda_i^{\frac{n-2}{2}}}) \text{ on } \Omega\setminus B_i \ ; \ |\nabla U_j| = 0(\frac{1}{\lambda_j^{\frac{n-2}{2}}}) \text{ on } \Omega\setminus B_j \ .$$

Moreover, using again (D.1), we have

$$\int_{B_i} |\nabla U_i| = 0\left(\lambda_i^{\frac{n+2}{2}} \int_{\mathbf{R}^n} \frac{|x|dx}{(1+\lambda_i^2|x|^2)^{\frac{n}{2}}}\right) = 0\left(\frac{1}{\lambda_i^{\frac{n}{2}}} \int_0^{+\infty} \frac{r^n dr}{(1+r^2)^{\frac{n}{2}}}\right) = 0\left(\frac{1}{\lambda_i^{\frac{n}{2}}}\right) \ .$$

Therefore

$$\int_\Omega \nabla U_i.\nabla U_j = 0(\frac{1}{\lambda^{n-2}}) \ .$$

The other integrals may be estimated in the same way.

Remarks.

When y^i and y^j are not assumed to be far from each other, and λ_i, λ_j are not necessarily of the same order, it follows from computations in [6, Part 1.1], that

$$(D.28) \qquad \int_\Omega \nabla U_i . \nabla U_j = 0\left[\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|y^i-y^j|^2\right)^{-\frac{n-2}{2}}\right] .$$

In other respects, assuming that T and $\frac{\lambda}{T}$ go to infinity, we have

$$(D.29) \qquad \begin{cases} \displaystyle\int_{\substack{\Omega \\ |x-y|\geq \frac{1}{4T}}} |\nabla U_{\lambda,y}|^2 = 0\left((\frac{T}{\lambda})^{n-2}\right) \\ \displaystyle\int_{\substack{\Omega \\ |x-y|\geq \frac{1}{4T}}} |\nabla \frac{\partial U_{\lambda,y}}{\partial \lambda}|^2 = 0\left(\frac{T^{n-2}}{\lambda^n}\right) \\ \displaystyle\int_{\substack{\Omega \\ |x-y|\geq \frac{1}{4T}}} |\nabla \frac{\partial U}{\partial y_j}|^2 = 0\left(\frac{T^n}{\lambda^{n-2}}\right) \end{cases}$$

Indeed, according to (D.1)

$$\begin{aligned} \int_{\substack{\Omega \\ |x-y|\geq \frac{1}{4T}}} |\nabla U_{\lambda,y}|^2 &= 0\left(\lambda^{n+2}\int_{|x|\geq\frac{1}{4T}} \frac{|x|^2dx}{(1+\lambda^2|x|^2)^n}\right) \\ &= 0(\int_{\frac{\lambda}{4T}}^{+\infty} \frac{r^{n+1}dr}{(1+r^2)^n}) \\ &= 0(\frac{T^{n-2}}{\lambda^{n-2}}) \end{aligned}$$

and comparable formula replacing $U_{\lambda,y}$ by $\frac{\partial U_{\lambda,y}}{\partial\lambda}$ and $\frac{\partial U_{\lambda,y}}{\partial y_j}$.

Finally, concerning the second derivatives of $U_{\lambda,y}$, we state

$$(D.30) \quad \int_\Omega |\nabla \frac{\partial^2 U_{\lambda,y}}{\partial\lambda^2}|^2 = 0(\frac{1}{\lambda^4}) \; ; \; \int_\Omega |\nabla \frac{\partial^2 U_{\lambda,y}}{\partial\lambda\partial y_j}|^2 = 0(1) \; ; \; \int_\Omega |\nabla \frac{\partial^2 U_{\lambda,y}}{\partial y_j \partial y_\ell}|^2 = 0(\lambda^4)$$

Indeed, it follows from (D.1) that

$$|\nabla \frac{\partial^2 U_{\lambda,y}}{\partial\lambda^2}| = 0\left(\frac{\lambda^{\frac{n-4}{2}}}{(1+\lambda^2|x-y|^2)^{\frac{n-1}{2}}}\right) ; |\nabla \frac{\partial^2 U_{\lambda,y}}{\partial\lambda\partial y_j}| = 0\left(\frac{\lambda^{\frac{n}{2}}}{(1+\lambda^2|x-y|^2)^{\frac{n}{2}}}\right)$$

hence the expected result.

Our last quadratic estimates are

$$(D.31)\quad \begin{cases} \int_\Omega U_i^2 = 0(\frac{1}{\lambda^2}) \quad \int_\Omega (\frac{\partial U_i}{\partial \lambda_i})^2 = 0(\frac{1}{\lambda^4}) \quad \int_\Omega (\frac{\partial U_i}{\partial y_j^i})^2 = 0(1) \\ \text{and, for} \quad i \neq j \\ \int_\Omega U_i U_j = 0(\frac{1}{\lambda^{n-2}}) \quad \int_\Omega U_i \frac{\partial U_j}{\partial \lambda_j} = 0(\frac{1}{\lambda^{n-1}}) \quad \int_\Omega U_i \frac{\partial U_j}{\partial y_\ell^j} = 0(\frac{1}{\lambda^{n-2}}) \ . \end{cases}$$

Proof.

The first estimates are a consequence of (D.21), noticing that $\frac{\partial U_i}{\partial \lambda_i} = 0(\frac{U_i}{\lambda_i})$ and $\frac{\partial U_i}{\partial y_j^i} = 0(\lambda_i U_i)$. With the same assumptions and notations as before, we deduce from (D.1) that

$$(D.32)\qquad U_i = 0(\frac{1}{\lambda_i^{\frac{n-2}{2}}}) \text{ on } \Omega \setminus B_i \qquad U_j = 0(\frac{1}{\lambda_j^{\frac{n-2}{2}}}) \text{ on } \Omega \setminus B_j \ .$$

On the other hand

$(D.33)$

$$\int_{B_i} U_i = 0(\lambda_i^{\frac{n-2}{2}} \int_{|x| \leq \frac{d_0}{2}} \frac{dx}{(1+\lambda^2|x|^2)^{\frac{n-2}{2}}}) = 0(\frac{1}{\lambda_i^{\frac{n+2}{2}}} \int_0^{\frac{\lambda d_0}{2}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n-2}{2}}}) = 0(\frac{1}{\lambda_i^{\frac{n-2}{2}}})$$

Therefore

$$\int_\Omega U_i U_j = 0(\frac{1}{\lambda^{n-2}}) \ .$$

The estimate of $\int_\Omega U_i \frac{\partial U_j}{\partial \lambda_j}$ follows from the fact that $\frac{\partial U_j}{\partial \lambda_j} = 0(\frac{U_j}{\lambda_j})$. Concerning the last integral, we remark that, according to (D.1)

$$(D.34)\qquad |\frac{\partial U_j}{\partial y_\ell^j}| = 0(\frac{1}{\lambda_j^{\frac{n-2}{2}}}) \text{ on } \Omega \setminus B_j$$

and

$(D.35)$

$$\int_{B_j} |\frac{\partial U_j}{\partial y_\ell^j}| = 0(\lambda_j^{\frac{n+2}{2}} \int_{|x| \leq \frac{d_0}{2}} \frac{|x|dx}{(1+\lambda_j^2|x|^2)^{\frac{n}{2}}}) = 0(\frac{1}{\lambda_j^{\frac{n}{2}}} \int_0^{\frac{\lambda_j d_0}{2}} \frac{r^n dr}{(1+r^2)^{\frac{n}{2}}}) = 0(\frac{1}{\lambda_j^{\frac{n-2}{2}}})$$

hence the announced result.

We collect now some estimates which turn out to be very useful in the arguments of the paper.

$$(D.36)\quad \left(\int_\Omega U_{\lambda,y}^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}} = 0\left(\frac{1}{\lambda^{3/2}} \text{ if } n=5\ ; \frac{(\text{Log}\lambda)^{\frac{2}{3}}}{\lambda^2} \text{ if } n=6\ ; \frac{1}{\lambda^2} \text{ if } n\geq 7\right)$$

$$(D.37)\quad \left(\int_{\Omega\cap B(y,\frac{1}{\lambda^{1/2}})} U_{\lambda,y}^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}} = 0(\tfrac{1}{\lambda^{7/4}}) \text{ if } n=5\ (p=\tfrac{7}{3})$$

$$(D.38)\quad \left(\int_\Omega U_{\lambda,y}^{\frac{(p-1)(p+1)}{p}}\right)^{\frac{p}{p+1}} = (\tfrac{1}{\lambda^{3/2}} \text{ if } n=5\ ; \tfrac{(\text{Log}\lambda)^{\frac{2}{3}}}{\lambda^2} \text{ if } n=6\ ; \tfrac{1}{\lambda^2} \text{ if } n\geq 7)$$

$$(D.39)\quad \left(\int_{\substack{\Omega\\ |x-y|\geq\frac{1}{4T}}} U_{\lambda,y}^{\frac{(p-2)(p+1)}{p-1}} |\tfrac{\partial U_{\lambda,y}}{\partial y_j}|^{\frac{p-1}{p+1}}\right)^{\frac{p+1}{p-1}} = 0(\tfrac{T^3}{\lambda^2}) \text{ as } T, \tfrac{\lambda}{T} \text{ go to infinity}$$

$$(D.40)\quad \int_\Omega |\tfrac{\partial U_{\lambda,y}}{\partial\lambda}|^{p+1} = 0(\tfrac{1}{\lambda^{p+1}}) \qquad \int_\Omega |\tfrac{\partial U_{\lambda,y}}{\partial y_j}|^{p+1} = 0(\lambda^{p+1})$$

$$(D.41)\quad (\int_\Omega |\tfrac{\partial U_{\lambda,y}}{\partial y_j}|^{\frac{p+1}{p}})^{\frac{p}{p+1}} = 0(\tfrac{1}{\lambda})\ .$$

Proof.

According to (D.1), we have

$$\int_\Omega U_{\lambda,y}^{\frac{p+1}{p}} = 0\left(\lambda^{\frac{n(n-2)}{n+2}} \int_{|x|\leq 2S} \frac{dx}{(1+\lambda^2|x|^2)^{\frac{n(n-2)}{n+2}}}\right)$$

where S is a fixed number such that $\Omega \subset B(0,S)$. Thus

$$\int_\Omega U_{\lambda,y}^{\frac{p+1}{p}} = 0\left(\lambda^{\frac{n(n-2)}{n+2}-n} \int_0^{2\lambda S} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n(n-2)}{n+2}}}\right)\ .$$

$\frac{r^{n-1}}{(1+r^2)^{\frac{n(n-2)}{n+2}}}$ is equivalent to $\frac{1}{r^{\frac{2n(n-2)}{n+2}-n+1}}$ as r goes to infinity, and $\frac{2n(n-2)}{n+2} - n + 1$ is larger than 1 for $n > 6$, is equal to 1 for $n = 6$, and is smaller than 1 for $n < 6$. Therefore

$$\int_0^{2\lambda S} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n(n-2)}{n+2}}} = 0\left(\lambda^{n-\frac{2n(n-2)}{n+2}} \text{ for } n<6\ ; \text{Log }\lambda \text{ for } n=6\ ; 1 \text{ for } n>6\right)$$

and (D.36) follows. Concerning (D.37), we have to change in the previous computation $2\lambda S$ by $\lambda^{1/2}$, and we obtain

$$\int_0^{\lambda^{1/2}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n(n-2)}{n+2}}} = 0\left(\lambda^{\frac{1}{2}(n-\frac{2n(n-2)}{n+2})}\right) \text{ for } n<6\ .$$

Taking $n = 5$, we get the result. Next, we write

$$\int_\Omega U_{\lambda,y}^{\frac{(p-1)(p+1)}{p}} = 0\left(\lambda^{\frac{4n}{n+2}} \int_{|x|\le 2S} \frac{dx}{(1+\lambda^2|x|^2)^{\frac{4n}{n+2}}}\right) = 0\left(\lambda^{\frac{4n}{n+2}-n} \int_0^{2\lambda S} \frac{r^{n-1}dr}{(1+r^2)^{\frac{4n}{n+2}}}\right).$$

$\frac{r^{n-1}}{(1+r^2)^{\frac{4n}{n+2}}}$ is equivalent to $\frac{1}{r^{\frac{8n}{n+2}-n+1}}$ as r goes to infinity, and $\frac{8n}{n+2} - n + 1$ is larger than 1 for $n < 6$, is equal to 1 for $n = 6$, and is smaller than 1 for $n > 6$. Therefore

$$\int_0^{2\lambda S} \frac{r^{n-1}dr}{(1+r^2)^{\frac{4n}{n+2}}} = 0\left(1 \text{ if } n = 5\ ;\ \text{Log } \lambda \text{ if } n = 6\ ; \lambda^{n-\frac{8n}{n+2}} \text{ if } n > 6\right)$$

and (D.38) follows. We have also

$$\begin{aligned}\int_{\substack{\Omega \\ |x-y|\ge \frac{1}{4T}}} U_{\lambda,y}^{\frac{(p-2)(p+1)}{p-1}} \left|\frac{\partial U_{\lambda,y}}{\partial y_j}\right|^{\frac{p+1}{p-1}} &= 0\left(\lambda^{\frac{n(6-n)}{4}+\frac{n(n+2)}{4}} \int_{|x|\ge\frac{1}{4T}} \frac{|x|^{\frac{n}{2}}dx}{(1+\lambda^2|x|^2)^{\frac{n(6-n)}{4}+\frac{n^2}{4}}}\right) \\ &= 0\left(\lambda^{\frac{n}{2}} \int_{\frac{\lambda}{4T}}^{+\infty} \frac{r^{\frac{3n}{2}-1}dr}{(1+r^2)^{\frac{3n}{2}}}\right) \\ &= 0\left(\lambda^{\frac{n}{2}}\left(\frac{T}{\lambda}\right)^{\frac{3n}{2}}\right)\end{aligned}$$

whence (D.39). (D.40) is a straightforward consequence of (D.17) and the fact that $\frac{\partial U_{\lambda,y}}{\partial \lambda} = 0(\frac{U_{\lambda,y}}{\lambda})$, $\frac{\partial U_{\lambda,y}}{\partial y_j} = 0(\lambda U_{x,y})$. Lastly, we have

$$\int_\Omega \left|\frac{\partial U_{\lambda,y}}{\partial y_j}\right|^{\frac{p+1}{p}} = 0\left(\lambda^n \int_{\mathbf{R}^n} \frac{|x|^{\frac{2n}{n+2}}dx}{(1+|x|^2)^{\frac{n^2}{n+2}}}\right) = 0\left(\frac{1}{\lambda^{\frac{2n}{n+2}}} \int_0^{+\infty} \frac{r^{\frac{2n}{n+2}+n-1}dr}{(1+r^2)^{\frac{n^2}{n+2}}}\right) = 0\left(\frac{1}{\lambda^{\frac{2n}{n+2}}}\right)$$

since, for every n, the integrals are convergent. Then, (D.41) holds.

We state some additional integral estimates of the same type, where both U_i and U_j, $i \neq j$, occur. Namely

(D.42) $\int_\Omega U_i^p U_j = 0(\frac{1}{\lambda^{n-2}})$

(D.43) $\int_\Omega U_i^{p-1} |\frac{\partial U_i}{\partial y_\ell^i}| U_j = 0(\frac{1}{\lambda^{n-3}})$

(D.44) $\int_\Omega U_i^{p-2} |\frac{\partial U_i}{\partial y_\ell^i}| U_j^2 = 0(\frac{1}{\lambda^4})$

(D.45) $\int_\Omega |\frac{\partial U_i}{\partial y_\ell^i}| U_j^p = 0(\frac{1}{\lambda^{n-2}})$

$$(D.46)\quad \int_\Omega U_i^{-\sigma} |\tfrac{\partial U_i}{\partial y_\ell^i}| U_j^{p+\sigma} = 0(\tfrac{1}{\lambda^{(n-2)(1-\sigma)}}) \qquad \text{for } n \geq 6\ ,\ 2-p < \sigma < 1$$

$$(D.47)\quad \left(\int_\Omega \inf(U_i^{p+1}, U_i^{\frac{(p-2)(p+1)}{p-1}} U_j^{\frac{p+1}{p-1}}) \right)^{\frac{p-1}{p+1}} = 0(\tfrac{1}{\lambda^2})\ .$$

Proof.

(D.32) yields

$$\int_\Omega U_i^p U_j = 0 \left(\frac{1}{\lambda_j^{\frac{n-2}{2}}} \int_{B_i} U_i^p + \frac{1}{\lambda_i^{\frac{n+2}{2}}} \int_{B_j} U_j + \frac{1}{\lambda_i^{\frac{n+2}{2}} \lambda_j^{\frac{n-2}{2}}} \right)$$

From (D.33), we know that

$$\int_{B_j} U_j = 0(\frac{1}{\lambda_j^{\frac{n-2}{2}}})$$

and

$$\begin{aligned} \int_{B_i} U_i^p &= 0 \left(\lambda_i^{\frac{n+2}{2}} \int_{|x| \leq \frac{d_0}{2}} \frac{dx}{(1+\lambda_i^2 |x|^2)^{\frac{n+2}{2}}} \right) \\ &= 0 \left(\frac{1}{\lambda_i^{\frac{n-2}{2}}} \int_0^{\lambda_i \frac{d_0}{2}} \frac{r^{n-1} dr}{(1+r^2)^{\frac{n+2}{2}}} \right) \end{aligned}$$

so that

$$(D.48)\qquad \int_{B_i} U_i^p = 0(\frac{1}{\lambda_i^{\frac{n-2}{2}}})\ .$$

This proves (D.42). (D.43) follows from (D.42) and the fact that $|\frac{\partial U_i}{\partial y_\ell^i}| = 0(\lambda_i U_i)$. Concerning (D.44), we write

$$\int_\Omega U_i^{p-2} |\frac{\partial U_i}{\partial y_\ell^i}| U_j^2 = 0 \left(\frac{1}{\lambda_j^{n-2}} \int_{B_i} U_i^{p-2} |\frac{\partial U_i}{\partial y_\ell^i}| + \frac{1}{\lambda_i^2} \int_{B_j} U_j^2 + \frac{1}{\lambda_i^2 \lambda_j^{n-2}} \right)$$

because of (D.32,34). Moreover, (D.21) ensures that

$$\int_{B_j} U_j^2 = 0(\frac{1}{\lambda_j^2})\ .$$

Lastly

$$\begin{aligned} \int_{B_i} U_i^{p-2} |\frac{\partial U_i}{\partial y_\ell^i}| &= 0 \left(\lambda_i^{\frac{6-n}{2} + \frac{n+2}{2}} \int_{|x| \leq \frac{d_0}{2}} \frac{|x| dx}{(1+\lambda_i^2 |x|^2)^{\frac{6-n}{2} + \frac{n}{2}}} \right) \\ &= 0 \left(\frac{1}{\lambda_i^{n-3}} \int_0^{\frac{\lambda_i d_0}{2}} \frac{r^n dr}{(1+r^2)^3} \right) \\ &= 0 \left(\frac{\mathrm{Log}\lambda_i}{\lambda_i^2} \text{ if } n = 5\ ; \frac{1}{\lambda_i^2} \text{ if } n \geq 6 \right) \end{aligned}$$

since $\frac{r^n}{(1+r^2)^3} \sim r^{n-6}$ as r goes to infinity, and (D.44) is proved. In the same way, we write

$$\int_\Omega |\frac{\partial U_i}{\partial y_\ell^i}| U_j^p = 0\left(\frac{1}{\lambda_j^{\frac{n+2}{2}}}\int_{B_i}|\frac{\partial U_i}{\partial y_\ell^i}| + \frac{1}{\lambda_i^{\frac{n-2}{2}}}\int_{B_j} U_j^p + \frac{1}{\lambda_i^{\frac{n-2}{2}}\lambda_j^{\frac{n+2}{2}}}\right) = 0(\frac{1}{\lambda^{n-2}})$$

because of (D.35, 48). We assume now that $n \geq 6$, and $2 - p = \frac{n-6}{n-2} < \sigma < 1$. From (D.32-34) we deduce that

$$\int_\Omega U_i^{-\sigma}|\frac{\partial U_i}{\partial y_\ell^i}| U_j^{p+\sigma} = 0\left(\frac{1}{\lambda_j^{\frac{n-2}{2}(p+\sigma)}}\int_{B_i} U_i^{-\sigma}|\frac{\partial U_i}{\partial y_\ell^i}| + \frac{1}{\lambda_i^{\frac{n-2}{2}(1-\sigma)}}\int_{B_j} U_j^{p+\sigma} + \frac{1}{\lambda_i^{\frac{n-2}{2}(1-\sigma)}\lambda_j^{\frac{n-2}{2}(p+\sigma)}}\right).$$

On one hand

$$\int_{B_j} U_j^{p+\sigma} = 0\left(\lambda_j^{\frac{n-2}{2}(p+\sigma)}\int_{|x|\leq\frac{d_0}{2}} \frac{dx}{(1+\lambda_j^2|x|^2)^{\frac{n-2}{2}(p+\sigma)}}\right)$$
$$= 0\left(\frac{1}{\lambda_j^{n-\frac{n-2}{2}(p+\sigma)}}\int_0^{\frac{\lambda_j d_0}{2}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n-2}{2}(p+\sigma)}}\right)$$

whence

$$\int_{B_j} U_j^{p+\sigma} = 0\left(\frac{1}{\lambda_j^{n-\frac{n-2}{2}(p+\sigma)}}\right)$$

since the integral of $\frac{r^{n-1}}{(1+r^2)^{\frac{n-2}{2}(p+\sigma)}}$ is in any cases convergent. On the other hand

$$\int_{B_i} U_i^{-\sigma}|\frac{\partial U_i}{\partial y_\ell^i}| = 0\left(\lambda_i^{-\frac{n-2}{2}\sigma+\frac{n+2}{2}}\int_{|x|\leq\frac{d_0}{2}} \frac{|x|dx}{(1+\lambda_i^2|x|^2)^{-\frac{n-2}{2}\sigma+\frac{n}{2}}}\right)$$
$$= 0\left(\frac{1}{\lambda_i^{\frac{n}{2}+\frac{n-2}{2}\sigma}}\int_0^{\frac{\lambda_i d_0}{2}} \frac{r^n dr}{(1+r^2)^{\frac{n}{2}-\frac{n-2}{2}\sigma}}\right)$$
$$= 0\left(\frac{1}{\lambda_i^{\frac{n-2}{2}(1-\sigma)}}\right)$$

since

$$\frac{r^n}{(1+r^2)^{\frac{n}{2}-\frac{n-2}{2}\sigma}} \sim r^{(n-2)\sigma} \quad \text{as} \quad r \quad \text{goes to infinity.}$$

Combining these results, we find (D.46). It only remains to show (D.47). We have, using (D.32)

$$\int_\Omega \inf\left(U_i^{p+1}, U_i^{\frac{(p-2)(p-1)}{p-1}} U_j^{\frac{p+1}{p-1}}\right) = 0\left(\frac{1}{\lambda_j^{\frac{n(n-2)}{4}}} \int_{B_i} U_i^{\frac{n(6-n)}{2(n-2)}} + \frac{1}{\lambda_i^n}\right)$$

and

$$\begin{aligned}\int_{B_i} U_i^{\frac{n(6-n)}{2(n-2)}} &= \lambda_i^{\frac{n(6-n)}{4}} \int_{|x|\le \frac{d_0}{2}} \frac{dx}{(1+\lambda_i|x|^2)^{\frac{n(6-n)}{4}}} \\ &= \frac{\sigma_{n-1}}{\lambda_i^{\frac{n(n-2)}{4}}} \int_0^{\frac{\lambda_i d_0}{4}} \frac{r^{n-1}dr}{(1+r^2)^{\frac{n(6-n)}{4}}} \\ &= 0\left(\frac{1}{\lambda_i^{\frac{n(6-n)}{4}}}\right)\end{aligned}$$

since

$$\frac{r^{n-1}}{(1+r^2)^{\frac{n(6-n)}{4}}} \sim r^{\frac{n(n-4)}{2}-1} \quad \text{as} \quad r \quad \text{goes to infinity.}$$

Then

$$\int_\Omega \inf\left(U_i^{p+1}, U_i^{\frac{(p-2)(p+1)}{p-1}} U_j^{\frac{p+1}{p-1}}\right) = 0(\frac{1}{\lambda^2})$$

hence (D.47).

It only remains to estimate the boundary integrals. We have

$$(D.49) \qquad \left(\int_{\partial\Omega} \left|\frac{\partial U_{\lambda,y}}{\partial \nu}\right|^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = 0\ (\frac{1}{\lambda})$$

$$(D.50) \qquad \left(\int_{\partial\Omega} \left|\frac{\partial^2 U_{\lambda,y}}{\partial \nu \partial \tau_i}\right|^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = 0(1)$$

(D.51)

$$\int_{\partial\Omega} \left|\frac{\partial U_{\lambda,y}}{\partial \nu}\right| U_{\lambda,y} = 0(\frac{1}{\lambda}); \int_{\partial\Omega} \left|\frac{\partial U_{\lambda,y}}{\partial \nu}\right| \frac{\partial U_{\lambda,y}}{\partial \lambda} = 0(\frac{1}{\lambda^2}); \int_{\partial\Omega} \left|\frac{\partial U_{\lambda,y}}{\partial \nu}\right| \frac{\partial U_{\lambda,y}}{\partial y_i} = 0(1)\ .$$

Proof.

With the nonrestrictive assumptions that $y = 0$ and that Ω is defined by (D.2) in a neighbourhood of 0, we write, for $\sigma = (x', f(x')) \in \partial\Omega$

$$\frac{\partial U_{\lambda,y}}{\partial \nu}(\sigma) = \frac{1}{(1+f^2(x'))^{1/2}} \left(\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} \frac{\partial U_{\lambda,y}}{\partial x_i} - \frac{\partial U_{\lambda,y}}{\partial x_n} \right)$$

Therefore, (D.1,3) yield

$$\frac{\partial U_{\lambda,y}}{\partial \nu}(\sigma) = -\frac{n-2}{(1+f'^2(x'))^{1/2}} \lambda^{\frac{n+2}{2}} \frac{\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} x_i - x_n}{(1+\lambda^2|x|^2)^{\frac{n}{2}}}$$

or

$$(D.52) \qquad \frac{\partial U_{\lambda,y}}{\partial \nu}(\sigma) = -\frac{n-2}{(1+f'^2(x'))^{1/2}} \lambda^{\frac{n+2}{2}} \frac{\sum_{i=1}^{n-1} f_i x_i^2 + 0(|x'|^3)}{(1+\lambda^2|x|^2)^{\frac{n}{2}}}.$$

Since

$$(D.53) \qquad \left|\frac{d\sigma}{dx'}\right| = (1+f'^2(x'))^{1/2}$$

we obtain

$$\int_{\partial\Omega\cap B(0,R)} \left| \frac{\partial U_{\lambda,y}}{\partial \nu} \right|^{\frac{2(n-1)}{n}} d\sigma = 0 \left(\lambda^{\frac{(n+2)(n-1)}{n}} \int_{|x'|\le R} \frac{|x'|^{\frac{4(n-1)}{n}} dx'}{(1+\lambda^2|x'|^2)^{n-1}} \right)$$

using the facts that on $\partial\Omega$, $|x|^2 = |x'|^2 + 0(|x'|^4)$ and $f'^2(x') = 0(|x'|^2)$. With rescaled spherical coordinates, this yields

$$(D.54) \qquad \int_{\partial\Omega\cap B(0,R)} \left| \frac{\partial U_{\lambda,y}}{\partial \nu} \right|^{\frac{2(n-1)}{n}} d\sigma = 0 \left(\frac{1}{\lambda^{\frac{2(n-1)}{n}}} \int_0^{\lambda R'} \frac{r^{n-2+\frac{4(n-1)}{n}} dr}{(1+r^2)^{n-1}} \right) .$$

$\frac{r^{n-2+\frac{4(n-1)}{n}}}{(1+r^2)^{n-1}} \sim \frac{1}{r^{\frac{(n-2)^2}{n}}}$ as r goes to infinity, and $\frac{(n-2)^2}{n} > 1$ for $n \ge 5$. The last integral is convergent, and

$$\int_{\partial\Omega\cap B(0,R)} \left| \frac{\partial U_{\lambda,y}}{\partial \nu} \right|^{\frac{2(n-1)}{n}} d\sigma = 0 \left(\frac{1}{\lambda^{\frac{2(n-1)}{n}}} \right) .$$

On the other hand, we deduce from (D.1) that

$$(D.55) \qquad \frac{\partial U_{\lambda,y}}{\partial \nu} = 0 \left(\sum_{i=1}^{n} \left| \frac{\partial U_{\lambda,y}}{\partial x_i} \right| \right) = 0 \left(\frac{\lambda^{\frac{n+2}{2}}|x|}{(1+\lambda^2|x|^2)^{n/2}} \right) = 0\left(\frac{1}{\lambda^{\frac{n-2}{2}}}\right)$$

on $\partial\Omega \setminus B(0,R)$. Consequently

$$\int_{\partial\Omega\setminus B(0,R)} \left|\frac{\partial U_{\lambda,y}}{\partial\nu}\right|^{\frac{2(n-1)}{n}} d\sigma = 0\left(\frac{1}{\lambda^{\frac{(n-2)(n-1)}{n}}}\right) = o(\frac{1}{\lambda^{\frac{2(n-1)}{n}}}) \quad \text{for} \quad n \geq 5$$

whence (D.49).

Remark.

From the previous computations - see (D.54) -, we deduce that

$$\int_{\substack{\partial\Omega\cap B(0,R)\\ |x-y|\geq\frac{1}{4T}}} \left|\frac{\partial U_{\lambda,y}}{\partial\nu}\right|^{\frac{2(n-1)}{n}} d\sigma = 0\left(\frac{1}{\lambda^{\frac{2(n-1)}{n}}}\int_{\frac{\lambda}{4T}}^{\lambda R} \frac{r^{n-2+\frac{4(n-1)}{n}}}{(1+r^2)^{n-1}}dr\right)$$

and

$$\int_{\frac{\lambda}{4T}}^{\lambda R} \frac{r^{n-2+\frac{4(n-1)}{n}}}{(1+r^2)^{n-1}}dr = 0\left(\left(\frac{T}{\lambda}\right)^{\frac{(n-1)(n-4)}{n}}\right) \quad \text{as} \quad T, \frac{\lambda}{T} \quad \text{go to infinity.}$$

Then, we have

$$(D.56) \qquad \left(\int_{\substack{\partial\Omega\\ |x-y|\geq\frac{1}{4T}}} \left|\frac{\partial U_{\lambda,y}}{\partial\nu}\right|^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = 0\left(\frac{1}{\lambda}\left(\frac{T}{\lambda}\right)^{\frac{n-4}{2}}\right).$$

The proofs of (D.50-51) are quite similar to that of (D.49), and we omit them. We notice that, like (D.56), we have

$$(D.57) \qquad \left(\int_{\substack{\partial\Omega\\ |x-y|\geq\frac{1}{4T}}} \left|\frac{\partial^2 U_{\lambda,y}}{\partial\nu\partial y_i}\right|^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = 0\left(\left(\frac{T}{\lambda}\right)^{\frac{n-2}{2}}\right)$$

We end this section with the following technical estimates

$$(D.58) \quad \left(\int_{\substack{\mathbf{R}^n\\ |z|\leq R}} \frac{|z|^6}{(1+\lambda^2|z|^2)^n}dz\right)^{1/2} = 0\left(\frac{(\mathrm{Log}\lambda)^{1/2}}{\lambda^6} \text{ if } n=6; \frac{1}{\lambda^{\frac{n+6}{2}}} \text{ if } n>6\right)$$

$$(D.59) \quad \left(\int_{\mathbf{R}^n} \frac{|z|^4}{(1+\lambda^2|z|^2)^n}dz\right)^{1/2} = 0\left(\frac{1}{\lambda^{\frac{n+4}{2}}}\right)$$

$$(D.60)\quad \left(\int_{\substack{\mathbf{R}^n \\ |z|\leq \frac{1}{T}}} \frac{|z|^8}{(1+\lambda^2|z|^2)^n}dz\right)^{1/2} = 0\left(\frac{T^{\frac{n-8}{2}}}{\lambda^n} \text{ if } n<8; \frac{(\mathrm{Log}\frac{\lambda}{T})}{\lambda^8} \text{ if} n=8; \frac{1}{\lambda^{\frac{n+8}{2}}} \text{ if } n>8\right)$$

$$(D.61)\quad \left(\int_{\mathbf{R}^n} \frac{|z|^8}{(1+\lambda^2|z|^2)^{n+4}}dz\right)^{1/2} = 0\left(\frac{1}{\lambda^{\frac{n+8}{2}}}\right)$$

$$(D.62)\quad \left(\int_{\mathbf{R}^n} \frac{|z|^{\frac{5n}{2}}}{(1+\lambda^2|z|^2)^{2n}}dz\right)^{\frac{2}{n}} = 0\left(\frac{1}{\lambda^7}\right)$$

$$(D.63)\quad \left(\int_{\mathbf{R}^{n-1}} \frac{|x'|^{\frac{2(n-1)}{n}}}{(1+\lambda^2|x'|^2)^{n-1}}dx'\right)^{\frac{n}{2(n-1)}} = 0\left(\frac{1}{\lambda^{\frac{n+2}{2}}}\right)$$

$$(D.64)\quad \left(\int_{\mathbf{R}^{n-1}} \frac{|z'|^{\frac{4(n-1)}{n}}}{(1+\lambda^2|z'|^2)^{n-1}}dz'\right)^{\frac{n}{2(n-1)}} = 0(\frac{1}{\lambda^{\frac{n+4}{2}}})$$

$$(D.65)\quad \left(\int_{\substack{\mathbf{R}^n \\ |z|\leq \frac{1}{T}}} \frac{|z|^8}{(1+\lambda^2|z|^2)^6}dz\right)^{\frac{1}{2}} = 0\left(\frac{1}{\lambda^6 T^{\frac{n-4}{2}}}\right) .$$

All these estimates follow from straighforward computations, using spherical coordinates and a rescaling $z \mapsto \frac{z}{\lambda}, x' \mapsto \frac{x'}{\lambda}, z' \mapsto \frac{z'}{\lambda}$.

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Olivier REY
URA 169 du CNRS
Centre de Mathématiques
Ecole Polytechnique
91128 - Palaiseau cedex France