# Multiple singular values of Hankel operators

Patrick Gerard, Sandrine Grellier

# MULTIPLE SINGULAR VALUES OF HANKEL OPERATORS

PATRICK GÉRARD AND SANDRINE GRELLIER

Abstract. The goal of this paper is to construct a nonlinear Fourier transformation on the space of symbols of compact Hankel operators on the circle. This transformation allows to solve a general inverse spectral problem involving singular values of a compact Hankel operator, with arbitrary multiplicities. The formulation of this result requires the introduction of the pair made with a Hankel operator and its shifted Hankel operator. As an application, we prove that the space of symbols of compact Hankel operators on the circle admits a singular foliation made of tori of finite or infinite dimensions, on which the flow of the cubic Szegő equation acts. In particular, we infer that arbitrary solutions of the cubic Szegő equation on the circle with finite momentum are almost periodic with values in $H^{1/2}(\mathbb{S}^1)$.

## Contents

1. Introduction 2
1.1. Inverse spectral theory of self-adjoint compact Hankel operators 3
1.2. Complexification and the Hardy space representation 5
1.3. Multiplicity and Blaschke products 7
1.4. Main results 9
1.5. Applications to the cubic Szegő equation 11
1.6. Organization of the paper 13
2. Spectral decomposition of the Hankel operators $H_u$ and $K_u$ 13
3. Multiplicity and Blaschke products. Proof of Proposition 1 16
3.1. Case of $\rho \in \Sigma_H(u)$ 16
3.2. Case of $\sigma \in \Sigma_K(u)$ 19
4. The inverse spectral theorem in the finite rank case 21
4.1. Continuity of $\Phi_n$ 22
4.2. The explicit formula, case $n$ even. 24
4.3. Surjectivity in the case $n$ even. 27

*Date*: February 7, 2014.

2010 *Mathematics Subject Classification.* 35B15, 47B35, 37K15.

The authors are grateful to T. Kappeler, V. Peller and A. Pushnitski for valuable discussions and comments about this paper, to L. Baratchart for pointing them references [1] and [3] in connection with Proposition 1, and to A. Nazarov for references concerning Bateman–type formulae. The first author is supported by ANR ANAE 13-BS01-0010-03.

4.4. $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty, $n$ even 33
4.5. The case $n$ odd. 38
4.6. $\mathcal{V}_{(d_1,\dots,d_n)}$ is a manifold 39
4.7. Proof of Theorem 1 in the finite rank case 39
5. Extension to compact Hankel operators 40
6. Evolution under the cubic Szegő flow 46
6.1. The theorem 46
6.2. Application: traveling waves revisited 48
6.3. Application to almost periodicity 50
7. Evolution under the Szegő hierarchy 51
8. Invariant tori of the Szegő hierarchy and unitary equivalence of pairs of Hankel operat
9. The symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$ 63
Appendix A. Some Bateman-type formulae 69
Appendix B. The structure of finite Blaschke products 73
Appendix C. Two results by Adamyan–Arov–Krein 74
References 79

## 1. Introduction

The theory of Hankel operators has many applications in various areas of mathematics, such as operator theory, approximation theory, control theory. We refer to the books [24] and [22] for a systematic presentation of this theory. More recently, spectral theory of Hankel operators arose as a key tool in the study of some completely integrable Hamiltonian system, called the cubic Szegő equation, see [8], [9], [11]. The goal of this paper is two-fold. On the one hand, we present the complete solution of some double inverse spectral problem for compact Hankel operators. On the other hand, we apply this theory in order to obtain qualitative results on the dynamics of the cubic Szegő equation.

We first recall the definition of a Hankel operator on the space $\ell^2(\mathbb{Z}_+)$. Given a sequence $c = (c_n)_{n\geq 0} \in \ell^2(\mathbb{Z}_+)$, the associated Hankel operator $\Gamma_c$ is formally defined by

$$\forall x = (x_n)_{n\geq 0} \in \ell^2(\mathbb{Z}_+) \ , \ \Gamma_c(x)_n = \sum_{k=0}^{\infty} c_{n+k} x_k \ .$$

Hankel operators are strongly related to the shift operator

$$\begin{array}{rccc} \Sigma : & \ell^2(\mathbb{Z}_+) & \longrightarrow & \ell^2(\mathbb{Z}_+) \\ & (x_0, x_1, x_2, \cdots) & \longmapsto & (0, x_0, x_1, x_2, \cdots) \ , \end{array}$$

and to its adjoint

$$\begin{array}{rccc} \Sigma^* : & \ell^2(\mathbb{Z}_+) & \longrightarrow & \ell^2(\mathbb{Z}_+) \\ & (x_0, x_1, x_2, \cdots) & \longmapsto & (x_1, x_2, \cdots) \ . \end{array}$$

Indeed, Hankel operators are those operators $\Gamma$ on $\ell^2(\mathbb{Z}_+)$ such that

$$\Sigma^*\Gamma = \Gamma\Sigma \ . \tag{1.1}$$

A famous result due to Nehari [21] characterizes the boundedness of $\Gamma_c$ on $\ell^2(\mathbb{Z}_+)$ by

$$\exists f \in L^\infty(\mathbb{T}) : \forall n \geq 0,\ c_n = \hat{f}(n) \ ,$$

where $\hat{f}$ denotes the sequence of Fourier coefficients of any distribution $f$ on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Using Fefferman's theorem [7], this is equivalent to $u_c \in BMO(\mathbb{T})$, where we define

$$u_c(\mathrm{e}^{ix}) := \sum_{n=0}^{\infty} c_n \mathrm{e}^{inx} \ ,\ x \in \mathbb{T}. \tag{1.2}$$

Throughout this paper, we shall focus on the special case where $\Gamma_c$ is compact, which corresponds to $u_c \in VMO(\mathbb{T})$ by a theorem due to Hartman [14].

1.1. **Inverse spectral theory of self-adjoint compact Hankel operators.** We first discuss the case of self-adjoint operators $\Gamma_c$, which corresponds to a real valued sequence $c$. Assume moreover that $\Gamma_c$ is compact. The spectrum of $\Gamma_c$ consists of 0 and of a finite or infinite sequence of real nonzero eigenvalues $(\lambda_j)_{j\geq 1}$, repeated according to their finite multiplicities. A natural question is the following : *given any finite or infinite sequence of nonzero real numbers $(\lambda_j)_{j\geq 1}$, does there exist a compact selfadjoint Hankel operator $\Gamma_c$ having this sequence as non zero eigenvalues, repeated according to their multiplicity?*

Of course, if the sequence $(\lambda_j)$ is infinite, it is necessary that $\lambda_j$ tends to 0. A much more subtle constraint was found by Megretskii-Peller–Treil in [19], who proved the following theorem, which we state only in the compact case.

**Theorem** (Megretskii, Peller, Treil)**.** *A finite or infinite sequence $(\lambda_j)$ of nonzero real numbers is the sequence of nonzero eigenvalues of a compact selfadjoint Hankel operator if and only if*

1. *If $(\lambda_j)$ is infinite, then $\lambda_j \underset{j\to\infty}{\longrightarrow} 0$ ;*
2. *For any $\lambda \in \mathbb{R}^*$, $|\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1$.*

Our first objective is to describe the set of solutions of this inverse problem, namely the isospectral sets for any given sequence $(\lambda_j)$. Observe that, even in the rank one case, there is no uniqueness to be expected. Indeed, $\Gamma_c$ is a selfadjoint rank one operator if and only if

$$c_n = \alpha p^n \ ,\ \alpha \in \mathbb{R}^* \ ,\ p \in (-1, 1) \ .$$

In this case, the only nonzero eigenvalue is

$$\lambda_1 = \frac{\alpha}{1 - p^2} \ .$$

Isospectral sets are therefore manifolds diffeomorphic to $\mathbb{R}$. Hence we need to introduce additional parameters. The study of the cubic Szegő equation led us to introduce a second Hankel operator $\Gamma_{\tilde{c}}$ in [9], where

$$\tilde{c}_n := c_{n+1} \ , \ n \in \mathbb{Z}_+ \ .$$

Notice that $\Gamma_{\tilde{c}}$ is quite a natural operator since it is precisely the operator arising in the identity (1.1) where $\Gamma = \Gamma_c$. Coming back to the rank one case, we observe that, if $p \neq 0$, the only nonzero eigenvalue of $\Gamma_{\tilde{c}}$ is

$$\mu_1 = \frac{\alpha p}{1-p^2} \ .$$

Notice that the knowledge of $\lambda_1$ and $\mu_1$ characterizes $\alpha$ and $p$, hence $c$. More generally, it is easy to check from (1.1) that

$$\Gamma_{\tilde{c}}^2 = \Gamma_c^2 - (\cdot|c)c \ . \tag{1.3}$$

Denoting by $(\lambda_j)$, $(\mu_k)$ the sequences of nonzero eigenvalues of $\Gamma_c$, $\Gamma_{\tilde{c}}$ respectively, labelled in decreasing order of their absolute values, this identity implies, from the min-max formula, the following interlacement inequalities,

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq \dots \tag{1.4}$$

Let us now investigate the inverse spectral problem for both operators $\Gamma_c, \Gamma_{\tilde{c}}$. In the special case where inequalities in (1.4) are strict, we proved in [9] and [10] that this problem admits a unique solution $c$. In the general case, one can prove that eigenspaces of $\Gamma_c^2$ and $\Gamma_{\tilde{c}}^2$ corresponding to the same positive eigenvalue, are such that one of them is of codimension 1 into the other one. As a consequence, in the sequence of inequalities (1.4), the length of every maximal string with consecutive equal terms is odd. Our first result is that this condition is optimal.

**Theorem 1.** *Let $(\lambda_j), (\mu_k)$ be two finite or infinite tending to zero sequences of nonzero real numbers satisfying*

1. $|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq \dots$
2. *In the above sequence of inequalities, the lengths of maximal strings with consecutive equal terms are odd. Denote them by $2d_r + 1$.*
3. *For any $\lambda \in \mathbb{R}^*$, $|\#\{j : \lambda_j = \lambda\} - \#\{j : \lambda_j = -\lambda\}| \leq 1$.*
4. *For any $\mu \in \mathbb{R}^*$, $|\#\{k : \mu_k = \mu\} - \#\{k : \mu_k = -\mu\}| \leq 1$.*

*Then there exists a sequence $c$ of real numbers such that $\Gamma_c$ is compact and the nonzero eigenvalues of $\Gamma_c$ and $\Gamma_{\tilde{c}}$ are respectively the $\lambda_j$'s and the $\mu_k$'s. Moreover, introduce*

$$M := \sum_r d_r \ \in \mathbb{Z}_+ \cup \{\infty\} \ .$$

*The isospectral set is a manifold diffeomorphic to $\mathbb{R}^M$ if $M < \infty$, and it is homeomorphic to $\mathbb{R}^\infty$ if $M = \infty$.*

Moreover, in the case of a finite sequence of nonzero eigenvalues, one can produce explicit formulae for $u_c$. For instance, given four real numbers $\lambda_1, \mu_1, \lambda_2, \mu_2$ such that

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > 0 \ ,$$

we get

$$u_c(\mathrm{e}^{ix}) = \frac{\dfrac{\lambda_1 - \mu_1 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_1^2} + \dfrac{\lambda_2 - \mu_2 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_2^2} - \dfrac{\lambda_1 - \mu_2 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_2^2} - \dfrac{\lambda_2 - \mu_1 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_1^2}}{\begin{vmatrix} \frac{\lambda_1 - \mu_1 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_1^2} & \frac{\lambda_2 - \mu_1 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_1^2} \\ \frac{\lambda_1 - \mu_2 \mathrm{e}^{ix}}{\lambda_1^2 - \mu_2^2} & \frac{\lambda_2 - \mu_2 \mathrm{e}^{ix}}{\lambda_2^2 - \mu_2^2} \end{vmatrix}}$$

If $|\lambda_1| > |\lambda_2| > 0$ and $\mu_1 = \lambda_2, \mu_2 = -\lambda_2$, then, there exists $p \in (-1, 1)$ such that

$$u_c(\mathrm{e}^{ix}) = (\lambda_1^2 - \lambda_2^2) \frac{1 - p\, \mathrm{e}^{ix}}{\lambda_1 - p\, \mathrm{e}^{ix}(\lambda_1 - \lambda_2) - \lambda_2\, \mathrm{e}^{2ix}} \ .$$

Finally, notice that, if $\lambda_1, \lambda_2$ are given such that $|\lambda_1| > |\lambda_2| > 0$, the corresponding isospectral set consists of sequences $c$ given by the above two formulae. Also notice that the second expression is obtained from the first one by making $\mu_1 \to \lambda_2$ , $\mu_2 \to -\lambda_2$ , and

$$\frac{2\lambda_2 + \mu_2 - \mu_1}{\mu_1 + \mu_2} \to p \ .$$

1.2. **Complexification and the Hardy space representation.** In the general case where $c$ is complex-valued, $\Gamma_c$ is no more selfadjoint, and the natural inverse spectral problem rather concerns singular values of $\Gamma_c$, namely square roots of nonzero eigenvalues of $\Gamma_c \Gamma_c^*$. In order to have a better understanding of the multiplicity phenomena, we are going to change the representation of these operators. A natural motivation for this new representation comes back to a celebrated paper by Beurling [5] characterizing the closed subspaces of $\ell^2(\mathbb{Z}_+)$ invariant by $\Sigma$. The connection with Hankel operators is made by the observation that, because of identity (1.1), the kernel of a Hankel operator is always such a space. According to Beurling's theorem, these spaces can be easily described by using the isometric Fourier isomorphism

$$\begin{array}{ccc} \ell^2(\mathbb{Z}_+) & \longrightarrow & L^2_+(\mathbb{T}) \\ c & \longmapsto & u_c \end{array}$$

where $L^2_+(\mathbb{T})$ denotes the closed subspace of $L^2(\mathbb{T})$ made of functions $u$ with

$$\forall n < 0 \ , \ \hat{u}(n) = 0 \ ,$$

endowed with the inner product

$$(f|g) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\mathrm{e}^{ix}) \overline{g(\mathrm{e}^{ix})}\, dx \ .$$

Notice that $L^2_+(\mathbb{T})$ is isomorphic to the Hardy space $\mathbb{H}^2(\mathbb{D})$ of holomorphic functions on the unit disc with $L^2$ traces on the unit circle. Under this representation, the shift operator $\Sigma$ becomes

$$\begin{array}{rccc} S: & \mathbb{H}^2(\mathbb{D}) & \longrightarrow & \mathbb{H}^2(\mathbb{D}) \\ & f & \longmapsto & zf \ , \end{array}$$

and its adjoint reads

$$\begin{array}{rccc} S^*: & \mathbb{H}^2(\mathbb{D}) & \longrightarrow & \mathbb{H}^2(\mathbb{D}) \\ & f & \longmapsto & \Pi(\overline{z}f) \ , \end{array}$$

where $\Pi$ is the orthogonal projector from $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T}) \simeq \mathbb{H}^2(\mathbb{D})$, usually referred as the Szegő projector. Using this representation, the Beurling theorem claims that non trivial closed subspaces of $\mathbb{H}^2(\mathbb{D})$ invariant by $S$ are exactly the spaces

$$\Psi\mathbb{H}^2(\mathbb{D}) \ ,$$

where $\Psi$ is an inner function, namely a bounded holomorphic function on $\mathbb{D}$ with modulus 1 on the unit circle.

Let us come back to Hankel operators. Using the above representation, $\Gamma_c$ corresponds to an operator $H_u$, $u = u_c$, defined by

$$H_u(h) = \Pi(u\overline{h}) \ , \ h \in L^2_+(\mathbb{T}) \ .$$

Precisely, we have the following identity, for every $u \in BMO_+(\mathbb{T}) := BMO(\mathbb{T}) \cap L^2_+(\mathbb{T})$ ,

$$\widehat{H_u(h)} = \Gamma_{\hat{u}}\left(\overline{\hat{h}}\right) \ , \ h \in L^2_+(\mathbb{T}) \ .$$

Notice that $H_u$ is an antilinear operator, satisfying the following self-adjointness property,

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1) \ , \ h_1, h_2 \in L^2_+(\mathbb{T}) \ . \tag{1.5}$$

Consequently, $H_u^2$ is a linear positive selfadjoint operator on $L^2_+(\mathbb{T})$, which is conjugated to $\Gamma_c\Gamma_c^*$ through the Fourier representation, hence the square roots of its positive eigenvalues are the singular values of $\Gamma_c$ or of $H_u$. In the same way, $\Gamma_{\tilde{c}}$ corresponds to

$$K_u := S^*H_u = H_uS = H_{S^*u} \ , \tag{1.6}$$

and identity (1.3) reads

$$K_u^2 = H_u^2 - (\,\cdot\,|u)u \ . \tag{1.7}$$

For every $s \geq 0$ and $u \in VMO_+(\mathbb{T}) := VMO(\mathbb{T}) \cap L^2_+(\mathbb{T})$, we set

$$E_u(s) := \ker(H_u^2 - s^2I) \ , \ F_u(s) := \ker(K_u^2 - s^2I) \ . \tag{1.8}$$

Notice that $E_u(0) = \ker H_u$ , $F_u(0) = \ker K_u$. Moreover, from the compactness of $H_u$, if $s > 0$, $E_u(s)$ and $F_u(s)$ are finite dimensional. Using (1.6) and (1.7), one can show the following result.

**Lemma 1.** *Let $s > 0$ such that $E_u(s) \neq \{0\}$ or $F_u(s) \neq \{0\}$. Then one of the following properties holds.*

(1) $\dim E_u(s) = \dim F_u(s) + 1$, $u \not\perp E_u(s)$, *and* $F_u(s) = E_u(s) \cap u^\perp$.
(2) $\dim F_u(s) = \dim E_u(s) + 1$, $u \not\perp F_u(s)$, *and* $E_u(s) = F_u(s) \cap u^\perp$.

We define

$$\Sigma_H(u) := \{s \geq 0;\ u \not\perp E_u(s)\}\,, \tag{1.9}$$

$$\Sigma_K(u) := \{s \geq 0;\ u \not\perp F_u(s)\}\,, \tag{1.10}$$

Remark that $0 \notin \Sigma_H(u)$, since $u = H_u(1)$ belongs to the range of $H_u$ hence, is orthogonal to its kernel. As a consequence of Lemma 1, $\Sigma_H(u)$ coincides with the set of $s > 0$ with $\dim E_u(s) = \dim F_u(s) + 1$. In contrast, it may happen that 0 belongs to $\Sigma_K(u)$.

1.3. **Multiplicity and Blaschke products.** Assume $u \in VMO_+(\mathbb{T})$ and $s \in \Sigma_H(u)$. Then $H_u$ acts on the finite dimensional vector space $E_u(s)$. It turns out that this action can be completely described by an inner function. A similar fact holds for the action of $K_u$ onto $F_u(s)$ when $s \in \Sigma_K(u)$, $s \neq 0$. In order to state this result, recall that a finite Blaschke product is an inner function of the form

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{k} \chi_{p_j}(z)\ ,\ \psi \in \mathbb{T}\ ,\ p_j \in \mathbb{D}\ ,\ \chi_p(z) := \frac{z-p}{1-\overline{p}z}\ ,\ p \in \mathbb{D}\ .$$

The integer $k$ is called the degree of $\Psi$. Alternatively, $\Psi$ can be written as

$$\Psi(z) = \mathrm{e}^{-i\psi} \frac{P(z)}{z^k \overline{P}\left(\frac{1}{z}\right)}\ ,$$

where $\psi \in \mathbb{T}$ is called the angle of $\Psi$ and $P$ is a monic polynomial of degree $k$ with all its roots in $\mathbb{D}$. Such polynomials are called Schur polynomials. We denote by $\mathcal{B}_k$ the set of Blaschke products of degree $k$. It is a classical result — see *e.g.* [15] or Appendix B — that $\mathcal{B}_k$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2k}$. Finally, we shall denote by

$$D(z) = z^k \overline{P}\left(\frac{1}{z}\right)$$

the normalized denominator of $\Psi$.

**Proposition 1.** *Let $s > 0$ and $u \in VMO_+(\mathbb{T})$.*

(1) *Assume $s \in \Sigma_H(u)$ and $m := \dim E_u(s) = \dim F_u(s) + 1$. Denote by $u_s$ the orthogonal projection of $u$ onto $E_u(s)$. There exists an inner function $\Psi_s \in \mathcal{B}_{m-1}$ such that*

$$s u_s = \Psi_s H_u(u_s)\ ,$$

*and if $D$ denotes the normalized denominator of $\Psi_s$,*

$$E_u(s) = \left\{ \frac{f}{D} H_u(u_s) \ , \ f \in \mathbb{C}_{m-1}[z] \right\} \ , \tag{1.11}$$

$$F_u(s) = \left\{ \frac{g}{D} H_u(u_s) \ , \ g \in \mathbb{C}_{m-2}[z] \right\} , \tag{1.12}$$

*and, for $a = 0, \ldots, m-1$ , $b = 0, \ldots, m-2$,*

$$H_u\left( \frac{z^a}{D} H_u(u_s) \right) = s\mathrm{e}^{-i\psi_s} \frac{z^{m-a-1}}{D} H_u(u_s) \ , \tag{1.13}$$

$$K_u\left( \frac{z^b}{D} H_u(u_s) \right) = s\mathrm{e}^{-i\psi_s} \frac{z^{m-b-2}}{D} H_u(u_s) \ , \tag{1.14}$$

*where $\psi_s$ denotes the angle of $\Psi_s$.*

(2) *Assume $s \in \Sigma_K(u)$ and $\ell := \dim F_u(s) = \dim E_u(s) + 1$. Denote by $u'_s$ the orthogonal projection of $u$ onto $F_u(s)$. There exists an inner function $\Psi_s \in \mathcal{B}_{\ell-1}$ such that*

$$K_u(u'_s) = s\Psi_s u'_s \ ,$$

*and if $D$ denotes the normalized denominator of $\Psi_s$,*

$$F_u(s) = \left\{ \frac{f}{D} u'_s \ , \ f \in \mathbb{C}_{\ell-1}[z] \right\} \ , \tag{1.15}$$

$$E_u(s) = \left\{ \frac{zg}{D} u'_s \ , \ g \in \mathbb{C}_{\ell-2}[z] \right\} , \tag{1.16}$$

*and, for $a = 0, \ldots, \ell-1$ , $b = 0, \ldots, \ell-2$,*

$$K_u\left( \frac{z^a}{D} u'_s \right) = s\mathrm{e}^{-i\psi_s} \frac{z^{\ell-a-1}}{D} u'_s \ , \tag{1.17}$$

$$H_u\left( \frac{z^{b+1}}{D} u'_s \right) = s\mathrm{e}^{-i\psi_s} \frac{z^{\ell-b-1}}{D} u'_s \ , \tag{1.18}$$

*where $\psi_s$ denotes the angle of $\Psi_s$.*

Notice that, if we come back to the case of selfadjoint Hankel operators, which corresponds to symbols $u$ with real Fourier coefficients, the angles $\psi_s$ belong to $\{0, \pi\}$, and condition (2) in the Megretskii–Peller–Treil Theorem is an elementary consequence of Proposition 1. Indeed, the identities in Proposition 1 provide very simple matrices for the action of $H_u$ and $K_u$ on $E_u(s)$ and $F_u(s)$, and one can easily check that the dimensions of the eigenspaces of these matrices associated to the eigenvalues $\pm s$ differ of at most 1.

Part of the content of Proposition 1 was in fact already proved by Adamyan-Arov-Krein in their famous paper [1]. Indeed, translating Theorem 1.2 of this paper, in the special case of finite multiplicity, into our normalization, one gets that, for every pair $(h, f) \in E_u(s) \times E_u(s)$ satisfying

$$H_u(h) = sf \ , \ H_u(f) = sh \ ,$$

there exists a polynomial $Q$ of degree at most $m-1$, where $m = \dim E_u(s)$, and a function $g \in L^2_+$ such that

$$h(z) = Q(z)g(z)\ ,\ \ f(z) = z^{m-1}\overline{Q}\left(\frac{1}{z}\right)g(z)\ .$$

We refer the reader to Appendix C for a self-contained proof of this property. It is easy to check that this property is a consequence of Proposition 1, which in fact says more about the structure of the space $E_u(s)$.

1.4. **Main results.** We now come to the main results of this paper. First let us introduce some additional notation. Given a positive integer $n$, we set

$$\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n\ .$$

Similarly $\Omega_\infty$ is the set of sequences $(s_r)_{r\geq 1}$ such that

$$s_1 > s_2 > \cdots > s_n \to 0\ .$$

We set

$$\Omega := \bigcup_{n=1}^{\infty} \Omega_n \cup \Omega_\infty\ ,\ \ \mathcal{B} = \bigcup_{k=0}^{\infty} \mathcal{B}_k\ ,$$

and

$$\mathcal{S}_n = \Omega_n \times \mathcal{B}^n\ ,\ \ \mathcal{S}_\infty = \Omega_\infty \times \mathcal{B}^\infty\ ,\ \ \ \mathcal{S} := \bigcup_{n=1}^{\infty} \mathcal{S}_n \cup \mathcal{S}_\infty\ .$$

Given $u \in VMO_+(\mathbb{T}) \setminus \{0\}$, one can define, according to proposition 1 and (1.7) combined with the min-max formula, a finite or infinite sequence $s = (s_1 > s_2 > \dots) \in \mathcal{S}$ such that

(1) The $s_{2j-1}$'s are the singular values of $H_u$ in $\Sigma_H(u)$.
(2) The $s_{2k}$'s are the singular values of $K_u$ in $\Sigma_K(u) \setminus \{0\}$.

For every $r \geq 1$, associate to each $s_r$ an inner function $\Psi_r$ by means of Proposition 1. This defines a mapping

$$\Phi : VMO_+(\mathbb{T}) \setminus \{0\} \longrightarrow \mathcal{S}\ .$$

**Theorem 2.** *The map $\Phi$ is bijective.*
*Moreover, we have the following explicit formula for $\Phi^{-1}$ on $\mathcal{S}_n$. If $n = 2q$ is even,*

$$\rho_j := s_{2j-1}\ ,\ \ \sigma_k := s_{2k}\ ,\ \ j,k = 1,\dots,q\ ,$$

*introduce the $q \times q$ matrix $\mathcal{C}(z)$ with coefficients*

$$c_{kj}(z) := \frac{\rho_j - \sigma_k z \Psi_{2k}(z)\Psi_{2j-1}(z)}{\rho_j^2 - \sigma_k^2}\ ,\ \ j,k = 1,\dots,q\ .$$

*Denote by $\Delta_{kj}(z)$ the minor determinant of this matrix corresponding to line $k$ and column $j$. We have*

$$u(z) = \sum_{1\le j,k\le q} (-1)^{j+k}\Psi_{2j-1}(z)\frac{\Delta_{kj}(z)}{\det(\mathcal{C}(z))} \ . \tag{1.19}$$

*If $n = 2q-1$ is odd, the same formula holds by setting $\sigma_q := 0$.*

Let us now come to the topological features of the mapping $\Phi$. We shall not describe the topology on $\mathcal{S}$ transported by $\Phi$ from $VMO_+(\mathbb{T})$, because it is a complicated matter. In the finite rank case, it is simpler to deal with the restrictions of $\Phi$ to the preimages of $\mathcal{S}_n$, which we denote by $\mathcal{U}_n$. We endow $\mathcal{U}_n$ with the topology induced by $VMO_+$, each $\Omega_n$ with the topology induced by $\mathbb{R}^n$, $\mathcal{B}$ with the disjoint sum of topologies of $\mathcal{B}_k$, and $\mathcal{S}_n$ with the product topology. In the infinite rank case, $\Omega_\infty$ is endowed with the topology induced by $c_0$, the Banach space of sequences tending to 0.

**Theorem 3.** *The following restriction maps of $\Phi$,*

$$\Phi_n : \mathcal{U}_n \to \mathcal{S}_n$$

*are homeomorphisms. Moreover, given a positive integer $n$, and a sequence $(d_1, \dots, d_n)$ of nonnegative integers, the map*

$$\Phi^{-1} : \Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r} \longrightarrow VMO_+(\mathbb{T})$$

*is a smooth embedding. Given a sequence $(d_r)_{r\ge1}$ of nonnegative integers, the map*

$$\Phi^{-1} : \Omega_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r} \longrightarrow VMO_+(\mathbb{T})$$

*is a continuous embedding.*

As a consequence of the second statement of Theorem 3, the set

$$\mathcal{V}_{(d_1,\dots,d_n)} := \Phi^{-1}\left(\Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r}\right)$$

is a submanifold of $VMO_+(\mathbb{T})$ of dimension

$$\dim \mathcal{V}_{(d_1,\dots,d_n)} = 2n + 2\sum_{r=1}^{n} d_r \ .$$

Notice that $\mathcal{V}_{(d_1,\dots,d_n)}$ is the set of symbols $u$ such that

(1) The singular values $s$ of $H_u$ in $\Sigma_H(u)$, ordered decreasingly, have respective multiplicities

$$d_1 + 1, d_3 + 1, \dots \ .$$

(2) The singular values $s$ of $K_u$ in $\Sigma_K(u)$, ordered decreasingly, have respective multiplicities

$$d_2+1, d_4+1, \dots \ .$$

In the last section of this paper, we shall investigate the properties of the manifold $\mathcal{V}_{(d_1,\dots,d_n)}$ with respect to the symplectic form.

1.5. **Applications to the cubic Szegő equation.** The cubic Szegő equation has been introduced in [8] as a toy model of Hamiltonian evolution PDEs with lack of dispersion. It can be formally described as the Hamiltonian equation on $L^2_+(\mathbb{T})$ associated to the energy

$$E(u) := \frac{1}{4}\int_{\mathbb{T}} |u|^4\, dx \ ,$$

and to the symplectic form

$$\omega(h_1,h_2) := \mathrm{Im}(h_1|h_2) \ .$$

It reads

$$i\dot{u} = \Pi(|u|^2 u) \ . \tag{1.20}$$

For every $s \geq 0$, we denote by $H^s(\mathbb{T})$ the Sobolev space of regularity $s$ on $\mathbb{T}$, and

$$H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T}) \ .$$

We first recall the wellposedness results from [8]. For every $u_0 \in H^s_+(\mathbb{T})$, $s \geq \frac{1}{2}$, there exists $u \in C(\mathbb{R}, H^s_+(\mathbb{T}))$ unique solution of equation (1.20) with $u(0) = u_0$. Moreover, we proved in [8] that equation (1.20) enjoys a Lax pair structure implying that $H_{u(t)}$ remains unitarily equivalent to $H_{u_0}$, and $K_{u(t)}$ remains unitarily equivalent to $K_{u_0}$. In particular, their singular values are preserved by the evolution. It is therefore natural to understand this evolution through the mapping $\Phi$ introduced in the previous subsection. This question was solved in [9] in the special case of generic states $u$ corresponding to simple singular values of $H_u$ and $K_u$. These states correspond through the map $\Phi$ to Blaschke products $\Psi_r$ of degree 0. In this case, one can write

$$\Psi_r = \mathrm{e}^{-i\psi_r} \ ,$$

and the evolution of the angle $\psi_r$ is given by

$$\frac{d\psi_r}{dt} = (-1)^{r-1} s_r^2 \ .$$

More precisely, using the notation introduced in [10], consider the set $\mathcal{V}(d)$ defined by

$$\mathcal{V}(d) = \left\{ u;\ \mathrm{rk}H_u = \left[\frac{d+1}{2}\right] \ ,\ \mathrm{rk}K_u = \left[\frac{d}{2}\right] \right\} \ . \tag{1.21}$$

One can prove that $\mathcal{V}(d)$ is a Kähler submanifold of $L^2_+(\mathbb{T})$, and that its open subset $\mathcal{V}(d)_{\rm gen}$ made of generic states is diffeomorphic through $\Phi$ to

$$\Omega_d \times \mathcal{B}_0^d \ .$$

The corresponding coordinates $(s_1, \dots, s_d) \in \Omega_d$ and $(\psi_1, \dots, \psi_d) \in \mathbb{T}^d$ are action angle variables on $\mathcal{V}(d)_{\rm gen}$, in the following sense, see [9],

$$\omega_{|\mathcal{V}(d)_{\rm gen}} = \sum_{r=1}^{d} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r \ , \ E = \frac{1}{4}\sum_{r=1}^{d} (-1)^{r-1} s_r^4 \ .$$

Our next result generalizes this fact to non generic states. Given $u \in H^{1/2}_+(\mathbb{T})$, we decompose the Blaschke products associated to $u$ by $\Phi$ as

$$\Psi_r := \mathrm{e}^{-i\psi_r}\chi_r \ ,$$

where $\chi_r$ is a Blaschke product built with a monic Schur polynomial.

**Theorem 4.** *The evolution of equation (1.20) on $H^{1/2}_+$ reads*

$$\frac{ds_r}{dt} = 0 \ , \ \frac{d\psi_r}{dt} = (-1)^{r-1} s_r^2 \ , \ \frac{d\chi_r}{dt} = 0 \ .$$

*Moreover, on the manifold $\mathcal{V}_{(d_1,\dots,d_n)}$ introduced in subsection 2, the restriction of the symplectic form $\omega$ and of the energy $E$ are given by*

$$\omega = \sum_{r=1}^{n} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r \ , \ E = \frac{1}{4}\sum_{r=1}^{n} (-1)^{r-1} s_r^4 \ .$$

*In particular, $\mathcal{V}_{(d_1,\dots,d_n)}$ is an involutive submanifold of the Kähler manifold $\mathcal{V}(d)$ with $d = n + 2\sum_{r=1}^{n} d_r$.*

This theorem shows that the cubic Szegő equation can be solved by using the inverse spectral transform provided by the mapping $\Phi^{-1}$. We refer for instance to the first part of the book [17] for a similar situation in the case of the Korteweg–de Vries equation.

As a corollary of Theorem 4, one gets the following qualitative information about all the trajectories of (1.20).

**Corollary 1.** *Every solution of equation (1.20) with initial data in $H^{1/2}_+(\mathbb{T})$ is an almost periodic function from $\mathbb{R}$ to $H^{1/2}_+(\mathbb{T})$.*

More precisely, we will show that the tori obtained as inverse images by the map $\Phi$ of the sets

$$\{(s_r)\} \times \prod_r \mathbb{S}^1 \Psi_r \ ,$$

where $((s_r);(\Psi_r)) \in \mathcal{S}$ is given, induce a singular foliation of the phase space $VMO_+(\mathbb{T}) \setminus \{0\}$. The cubic Szegő flow acts on those tori which are included in $H^{1/2}_+(\mathbb{T})$. In the generic case where all the $\Psi_r$ have

degree 0, it is easy to check that these tori are classes of unitary equivalence for the pair of operators $(H_u, K_u)$. In the general case, we introduce a more stringent unitary equivalence of which these tori are the classes. This answers a question asked to us by T. Kappeler.

Let us end this introduction by mentioning that a natural generalization of the results of this paper would concern bounded — not necessarily compact — Hankel operators. A first step in this direction was recently made in the paper [13].

1.6. **Organization of the paper.** In section 2, we prove Lemma 1 about the eigenspaces of $H_u^2$ and $K_u^2$. Section 3 is devoted to Proposition 1 which introduces Blaschke products encoding the action of $H_u$ and $K_u$ on these eigenspaces. Section 4 gives the proof of the main theorem 2, as well as the proof of Theorems 3 and 1, in the special case $n < \infty$ of finite rank Hankel operators. Section 5 deals with the case $n = \infty$ of infinite rank compact Hankel operators. Sections 6, 7, 9 contain the proof of Theorem 4, as well as applications to almost periodicity — Corollary 1 — , and to a new proof of the classification of traveling waves for the cubic Szegő equation. Finally, section 8 provides the description of the singular foliation of $VMO_+(\mathbb{T}) \setminus \{0\}$ in terms of equivalent classes for some special unitary equivalence for the pair of operators $(H_u, K_u)$. The paper ends with three appendices. The first one is devoted to a classical set of formulae connected to the relative determinant of two selfadjoint compact operators with a rank one difference. The second one specifies the structure of the set of Blaschke products of a given degree. The third one gives a self-contained proof of two important results from the paper [1] by Adamyan-Arov-Krein, which are used throughout the paper.

## 2. Spectral decomposition of the Hankel operators $H_u$ and $K_u$

We begin with a precise spectral analysis of operators $H_u^2$ and $K_u^2$ on the closed range of $H_u$.

We prove a more precise version of Lemma 1, namely

**Proposition 2.** *Let $u \in VMO_+(\mathbb{T}) \setminus \{0\}$ and $s > 0$ such that*

$$E_u(s) \neq \{0\} \quad \text{or} \quad F_u(s) \neq \{0\} \ .$$

*Then one of the following properties holds.*

(1) $\dim E_u(s) = \dim F_u(s) + 1$, $u \not\perp E_u(s)$, *and* $F_u(s) = E_u(s) \cap u^\perp$.
(2) $\dim F_u(s) = \dim E_u(s) + 1$, $u \not\perp F_u(s)$, *and* $E_u(s) = F_u(s) \cap u^\perp$.

*Moreover, if $u_\rho$ and $u'_\sigma$ denote respectively the orthogonal projections of $u$ onto $E_u(\rho)$, $\rho \in \Sigma_H(u)$, and onto $F_u(\sigma)$, $\sigma \in \Sigma_K(u)$, then*

(1) *$\Sigma_H(u)$ and $\Sigma_K(u)$ are disjoint, with the same cardinality;*

(2) *if* $\rho \in \Sigma_H(u)$,

$$u_\rho = \|u_\rho\|^2 \sum_{\sigma \in \Sigma_K(u)} \frac{u'_\sigma}{\rho^2 - \sigma^2} \ , \tag{2.1}$$

(3) *if* $\sigma \in \Sigma_K(u)$,

$$u'_\sigma = \|u'_\sigma\|^2 \sum_{\rho \in \Sigma_H(u)} \frac{u_\rho}{\rho^2 - \sigma^2} \ . \tag{2.2}$$

(4) *A nonnegative number* $\sigma$ *belongs to* $\Sigma_K(u)$ *if and only if it does not belong to* $\Sigma_H(u)$ *and*

$$\sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2 - \sigma^2} = 1 \ . \tag{2.3}$$

*Proof.* Let $s > 0$ be such that $E_u(s) + F_u(s) \neq \{0\}$. We first claim that either $u \perp E_u(s)$ or $u \perp F_u(s)$. Assume first $E_u(s) \neq \{0\}$ and $u \not\perp E_u(s)$, then there exists $h \in E_u(s)$ such that $(h|u) \neq 0$. From equation (1.7),

$$-(h|u)u = (K_u^2 - s^2 I)h \in (F_u(s))^\perp \ ,$$

hence $u \perp F_u(s)$. Similarly, if $F_u(s) \neq \{0\}$ and $u \not\perp F_u(s)$, then $u \perp E_u(s)$.
Let $s$ be such that $F_u(s) \neq \{0\}$. Assume $u \perp F_u(s)$. Then, for any $h \in F_u(s)$, as $K_u^2 = H_u^2 - (\cdot|u)u$, $H_u^2(h) = K_u^2(h) = s^2 h$, hence $F_u(s) \subset E_u(s)$. We claim that this inclusion is strict. Indeed, suppose it is an equality. Then $H_u$ and $K_u$ are both automorphisms of the vector space

$$N := F_u(s) = E_u(s) \ .$$

Consequently, since $K_u = S^* H_u$, $S^*(N) \subset N$. On the other hand, since every $h \in N$ is orthogonal to $u$, we have

$$0 = (H_u(h)|u) = (1|H_u^2 h) = \sigma^2 (1|h) \ ,$$

hence $N \perp 1$. Therefore, for every $h \in N$, for every integer $k$, $(S^*)^k(h) \perp 1$. Since $S^k(1) = z^k$, we conclude that all the Fourier coefficients of $h$ are 0, hence $N = \{0\}$, a contradiction. Hence, the inclusion of $F_u(s)$ in $E_u(s)$ is strict and, necessarily $u \not\perp E_u(s)$ and $F_u(s) = E_u(s) \cap u^\perp$. One also has $\dim E_u(s) = \dim F_u(s) + 1$.
One proves as well that if $E_u(s) \neq \{0\}$ and $u \perp E_u(s)$ then $u \not\perp F_u(s)$, $E_u(s) = F_u(s) \cap u^\perp$ and $\dim F_u(s) = \dim E_u(s) + 1$. This gives the first part of Proposition (2).

For the second part, we first observe that $u = H_u(1) \in E_u(0)^\perp$, hence $0 \notin \Sigma_H(u)$. From what we just proved, we conclude that $\Sigma_H(u)$ and $\Sigma_K(u)$ are disjoint. Furthermore, by the spectral theory of $H_u^2$ and of $K_u^2$, we have the orthogonal decompositions,

$$L^2_+ = \overline{\oplus_{s \geq 0} E_u(s)} = \overline{\oplus_{s \geq 0} F_u(s)} \ .$$

Writing $u$ according to these two orthogonal decompositions yields

$$u = \sum_{\rho \in \Sigma_H(u)} u_\rho = \sum_{\sigma \in \Sigma_K(u)} u'_\sigma \ .$$

Consequently, the cyclic spaces generated by $u$ under the action of $H^2_u$ and $K^2_u$ are given by

$$\langle u \rangle_{H^2_u} = \overline{\oplus_{\rho \in \Sigma_H(u)} \mathbb{C} u_\rho} \ , \ \langle u \rangle_{K^2_u} = \overline{\oplus_{\sigma \in \Sigma_K(u)} \mathbb{C} u'_\sigma} \ .$$

Since $K^2_u = H^2_u - (\,.\,|u)u$, these cyclic spaces are equal. This proves that $\Sigma_H(u)$ and $\Sigma_K(u)$ have the same — possibly infinite — number of elements.

Let us prove (2.1) and (2.2). Observe that, by the Fredholm alternative, for $\sigma > 0$, $H^2_u - \sigma^2 I$ is an automorphism of $E_u(\sigma)^\perp$. Consequently, if moreover $\sigma \in \Sigma_K(u)$, $u \in E_u(\sigma)^\perp$ and there exists $v \in E_u(\sigma)^\perp$ unique such that

$$(H^2_u - \sigma^2 I)v = u \ .$$

We set $v := (H^2_u - \sigma^2 I)^{-1}(u)$. If $\sigma = 0 \in \sigma_K(u)$, of course $H^2_u$ is no more a Fredholm operator, however there still exists $w \in E_u(0)^\perp$ such that

$$H^2_u(w) = u \ .$$

Indeed, since $K_u = S^* H_u$, $E_u(0) \subset F_u(0)$, and the hypothesis $u \not\perp F_u(0)$ implies that the latter inclusion is strict. This means that there exists $w \in E_u(0)^\perp$ such that $H_u(w) = 1$, whence $H^2_u(w) = u$. Again, we set $w := (H^2_u)^{-1}(u)$ .

For every $\sigma \in \Sigma_K(u)$, the equation

$$K^2_u h = \sigma^2 h$$

is equivalent to

$$(H^2_u - \sigma^2 I)h = (h|u)u \ ,$$

or $h \in \mathbb{C}(H^2_u - \sigma^2 I)^{-1}(u) \oplus E_u(\sigma)$ , with

$$((H^2_u - \sigma^2 I)^{-1}(u)|u) = 1 \ . \tag{2.4}$$

Since $u'_\sigma \in E_u(\sigma)^\perp$, this leads to

$$\frac{u'_\sigma}{\|u'_\sigma\|^2} = (H^2_u - \sigma^2 I)^{-1}(u) \ .$$

In particular, if $\rho \in \Sigma_H(u)$, $\sigma \in \Sigma_K(u)$,

$$\left( \frac{u'_\sigma}{\|u'_\sigma\|^2} \middle| \frac{u_\rho}{\|u_\rho\|^2} \right) = \frac{1}{\rho^2 - \sigma^2} \ .$$

This leads to equations (2.1) and (2.2). Finally, equation (2.3) is nothing but the expression of (2.4) in view of equation (2.1).

This completes the proof of Proposition 2. □

## 3. Multiplicity and Blaschke products. Proof of Proposition 1

In this section, we prove Proposition 1.

3.1. **Case of $\rho \in \Sigma_H(u)$.** Let $u \in VMO_+(\mathbb{T})$. Assume that $\rho \in \Sigma_H(u)$ and $m := \dim E_u(\rho)$. We may assume $\rho = 1$ and write $u_\rho = u_1$,

$$E = E(1) = \ker(H_u^2 - I),\ F = F(1) = \ker(K_u^2 - I)$$

for simplicity. By Proposition 2,

$$F = E \cap u_1^\perp \ .$$

3.1.1. *Definition of $\Psi$.* We claim that, at every point of $\mathbb{T}$,

$$|u_1|^2 = |H_u(u_1)|^2 \ .$$

Indeed, denoting by $S$ the shift operator, for every integer $n \geq 0$,

$$\begin{aligned}(|u_1|^2|z^n) &= (u_1|S^n u_1) = (H_u^2(u_1)|S^n u_1) = (H_u(S^n u_1)|H_u(u_1)) \\ &= ((S^*)^n H_u(u_1)|H_u(u_1)) = (H_u(u_1)|S^n H_u(u_1)) = (|H_u(u_1)|^2|z^n) \ .\end{aligned}$$

Since $|u_1|^2$ and $|H_u(u_1)|^2$ are real valued, this proves the claim.

We thus define

$$\Psi := \frac{u_1}{H_u(u_1)} \ .$$

3.1.2. *The function $\Psi$ is an inner function.* We know that $\Psi$ is of modulus 1 at every point of $\mathbb{T}$. Let us show that $\Psi$ is in fact an inner function. By part (1) of the Adamyan–Arov–Krein theorem in Appendix C, we already know that $\Psi$ is a rational function with no poles on the unit circle. Therefore, it is enough to prove that $\Psi$ has no pole in the open unit disc. Assume that $q \in \overline{\mathbb{D}}$ is a zero of $H_u(u_1)$, and let us show that $q$ is a zero of $u_1$ with at least the same multiplicity.

Denote by $(e_1, \dots, e_m)$ an orthonormal basis of $E$, such that

$$H_u(e_j) = e_j \ ,\ j = 1, \dots, m \ .$$

Such a basis always exists, in view of the antilinearity of $H_u$. Since, for every $f \in F$, $(u|f) = 0$, $(H_u(f)|1) = 0$, hence

$$H_u(f) = SK_u(f) \ . \tag{3.1}$$

Assume $H_u(u_1)(q) = 0$ and consider

$$f := \sum_{j=1}^m e_j(q) e_j \ .$$

Then

$$(f|u_1) = \sum_{j=1}^m e_j(q)(e_j|u_1) = \sum_{j=1}^m H_u(e_j)(q)(e_j|u_1) = H_u(u_1)(q) = 0 \ ,$$

therefore $f$ belongs to $E \cap u_1^\perp = F$ as well as $K_u(f)$. Hence, by (3.1),

$$H_u K_u(f) = SK_u^2(f) = S(f)$$

from which we get

$$K_u(f) = \sum_{j=1}^{m} (K_u(f)|e_j)e_j = \sum_{j=1}^{m} (K_u(f)|H_u(e_j))e_j = \sum_{j=1}^{m} (e_j|Sf)e_j \ ,$$

hence

$$K_u(f)(q) = (f|Sf) \ .$$

Therefore, using again (3.1),

$$\|f\|^2 = H_u(f)(q) = qK_u(f)(q) = q(f|Sf) \ .$$

Since $\|Sf\| = \|f\|$ and $|q| < 1$, we infer $f = 0$, hence

$$e_j(q) = 0 \ , \ j = 1, \dots, m \ ,$$

in particular $u_1(q) = 0$.

Assume now that $q$ is a zero of order $r$ of $H_u(u_1)$, so that, for every $a \leq r-1$,

$$(H_u(u_1))^{(a)}(q) = \sum_{j=1}^{m} (u_1|e_j)e_j^{(a)}(q) = 0 \ .$$

Let us prove by induction that

$$e_j^{(a)}(q) = 0 \ , \ a \leq r-1 \ , \ j = 1, \dots, m \ .$$

Assuming we have this property for $a < r-1$, we consider

$$f := \sum_{j=1}^{m} e_j^{(r-1)}(q)e_j \ .$$

As before, $f$ belongs to $F$ as well as $K_u(f)$ so that, as above

$$(H_u(f))^{(r-1)}(q) = \|f\|^2 \ , \ (K_u(f))^{(r-1)}(q) = (f|Sf) \ .$$

We then derive $r-1$ times identity (3.1) at $z = q$ and use the induction hypothesis. We obtain

$$(H_u(f))^{(r-1)}(q) = q(K_u(f))^{(r-1)}(q) \ ,$$

hence $\|f\|^2 = q(f|Sf)$, and we conclude as before.

3.1.3. *The function $\Psi$ is a Blaschke product of degree $m-1$, $m = \dim E$.* In fact this is a consequence of what we have just done, and of part (1) of the Adamyan–Arov–Krein theorem in Appendix C. However it is useful to give another proof. We start with proving the following lemma.

**Lemma 2.** *Let $f \in \mathbb{H}^\infty(\mathbb{D})$ such that $\Pi(\Psi\overline{f}) = \Psi\overline{f}$ . Then*

$$H_u(fH_u(u_1)) = \Psi\overline{f}H_u(u_1) \ .$$

The proof of the lemma is straightforward,

$$\begin{aligned} H_u(fH_u(u_1)) &= \Pi(u\overline{fH_u(u_1)}) = \Pi(\overline{f}H_u^2(u_1)) = \Pi(\overline{f}u_1) = \Pi(\overline{f}\Psi H_u(u_1)) \\ &= \Psi\overline{f}H_u(u_1)\ . \end{aligned}$$

As a first consequence of the lemma, we observe that, if $\Psi = \Psi_a\Psi_b$, where $\Psi_a, \Psi_b$ are inner functions, then

$$H_u(\Psi_a H_u(u_1)) = \Psi_b H_u(u_1)\ .$$

In particular, $\Psi_a H_u(u_1)$ belongs to $E$, and the number of inner divisors of $\Psi$ is at most equal to the dimension of $E$. Thus $\Psi$ is a Blaschke product of degree at most $m-1$.

We now show that $\Psi \in \mathcal{B}_{m-1}$. Write

$$\Psi(z) = \mathrm{e}^{-i\psi}\frac{z^k\overline{D}\left(\frac{1}{z}\right)}{D(z)}$$

where $D$ is a normalized polynomial of degree $k$. Using again the lemma, we have, for any $0 \le a \le k$,

$$H_u\left(\frac{z^a}{D}H_u(u_1)\right) = \mathrm{e}^{-i\psi}\frac{z^{k-a}}{D}H_u(u_1)\ .$$

Let us set

$$V := \mathrm{span}\left(\frac{z^a}{D}H_u(u_1),\ 0 \le a \le k\right)\ .$$

Notice that

$$V \subset E\ ,\ H_u(V) = V\ .$$

Since $\dim V = k+1$, this imposes $k \le m-1$. In order to prove $k = m-1$, we introduce

$$G := V^\perp \cap E\ .$$

The proof will be complete if we establish that $G = \{0\}$. It is enough to prove that $G \subset 1^\perp$ and that $S^*(G) \subset G$ (see the argument in the proof of Proposition 2).

Since $H_u(V) = V$, then $H_u(G) = G$. On the other hand, as $u_1 = \Psi H_u(u_1) \in V$, $G \subset u_1^\perp \cap E \subset u^\perp$, hence $H_u(G) \subset 1^\perp$. This proves the first fact. Remark also that, since $K_u^2 = H_u^2 - (\cdot|u)u$, one gets that $K_u^2 = H_u^2$ on $G$ and $G \subset F$.

As for the second fact, it is enough to prove that $H_u(G) \subset S(G)$. Let $g \in G$. By (3.1), since $g \in F$, $SK_u(g) = H_u(g)$ so it suffices to prove that $K_u(g)$ belongs to $G$. We set

$$v_a := \frac{z^a}{D}H_u(u_1),\ 0 \le a \le k\ ,$$

and we prove that $(K_u(g)|v_a) = 0$ for $0 \le a \le k$.
For $1 \le l \le k$, we write

$$0 = (H_u(g)|v_l) = (SK_u(g)|v_l) = (K_u(g)|S^*v_l) = (K_u(g)|v_{l-1})\ .$$

For the scalar product with $v_k$ we remark that $v_k$ is a linear combination of the $v_j$'s, $0 \le j \le k-1$, and of $u_1 = \Psi H_u(u_1)$. As $K_u(g) \in F$, $(K_u(g)|u_1) = 0$ hence finally $(K_u(g)|v_k) = 0$ .

This proves that $\Psi$ is of degree $m-1$, that $E = \dfrac{\mathbb{C}_{m-1}[z]}{D} H_u(u_1)$ and that the action of $H_u$ on $E$ is as expected in Equation (1.13). It remains to prove that

$$F = \frac{\mathbb{C}_{m-2}[z]}{D} H_u(u_1)$$

and that the action of $K_u$ is described as in (1.13). We have, for $0 \le b \le m-2$,

$$\begin{aligned} K_u\left(\frac{z^b}{D} H_u(u_1)\right) &= H_u S\left(\frac{z^b}{D} H_u(u_1)\right) = H_u\left(\frac{z^{b+1}}{D} H_u(u_1)\right) \\ &= \mathrm{e}^{-i\psi} \frac{z^{m-2-b}}{D} H_u(u_1) \end{aligned}$$

In particular, it proves that $\frac{\mathbb{C}_{m-2}[z]}{D} H_u(u_1) \subset F$. As the dimension of $F$ is $m-1$ by assumption, we get the equality.

3.2. **Case of $\sigma \in \Sigma_K(u)$.** The second part of the proposition, concerning the case of $\sigma \in \Sigma_K(u)$, can be proved similarly. We just give the main lines of the argument. As before, we assume that $\sigma = 1$ for simplicity and denote by $u_1'$ the function $u_\sigma'$. The first step is to prove that

$$\frac{K_u(u_1')}{u_1'}$$

is an inner function. The same argument as the one used above proved that it has modulus one. To prove that it is an inner function, we argue as before. Namely, using again part (1) of the Adamyan-)Arov–Krein theorem in Appendix C, for $S^*u$ in place of $u$, we prove that if $u_1'$ vanishes at some $q \in \mathbb{D}$, $K_u(u_1')$ also vanishes at $q$ at the same order. We introduce an orthonormal basis $\{f_1, \dots, f_\ell\}$ of $F := F_u(1)$ such that

$$K_u(f_j) = f_j, \ j = 1, \dots, m.$$

Assume $u_1'(q) = 0$ and consider

$$e := \sum_{k=1}^{\ell} \overline{f_k(q)} f_k \ .$$

Let us prove that $e = 0$. Observe first that $e$ belongs to $E := E_u(1)$ since

$$(u_1'|e) = \sum_{k=1}^{\ell} f_k(q)(u_1'|f_k) = \sum_{k=1}^{\ell} K_u^2(f_k)(q)(u_1'|f_k) = K_u^2(u_1')(q) = u_1'(q) = 0 \, .$$

We infer that $H_u(e) \in E$ as well, and therefore

$$(e|1) = (H_u(H_u(e))|1) = (u|H_u(e)) = 0 \ ,$$

which implies

$$e = SS^*e = SS^*H_u^2(e) = SK_uH_u(e) \ .$$

Consequently,

$$\begin{aligned} \|e\|^2 &= e(q) = qK_uH_u(e)(q) = q\sum_{k=1}^{\ell}(K_uH_u(e)|f_k)f_k(q) \\ &= q\sum_{k=1}^{\ell}(f_k|H_u(e))f_k(q) = q(K_u(e)|H_u(e)) = q(H_u(e)|SH_u(e)) \ . \end{aligned}$$

Since $\|H_u(e)\| = \|e\|$, we conclude as before that $SH_u(e) = qH_u(e)$ and finally $H_u(e) = 0 = e$. One proves as well that if $q$ is a zero of order $r$ of $(u_1')$, each $f_k$, $1 \leq k \leq \ell$, vanishes at $q$ with the same order.

We now come to the third part of the proof to get

$$\Psi := \frac{K_u(u_1')}{u_1'} \in \mathcal{B}_{\ell-1}.$$

We start with a lemma analogous to Lemma 2.

**Lemma 3.** *Let $f \in \mathbb{H}^\infty(\mathbb{D})$ such that $\Pi(\Psi\overline{f}) = \Psi\overline{f}$. Then*

$$K_u(fu_1') = \Psi\overline{f}u_1' \ .$$

The proof of the lemma is similar to the one of Lemma 2. In particular, for every inner divisor $\Psi_a$ of $\Psi$, $\Psi_a u_1'$ belongs to $F$, and therefore the number of inner divisors of $\Psi$ is at most the dimension $\ell$ of $F$. In order to prove the equality, write

$$\Psi = \mathrm{e}^{-i\psi}\frac{z^k\overline{D}\left(\frac{1}{z}\right)}{D(z)},$$

where $D$ is some normalized polynomial of degree $k$. From the above lemma, for $0 \leq a \leq k$,

$$K_u\left(\frac{z^a}{D}u_1'\right) = \mathrm{e}^{-i\psi}\frac{z^{k-a}}{D}u_1' \ .$$

Let us set

$$W := \mathrm{span}\left(\frac{z^a}{D}u_1' \ , 0 \leq a \leq k\right) \ ,$$

so that

$$W \subset F \ , \ K_u(W) = W \ .$$

To prove $k = \ell - 1$, we introduce as before

$$H := W^\perp \cap F$$

and we prove that $H = \{0\}$ by proving that $H_u(H) \subset 1^\perp$ and that $S^*(H_u(H)) \subset H_u(H)$. It would imply $H_u(H) = \{0\}$ hence $H = \{0\}$ since $H$ is a subset of the range of $H_u$ by assumption.

First, remark that $H \subset u^\perp$ since $H \subset {u'_1}^\perp$ as $u'_1 \in W$, hence $H_u(H) \subset 1^\perp$.

For the second fact, take $h \in H$ and write $S^* H_u(h) = K_u(h) = H_u(S(h))$ so, it suffices to prove that $S(h)$ belongs to $H$. Let us first prove that $S(h)$ belongs to $E$. By (3.1), since $K_u(h)$ belongs to $H$, one has

$$H_u^2(Sh) = H_u(K_u(h)) = SK_u^2(h) = Sh \ .$$

It remains to prove that $Sh \in W^\perp$.

Let $w_j := \dfrac{z^j}{D} u'_1$, $0 \leq j \leq k$. For $1 \leq j \leq k$, we have

$$(Sh|w_j) = (h|S^* w_j) = (h|w_{j-1}) = 0.$$

It remains to prove that $(Sh|w_0) = 0$. It is easy to check that $w_0$ is a linear combination of the $w_j$'s, $1 \leq j \leq k$ and of $u'_1$. As $S(h)$ belongs to $H$, $(S(h)|u'_1) = 0$ hence $(h|w_0) = 0$.

In order to complete the proof, we just need to describe $E$ as the subspace of $F$ made with functions which vanish at $z = 0$, or equivalently are orthogonal to 1. We already know that vectors of $E$ are orthogonal to $u$, and that $H_u$ is a bijection from $E$ onto $E$. We infer that vectors of $E$ are orthogonal to 1. A dimension argument allows to conclude.

## 4. The inverse spectral theorem in the finite rank case

In this section, we prove Theorem 2, Theorem 3, and Theorem 1 in the case of finite rank Hankel operators. Let $u$ be such that $H_u$ has finite rank. Then the sets $\Sigma_H(u)$ and $\Sigma_K(u)$ are finite. We set

$$q := |\Sigma_H(u)| = |\Sigma_K(u)| \ .$$

If

$$\Sigma_H(u) := \{\rho_j, j = 1, \dots, q\}, \ \rho_1 > \cdots > \rho_q > 0 \ ,$$
$$\Sigma_K(u) := \{\sigma_j, j = 1, \dots, q\}, \ \sigma_1 > \cdots > \sigma_q \geq 0 \ ,$$

we know from (2.3) that

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_q > \sigma_q \geq 0 \ . \tag{4.1}$$

We set $n := 2q$ if $\sigma_q > 0$ and $n := 2q - 1$ if $\sigma_q = 0$. For $2j \leq n$, we set

$$s_{2j-1} := \rho_j \ , \ s_{2j} := \sigma_j \ ,$$

so that the positive elements in the list (4.1) read

$$s_1 > s_2 > \cdots > s_n > 0 \ . \tag{4.2}$$

Recall that we denote by $\mathcal{U}_n$ the set of symbols $u$ such the number of non zero elements of $\Sigma_H(u) \cup \Sigma_K(u)$ is exactly $n$, and that $\Omega_n$ denotes

the open subset of $\mathbb{R}^n$ defined by inequalities (4.2). Using Proposition 1, we define $n$ finite Blaschke products $\Psi_1, \dots, \Psi_n$ by

$$\rho_j u_j = \Psi_{2j-1} H_u(u_j) \ , \ K_u(u'_j) = \sigma_j \Psi_{2j} u'_j \ , \ 2j \leq n \ ,$$

where $u_j$ denotes the orthogonal projection of $u$ onto $E_u(\rho_j)$, and $u'_j$ denotes the orthogonal projection of $u$ onto $F_u(\sigma_j)$. Our goal in this section is to prove the following statement.

**Theorem 5.** *The mapping*

$$\begin{array}{rcl} \Phi_n : \mathcal{U}_n & \longrightarrow & \mathcal{S}_n = \Omega_n \times \mathcal{B}^n \\ u & \longmapsto & ((s_r)_{1\leq r\leq n}, (\Psi_r)_{1\leq r\leq n}) \end{array}$$

*is a homeomorphism.*

*Proof.* The proof of Theorem 5 involves several steps. Firstly, we prove the continuity of $\Phi_n$, and we prove that, for $r = 1, \dots, n$, the degree of $\Psi_r$ is locally constant. We then consider, for each $n$-uple $(d_1, \dots, d_n)$ of nonnegative integers, the open set of $\mathcal{U}_n$

$$\mathcal{V}_{(d_1,\dots,d_n)} := \Phi_n^{-1}(\Omega_n \times \mathcal{B}_{d_1} \times \cdots \times \mathcal{B}_{d_n}) \ ,$$

and we just have to prove that $\Phi_n$ is a homeomorphism from $\mathcal{V}_{(d_1,\dots,d_n)}$ onto $\Omega_n \times \mathcal{B}_{d_1} \times \cdots \times \mathcal{B}_{d_n}$.

We first prove this fact in the case $n$ even, along the following lines :

- $\Phi_n$ is injective, with an explicit formula for its left inverse.
- $\Phi_n$ is an open mapping.
- $\Phi_n$ is a proper mapping.
- $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty.

Since the target space $\Omega_n \times \mathcal{B}_{d_1} \times \cdots \times \mathcal{B}_{d_n}$ is connected, these four items trivially lead to the result. The fourth item is proved by an induction argument on $\sum_r d_r$.

Finally, the case $n$ odd is deduced from a simple limiting argument.

As a complementary information, we prove that $\Phi_n^{-1}$ is a smooth embedding of the manifold $\Omega_n \times \mathcal{B}_{d_1} \times \cdots \times \mathcal{B}_{d_n}$, which implies that $\mathcal{V}_{(d_1,\dots,d_n)}$ is a manifold.

4.1. **Continuity of $\Phi_n$.** In this part, we prove that $\Phi_n$ is continuous from $\mathcal{U}_n$ into $\mathcal{S}_n$. Fix $u_0 \in \mathcal{U}_n$. We prove that, in a neighborhood $V_0$ of $u_0$ in $\mathcal{U}_n$, the degrees of the $\Psi_r$'s are constant.

Let $\rho \in \Sigma_H(u_0)$. The orthogonal projector $P_\rho$ on the eigenspace $E_{u_0}(\rho)$ is given by

$$P_\rho = \int_{\mathcal{C}_\rho} (zI - H_{u_0}^2)^{-1} \frac{dz}{2i\pi}$$

where $\mathcal{C}_\rho$ is a circle, centered at $\rho^2$ whose radius is small enough so that the closed disc $\overline{D}_\rho$ delimited by $\mathcal{C}_\rho$ is at positive distance to the rest of

the spectrum of $H^2_{u_0}$. For $u$ in a neighborhood $V_0$ of $u_0$ in $VMO_+$, $C_\rho$ does not meet the spectrum of $H^2_u$, and one may consider

$$P^{(u)}_\rho := \int_{\mathcal{C}_\rho} (zI - H^2_u)^{-1} \frac{dz}{2i\pi}$$

which is a finite rank orthogonal projector smoothly dependent on $u$. Hence, $P^{(u)}_\rho(u)$ is well defined and smooth. Since this vector is not zero for $u = u_0$, it is still not zero for every $u$ in $V_0$. This implies in particular that $\Sigma_H(u)$ meets the open disc $D_\rho$.

We can do the same construction with any $\sigma \in \Sigma_K(u_0)\backslash\{0\}$. We have therefore constructed $n$ smooth functions $u \in V_0 \mapsto P^{(u)}_r$ , $r = 1, \dots, n$, valued in the finite orthogonal projectors, and satisfying

$$P^{(u)}_r(u) \neq 0 \ , \ r = 1, \dots, n \ .$$

Moreover, by continuity,

$$\mathrm{rk} P^{(u)}_r = \mathrm{rk} P^{(u_0)}_r := d_r + 1 \ .$$

If we assume moreover that $u \in \mathcal{U}_n$, we conclude that $\Sigma_H(u)$ has exactly one element in each $D_{s_{2j-1}}$, and that $\Sigma_K(u)$ has exactly one element in each $D_{s_{2k}}$, and that the dimensions of the corresponding eigenspaces are independent of $u$, hence equal to $d_r+1$. In other words, the degrees of the corresponding Blaschke products are $d_r$. In other words, $V_0 \cap \mathcal{U}_n$ is contained into $\mathcal{V}_{(d_1,\dots,d_n)}$.

Since, for every $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, we have

$$\begin{aligned} |\Sigma_H(u)| &= \sum_{2j-1\leq n} (d_{2j-1}+1) + \sum_{2k\leq n} d_{2k} \ , \\ |\Sigma_K(u)\setminus\{0\}| &= \sum_{2j-1\leq n} d_{2j-1} + \sum_{2k\leq n} (d_{2k}+1) \ , \end{aligned}$$

we conclude that

$$\mathrm{rk} H_u = \left[\frac{d+1}{2}\right] \ , \ \mathrm{rk} K_u = \left[\frac{d}{2}\right] \ , \ d := 2\sum_{r=1}^n d_r + n \ ,$$

namely that $u \in \mathcal{V}(d)$, with the notation of [10], [11]. Recall that $\mathcal{V}(d)$ is a complex manifold of dimension $d$. We then define a map $\tilde{\Phi}_n$ on $V_0 \cap \mathcal{V}(d)$ by setting

$$\tilde{\Phi}_n(u) = ((s_r(u))_{1\leq r\leq n}; (\Psi_r(u))_{1\leq r\leq n}) \ ,$$

with

$$\begin{aligned} s_{2j-1}(u) &:= \frac{\|H_u(P^{(u)}_{2j-1}(u))\|}{\|P^{(u)}_{2j-1}(u)\|} \quad , \quad s_{2k}(u) := \frac{\|K_u(P^{(u)}_{2k}(u))\|}{\|P^{(u)}_{2k}(u)\|} \ , \\ \Psi_{2j-1}(u) &:= \frac{s_{2j-1}(u)P^{(u)}_{2j-1}(u)}{H_u(P^{(u)}_{2j-1}(u))} \quad , \quad \Psi_{2k}(u) = \frac{K_u(P^{(u))}_{2k}(u)}{s_{2k}(u)P^{(u)}_{2k}(u)} \ . \end{aligned}$$

The mapping $\tilde{\Phi}_n$ is smooth from $\mathcal{V}(d)$ into $\Omega_n \times \mathcal{R}_d^n$, where $\mathcal{R}_d$ denotes the manifold of rational functions with numerators and denominators of degree at most $\left[\frac{d+1}{2}\right]$. Moreover, the restriction of $\tilde{\Phi}_n$ to $V_0 \cap \mathcal{V}_{(d_1,\dots,d_n)}$ coincides with $\Phi_n$. This proves in particular that $\Phi_n$ is continuous. For future reference, let us state more precisely what we have proved.

**Lemma 4.** *For every $u_0 \in \mathcal{V}_{(d_1,\dots,d_n)}$, there exists a neighborhood $V$ of $u_0$ in $\mathcal{V}(d)$, $d = n + 2\sum_{r=1}^n d_r$ , and a smooth mapping $\tilde{\Phi}_n$ from this neighborhood into some manifold, such that the restriction of $\tilde{\Phi}_n$ to $V \cap \mathcal{V}_{(d_1,\dots,d_n)}$ coincides with $\Phi_n$.*

4.2. **The explicit formula, case $n$ even.** Assume that $n = 2q$ is an even integer.
The fact that the mapping $\Phi_n$ is one-to-one follows from an explicit formula giving $u$ in terms of $\Phi_n(u)$, which we establish in this subsection.

We use the expected description of elements of $\Phi^{-1}(\mathcal{S}_n)$ suggested by the action of $H_u, K_u$ onto the orthogonal projections $u_j, u'_k$ of $u$ onto the corresponding eigenspaces of $H_u^2, K_u^2$ respectively, namely

$$\rho_j u_j = \Psi_{2j-1} H_u(u_j) \ , \ K_u(u'_k) = \sigma_k \Psi_{2k} u'_k \ , \ j,k = 1,\dots,q \ , \tag{4.3}$$

where the $\Psi_r$'s are Blaschke products.

We then define $\tau_j, \kappa_k > 0$ by

$$\prod_{j=1}^q \frac{1 - x\sigma_j^2}{1 - x\rho_j^2} = 1 + x \sum_{j=1}^q \frac{\tau_j^2}{1 - x\rho_j^2} \tag{4.4}$$

$$\prod_{j=1}^q \frac{1 - x\rho_j^2}{1 - x\sigma_j^2} = 1 - x \left( \sum_{j=1}^q \frac{\kappa_j^2}{1 - x\sigma_j^2} \right) \tag{4.5}$$

From Appendix A, we have

$$\|u_j\|^2 = \tau_j^2 \ , \ \|u'_k\|^2 = \kappa_k^2 \ , \ j,k = 1,\dots,q \ .$$

Applying the operator $S$ of multiplication by $z$ to the second set of equations in (4.3), and using $SS^* = I - (\,.\,|1)$ , we get

$$H_u(u'_k)(z) = \sigma_k z \Psi_{2k}(z) u'_k(z) + \kappa_k^2 \ .$$

We use the identities (2.1), (2.2) in this setting

$$u_j = \tau_j^2 \sum_{k=1}^q \frac{1}{\rho_j^2 - \sigma_k^2} u'_k \ , \tag{4.6}$$

$$u'_k = \kappa_k^2 \sum_{j=1}^q \frac{1}{\rho_j^2 - \sigma_k^2} u_j \ , \tag{4.7}$$

and we introduce the new unknowns $h_1, \dots, h_q$ defined by

$$u_j = \Psi_{2j-1} h_j \ , \ \text{or} \ h_j := \frac{1}{\rho_j} H_u(u_j) \ .$$

For the vector valued function

$$\mathcal{H}(z) := (h_j(z))_{1 \le j \le q} \ ,$$

we finally obtain the following linear system,

$$\mathcal{H}(z) = \mathcal{F}(z) + \mathcal{A}(z)\mathcal{H}(z) \ , \tag{4.8}$$

where, thanks to equation (A.11)

$$\begin{aligned} \mathcal{F}(z) &:= \left( \frac{\tau_j^2}{\rho_j} \sum_{k=1}^{q} \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \right)_{1 \le j \le q} = \left( \frac{\tau_j^2}{\rho_j} \right)_{1 \le j \le q} \ , \\ \mathcal{A}(z) &:= \left( \frac{\tau_j^2}{\rho_j} \sum_{k=1}^{q} \frac{\kappa_k^2 \sigma_k z \Psi_{2k}(z) \Psi_{2\ell-1}(z)}{(\rho_j^2 - \sigma_k^2)(\rho_\ell^2 - \sigma_k^2)} \right)_{1 \le j, \ell \le q} \ . \end{aligned}$$

Notice that the matrix $\mathcal{A}(z)$ depends holomorphically on $z \in \mathbb{D}$ and satisfies $\mathcal{A}(0) = 0$. Hence $I - \mathcal{A}(z)$ is invertible at least for $z$ in a neighborhood of 0, which characterizes $\mathcal{H}(z)$, hence characterizes

$$u(z) = \sum_{j=1}^{q} \Psi_{2j-1}(z) h_j(z) \ .$$

This is enough for proving the injectivity of $\Phi_n$. However, we are going to transform the expression of $\mathcal{H}(z)$ into a simpler one, which will be very useful in the sequel.

Introduce the matrix $\mathcal{B} = (b_{jk})_{1 \le j,k \le q}$ defined by

$$b_{jk} := \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \ .$$

From the identities (A.13) and (A.12) in Appendix A, we know that $\mathcal{B}$ is invertible, with

$$\mathcal{B}^{-1} = \left( \frac{\tau_j^2}{\rho_j^2 - \sigma_k^2} \right)_{1 \le k, j \le q} \ .$$

In view of these identities, we observe that

$$I - \mathcal{A}(z) = \operatorname{diag}\left( \frac{\tau_j^2}{\rho_j} \right) \mathcal{B}\mathcal{C}(z) \ ,$$

where $\mathcal{C}(z) = (c_{k\ell}(z))_{1 \le k, \ell \le q}$ is defined by

$$c_{k\ell}(z) := \frac{\rho_\ell - \sigma_k z \Psi_{2k}(z) \Psi_{2\ell-1}(z)}{\rho_\ell^2 - \sigma_k^2} \ . \tag{4.9}$$

Consequently, Equation (4.8) above reads

$$\operatorname{diag}\left(\frac{\tau_j^2}{\rho_j}\right)\mathcal{BC}(z)\mathcal{H}(z) = \mathcal{F}(z) = \operatorname{diag}\left(\frac{\tau_j^2}{\rho_j}\right)\mathcal{B}(\mathbf{1}) \ ,$$

where

$$\mathbf{1} := \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \ .$$

Notice that we again used (A.11) under the form $\mathcal{B}(\mathbf{1}) = \mathbf{1}$ . Finally, equation (4.8) is equivalent to

$$\mathcal{C}(z)\mathcal{H}(z) = \mathbf{1} \ . \tag{4.10}$$

Using the Cramer formulae, we get

$$h_j(z) = \frac{\sum_{k=1}^{q}(-1)^{j+k}\Delta_{kj}(z)}{\det(\mathcal{C}(z))} \ ,$$

where $\Delta_{kj}(z)$ is the minor determinant of $\mathcal{C}(z)$ corresponding to line $k$ and column $j$. This provides formula (1.19) of Theorem 2.

For future reference, we shall rewrite the above formula in a slightly different manner. Recall that

$$\Psi_r(z) = \mathrm{e}^{-i\psi_r}\frac{P_r(z)}{D_r(z)} \ , \ D_r(z) := z^{d_r}\overline{P}_r\left(\frac{1}{z}\right) \ , \tag{4.11}$$

where $P_r$ is a monic polynomial of degree $d_r$. Introduce the matrix $\mathcal{C}^{\#}(z) = (c_{k\ell}^{\#}(z))_{1\le k,\ell\le q}$ as

$$c_{k\ell}^{\#}(z) = \frac{\rho_\ell D_{2k}(z)D_{2\ell-1}(z) - \sigma_k z\mathrm{e}^{-i(\psi_{2k}+\psi_{2\ell-1})}P_{2k}(z)P_{2\ell-1}(z)}{\rho_\ell^2 - \sigma_k^2} \ , \tag{4.12}$$

denote by $Q(z)$ its determinant and by $\Delta_{k\ell}^{\#}(z)$ the corresponding minor determinant. Then

$$h_j(z) = D_{2j-1}(z)R_{2j-1}(z) \ , \tag{4.13}$$

with

$$R_{2j-1}(z) := \frac{\sum_{k=1}^{q}(-1)^{k+j}D_{2k}(z)\Delta_{kj}^{\#}(z)}{Q(z)} \ . \tag{4.14}$$

Notice that $Q$ is a polynomial of degree at most

$$N := q + \sum_{r=1}^{n} d_r \ ,$$

and the numerator of $R_{2j-1}$ is a polynomial of degree at most $N-1-d_{2j-1}$. Consequently,

$$u(z) = \sum_{j=1}^{q} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}(z) R_{2j-1}(z) \ ,$$

is a rational function with denominator $Q$ and with a numerator of degree at most $N-1$. Since the rank of $H_u$ is exactly $N$, we infer that the degree of $Q$ is exactly $N$, and that $Q$ has no zero in the closed unit disc. Indeed, otherwise the numerator of $u$ would have the same zero in order to preserve the analyticity, and, by simplification, $u$ could be written as a quotient of polynomials of degrees smaller than $N-1$ and $N$ respectively, so that the rank of $H_u$ would be smaller.

We close this section by giving similar formulae for $u'_k, k = 1, \dots, q$. The main ingredient is the following algebraic lemma.

**Lemma 5.** *For every $z \in \overline{\mathbb{D}}$,*

$$^t\mathcal{C}(z)\ ^t\mathcal{B}\ \mathrm{diag}(\Psi_{2\ell-1}(z))_{1\le\ell\le q} = \mathrm{diag}(\Psi_{2j-1}(z))_{1\le j\le q}\ \mathcal{B}\ \mathcal{C}(z) \ .$$

The proof of this lemma is straightforward, using identity (A.13). As a consequence of this lemma and of the identities (4.7), (4.10), we infer that $\mathcal{U}'(z) := (u'_k(z))_{1\le k\le q}$ satisfies

$$^t\mathcal{C}(z)\mathcal{U}'(z) = (\Psi_{2j-1}(z))_{1\le j\le q} \ . \tag{4.15}$$

Using Cramer's formulae, we infer

$$u'_k(z) = \frac{\sum_{j=1}^{q}(-1)^{j+k}\Delta_{kj}(z)\Psi_{2j-1}(z)}{\det(\mathcal{C}(z))} = D_{2k}(z)R_{2k}(z) \ , \tag{4.16}$$

where

$$R_{2k}(z) = \frac{\sum_{j=1}^{q}(-1)^{k+j}\mathrm{e}^{-i\psi_{2j-1}}P_{2j-1}(z)\Delta^{\#}_{kj}(z)}{Q(z)} \ . \tag{4.17}$$

Notice that the numerator of $R_{2k}$ has degree at most $N-1-d_{2k}$. Moreover, (4.7) now reads

$$D_{2k}R_{2k} = \sum_{j=1}^{q} \frac{\kappa_k^2}{\rho_j^2-\sigma_k^2}\mathrm{e}^{-i\psi_{2j-1}}P_{2j-1}R_{2j-1} \ . \tag{4.18}$$

4.3. **Surjectivity in the case $n$ even.** Our purpose is now to prove that the mapping $\Phi_n$ is onto. Since we got a candidate from the formula giving $u$ in the latter section, it may seem natural to try to check that this formula indeed provides an element $u$ of $\mathcal{V}_{(d_1,\dots,d_n)}$ with the required $\Phi_n(u)$. However, in view of the complexity of the formulae (4.13), (4.16), it seems difficult to infer from them the spectral properties of $H_u$ and $K_u$. We shall therefore use an indirect method, by proving that the mapping $\Phi_n$ on $\mathcal{V}_{(d_1,\dots,d_n)}$ is open, closed, and that the source space $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty. Since the target space $\Omega_n \times \prod_{j=1}^{n}\mathcal{B}_{d_j}$ is clearly

connected, this will imply the surjectivity. A first step in proving that $\Phi_n$ is an open mapping, consists in the construction of an anti-linear operator $H$ satisfying the required spectral properties, and which will be finally identified as $H_u$.

4.3.1. *Construction of the operator $H$.* Let

$$\mathcal{P} = ((s_r)_{1\leq r\leq n}, (\Psi_r)_{1\leq r\leq n})$$

be an arbitrary element of

$$\mathcal{P} \in \Omega_n \times \prod_{j=1}^{n} \mathcal{B}_{d_j} \text{ for some non negative integers } d_r \ .$$

We look for $u \in \mathcal{V}_{(d_1,\ldots,d_n)}$, $\Phi_n(u) = \mathcal{P}$. We set $\rho_j := s_{2j-1}$ , $\sigma_k := s_{2k}$ , $1 \leq j,k \leq q$ .

Firstly, we define matrices $\mathcal{C}(z)$ and $\mathcal{C}^{\#}(z)$ using formulae (4.9), (4.11), (4.12). We assume moreover the following open properties,

$$\det \mathcal{C}(z) \neq 0 \ , \ z \in \overline{\mathbb{D}} \ , \ \deg(Q) = N := q + \sum_{r=1}^{n} d_r \ . \tag{4.19}$$

We then define $R_r(z), r = 1,\ldots,n$ by formulae (4.14) and (4.17). Setting $\mathcal{H}(z) := (D_{2j-1}(z)R_{2j-1}(z))_{1\leq j\leq q}$ and $\mathcal{U}'(z) := (D_{2k}(z)R_{2k}(z))_{1\leq k\leq q}$, this is equivalent to equations (4.10) and (4.15). Moreover, by Lemma 5, one checks that the column

$$\mathcal{U}''(z) := \left( \sum_{j=1}^{q} \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}(z) R_{2j-1}(z) \right)_{1\leq k\leq q}$$

satisfies

$${}^t\mathcal{C}(z)\mathcal{U}''(z) = (\Psi_{2j-1}(z))_{1\leq j\leq q} \ ,$$

and therefore $\mathcal{U}'' = \mathcal{U}'$, which is (4.18).

We are going to define an antilinear operator on $W = \frac{\mathbb{C}_{N-1}[z]}{Q(z)}$. For this, we define the following vectors of $W$,

$$\begin{aligned} e_{2j-1,a}(z) &:= z^a R_{2j-1}(z), \ 0 \leq a \leq d_{2j-1} \ , \\ e_{2k,b}(z) &:= z^b R_{2k}(z) \ , \ 1 \leq b \leq d_{2k} \ , \end{aligned}$$

for $1 \leq j,k \leq q$. We need a second open assumption.

$$\mathcal{E} := \left( (e_{2j-1,a})_{0\leq a\leq d_{2j-1}} \ , \ (e_{2k,b})_{1\leq b\leq d_{2k}} \right)_{1\leq j,k\leq q} \text{ is a basis of } W. \tag{4.20}$$

We define an antilinear operator $H$ on $W$ by

$$\begin{aligned} H(e_{2j-1,a}) &:= \rho_j \mathrm{e}^{-i\psi_{2j-1}} e_{2j-1,d_{2j-1}-a} \ , \ 0 \leq a \leq d_{2j-1} \ , \\ H(e_{2k,b}) &:= \sigma_k \mathrm{e}^{-i\psi_{2k}} e_{2k,d_{2k}+1-b}, \ 1 \leq b \leq d_{2k} \ , \end{aligned}$$

for $1 \leq j,k \leq q$.

From this definition, $H$ satisfies

$$H(AR_{2j-1}) = \rho_j \mathrm{e}^{-i\psi_{2j-1}} z^{d_{2j-1}} \overline{A}\left(\frac{1}{z}\right) \tag{4.21}$$

$$H(zBR_{2k}) = \sigma_k \mathrm{e}^{-i\psi_{2k}} z^{d_{2k}} \overline{B}\left(\frac{1}{z}\right) . \tag{4.22}$$

for any $A \in \mathbb{C}_{d_{2j-1}}[z]$ and any $B \in \mathbb{C}_{d_{2k}-1}[z]$.

4.3.2. *Identifying $H$ and $H_u$.* Notice that $W$ is invariant by $S^*$. The key of the proof of $H = H_u$ is the following lemma.

**Lemma 6.**

$$S^*HS^* = H - (1|\cdot)u \text{ on } W , \tag{4.23}$$

*where*

$$u := \sum_{j=1}^{q} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}R_{2j-1} .$$

*Proof.* We check the above identity on all the elements of the basis $\mathcal{E}$ of $W$. The only non trivial cases correspond to $e_{2j-1,0}$ and $e_{2k,1}$. In other words, we have to prove

$$S^*HS^*(R_{2j-1}) = H(R_{2j-1}) - (1|R_{2j-1})u , \tag{4.24}$$

$$S^*H(R_{2k}) = H(SR_{2k}) , \tag{4.25}$$

for $1 \le j, k \le q$ . We start with (4.25). We set

$$D_{2k}(z) = 1 + zF_{2k}(z) ,$$

so that, from (4.18),

$$R_{2k} = \sum_{j=1}^{q} \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}R_{2j-1} - SF_{2k}R_{2k} .$$

Using (4.21), we infer

$$H(R_{2k}) = \sum_{j=1}^{q} \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \rho_j D_{2j-1}R_{2j-1} - \sigma_k \mathrm{e}^{-i\psi_{2k}} G_{2k}R_{2k} , \tag{4.26}$$

where

$$G_{2k}(z) = z^{d_{2k}} \overline{F}_{2k}\left(\frac{1}{z}\right) = z(P_{2k}(z) - z^{d_{2k}}) .$$

In view of equation (4.10),

$$\sum_{j=1}^{q} \frac{\rho_j}{\rho_j^2 - \sigma_k^2} D_{2j-1}(z)R_{2j-1}(z) = 1 + z \sum_{j=1}^{q} \frac{\sigma_k \Psi_{2k}(z)}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}(z)R_{2j-1}(z) .$$

Multiplying by $\kappa_k^2$ and applying $S^*$ to both sides, we obtain

$$\sum_{j=1}^{q} \frac{\kappa_k^2 \rho_j}{\rho_j^2 - \sigma_k^2} S^*(D_{2j-1}R_{2j-1}) = \sum_{j=1}^{q} \frac{\kappa_k^2 \sigma_k \Psi_{2k}}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}R_{2j-1} .$$

Using (4.18), we conclude

$$\sum_{j=1}^{q} \frac{\kappa_k^2 \rho_j}{\rho_j^2 - \sigma_k^2} S^*(D_{2j-1} R_{2j-1}) = \sigma_k \mathrm{e}^{-i\psi_{2k}} P_{2k} R_{2k} \ . \tag{4.27}$$

Coming back to (4.26), we conclude

$$S^* H(R_{2k})(z) = \sigma_k \mathrm{e}^{-i\psi_{2k}} z^{d_{2k}} R_{2k}(z) = H(S R_{2k})(z) \ .$$

This proves (4.25).

Let us establish (4.24). Let us set $D_{2j-1} = 1 + S F_{2j-1}$, so that

$$\begin{aligned} S^* H S^*(R_{2j-1}) - H(R_{2j-1}) \ &= \\ S^* H S^*(D_{2j-1} R_{2j-1}) \ &- \ \rho_j \mathrm{e}^{-i\psi_{2j-1}} (z^{d_{2j-1}} R_{2j-1} + S^* G_{2j-1} R_{2j-1}) \ , \end{aligned}$$

with

$$G_{2j-1}(z) = z^{d_{2j-1}} \overline{F}_{2j-1}\left(\frac{1}{z}\right) = z(P_{2j-1}(z) - z^{d_{2j-1}}) \ .$$

This yields

$$S^* H S^*(R_{2j-1}) - H(R_{2j-1}) = S^* H S^*(D_{2j-1} R_{2j-1}) - \rho_j \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1} \ .$$

Using (4.27), we obtain

$$\begin{aligned} T_k \ &:= \ \sum_{j=1}^{q} \frac{\kappa_k^2 \rho_j}{\rho_j^2 - \sigma_k^2} \left( S^* H S^*(R_{2j-1}) - H(R_{2j-1}) \right) \\ &= \ \sigma_k \mathrm{e}^{i\psi_{2k}} S^* H(P_{2k} R_{2k}) - \sum_{j=1}^{q} \frac{\kappa_k^2 \rho_j^2}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1} \ . \end{aligned}$$

At this stage, notice that, in view of (4.25) and of (4.22), we have, for every $A \in \mathbb{C}_{d_{2k}}[z]$,

$$S^* H(A R_{2k}) = \sigma_k \mathrm{e}^{-i\psi_{2k}} B R_{2k} \ , \ B(z) := z^{d_{2k}} \overline{A}\left(\frac{1}{z}\right) \ .$$

Applying this formula to $A = P_{2k}$ and using (4.18), we finally get

$$T_k = -\kappa_k^2 \sum_{j=1}^{q} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1} = -\kappa_k^2 u \ .$$

On the other hand, using equation (4.10) at $z = 0$, we have

$$\sum_{j=1}^{q} \frac{\rho_j}{\rho_j^2 - \sigma_k^2} R_{2j-1}(0) = 1 \ .$$

Since the matrix $\mathcal{B}$ is invertible, we infer that $R_{2j-1}(0) \in \mathbb{R}$, hence equals $(1|R_{2j-1})$. In other words,

$$\sum_{j=1}^{q} \frac{\kappa_k^2 \rho_j}{\rho_j^2 - \sigma_k^2} (S^* H S^*(R_{2j-1}) - H(R_{2j-1}) - (1|R_{2j-1}) u) = 0 \ .$$

This completes the proof. □

We now prove that an operator satisfying equality (6) is actually a Hankel operator.

**Lemma 7.** *Let $N$ be a positive integer. Let*

$$Q(z) := 1 - c_1 z - c_2 z^2 - \cdots - c_N z^N$$

*be a complex valued polynomial with no roots in the closed unit disc. Set*

$$W := \frac{\mathbb{C}_{N-1}[z]}{Q(z)} \subset L^2_+ \ .$$

*Let $H$ be an antilinear operator on $W$ satisfying*

$$S^* H S^* = H - (1|\cdot)u$$

*on $W$, for some $u \in W$. Then $H$ coïncides with the Hankel operator of symbol $u$ on $W$.*

*Proof.* Consider the operator $\tilde{H} := H - H_u$, then $S^*\tilde{H}S^* = \tilde{H}$ on $W$ and hence, it suffices to show that, if $H$ is an antilinear operator on $W$ such that $S^*HS^* = H$, then $H = 0$.

The family $(e_j)_{1\leq j\leq N}$ where

$$e_0(z) = \frac{1}{Q(z)} \ , \ e_j(z) = S^j e_0(z), \ j = 1, \ldots, N-1$$

is a basis of $W$. Using that

$$S^* H S^* = H$$

we get on the one hand that $He_k = (S^*)^k He_0$. On the other hand, since

$$S^* e_0 = S^*\left(\frac{1}{Q}\right) = \sum_{j=1}^{N} c_j e_{j-1} \ ,$$

this implies

$$He_0 = S^* H S^* e_0 = \sum_{j=1}^{N} \overline{c_j} (S^*)^j He_0 \ ,$$

hence $\overline{Q}(S^*)H(e_0) = 0$ . Observe that, by the spectral mapping theorem, the spectrum of $\overline{Q}(S^*))$ is contained into $\overline{Q}(\mathbb{D})$, hence $\overline{Q}(S^*)$ is one-to-one. We conclude that $H(e_0) = 0$, and finally that $H = 0$.

□

Applying Lemma 7 to our vector space $W$, we conclude that $H = H_u$.

It remains to check that $\Phi_n(u) = \mathcal{P}$.

4.3.3. *The function $u$ has the required properties.* Using the definition of $H = H_u$, we observe that the restriction of $H_u^2$ to the space $\mathbb{C}_{d_{2j-1}}[z]R_{2j-1}$ is $\rho_j^2 I$. Similarly, the restriction of $H_u^2$ to the space $z\mathbb{C}_{d_{2k}}[z]R_{2k}$ is $\sigma_k^2 I$. Since the range of $H_u$ is contained into $W$, this provides a complete diagonalization of $H_u^2$. Moreover,

$$u = \sum_{j=1}^{q} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1} \ .$$

This implies that

$$\Sigma_H(u) = \{\rho_1, \dots, \rho_q\} \ , \ u_j = \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1}, \ j = 1, \dots, q \ .$$

We argue similarly for $K_u^2$, noticing that

$$\sum_{k=1}^{q} D_{2k} R_{2k} = \sum_{1 \le j,k \le q} \frac{\kappa_k^2}{\rho_j^2 - \sigma_k^2} \mathrm{e}^{-i\psi_{2j-1}} P_{2j-1} R_{2j-1} \ ,$$

from (4.18), we conclude, using again (A.11), that

$$\sum_{k=1}^{q} D_{2k} R_{2k} = u \ .$$

This shows that

$$\Sigma_K(u) = \{\sigma_1, \dots, \sigma_q\} \ , \ u_k' = D_{2k} R_{2k} \ , \ j = 1, \dots, q \ .$$

Finally, from the definition of $H$, we recover exactly identities (4.3).

4.3.4. *The mapping $\Phi_n$ is open from $\mathcal{V}_{(d_1,\dots,d_n)}$ to $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$.* Notice that we have not yet completed the proof of Theorem 5 since the previous calculations were made under the assumptions (4.19) and (4.20). In other words, we proved that an element $\mathcal{P}$ of the target space satisfying (4.19) and (4.20) is in the range of $\Phi_n$. On the other hand, in section 4.2, we proved that these properties are satisfied by the elements of the range of $\Phi_n$. Since these hypotheses are clearly open in the target space, we infer that the range of $\Phi_n$ is open.

4.3.5. *The mapping $\Phi_n$ is closed.* Let $(u^\varepsilon)$ be a sequence of $\mathcal{V}_{(d_1,\dots,d_n)}$ such that $\Phi_n(u^\varepsilon) := \mathcal{P}^\varepsilon$ converges to some $\mathcal{P}$ in $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}$ as $\varepsilon$ goes to 0. In other words,

$$\mathcal{P}^\varepsilon = ((s_r^\varepsilon)_{1\le r\le 2q}, (\Psi_r^\varepsilon)_{1\le r\le 2q}) \longrightarrow \mathcal{P} = ((s_r)_{1\le r\le 2q}, (\Psi_r)_{1\le r\le 2q})$$

in $\Omega_n \times \prod_{j=1}^n \mathcal{B}_{d_j}$ as $\varepsilon \to 0$. We have to find $u$ such that $\Phi(u) = \mathcal{P}$. Since

$$\|u^\varepsilon\|_{H^{1/2}} \simeq \mathrm{Tr}(H_{u^\varepsilon}^2) = \sum_{r=1}^{2q} d_r (s_r^\varepsilon)^2 + \sum_{j=1}^{q} (s_{2j-1}^\varepsilon)^2$$

is bounded, we may assume, up to extracting a subsequence, that $u^\varepsilon$ is weakly convergent to some $u$ in $H^{1/2}$. Moreover, the rank of $H_u$ is at most $N = q + \sum_{r=1}^{2q} d_r$.

Denote by $u_j^\varepsilon$ the orthogonal projection of $u^\varepsilon$ onto $\ker(H_u^2-(s_{2j-1}^\varepsilon)^2 I)$, $j=1,\dots,q$, and by $(u_k^\varepsilon)'$ the orthogonal projection of $u^\varepsilon$ onto $\ker(K_u^2-(s_{2k}^\varepsilon)^2 I)$, $k=1,\dots,q$. Since all these functions are bounded in $L^2_+$, we may assume that, for the weak convergence in $L^2_+$,

$$u_j^\varepsilon \rightharpoonup v_j \ , \ (u_k^\varepsilon)' \rightharpoonup v_k' \ .$$

Taking advantage of the strong convergence of $u^\varepsilon$ in $L^2_+$ due to the Rellich theorem, we can pass to the limit in

$$(u^\varepsilon|u_j^\varepsilon) = (\tau_j^\varepsilon)^2 \ , \ (u^\varepsilon|(u_k^\varepsilon)') = (\kappa_k^\varepsilon)^2 \ ,$$

and obtain, thanks to the explicit expressions (A.5), (A.6) of $\tau_j^2, \kappa_k^2$ in terms of the $s_r$,

$$(u|v_j) = \tau_j^2 > 0 \ , \ (u|v_k') = \kappa_k^2 > 0 \ ,$$

in particular $v_j \neq 0, v_k' \neq 0$ for every $j,k$.

On the other hand, passing to the limit in

$$\begin{aligned} s_{2j-1}^\varepsilon u_j^\varepsilon &= \Psi_{2j}^\varepsilon \ H_{u^\varepsilon} u_j^\varepsilon \ , \ H_{u^\varepsilon}^2(u_j^\varepsilon) = (s_{2j-1}^\varepsilon)^2 u_j^\varepsilon \ , \\ K_{u^\varepsilon}(u_k^\varepsilon)' &= s_{2k}^\varepsilon \Psi_{2k}^\varepsilon \ (u_k^\varepsilon)' \ , \ K_{u^\varepsilon}^2(u_k^\varepsilon)' = (s_{2k}^\varepsilon)^2 (u_k^\varepsilon)' \ , \\ u^\varepsilon &= \sum_{j=1}^q u_j^\varepsilon = \sum_{k=1}^q (u_k^\varepsilon)' \ , \end{aligned}$$

we obtain

$$\begin{aligned} s_{2j-1} v_j &= \Psi_{2j} \ H_u v_j \ , \ H_u^2(v_j) = s_{2j-1}^2 v_j \ , \\ K_u v_k' &= s_{2k} \Psi_{2k} \ v_k' \ , \ K_u^2(v_k') = s_{2k}^2 v_k' \ , \\ u &= \sum_{j=1}^q v_j = \sum_{k=1}^q v_k' \ . \end{aligned}$$

This implies that $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, $v_j = u_j, v_k' = u_k'$, and $\Phi(u) = \mathcal{P}$. The proof of Theorem 5 is thus complete in the case $n = 2q$, under the assumption that $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty.

4.4. $\mathcal{V}_{(d_1,\dots,d_n)}$ **is non empty,** $n$ **even.** Let $n$ be a positive even integer. The aim of this section is to prove that $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty for any multi-index $(d_1,\dots,d_n)$ of non negative integers.

The preceding section implies that, as soon as $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty, it is homeomorphic to $\Omega_n \times \prod_{j=1}^n \mathcal{B}_{d_j}$, via the explicit formula (1.19) of Theorem 2. We argue by induction on the integer $d_1+\cdots+d_n$. In the generic case consisting of simple eigenvalues (see [8]), we proved that for any positive integer $q$, $\mathcal{V}_{(0,\dots,0)} (= \mathcal{V}_{\rm gen}(2q))$ is non empty. As a consequence, to any given sequence $((s_r),(\Psi_r)) \in \Omega_{2q} \times \mathbb{T}^{2q}$ corresponds a unique $u \in \mathcal{V}_{(0,\dots,0)}$, the $s_{2j-1}^2$ being the simple eigenvalues of $H_u^2$ and the $s_{2j}^2$ the simple eigenvalues of $K_u^2$. This gives the theorem in the case $(d_1,\dots,d_n) = (0,\dots,0)$ for every $n$, which is one of the main

theorems of [9]. Let us turn to the induction argument, which is clearly a consequence of the following lemma.

**Lemma 8.** *Let $n = 2q$, $(d_1, \dots, d_n)$ and $1 \leq r \leq n-1$. Assume*

$$\mathcal{V}_{(d_1,\dots,d_r,0,0,d_{r+1},\dots,d_n)} \text{ is non empty,}$$

*then*

$$\mathcal{V}_{(d_1,\dots,d_{r-1},d_r+1,d_{r+1},\dots,d_n)} \text{ is non-empty.}$$

*Proof.* We consider the case $r = 1$. The proof in the case $r$ odd follows the same lines. Write $m_j := d_{2j-1} + 1$ and $\ell_k = d_{2k} + 1$. From the assumption,

$$\mathcal{V} := \mathcal{V}_{(d_1,0,0,d_2,\dots,d_n)} \text{ is non empty,}$$

hence $\Phi$ establishes a diffeomorphism from $\mathcal{V}$ into

$$\Omega_{n+2} \times \mathcal{B}_{d_1} \times \mathcal{B}_0 \times \mathcal{B}_0 \times \prod_{r=2}^{n} \mathcal{B}_{d_r} \ .$$

Therefore, given $\rho > \sigma_2 > \rho_3 > \sigma_3 > \cdots > \rho_{q+1} > \sigma_{q+1} > 0$, and $\Psi_1, \theta_1, \varphi_2, \Psi_4, \dots, \Psi_{n+2}$, for every $\eta > 0$, for every $\varepsilon > 0$ small enough, we define $u^\varepsilon$ to be the inverse image by $\Phi$ of

$$\big((\rho+\varepsilon, \rho, \rho-\eta\varepsilon, \sigma_2, \rho_3, \sigma_3, \dots, \rho_{q+1}, \sigma_{q+1}), (\Psi_1, \mathrm{e}^{-i\theta_1}, \mathrm{e}^{-i\varphi_2}, \Psi_4, \dots, \Psi_{n+2})\big) \ .$$

By making $\varepsilon$ go to 0, we are going to construct $u$ in $\mathcal{V}_{(d_1+1,\dots,d_n)}$, such that $\rho_1(u) = \rho$ is of multiplicity $m_1 + 1 = d_1 + 2$, $\rho_j(u) = \rho_{j+1}$, $j = 2, \dots, q$, is of multiplicity $m_j$ and $\sigma_k(u) = \sigma_{k+1}$ for $k = 1, \dots, q$, is of multiplicity $\ell_k$.

First of all, observe that $u^\varepsilon$ is bounded in $H^{1/2}_+$, since its norm is equivalent to $\mathrm{Tr}(H^2_{u^\varepsilon})$. Hence, by the Rellich theorem, up to extracting a subsequence, $u^\varepsilon$ strongly converges in $L^2_+$ to some $u \in H^{1/2}_+$. Similarly, the orthogonal projections $u^\varepsilon_j$ and $(u^\varepsilon_k)'$ are bounded in $L^2_+$, hence are weakly convergent to $v_j$, $v'_k$. Arguing as in the previous subsection, we have

$$\begin{aligned}
(u|v_1) &= \lim_{\varepsilon\to 0} \|u^\varepsilon_1\|^2 = \lim_{\varepsilon\to 0} \frac{(\rho+\varepsilon)^2 - \rho^2}{(\rho+\varepsilon)^2 - (\rho-\eta\varepsilon)^2} \frac{\prod_{k\geq 2}((\rho+\varepsilon)^2 - \sigma_k^2)}{\prod_{k\geq 3}((\rho+\varepsilon)^2 - \rho_k^2)}, \\
(u|v_2) &= \lim_{\varepsilon\to 0} \|u^\varepsilon_2\|^2 = \lim_{\varepsilon\to 0} \frac{(\rho-\eta\varepsilon)^2 - \rho^2}{(\rho-\eta\varepsilon)^2 - (\rho+\varepsilon)^2} \frac{\prod_{k\geq 2}((\rho-\eta\varepsilon)^2 - \sigma_k^2)}{\prod_{k\geq 3}((\rho-\eta\varepsilon)^2 - \rho_k^2)}, \\
(u|v_j) &= \lim_{\varepsilon\to 0} \|u^\varepsilon_j\|^2 \\
&= \lim_{\varepsilon\to 0} \frac{\rho_j^2 - \rho^2}{(\rho_j^2 - (\rho-\eta\varepsilon)^2)(\rho_j^2 - (\rho+\varepsilon)^2)} \frac{\prod_{k\geq 2}(\rho_j^2 - \sigma_k^2)}{\prod_{k\geq 3, k\neq j}(\rho_j^2 - \rho_k^2)}, j \geq 3,
\end{aligned}$$

and

$$\begin{aligned}
(u|v_1') &= \lim_{\varepsilon\to 0} \|(u_1^\varepsilon)'\|^2 \\
&= \lim_{\varepsilon\to 0} (\rho^2-(\rho+\varepsilon)^2)(\rho^2-(\rho-\eta\varepsilon)^2)\frac{\prod_{k\geq 3}(\rho^2-\rho_k^2)}{\prod_{k\geq 2}(\rho^2-\sigma_k^2)}, \\
(u|v_k') &= \lim_{\varepsilon\to 0} \|(u_k^\varepsilon)'\|^2 \\
&= \lim_{\varepsilon\to 0} \frac{(\sigma_k^2-(\rho+\varepsilon)^2)(\sigma_k^2-(\rho-\eta\varepsilon)^2)}{\sigma_k^2-\rho^2}\frac{\prod_{j\geq 3}(\sigma_k^2-\rho_j^2)}{\prod_{j\geq 2, j\neq k}(\sigma_k^2-\sigma_j^2)},\ k\geq 2\ .
\end{aligned}$$

In view of these identities, we infer that $v_j, j\geq 1$ and $v_k', k\geq 2$ are not 0. Passing to the limit into the identities

$$\begin{aligned}
s_{2j-1}^\varepsilon u_j^\varepsilon &= \Psi_{2j}^\varepsilon\ H_{u^\varepsilon} u_j^\varepsilon\ ,\ H_{u^\varepsilon}^2(u_j^\varepsilon) = (s_{2j-1}^\varepsilon)^2 u_j^\varepsilon\ , \\
K_{u^\varepsilon}(u_k^\varepsilon)' &= s_{2k}^\varepsilon \Psi_{2k}^\varepsilon\ (u_k^\varepsilon)'\ ,\ K_{u^\varepsilon}^2(u_k^\varepsilon)' = (s_{2k}^\varepsilon)^2 (u_k^\varepsilon)'\ , \\
u^\varepsilon &= \sum_{j=1}^q u_j^\varepsilon = \sum_{k=1}^q (u_k^\varepsilon)'\ ,
\end{aligned}$$

we obtain

$$\begin{aligned}
s_{2j-1} v_j &= \Psi_{2j}\ H_u v_j\ ,\ H_u^2(v_j) = s_{2j-1}^2 v_j\ , \\
K_u v_k' &= s_{2k}\Psi_{2k}\ v_k'\ ,\ K_u^2(v_k') = s_{2k}^2 v_k'\ , \\
u &= \sum_{j=1}^q v_j = \sum_{k=1}^q v_k'\ ,
\end{aligned}$$

hence

$$\dim E_u(\rho_j)\geq m_j\ ,\ j\geq 3,\ \dim F_u(\sigma_k)\geq \ell_k\ ,\ k\geq 2\ .$$

In order to conclude that $u\in \mathcal{V}_{(d_1+1,d_2,\dots,d_n)}$, it remains to prove that

$$\dim E_u(\rho)\geq m_1+1\ .$$

We use the explicit formulae obtained in section 4.2. We set

$$\Psi_1(z) = \mathrm{e}^{-i\varphi_1}\chi_1(z)\ .$$

We start with $\det \mathcal{C}(z)$ defined by (4.9). Notice that elements $c_{11}(z)$ and $c_{12}(z)$ in formulae (4.9) are of order $\varepsilon^{-1}$, hence we compute

$$\begin{aligned}
&\lim_{\varepsilon\to 0} 2\varepsilon \det \mathcal{C}(z) = \\
&= \begin{vmatrix}
1-z\mathrm{e}^{-i\theta_1}\Psi_1 & -\dfrac{1-z\mathrm{e}^{-i(\theta_1+\varphi_2)}}{\eta} & 0\dots \\
\dfrac{\rho-z\sigma_k\Psi_1\Psi_{2k}}{\rho^2-\sigma_k^2} & \dfrac{\rho-z\sigma_k\mathrm{e}^{-i\varphi_2}\Psi_{2k}}{\rho^2-\sigma_k^2} & \dfrac{\rho_3-z\sigma_k\Psi_5\Psi_{2k}}{\rho_3^2-\sigma_k^2},\dots,k\geq 2
\end{vmatrix}\ .
\end{aligned}$$

Let us add $\xi(z)$ times the first column to the second column in the above determinant, with

$$\xi(z) := \frac{1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}}{\eta(1 - z\mathrm{e}^{-i\theta_1}\Psi_1(z))} \ .$$

We get

$$\lim_{\varepsilon\to 0} 2\varepsilon \det \mathcal{C}(z) = (1 - z\mathrm{e}^{-i\theta_1}\Psi_1) \det \left( \ \zeta_k(z), \quad \frac{\rho_\ell - z\sigma_k\Psi_{2\ell-1}\Psi_{2k}}{\rho_\ell^2 - \sigma_k^2}, k \geq 2, \ell \geq 3 \ \right) \ ,$$

with

$$\begin{aligned} \zeta_k(z) &= \frac{(1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)})(\rho - z\sigma_k\Psi_1\Psi_{2k})}{\eta(1 - z\mathrm{e}^{-i\theta_1}\Psi_1)(\rho^2 - \sigma_k^2)} + \frac{(1 - z\mathrm{e}^{-i\theta_1}\Psi_1)(\rho - z\sigma_k\mathrm{e}^{-i\varphi_2}\Psi_{2k})}{(1 - z\mathrm{e}^{-i\theta_1}\Psi_1)(\rho^2 - \sigma_k^2)} \\ &= \left(\frac{1}{\eta} + 1\right)(1 - q(z)z)\,\frac{\rho - z\sigma_k\mathrm{e}^{-i\psi}\chi_\psi(z)\Psi_{2k}(z)}{(1 - z\mathrm{e}^{-i\theta_1}\Psi_1)(\rho^2 - \sigma_k^2)} \ , \end{aligned}$$

where

$$\begin{aligned} q(z) &:= \frac{\mathrm{e}^{-i\theta_1}(\eta\Psi_1(z) + \mathrm{e}^{-i\varphi_2})}{1+\eta} \ , \\ \chi_\psi(z) : &= \frac{\chi_1(z)z - \mathrm{e}^{i\theta_1}\frac{\eta\mathrm{e}^{i\varphi_1} + \mathrm{e}^{i\varphi_2}\chi_1(z)}{1+\eta}}{1 - q(z)z} \ , \ \psi := \theta_1 + \varphi_1 + \varphi_2 + \pi \ . \end{aligned}$$

We know that $\chi_1$ is a Blaschke product of degree $m_1 - 1$. Let us verify that it is possible to choose $\varphi_2$ so that $\chi_\psi$ is a Blaschke product of degree $m_1$. We first claim that it is possible to choose $\varphi_2$ so that $1 - q(z)z \neq 0$ for $|z| \leq 1$. Write $\alpha := \frac{1}{1+\eta}$, $\psi_1 := \varphi_1 + \theta_1$ and $\psi_2 := \varphi_2 + \theta_1$ and assume $1 - q(z)z = 0$. Then

$$1 = (1 - \alpha)\mathrm{e}^{-i\psi_1}\chi_1(z)z + \alpha\mathrm{e}^{-i\psi_2}z \ . \tag{4.28}$$

First notice that this clearly imposes $|z| = 1$. Furthermore, this implies equality in the Minkowski inequality, therefore there exists $\lambda > 0$ so that $\chi_1(z) = \lambda\mathrm{e}^{-i(\psi_2-\psi_1)}$ and, eventually, that $\chi_1(z) = \mathrm{e}^{-i(\psi_2-\psi_1)}$ since $|\chi_1(z)| = 1$. Inserting this in equation (4.28) gives $z = \mathrm{e}^{i\psi_2}$ so that $\chi_1(\mathrm{e}^{i\psi_2}) = \mathrm{e}^{-i(\psi_2-\psi_1)}$. If this equality holds true for any choice of $\psi_2$, by analytic continuation inside the unit disc, we would have

$$\chi_1(z) = \frac{\mathrm{e}^{i\psi_1}}{z}$$

which is not possible since $\chi_1$ is a holomorphic function in the unit disc. Hence, one can choose $\psi_2$, hence $\varphi_2$, in order to have $1 - q(z)z \neq 0$ for any $|z| \leq 1$. It implies that $\chi_\psi$ is a holomorphic rational function in the unit disc. Moreover, one can easily check that it has modulus one on the unit circle, hence it is a Blaschke product. Finally, its degree is $\deg(\chi_1) + 1 = m_1$.

Summing up,

$$\lim_{\varepsilon\to 0} 2\varepsilon \det \mathcal{C}(z) = \left(1 + \frac{1}{\eta}\right)(1 - q(z)z) \det((\tilde{c}_{k\ell})_{2\leq k,\ell\leq q+1})$$

where, for $k \geq 2$, $\ell \geq 3$,

$$\begin{aligned} \tilde{c}_{k2} &= \frac{\rho - z\sigma_k \mathrm{e}^{-i\psi}\chi_\psi(z)\Psi_{2k}(z)}{(\rho^2 - \sigma_k^2)} \\ \tilde{c}_{k\ell} &= c_{k\ell} = \frac{\rho_\ell - z\sigma_k \Psi_{2\ell-1}(z)\Psi_{2k}(z)}{\rho_\ell^2 - \sigma_k^2} \ . \end{aligned}$$

Next, we perform the same calculation with the numerator of $u_j^\varepsilon(z)$, $j = 1, 2$. Recall that

$$u_j^\varepsilon(z) = \Psi_{2j-1}(z)\frac{\det \mathcal{C}_j(z)}{\det \mathcal{C}(z)}$$

where $\mathcal{C}_j(z)$ denotes the matrix deduced from $(c_{k\ell}(z))_{1\leq k,\ell\leq q+1}$ by replacing the column $j$ by

$$\mathbf{1} := \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \ .$$

We compute

$$\begin{aligned} &\lim_{\varepsilon\to 0} 2\varepsilon \det \mathcal{C}_1(z) = \\ &= \begin{vmatrix} 0 & -\dfrac{1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}}{\eta} & 0 \dots \\ \mathbf{1} & \dfrac{\rho - z\sigma_k \mathrm{e}^{-i\varphi_2}\Psi_{2k}}{\rho^2 - \sigma_k^2} & \dfrac{\rho_3 - z\sigma_k \Psi_5 \Psi_{2k}}{\rho_3^2 - \sigma_k^2}, \dots, k \geq 2 \end{vmatrix} \\ &= \frac{1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}}{\eta} \det\left(\mathbf{1}, (c_{k\ell})_{k\geq 2, \ell \geq 3}\right) \end{aligned}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0} 2\varepsilon \det \mathcal{C}_2(z) = \\ &= \begin{vmatrix} 1 - z\mathrm{e}^{-i\theta_1}\Psi_1 & 0 & 0 \dots \\ \dfrac{\rho - z\sigma_k \Psi_1 \Psi_{2k}}{\rho^2 - \sigma_k^2} & \mathbf{1} & \dfrac{\rho_3 - z\sigma_k \Psi_5 \Psi_{2k}}{\rho_3^2 - \sigma_k^2}, \dots, k \geq 2 \end{vmatrix} \\ &= (1 - z\mathrm{e}^{-i\theta_1}\Psi_1) \det\left(\mathbf{1}, (c_{k\ell})_{k\geq 2, \ell \geq 3})\right) \end{aligned}$$

Hence we have, for the weak convergence in $L^2_+$,

$$\begin{aligned} v_1 &:= \lim_{\varepsilon\to 0} u_1^\varepsilon = \Psi_1 \frac{(1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)})}{(1+\eta)(1 - q(z)z)} \cdot \frac{\det\left((\mathbf{1}, (c_{k\ell})_{k\geq 2, \ell\geq 3})\right)}{\det\left((\tilde{c}_{k\ell})_{2\leq k,\ell\leq q+1}\right)} \\ v_2 &:= \lim_{\varepsilon\to 0} u_2^\varepsilon = \eta \mathrm{e}^{-i\varphi_2} \frac{(1 - z\mathrm{e}^{-i\theta_1}\Psi_1)}{(1+\eta)(1 - q(z)z)} \cdot \frac{\det\left((\mathbf{1}, (c_{k\ell})_{k\geq 2, \ell\geq 3})\right)}{\det\left((\tilde{c}_{k\ell})_{2\leq k,\ell\leq q+1}\right)} \ . \end{aligned}$$

Furthermore, if $D_1$ denotes the normalized denominator of $\Psi_1$, we have

$$\begin{aligned} H^2_{u^\varepsilon}\left(\frac{z^a}{D_1(z)}\frac{u_1^\varepsilon}{\Psi_1}\right) &= (\rho+\varepsilon)^2\frac{z^a}{D_1(z)}\frac{u_1^\varepsilon}{\Psi_1},\ 0\le a\le m_1-1,\\ H^2_{u^\varepsilon}(u_2^\varepsilon) &= (\rho-\eta\varepsilon)^2 u_2^\varepsilon, \end{aligned}$$

Passing to the limit in these identities as $\varepsilon$ goes to 0, we get

$$\begin{aligned} H^2_{u}\left(\frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1}\right) &= \rho^2\frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1},\ 0\le a\le m_1-1,\\ H^2_{u}(v_2) &= \rho^2 v_2\ . \end{aligned}$$

It remains to prove that the dimension of the vector space generated by

$$\frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1},\ 0\le a\le m_1-1, v_2,$$

is $m_1+1$. From the expressions of $v_1$ and $v_2$, it is equivalent to prove that the dimension of the vector space spanned by

$$\frac{z^a}{D_1(z)}(1-\mathrm{e}^{-i\psi_2}z),\ 0\le a\le m_1-1,\ (1-\mathrm{e}^{-i\psi_1}z\chi_1(z))$$

is $m_1+1$. Indeed, we claim that our choice of $\psi_2$ implies that this family is free. Assume that for some $\lambda_a$, $0\le a\le m_1-1$ we have

$$\sum_{a=0}^{m_1-1}\lambda_a\frac{z^a}{D_1(z)} = \frac{1-\mathrm{e}^{-i\psi_1}z\chi_1(z)}{1-\mathrm{e}^{-i\psi_2}z}$$

then, as the left hand side is a holomorphic function in $\overline{\mathbb{D}}$, it would imply $\chi_1(\mathrm{e}^{i\psi_2}) = \mathrm{e}^{i(\psi_1-\psi_2)}$ but $\psi_2$ has been chosen so that this does not hold. Eventually, we have constructed $u$ in $\mathcal{V}_{(d_1+1,\dots,d_n)}$. An analogous procedure would allow to construct $u$ in $\mathcal{V}_{(d_1,\dots,d_{r-1},d_r+1,\dots,d_n)}$ for any $r$ odd. The case $r$ even can be handled similarly, by collapsing two variables $\sigma$ and one variable $\rho$. □

4.5. **The case $n$ odd.** The proof of the fact that $\Phi_n$ is one-to-one is the same as in the case $n$ even. One has to prove that $\Phi_n$ is onto. We shall proceed by approximation from the case $n$ even. We define $q=\frac{n+1}{2}$.
Let

$$\mathcal{P} = ((\rho_1,\sigma_1,\dots,\rho_q),(\Psi_r)_{1\le r\le n})$$

be an arbitrary element of $\Omega_n\times\prod_{r=1}^n\mathcal{B}_{d_r}$. We look for $u\in\mathcal{V}_{(d_1,\dots,d_n)}$ such that $\Phi_n(u)=\mathcal{P}$. Consider, for every $\varepsilon$ such that $0<\varepsilon<\rho_q$,

$$\mathcal{P}_\varepsilon = ((\rho_1,\sigma_1,\dots,\rho_q,\varepsilon),((\Psi_r)_{1\le r\le n},1))\in\Omega_{n+1}\times\prod_{r=1}^{n+1}\mathcal{B}_{d_r}$$

with $d_{n+1}:=0$ – we take $\Psi_{2q}=1\in\mathcal{B}_0$. From Theorem 5, we get $u_\varepsilon\in\mathcal{V}_{(d_1,\dots,d_{n+1})}$ such that $\Phi(u_\varepsilon)=\mathcal{P}_\varepsilon$. As before, we can prove by

a compactness argument that a subsequence of $u_\varepsilon$ has a limit $u \in \mathcal{V}_{(d_1,\dots,d_n)}$ as $\varepsilon$ tends to 0 with $\Phi_n(u) = \mathcal{P}$. We leave the details to the reader. □

4.6. $\mathcal{V}_{(d_1,\dots,d_n)}$ **is a manifold.** Let $d = n + 2\sum_r d_r$. We consider the map

$$\Theta := \left\{ \begin{array}{ccc} \Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r} & \longrightarrow & \mathcal{V}(d) \\ ((s_r),(\Psi_r)) & \longmapsto & u \end{array} \right.$$

where $u$ is given by the explicit formula obtained in section 4.2. This map is well defined and $\mathcal{C}^\infty$ on $\Omega_n \times \prod_{j=1}^n \mathcal{B}_{d_j}$. Moreover, from the previous section, it is a homeomorphism onto its range $\mathcal{V}_{(d_1,\dots,d_n)}$. In order to prove that $\mathcal{V}_{(d_1,\dots,d_n)}$ is a manifold of dimension $2n+2\sum_{j=1}^n d_j$, it is enough to check that the differential of $\Theta$ is injective at every point. From Lemma 4, near every point $u_0 \in \mathcal{V}_{(d_1,\dots,d_n)}$, there exists a smooth function $\tilde{\Phi}_n$, defined on a neighborhood $V$ on $u_0$ in $\mathcal{V}(d)$, such that $\tilde{\Phi}_n$ coincides with $\Phi_n$ on $V \cap \mathcal{V}_{(d_1,\dots,d_n)}$. Consequently, $\tilde{\Phi}_n \circ \Theta$ is the identity on a neighborhood of $\mathcal{P}_0 := \Phi_n(u_0)$. In particular, the differential of $\Theta$ at $\mathcal{P}_0$ is injective.

4.7. **Proof of Theorem 1 in the finite rank case.** Denote by $L^2_{+,r}$ the real subspace of $L^2_+$ made of functions with real Fourier coefficients. If $u \in (VMO_+ \setminus \{0\}) \cap L^2_{+,r}$, then $H_u$ acts on $L^2_{+,r}$ as a compact self adjoint operator, which is unitarily equivalent to $\Gamma_c$ if $u = u_c$. Consequently, for every Borel real valued function $f$, $f(H_u^2)$ acts on $L^2_{+,r}$. In particular, the orthogonal projections $u_j$, $u'_k$ belong to $L^2_{+,r}$. Therefore, for every $r$, the Blaschke product $\Psi_r(u)$ belongs to $L^2_{+,r}$, which means that its coefficients are real, in particular $\psi_r \in \{0,\pi\}$. Moreover, by Proposition 1, for every $r$, there exists bases of $E_u(s_r) \cap L^2_{+,r}$ and $F_u(s_r) \cap L^2_{+,r}$ on which the respective actions of $H_u$ and $K_u$ are described by matrices of the type $\varepsilon_r s_r A$, where $\varepsilon_r = \mathrm{e}^{-i\psi_r} = \pm 1$ and

$$A = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 1 & 0 \\ \vdots & \dots & \diagup & \dots & \vdots \\ 0 & 1 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

being of dimension $d_r + 1$ or $d_r$. By an elementary observation, the eigenvalues of $A$ are $\pm 1$, with equal multiplicities if the dimension of $A$ is even, and where the multiplicity of 1 is one unit greater than the multiplicity of $-1$ if the dimension of $A$ is odd. Consequently, if we denote by $(\lambda_j),(\mu_k)$ the respective sequences of non zero eigenvalues of $H_u$ and of $K_u$ on $L^2_{+,r}$, repeated according to their multiplicities, and ordered following

$$|\lambda_1| \geq |\mu_1| \geq |\lambda_2| \geq \dots ,$$

each $s_r$ correspond to a maximal string with consecutive equal terms, of length $2d_r + 1$, where $d_r$ is the degree of $\Psi_r$. Moreover, we have

$$|\#\{j : \lambda_j = s_r\} - \#\{j : \lambda_j = -s_r\}| \leq 1 \ ,$$
$$|\#\{k : \mu_k = s_r\} - \#\{k : \mu_k = -s_r\}| \leq 1.$$

This is the Megretskii–Peller–Treil condition. Moreover, according to the parity of $r$ and $d_r$, if one of the above integers is 0, the other one is 1, and the eigenvalue with the greatest multiplicity is then $\varepsilon_r s_r$.

Conversely, given two sequences $(\lambda_j), (\mu_k)$ satisfying the assumptions of Theorem 1, the above considerations imply that the set of solutions $u$ to the inverse spectral problem is exactly

$$\Phi^{-1}\left(\{(s_r)\} \times \varepsilon_1 \mathcal{B}^\sharp_{d_1,r} \times \cdots \times \varepsilon_n \mathcal{B}^\sharp_{d_n,r}\right)$$

where $\mathcal{B}^\sharp_{d,r}$ denotes the set of Blaschke products of degree $d$, with real coefficients and with angle 0, which is diffeomorphic to $\mathbb{R}^d$ in view of the result of Appendix B. Notice that the explicit formula (1.19) allows to check that $u$ belongs to $L^2_{+,r}$. In view of Theorem 3, we conclude that the set of solutions $u$ to the inverse spectral problem is diffeomorphic to $\mathbb{R}^M$, with $M = \sum_r d_r$.

## 5. Extension to compact Hankel operators

In this section, we prove the parts of Theorems 2, 3 corresponding to infinite rank Hankel operators. Given an arbitrary sequence $(d_r)_{r\geq 1}$ of nonnegative integers, we set

$$\mathcal{V}_{(d_r)_{r\geq 1}} := \Phi^{-1}\left(\Omega_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}\right) \ .$$

**Theorem 6.** *The mapping*

$$\begin{array}{rcl} \Phi : \mathcal{V}_{(d_r)_{r\geq 1}} & \longrightarrow & \Omega_\infty \times \displaystyle\prod_{r=1}^{\infty} \mathcal{B}_{d_r} \\ u & \longmapsto & ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \end{array}$$

*is a homeomorphism.*

Before giving the proof of this theorem, notice that it implies Theorem 1 in the infinite case, following the same considerations as in the finite rank case above.

*Proof.* The fact that $\Phi$ is one-to-one follows from an explicit formula analogous to the one obtained in the finite rank case, see section 4.2. However, in this infinite rank situation, we have to deal with the continuity of infinite rank matrices on appropriate $\ell^2$ spaces.

Indeed, we still have

$$u = \sum_{j=1}^{\infty} \Psi_{2j-1} h_j$$

where $\mathcal{H}(z) := (h_j(z))_{j\geq 1}$ satisfies, for every $z \in \mathbb{D}$,

$$\mathcal{H}(z) = \mathcal{F}(z) + z\mathcal{D}(z)\mathcal{H}(z) \tag{5.1}$$

with

$$\begin{aligned} \mathcal{F}(z) &:= \left(\frac{\tau_j^2}{\rho_j}\right)_{j\geq 1} , \\ \mathcal{D}(z) &:= \left(\frac{\tau_j^2}{\rho_j} \sum_{k=1}^{\infty} \frac{\kappa_k^2 \sigma_k \Psi_{2k}(z)\Psi_{2\ell-1}(z)}{(\rho_j^2 - \sigma_k^2)(\rho_\ell^2 - \sigma_k^2)}\right)_{j,\ell\geq 1} . \end{aligned}$$

Notice that the coefficients of the infinite matrix $\mathcal{D}(z)$ depend holomorphically on $z \in \mathbb{D}$. We are going to prove that, for every $z \in \mathbb{D}$, $\mathcal{D}(z)$ defines a contraction on the space $\ell^2_\tau$ of sequences $(v_j)_{j\geq 1}$ satisfying

$$\sum_{j=1}^{\infty} \frac{|v_j|^2}{\tau_j^2} < \infty .$$

From the maximum principle, we may assume that $z$ belongs to the unit circle. Then $z$ and $\Psi_r(z)$ have modulus 1. We then compute $\mathcal{D}(z)\mathcal{D}(z)^*$, where the adjoint is taken for the inner product associated to $\ell^2_\tau$. We get, using identities (A.12), (A.11) and (A.13),

$$\begin{aligned} [\mathcal{D}(z)\mathcal{D}(z)^*]_{jn} &= \frac{\tau_j^2}{\rho_j\rho_n} \sum_{k,\ell,m} \frac{\kappa_k^2\sigma_k \Psi_{2k}(z) \tau_\ell^2 \kappa_m^2 \sigma_m \overline{\Psi_{2m}(z)}}{(\rho_j^2 - \sigma_k^2)(\rho_\ell^2 - \sigma_k^2)(\rho_n^2 - \sigma_m^2)(\rho_\ell^2 - \sigma_m^2)} \\ &= \frac{\tau_j^2}{\rho_j\rho_n} \sum_{k} \frac{\kappa_k^2\sigma_k^2}{(\rho_j^2 - \sigma_k^2)(\rho_n^2 - \sigma_k^2)} \\ &= -\frac{\tau_j^2}{\rho_j\rho_n} + \delta_{jn} . \end{aligned}$$

Since, from the identity (A.3) in Appendix A,

$$\sum_{j=1}^{\infty} \frac{\tau_j^2}{\rho_j^2} \leq 1 ,$$

we conclude that $\mathcal{D}(z)\mathcal{D}(z)^* \leq I$ on $\ell^2_\tau$, and consequently that

$$\|\mathcal{D}(z)\|_{\ell^2_\tau \to \ell^2_\tau} \leq 1 .$$

From the Cauchy inequalities, this implies

$$\|\mathcal{D}^{(n)}(0)\|_{\ell^2_\tau \to \ell^2_\tau} \leq n! .$$

Coming back to equation (5.1), we observe that $\mathcal{H}(0) = \mathcal{F}(0) \in \ell^2_\tau$, and that, for every $n \geq 0$,

$$\mathcal{H}^{(n+1)}(0) = (n+1) \sum_{p=0}^{n} \binom{n}{p} \mathcal{D}^{(p)}(0) \mathcal{H}^{(n-p)}(0) \ .$$

By induction on $n$, this determines $\mathcal{H}^{(n)}(0) \in \ell^2_\tau$, whence the injectivity of $\Phi$.

Next, we prove that $\Phi$ is onto. We pick an element

$$\mathcal{P} \in \Omega_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}$$

and we construct $u \in VMO_+$ so that $\Phi(u) = \mathcal{P}$. Set

$$\mathcal{P} = ((\rho_1, \sigma_1, \rho_2, \dots), (\Psi_r)_{r \geq 1})$$

and consider, for any integer $N$,

$$\mathcal{P}_N := ((\rho_1, \sigma_1, \dots, \rho_N, \sigma_N), (\Psi_r)_{1 \leq r \leq 2N})$$

in

$$\Omega_{2N} \times \prod_{j=1}^{2N} \mathcal{B}_{d_j} \ .$$

From Theorem 5, there exists $u_N \in \mathcal{V}_{(d_1,\dots,d_{2N})}$ with $\Phi(u_N) = \mathcal{P}_N$. As $u_N$ is bounded in $L^2_+$, there exists a subsequence converging weakly to some $u$ in $L^2_+$. Let $u_{N,j}$ and $u'_{N,k}$ denote the orthogonal projections of $u_N$ respectively on $E_{u_N}(\rho_j)$ and on $F_{u_N}(\sigma_k)$ so that we have the orthogonal decompositions,

$$u_N = \sum_{j=1}^{N} u_{N,j} = \sum_{k=1}^{N} u'_{N,k} \ .$$

After a diagonal extraction procedure, one may assume that $u_{N,j}$ and $u'_{N,k}$ converge weakly in $L^2_+$ respectively to some $v^{(j)}$ and to some $v'^{(k)}$. In fact one may assume that $u_N$ converges strongly to $u$ in $L^2_+$. The proof is along the same lines as the one developed for Proposition 2 in [10], and is based on the Adamyan-Arov-Krein (AAK)theorem [1], [24]. Let us recall the argument.

First we recall that the AAK theorem states that the $(p+1)$-th singular value of a Hankel operator, as the distance of this operator to operators of rank at most $p$, is exactly achieved by some Hankel operator of rank at most $p$, hence, with a rational symbol. We refer to part (2) of the theorem in Appendix C. We set, for every $m \geq 1$,

$$p_m = m + \sum_{r \leq 2m} d_r \ .$$

With the notation of Appendix C, one easily checks that, for every $m$,

$$\underline{s}_{p_{m-1}}(u) > \underline{s}_{p_m}(u) = \rho_{m+1}(u) \ .$$

By part (1) of the AAK theorem in Appendix C, for every $N$ and every $m = 1, \dots, N$, there exists a rational symbol $u_N^{(m)}$, defining a Hankel operator of rank $p_m$, namely $u_N^{(m)} \in \mathcal{V}(2p_m) \cup \mathcal{V}(2p_m - 1)$, such that

$$\|H_{u_N} - H_{u_N^{(m)}}\| = \rho_{m+1}(u_N) = \rho_{m+1}.$$

In particular, we get

$$\|u_N - u_N^{(m)}\|_{L^2} \leq \rho_{m+1}.$$

On the other hand, one has

$$\|H_{u_N^{(m)}}\| \geq \frac{1}{\sqrt{p_m}}(Tr(H^2_{u_N^{(m)}}))^{1/2} \geq \frac{1}{\sqrt{p_m}}\|u_N^{(m)}\|_{H^{1/2}_+}.$$

Hence, for fixed $m$, the sequence $(u_N^{(m)})_N$ is bounded in $H^{1/2}_+$. Our aim is to prove that the sequence $(u_N)$ is precompact in $L^2_+$. We show that, for any $\varepsilon > 0$ there exists a finite sequence $v_k \in L^2_+$, $1 \leq k \leq M$ so that

$$\{u_N\}_N \subset \bigcup_{k=1}^{M} B_{L^2_+}(v_k, \varepsilon).$$

Let $m$ be fixed such that

$$\rho_{m+1} \leq \varepsilon/2.$$

Since the sequence $(u_N^{(m)})_N$ is uniformly bounded in $H^{1/2}_+$, it is precompact in $L^2_+$, hence there exists $v_k \in L^2_+$, $1 \leq k \leq M$, such that

$$\{u_N^{(m)}\}_N \subset \bigcup_{k=1}^{M} B_{L^2_+}(v_k, \varepsilon/2) \ .$$

Then, for every $N$ there exists some $k$ such that

$$\|u_N - v_k\|_{L^2} \leq \rho_{m+1} + \|u_N^{(m)} - v_k\|_{L^2} \leq \varepsilon.$$

Therefore $\{u_N\}$ is precompact in $L^2_+$ and, since $u_N$ converges weakly to $u$, it converges strongly to $u$ in $L^2_+$. Since $\|H_{u_N}\| = \rho_1$ is bounded, we infer the strong convergence of operators,

$$\forall h \in L^2_+, H_{u_N}(h) \underset{p\to\infty}{\longrightarrow} H_u(h) \ .$$

We now observe that if $\rho^2$ is an eigenvalue of $H^2_{u_N}$ of multiplicity $m$ then $\rho^2$ is an eigenvalue of $H^2_u$ of multiplicity at most $m$. Let $(e_N^{(l)})_{1\leq l\leq m}$ be an orthonormal family of eigenvectors of $H^2_{u_N}$ associated to the eigenvector $\rho^2$. Let $h$ be in $L^2_+$ and write

$$h = \sum_{l=1}^{m}(h|e_N^{(l)})e_N^{(l)} + h_{0,N}$$

where $h_{0,N}$ is the orthogonal projection of $h$ on the orthogonal complement of $E_{u_N}(\rho)$ so that

$$\begin{aligned}\|(H^2_{u_N} - \rho^2 I)h\|^2 &= \|(H^2_{u_N} - \rho^2 I)h_{0,N}\|^2 \\ &\geq d_{\rho^2}\|h_{0,N}\|^2 = d_{\rho^2}(\|h\|^2 - \sum_{l=1}^{m} |(h|e_N^{(l)})|^2) \ ,\end{aligned}$$

here $d_{\rho^2}$ denotes the distance to the other eigenvalues of $H^2_{u_N}$. By taking the limit as $N$ tends to $\infty$ one gets

$$\|(H^2_u - \rho^2 I)h\|^2 \geq d_{\rho^2}(\|h\|^2 - \sum_{l=1}^{m} |(h|e^{(l)})|^2)$$

where $e^{(l)}$ denotes a weak limit of $e_N^{(l)}$. Assume now that the dimension of $E_u(\rho)$ is larger than $m+1$ then we could construct $h$ orthogonal to $(e^{(1)}, \dots e^{(m)})$ with $H^2_u(h) = \rho^2 h$, a contradiction. The same argument allows to obtain that if $\rho^2$ is not an eigenvalue of $H^2_{u_N}$, $\rho^2$ is not an eigenvalue of $H^2_u$.

We now argue as in section 4.3.5 above. We may assume, up to extracting a subsequence, that $u_{N,j}$ weakly converges to $v_j$, and that $u'_{N,k}$ weakly converges to $v'_k$ in $L^2_+$, with the identities

$$\rho_j v_j = \Psi_{2j-1} H_u(v_j) \ , \ H^2_u(v_j) = \rho_j^2 v_j \ , \ K_u(v'_k) = \sigma_k \Psi_{2k} v'_k \ , \ K^2_u(v'_k) = \sigma_k^2 v'_k \ .$$

and

$$(u|v_j) = \tau_j^2 \ , \ (u|v'_k) = \kappa_k^2 \ .$$

This already implies that $v_j$, $v'_k$ are not 0, and hence, in view of Lemmas 2 and 3, that

$$\dim E_u(\rho_j) = m_j \ , \ \dim F_u(\sigma_k) = \ell_k \ .$$

We infer that $u \in \mathcal{V}_{(d_r)_{r\geq 1}}$ and that $\rho_j = s_{2j-1}(u)$ , $\sigma_k = s_{2k}(u)$ . It remains to identify $v_j$ with the orthogonal projection $u_j$ of $u$ onto $E_u(\rho_j)$, and $v'_k$ with the orthogonal projection $u'_k$ of $u$ onto $F_u(\sigma_k)$. The strategy of passing to the limit, as $N$ tends to infinity, in the decompositions

$$u_N = \sum_{j=1}^{N} u_{N,j} = \sum_{k=1}^{N} u'_{N,k}$$

is not easy to apply because of infinite sums. Hence we argue as follows. From the identity

$$\|u_{N,j}\|^2 = (u_N|u_{N,j})$$

we get

$$\|v_j\|^2 \leq (u|v_j) = (u_j|v_j) \ ,$$

and, by the Cauchy-Schwarz inequality, since $v_j \neq 0$,

$$\|v_j\| \leq \|u_j\| \ .$$

On the other hand, we know from the general formulae of Appendix A that

$$\|u_j\|^2 = \tau_j^2 \ .$$

Since $\tau_j^2 = (u_j|v_j)$, we get $\|u_j\| \leq \|v_j\|$ and finally infer

$$(u_j|v_j) = \|v_j\|^2 = \|u_j\|^2 \ ,$$

hence

$$\|v_j - u_j\|^2 = 0 \ .$$

Similarly, $v_k' = u_k'$. This completes the proof of the surjectivity.

The continuity of $\Phi$ follows as in section 4.1. As for the continuity of $\Phi^{-1}$, we argue exactly as for surjectivity above, except that we have to prove the convergence of $u_N$ to $u$ in $VMO_+$. This can be achieved exactly as in the proof of Proposition 2 of [10] : the Adamyan-Arov-Krein theorem allows to reduce to the following statement : if $w_N \in \mathcal{V}(2p) \cup \mathcal{V}(2p-1)$ strongly converges to $w \in \mathcal{V}(2p) \cup \mathcal{V}(2p-1)$, then the convergence takes place in $VMO$ — in fact in $C^\infty$. See Lemma 3 of [10]. □

For future reference, we state a similar result in the case of Hilbert–Schmidt operators. We set

$$\mathcal{V}^{(2)}_{(d_r)_{r\geq 1}} := \mathcal{V}_{(d_r)_{r\geq 1}} \cap H^{1/2}_+ \ ,$$

and

$$\Omega^{(2)}_\infty((d_r)_{r\geq 1}) := \{(s_r)_{r\geq 1} \in \Omega_\infty : \sum_{r=1}^\infty (d_r+1)s_r^2 < \infty\} \ ,$$

endowed with the topology induced by the above weighted $\ell^2$ norm.

**Theorem 7.** *The mapping*

$$\begin{array}{rcl} \Phi : \mathcal{V}^{(2)}_{(d_r)_{r\geq 1}} & \longrightarrow & \Omega^{(2)}_\infty((d_r)_{r\geq 1}) \times \prod_{r=1}^\infty \mathcal{B}_{d_r} \\ u & \longmapsto & ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \end{array}$$

*is a homeomorphism.*

The proof is essentially the same as the one of Theorem 6, except that the argument based on the AAK theorem is simplified by the identity

$$\mathrm{Tr}(H_u^2) = \sum_{r=1}^\infty d_r s_r^2(u) + \sum_{j=1}^\infty s_{2j-1}^2(u) \ ,$$

which provides bounds in $H^{1/2}_+$, hence strong convergence in $L^2_+$, and finally strong convergence in $H^{1/2}_+$. We leave the easy details to the reader.

## 6. Evolution under the cubic Szegő flow

6.1. **The theorem.** In this section, we prove the following result.

**Theorem 8.** *Let $u_0 \in H^{1/2}_+$ with*

$$\Phi(u_0) = ((s_r), (\Psi_r)).$$

*The solution of*

$$i\partial_t u = \Pi(|u|^2 u),\ u(0) = u_0$$

*is characterized by*

$$\Phi(u(t)) = ((s_r), (\mathrm{e}^{i(-1)^r s_r^2 t}\Psi_r)) \ .$$

**Remark 1.** *It is in fact possible to define the flow of the cubic Szegő on $BMO_+ = BMO(\mathbb{T}) \cap L^2_+$, see [12]. The above theorem then extends to the case of an initial datum $u_0$ in $VMO_+$ .*

*Proof.* In view of the continuity of the flow map on $H^{1/2}_+$, see [8], we may assume that $H_{u_0}$ is of finite rank. Let $u$ be the corresponding solution of the cubic Szegő equation. Let $\rho$ be a singular value of $H_u$ in $\Sigma_H(u)$ such that $m := \dim E_u(\rho) = \dim F_u(\rho) + 1$ and denote by $u_\rho$ the orthogonal projection of $u$ on $E_u(\rho)$. Hence, $u_\rho = \mathbb{1}_{\{\rho^2\}}(H_u^2)(u)$. Let us differentiate this equation with respect to time. Recall [8], [11] that

$$\frac{dH_u}{dt} = [B_u, H_u] \text{ with } B_u = \frac{i}{2}H_u^2 - iT_{|u|^2} \ . \tag{6.1}$$

Here $T_b$ denotes the Toeplitz operator of symbol $b$,

$$T_b(h) = \Pi(bh) \ ,\ h \in L^2_+ \ ,\ b \in L^\infty \ . \tag{6.2}$$

Equation (6.1) implies, for every Borel function $f$,

$$\frac{df(H_u^2)}{dt} = -i[T_{|u|^2}, f(H_u^2)] \ .$$

We get from this Lax pair structure

$$\begin{aligned} \frac{du_\rho}{dt} &= -i[T_{|u|^2}, \mathbb{1}_{\{\rho^2\}}(H_u^2)](u) + \mathbb{1}_{\{\rho^2\}}(H_u^2)\left(\frac{du}{dt}\right) \\ &= -i[T_{|u|^2}, \mathbb{1}_{\{\rho^2\}}(H_u^2)](u) + \mathbb{1}_{\{\rho^2\}}(H_u^2)\left(-iT_{|u|^2}u\right) \ , \end{aligned}$$

and eventually

$$\frac{du_\rho}{dt} = -iT_{|u|^2}u_\rho \ . \tag{6.3}$$

On the other hand, differentiating the equation

$$\rho u_\rho = \Psi H_u(u_\rho)$$

one obtains

$$\rho\frac{du_\rho}{dt} = \dot{\Psi}H_u(u_\rho) + \Psi\left([B_u, H_u](u_\rho) + H_u\left(\frac{du_\rho}{dt}\right)\right)$$

Hence, using the expression (6.3), we get

$$-i\rho T_{|u|^2}(u_\rho) = \dot\Psi H_u(u_\rho) + \Psi\left(-iT_{|u|^2}H_u(u_\rho) + i\rho^2 H_u(u_\rho)\right) \ ,$$

hence

$$-i[T_{|u|^2},\Psi]H_u(u_\rho) = (\dot\Psi + i\rho^2\Psi)H_u(u_\rho) \ .$$

We claim that the left hand side of this equality is zero. Assume this claim proved, we get, as $H_u(u_\rho)$ is not identically zero, that $\dot\Psi + i\rho^2\Psi = 0$, whence

$$\Psi(t) = \mathrm{e}^{-it\rho^2}\Psi(0) \ .$$

It remains to prove the claim. We first prove that, for any $p \in \mathbb{D}$ such that $\chi_p$ is a factor of $\chi$,

$$[T_{|u|^2},\chi_p](e) = 0$$

for any $e \in E_u(\rho)$ such that $\chi_p e \in E_u(\rho)$. Recall that

$$\chi_p(z) = \frac{z-p}{1-\overline{p}z} \ .$$

For any $L^2$ function $f$,

$$\Pi(\chi_p f) - \chi_p\Pi(f) = K_{\chi_p}(g) = (1-|p|^2)H_{1/(1-\overline{p}z)}(g) \ ,$$

where $\overline{(I-\Pi)f} = Sg$ . Consequently, the range of $[\Pi,\chi_p]$ is one dimensional, directed by $\frac{1}{1-\overline{p}z}$. In particular, $[T_{|u|^2},\chi_p](e)$ is proportional to $\frac{1}{1-\overline{p}z}$. On the other hand,

$$\begin{aligned}([T_{|u|^2},\chi_p](e)|1) &= (T_{|u|^2}(\chi_p e) - \chi_p T_{|u|^2}(e))|1) \\ &= (\chi_p(e)|H_u^2(1)) - (\chi_p|1)(e|H_u^2(1)) \\ &= (H_u^2(\chi_p(e))|1) - (\chi_p|1)(H_u^2(e)|1) = 0 \ .\end{aligned}$$

This proves that $[T_{|u|^2},\chi_p](e) = 0$.

For the general case, we write $\Psi = \mathrm{e}^{-i\psi}\chi_{p_1}\cdots\chi_{p_{m-1}}$ and

$$[T_{|u|^2},\Psi]H_u(u_\rho) = \mathrm{e}^{-i\psi}\sum_{j=1}^{m-1}\prod_{k=1}^{j-1}\chi_{p_k}[T_{|u|^2},\chi_{p_j}]\prod_{k=j+1}^{m-1}\chi_{p_k}H_u(u_\rho) = 0 \ .$$

It remains to consider the evolution of the $\Psi_{2k}$'s. Let $\sigma$ be a singular value of $K_u$ in $\Sigma_K(u)$ such that $\dim F_u(\sigma) = \dim E_u(\sigma) + 1$ and denote by $u'_\sigma$ the orthogonal projection of $u$ onto $F_u(\sigma)$. Recall [11] that

$$\frac{dK_u}{dt} = [C_u,K_u] \text{ with } C_u = \frac{i}{2}K_u^2 - iT_{|u|^2} \ .$$

As before, we compute the derivative in time of $u'_\sigma = \mathbb{1}_{\{\sigma^2\}}(K_u^2)(u)$, and get

$$\frac{du'_\sigma}{dt} = -iT_{|u|^2}u'_\sigma \ . \tag{6.4}$$

On the other hand, differentiating the equation

$$K_u(u'_\sigma) = \sigma\Psi u'_\sigma$$

one obtains

$$-i[T_{|u|^2}, \Psi]u'_\sigma = (\dot{\Psi} - i\sigma^2\Psi)u'_\sigma \ .$$

As before, we prove that the left hand side of the latter identity is 0, by checking that, for every factor $\chi_p$ of $\Psi$, for any $f \in F_u(\sigma)$ such that $\chi_p f \in F_u(\sigma)$,

$$([T_{|u|^2}, \chi_p](f)|1) = 0 \ .$$

The calculation leads to

$$\begin{aligned} ([T_{|u|^2}, \chi_p](f)|1) &= (H^2_u(\chi_p f) - (\chi_p|1)H^2_u(f)|1) \\ &= ((\chi_p - (\chi_p|1))f|u)(u|1), \end{aligned}$$

where we have used (1.7). Now $(\chi_p - (\chi_p|1))f \in F_u(\sigma)$ is orthogonal to 1, hence, from Proposition 1, it belongs to $E_u(\sigma)$, hence it is orthogonal to $u$. This completes the proof.

□

6.2. **Application: traveling waves revisited.** As an application of Theorem 2 and of the previous section, we revisit the traveling waves of the cubic Szegő equation. These are the solutions of the form

$$u(t, e^{ix}) = e^{-i\omega t}u_0(e^{i(x-ct)}) \ , \ \omega, c \in \mathbb{R} \ .$$

For $c = 0$, it is easy to see [8] that this condition for $u_0 \in H^{1/2}_+$ corresponds to finite Blaschke product. The problem of characterizing traveling waves with $c \neq 0$ is more delicate, and was solved in [8] by the following result.

**Theorem.** [8] *A function $u$ in $H^{1/2}_+$ is a traveling wave with $c \neq 0$ and $\omega \in \mathbb{R}$ if and only if there exist non negative integers $\ell$ and $N$, $0 \leq \ell \leq N-1$, $\alpha \in \mathbb{R}$ and a complex number $p \in \mathbb{C}$ with $0 < |p| < 1$ so that*

$$u(z) = \frac{\alpha z^\ell}{1 - pz^N}$$

Here we give an elementary proof of this theorem.

*Proof.* The idea is to keep track of the Blaschke products associated to $u$ through the following unitary transform on $L^2(\mathbb{T})$,

$$\tau_\alpha f(\mathrm{e}^{ix}) := f(\mathrm{e}^{i(x-\alpha)}) \ , \ \alpha \in \mathbb{R}.$$

Since $\tau_\alpha$ commutes to $\Pi$, notice that

$$\tau_\alpha(H_u(h)) = H_{\tau_\alpha(u)}(\tau_\alpha(h)) \ .$$

Consequently, $\tau_\alpha$ sends $E_u(\rho)$ onto $E_{\tau_\alpha(u)}(\rho)$, and

$$\tau_\alpha(u_\rho) = [\tau_\alpha(u)]_\rho \ .$$

Applying $\tau_\alpha$ to the identity

$$\rho u_\rho = \Psi_\rho H_u(u_\rho) \ ,$$

we infer

$$\rho[\tau_\alpha(u)]_\rho = \tau_\alpha(\Psi_\rho) H_{\tau_\alpha(u)}\left([\tau_\alpha(u)]_\rho\right) \ ,$$

and similarly

$$\rho[\mathrm{e}^{-i\beta}\tau_\alpha(u)]_\rho = \mathrm{e}^{-i\beta}\tau_\alpha(\Psi_\rho) H_{\mathrm{e}^{-i\beta}\tau_\alpha(u)}\left([\mathrm{e}^{-i\beta}\tau_\alpha(u)]_\rho\right) \ .$$

This leads, for every $\rho \in \Sigma_H(u)$, to

$$\Psi_\rho(\mathrm{e}^{-i\beta}\tau_\alpha(u)) = \mathrm{e}^{-i\beta}\tau_\alpha(\Psi_\rho(u)) \ .$$

Applying this identity to $u = u_0$, $\alpha = ct$ and $\beta = \omega t$, and comparing with Theorem 4, we conclude

$$\mathrm{e}^{-it\rho^2}\Psi_\rho(u_0) = \mathrm{e}^{-i\omega t}\tau_{ct}(\Psi_\rho(u_0)) \ .$$

Writing

$$\Psi_\rho(u_0) = \mathrm{e}^{-i\varphi} \prod_{1\le j\le m-1} \chi_{p_j},$$

we get, for every $t$,

$$\mathrm{e}^{-it\rho^2} \prod_{1\le j\le m-1} \chi_{p_j} = \mathrm{e}^{-it(\omega+c(m-1))} \prod_{1\le j\le m-1} \chi_{\mathrm{e}^{ict}p_j} \ .$$

This imposes, since $c \neq 0$,

$$\rho^2 = \omega + (m-1)c \ , \ p_j = 0 \ ,$$

for every $\rho \in \Sigma_H(u_0)$. In other words, $\Psi_\rho(u_0)(z) = \mathrm{e}^{-i\varphi} z^{m-1}$ .

We repeat the same argument for $\sigma \in \Sigma_K(u)$, with $\ell = \dim F_u(\sigma) = \dim E_u(\sigma) + 1$ and

$$K_u(u'_\sigma) = \sigma \Psi_\sigma u'_\sigma \ ,$$

using this time

$$\tau_\alpha(K_u(h)) = \mathrm{e}^{i\alpha} K_{\tau_\alpha(u)}(\tau_\alpha(h)) \ .$$

We get

$$\sigma^2 = \omega - \ell c \ ,$$

and

$$\Psi_\sigma(u_0)(z) = \mathrm{e}^{-i\theta} z^{\ell-1} \ .$$

If we assume that there exists at least two elements $\rho_1 > \rho_2$ in $\Sigma_H(u_0)$, with $m_j = \dim E_{u_0}(\rho_j)$ for $j = 1, 2$, from Proposition 2, there is at least one element $\sigma_1$ in $\Sigma_K(u_0)$, satisfying

$$\rho_1 > \sigma_1 > \rho_2.$$

Set $\ell_1 := \dim F_{u_0}(\sigma_1)$, we get

$$(m_1 - 1)c > -\ell_1 c > (m_2 - 1)c$$

which is impossible since $m_1, \ell_1, m_2$ are positive integers. Therefore, there is only one element $\rho$ in $\Sigma_H(u_0)$, with $m = \dim E_{u_0}(\rho)$ and at

most one element $\sigma$ in $\Sigma_K(u_0)$, of multiplicity $\ell$. Applying the results of section 4.2, we obtain

$$u_0(z) = \frac{(\rho^2 - \sigma^2)\mathrm{e}^{-i\varphi}}{\rho} \frac{z^{m-1}}{1 - \frac{\sigma}{\rho}\mathrm{e}^{-i(\varphi+\theta)} z^{\ell+m-1}} \ .$$

This completes the proof. □

6.3. **Application to almost periodicity.** As a second application of our main result, we prove that the solutions of the Szegő equation are almost periodic. Let us recall a definition. Let $X$ be a Banach space. A function

$$f : \mathbb{R} \longrightarrow X$$

is almost periodic if it is the uniform limit of quasi-periodic functions, namely finite linear combinations of functions

$$t \longmapsto \mathrm{e}^{i\omega t} x \ ,$$

where $x \in X$ and $\omega \in \mathbb{R}$. Of course, from the explicit formula obtained in Theorem 2 and from the evolution under the cubic Szegő flow, for any $u_0 \in \mathcal{V}(d)$, the solution $u(t)$ is quasi-periodic. This is also a consequence of the results of [11]. It remains to consider data in $H^{1/2}_+$ corresponding to infinite rank Hankel operators. We are going to use Bochner's criterion, see chapters 1, 2 of [18], namely that $f \in C(\mathbb{R}, X)$ is almost periodic if and only if it is bounded and the set of functions

$$f_h : t \in \mathbb{R} \longmapsto f(t+h) \in X \ , \ h \in \mathbb{R} \ ,$$

is relatively compact in the space of bounded continuous functions valued in $X$.

Let $u_0 \in \mathcal{V}^{(2)}_{(d_r)_{r\geq 1}}$. Set

$$\Phi(u_0) = ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \ .$$

Then, from Theorem 4,

$$\Phi(u(t)) = ((s_r)_{r\geq 1}, (\mathrm{e}^{i(-1)^r s_r^2 t}\Psi_r)_{r\geq 1}) \ .$$

By Theorem 7, it is enough to prove that the set of functions

$$t \in \mathbb{R} \longmapsto \Phi(u(t+h)) \in \Omega^{(2)}_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}$$

is relatively compact in $C(\mathbb{R}, \Omega^{(2)}_\infty \times \prod_{r=1}^\infty \mathcal{B}_{d_r})$. This is equivalent to the relative compactness of the family $(\mathrm{e}^{i(-1)^r s_r^2 h})_{r\geq 1}$ in $(\mathbb{S}^1)^\infty$, $h \in \mathbb{R}$, which is trivial.

## 7. Evolution under the Szegő hierarchy

The Szegő hierarchy was introduced in [8] and used in [9] and [11]. In [9], it was used to identify the symplectic form on the generic part of $\mathcal{V}(d)$. Similarly, our purpose in this section is to establish preliminary formulae, towards the identification of the symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$ in the next section.

For the convenience of the reader, we recall the main properties of the hierarchy. For $y > 0$ and $u \in H^{\frac{1}{2}}_+$, we set

$$J^y(u) = ((I + yH_u^2)^{-1}(1)|1) \ .$$

Notice that the connection with the Szegő equation is made by

$$E(u) = \frac{1}{4}(\partial_y^2 J^y_{|y=0} - (\partial_y J^y_{|y=0})^2) \ .$$

Thanks to formula (A.7) in Appendix A, $J^y(u)$ is a function of the singular values $s_r(u)$. For every $s > \frac{1}{2}$, $J^y$ is a smooth real valued function on $H^s_+$, and its Hamiltonian vector field is given by

$$X_{J^y}(u) = 2iyw^y H_u w^y \ , \ w^y := (I + yH_u^2)^{-1}(1) \ ,$$

which is a Lipschitz vector field on bounded subsets of $H^s_+$. By the Cauchy–Lipschitz theorem, the evolution equation

$$\dot{u} = X_{J^y}(u) \tag{7.1}$$

admits local in time solutions for every initial data in $H^s_+$ for $s > 1$, and the lifetime is bounded from below if the data are bounded in $H^s_+$. We recall that this evolution equation admits a Lax pair structure ([11]).

**Theorem 9.** *For every $u \in H^s_+$, we have*

$$\begin{aligned} H_{iX_{J^y}(u)} &= H_u F^y_u + F^y_u H_u \ , \\ K_{iX_{J^y}(u)} &= K_u G^y_u + G^y_u K_u \ , \\ G^y_u(h) &:= -yw^y \, \Pi(\overline{w^y}\, h) + y^2 H_u w^y \, \Pi(\overline{H_u w^y}\, h) \ , \\ F^y_u(h) &:= G^y_u(h) - y^2 (h|H_u w^y) H_u w^y \ . \end{aligned}$$

*If $u \in C^\infty(\mathcal{I}, H^s_+)$ is a solution of equation (7.1) on a time interval $\mathcal{I}$, then*

$$\begin{aligned} \frac{dH_u}{dt} &= [B^y_u, H_u] \ , \ \frac{dK_u}{dt} = [C^y_u, K_u] \ , \\ B^y_u &= -iF^y_u \ , \ C^y_u = -iG^y_u \ . \end{aligned}$$

In particular, $\Sigma_H(u_0) = \Sigma_H(u(t))$ and $\Sigma_K(u_0) = \Sigma_K(u(t))$ for every $t$, therefore $J^y(u(t))$ is a constant $J^y$. We now state the main result of this section.

**Theorem 10.** *Let $u_0 \in H^s_+$ , $s > 1$, with*

$$\Phi(u_0) = ((s_r), (\Psi_r)).$$

*The solution of*

$$\dot{u} = X_{J^y}(u) \ , \ u(0) = u_0 \ ,$$

*is characterized by*

$$\Phi(u(t)) = ((s_r), (\mathrm{e}^{i\omega_r t}\Psi_r)) \ , \ \omega_r := (-1)^{r-1}\frac{2yJ^y}{1+ys_r^2} \ .$$

*Proof.* Let $\rho \in \Sigma_H(u_0)$. Denote by $u_\rho$ the orthogonal projection of $u$ on $E_u(\rho)$. Hence, $u_\rho = 1\!\!1_{\{\rho^2\}}(H_u^2)(u)$. Let us differentiate this equation with respect to time. We get from the Lax pair structure

$$\begin{aligned}\frac{du_\rho}{dt} &= [B_u^y, 1\!\!1_{\{\rho^2\}}(H_u^2)](u) + 1\!\!1_{\{\rho^2\}}(H_u^2)[B_u^y, H_u](1) \\ &= B_u^y(u_\rho) - 1\!\!1_{\{\rho^2\}}(H_u^2)(H_u(B_u^y(1))) \ .\end{aligned}$$

Since $B_u^y(1) = iyJ^y w^y$, and since $1\!\!1_{\{\rho^2\}}(H_u^2)(H_u w^y) = \frac{1}{1+y\rho^2}u_\rho$, we get

$$\frac{du_\rho}{dt} = B_u^y(u_\rho) + i\frac{yJ^y}{1+y\rho^2}u_\rho \ . \tag{7.2}$$

On the other hand, differentiating the equation

$$\rho u_\rho = \Psi H_u(u_\rho)$$

one obtains

$$\rho\frac{du_\rho}{dt} = \dot{\Psi}H_u(u_\rho) + \Psi\left([B_u^y, H_u](u_\rho) + H_u\left(\frac{du_\rho}{dt}\right)\right)$$

Hence, using the expression (7.2), we get

$$\rho\left(B_u^y(u_\rho) + i\frac{yJ^y}{1+y\rho^2}u_\rho\right) = \left(\dot{\Psi} - i\frac{yJ^y}{1+y\rho^2}\Psi\right)H_u(u_\rho) + \Psi B_u^y H_u(u_\rho) \ ,$$

hence

$$[B_u^y, \Psi]H_u(u_\rho) = \left(\dot{\Psi} - 2i\frac{yJ^y}{1+y\rho^2}\Psi\right)H_u(u_\rho).$$

It remains to prove that the left hand side of this equality is zero. We first show that, for any $p \in \mathbb{D}$ such that $\chi_p$ is a factor of $\chi$, for every $e \in E_u(\rho)$ such that $\chi_p e \in E_u(\rho)$, $[B_u^y, \chi_p](e) = 0$. We write

$$\begin{aligned}i[B_u^y, \chi_p](e) &= -yw^y\left(\Pi(\overline{w^y}\,\chi_p e) - \chi_p\Pi(\overline{w^y}\,e)\right) \\ &+ y^2 H_u w^y\left(\Pi(\overline{H_u w^y}\,\chi_p e) - \chi_p\Pi(\overline{H_u w^y}\,e)\right) \\ &- y^2((\chi_p e|H_u w^y)H_u w^y - \chi_p(e|H_u w^y)H_u w^y))\end{aligned}$$

We already used that, for any function $f \in L^2$, $\Pi(\chi_p f) - \chi_p \Pi(f)$ is proportional to $\frac{1}{1-\overline{p}z}$. Hence, we obtain

$$\begin{aligned} i[B_u^y, \chi_p](e) &= -yw^y \frac{c}{1-\overline{p}z} + y^2 H_u w^y \frac{\tilde{c}}{1-\overline{p}z} \\ &- y^2((\chi_p e|H_u w^y)H_u w^y - \chi_p(e|H_u w^y)H_u w^y)) \end{aligned}$$

with

$$c = (\Pi(\overline{w^y}\,\chi_p e) - \chi_p \Pi(\overline{w^y}\,e)|1) = (\chi_p e|w^y) - (\chi_p|1)(e|w^y)$$

and

$$\tilde{c} = (\Pi(\overline{H_u w^y}\,\chi_p e) - \chi_p \Pi(\overline{H_u w^y}\,e)|1) = (\chi_p e|H_u(w^y)) - (\chi_p|1)(e|H_u(w^y))\ .$$

Now, for any $v \in E_u(\rho)$

$$(v|w^y) = (v|\mathbb{1}_{\{\rho^2\}}(H_u^2)(w^y)) = \frac{1}{1+y\rho^2}(v|1)$$

hence $c = 0$. On the other hand,

$$\begin{aligned} (v|H_u w^y) &= (v|\mathbb{1}_{\{\rho^2\}}(H_u^2)(H_u(w^y))) = \frac{1}{1+y\rho^2}(v|u_\rho) \\ &= \frac{1}{1+y\rho^2}(v|H_u(1)) = \frac{1}{1+y\rho^2}(1|H_u(v))\ . \end{aligned}$$

We infer

$$i[B_u^y, \chi_p](e) = C(z)\frac{1}{1+y\rho^2} y^2 H_u w^y$$

where

$$\begin{aligned} C(z) &= \frac{1}{1-\overline{p}z}\left((1|H_u(\chi_p e)) - (\chi_p|1)(1|H_u(e)) - (1|H_u(\chi_p e) + \chi_p(1|H_u(e))\right. \\ &= (1|H_u(\chi_p e))(\frac{1}{1-\overline{p}z} - 1) + (1|H_u(e))(\chi_p + \frac{p}{1-\overline{p}z}) \\ &= \frac{z}{1-\overline{p}z}(\overline{p}(1|H_u(\chi_p e)) + (1|H_u(e)))\ . \end{aligned}$$

We claim that $H_u(e) = \chi_p H_u(\chi_p e)$. Indeed, from the assumption $e \in E_u(\rho)$ and $\chi_p e \in E_u(\rho)$, we can write $e = f H_u(u_\rho)$ with $\Pi(\Psi\overline{f}) = \Psi\overline{f}$ and $\Pi(\Psi\overline{\chi_p f}) = \Psi\overline{\chi_p f}$. From Lemma 2, we infer

$$H_u(\chi_p e) = \rho\Psi\overline{\chi_p f}H_u(u_\rho)\ ,\ H_u(e) = \rho\Psi\overline{f}H_u(u_\rho)\ .$$

This proves the claim. Since $(1|\chi_p) = -\overline{p}$, we conclude that $C(z) = 0$. Hence $[B_u^y, \chi_p](e) = 0$. Arguing as in the previous section, we conclude that $[B_u, \chi]H_u(u_\rho) = 0$.

It remains to consider the other eigenvalues. Let $\sigma \in \Sigma_K(u_0)$. Denote by $u'_\sigma$ the orthogonal projection of $u$ on $F_u(\sigma)$. We compute the derivative of $u'_\sigma = \mathbb{1}_{\{\sigma^2\}}(K_u^2)(u)$ as before. From the Lax pair formula, we

get

$$\begin{aligned}\frac{du'_\sigma}{dt} &= [C^y_u, \mathbb{1}_{\{\sigma^2\}}(K^2_u)](u) + \mathbb{1}_{\{\sigma^2\}}(K^2_u)[B^y_u, H_u](1)\\ &= C^y_u(u'_\sigma) + \mathbb{1}_{\{\sigma^2\}}(K^2_u)(B^y_u(u) - C^y_u(u) - H_u(B^y_u(1)))\\ &= C^y_u(u'_\sigma) + \mathbb{1}_{\{\sigma^2\}}(K^2_u)(iy^2(u|H_uw^y)H_uw^y + iyJ^yH_uw^y)\\ &= C^y_u(u'_\sigma) + iy\mathbb{1}_{\{\sigma^2\}}(K^2_u)H_uw^y\end{aligned}$$

since $(B^y_u - C^y_u)(h) = iy^2(h|H_uw^y)H_uw^y$ and $-yH^2_uw^y = w^y - 1$ so that $(u| - yH_uw^y) = (-yH^2_uw^y|1) = J^y - 1$.

We claim that

$$\mathbb{1}_{\{\sigma^2\}}(K^2_u)(H_uw^y) = \frac{J^y}{1+y\sigma^2}u'_\sigma \ . \tag{7.3}$$

Using $K^2_u = H^2_u - (\cdot|u)u$ one gets, for any $f \in L^2_+$

$$(I+yH^2_u)^{-1}f = (I+yK^2_u)^{-1}f - y((I+yH^2_u)^{-1}f|u)(I+yK^2_u)^{-1}u \ . \tag{7.4}$$

Applying formula (7.4) to $f = u$, we get

$$H_uw^y = (I+yH^2_u)^{-1}(u) = (I+yK^2_u)^{-1}(u) - y((I+yH^2_u)^{-1}(u)|u)(I+yK^2_u)^{-1}(u) \ ,$$

hence

$$H_uw^y = J^y(I+yK^2_u)^{-1}(u) \ . \tag{7.5}$$

Formula (7.3) follows by taking the orthogonal projection on $F_u(\sigma)$. Using Formula (7.3), we get

$$\frac{du'_\sigma}{dt} = C^y_u(u'_\sigma) + iy\frac{J^y}{1+y\sigma^2}u'_\sigma \ . \tag{7.6}$$

On the other hand, differentiating the equation

$$K_u(u'_\sigma) = \sigma\Psi u'_\sigma$$

one obtains

$$[C^y_u, K_u](u'_\sigma) + K_u\left(\frac{du'_\sigma}{dt}\right) = \sigma\dot{\Psi}u'_\sigma + \sigma\Psi\frac{du'_\sigma}{dt}.$$

From identity (7.5), we use the expression of $\frac{du'_\sigma}{dt}$ obtained above to get

$$\left(\dot{\Psi} + 2i\frac{yJ^y}{1+\sigma^2y}\Psi\right)u'_\sigma = \sigma[C^y_u, \Psi](u'_\sigma) \ .$$

The result follows once we prove that $[C^y_u, \Psi](u'_\sigma) = 0$.

From the arguments developed before, it is sufficient to prove that $[C^y_u, \chi_p](f) = 0$ for any $f \in F_u(\sigma)$ such that $\chi_p f \in F_u(\sigma)$. As before

$$[C^y_u, \chi_p](f) = i\frac{c}{1-\overline{p}z}yw^y - iy^2H_uw^y\frac{\tilde{c}}{1-\overline{p}z}$$

where

$$c = ((\chi_p - (\chi_p|1))f|w^y)$$

and

$$\tilde{c} = ((\chi_p - (\chi_p|1))f|H_u w^y).$$

Notice that $w^y = 1 - yH_u w^y$, hence $c = -y\tilde{c}$. Let us first prove that $\tilde{c} = 0$. Using formula (7.5),

$$\begin{aligned} \tilde{c} &= (\chi_p f|\mathbb{1}_{\{\sigma^2\}}(K_u^2)H_u w^y) - (\chi_p|1)(f|\mathbb{1}_{\{\sigma^2\}}(K_u^2)H_u w^y) \\ &= \frac{J^y}{1+y\sigma^2}((\chi_p - (\chi_p|1))f|u) = 0 \ , \end{aligned}$$

since, as we already observed at the end of the proof of Theorem 8,

$$F_u(\sigma) \cap 1^\perp = E_u(\sigma) = F_u(\sigma) \cap u^\perp \ .$$

This completes the proof. □

We close this section by stating a corollary which will be useful for describing the symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$.

**Corollary 2.** *On $\mathcal{V}_{(d_1,\dots,d_n)}$, we have*

$$X_{J^y} = \sum_{r=1}^{n} (-1)^r \frac{2yJ^y}{1+ys_r^2}\frac{\partial}{\partial\psi_r} \ . \tag{7.7}$$

*The vector fields $X_{J^y}, y \in \mathbb{R}_+$, generate an integrable sub-bundle of rank $n$ of the tangent bundle of $\mathcal{V}_{(d_1,\dots,d_n)}$. The leaves of the corresponding foliation are the isotropic tori*

$$\mathcal{T}((s_1,\dots,s_n),(\Psi_1,\dots,\Psi_n)) := \Phi^{-1}\left(\{(s_1,\dots,s_n)\} \times \mathbb{S}^1\Psi_1 \times \cdots \times \mathbb{S}^1\Psi_n\right) \ ,$$

*where $(s_1,\dots,s_n) \in \Omega$ and $(\Psi_1,\dots,\Psi_n) \in \mathcal{B}_{d_1}^\sharp \times \cdots \times \mathcal{B}_{d_n}^\sharp$ are given.*

*Proof.* For every $y \in \mathbb{R}_+$, Theorem 10 can be rephrased as the following identities for Lie derivatives along $X_{J^y}$.

$$X_{J^y}(s_r) = 0 \ , \ X_{J^y}(\chi_r) = 0 \ , \ X_{J^y}(\psi_r) = (-1)^r \frac{2yJ^y}{1+ys_r^2} \ , \ r = 1,\dots,n \ .$$

This implies identity (7.7) on $\mathcal{V}_{(d_1,\dots,d_n)}$. Given $n$ positive numbers $y_1 > \cdots > y_n$, the matrix

$$\left(\frac{1}{1+y_\ell s_r^2}\right)_{1\le\ell,r\le n}$$

is invertible. This implies that, for every $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, the vector subspace of $T_u\mathcal{V}_{(d_1,\dots,d_n)}$ spanned by the $X_{J^y}(u), y \in \mathbb{R}_+$ is exactly

$$\operatorname{span}\left(\frac{\partial}{\partial\psi_r}, r = 1,\dots,n\right) \ .$$

The integrability follows, as well as the identification of the leaves, while the isotropy of the tori comes from the identity

$$\{J^y, J^{y'}\} = 0$$

which was proved in [8] and is also a consequence of identity A.1 and of the conservation of the $s_r$'s along the Hamiltonian curves of $J^y$, as stated in Theorem 10. □

In the next section, we identify the tori $\mathcal{T}((s_r),(\Psi_r))$ above as classes of some special unitary equivalence for the pair of operators $(H_u, K_u)$.

## 8. Invariant tori of the Szegő hierarchy and unitary equivalence of pairs of Hankel operators

In this section, we identify the sets of symbols $u \in VMO_+ \setminus \{0\}$ having the same list of singular values $(s_r)$ and the same list $(\chi_r)$ of monic Blaschke products, for the pair $(H_u, K_u)$. In view of Theorem 3, these sets are tori. Moreover, $VMO_+ \setminus \{0\}$ is the disjoint union of these tori, and, from sections 6 and 7, the Hamilton flows of the Szegő hierarchy act on them. We prove that they are classes of some specific unitary equivalence between the pairs $(H_u, K_u)$, which we now define.

**Definition 1.** *Given $u, \tilde{u} \in VMO_+ \setminus \{0\}$, we set $u \sim \tilde{u}$ if there exist unitary operators $U, V$ on $L^2_+$ such that*

$$H_{\tilde{u}} = U H_u U^* \ , \ K_{\tilde{u}} = V K_u V^* \ ,$$

*and there exists a Borel function $F : \mathbb{R}_+ \to \mathbb{S}^1$ such that*

$$U(u) = F(H^2_{\tilde{u}})\tilde{u} \ , \ V(u) = F(K^2_{\tilde{u}})\tilde{u} \ , \ U^*V = \overline{F}(H^2_u)F(K^2_u) \ .$$

It is easy to check that the above definition gives rise to an equivalence relation.

**Theorem 11.** *Given $u, \tilde{u} \in VMO_+ \setminus \{0\}$, the following assertions are equivalent.*

(1) $u \sim \tilde{u}$.
(2) $\forall r \geq 1, s_r(u) = s_r(\tilde{u})$ *and* $\exists \gamma_r \in \mathbb{T} : \Psi_r(\tilde{u}) = \mathrm{e}^{i\gamma_r}\Psi_r(u)$ .

*Proof.* Assume that (1) holds. Then $H^2_{\tilde{u}}$ is unitarily equivalent to $H^2_u$, and $K^2_{\tilde{u}}$ is unitarily equivalent to $K^2_u$. This clearly implies $\Sigma_H(\tilde{u}) = \Sigma_H(u)$ and $\Sigma_K(\tilde{u}) = \Sigma_K(u)$, so that $s_r(\tilde{u}) = s_r(u)$ for every $r$. Let us show that, for every $r$, $\Psi_r(u)$ and $\Psi_r(\tilde{u})$ only differ by a phase factor. Of course the only cases to be addressed are $d_r \geq 1$. We start with $r = 2j-1$. From the hypothesis, we infer

$$U(u_j) = U(\mathbf{1}_{\{\rho_j^2\}}(H^2_u)(u)) = \mathbf{1}_{\{\rho_j^2\}}(H^2_{\tilde{u}})(U(u)) = F(\rho_j^2)\tilde{u}_j \ ,$$

and, consequently,

$$U(H_u(u_j)) = H_{\tilde{u}}(U(u_j)) = \overline{F}(\rho_j^2)H_{\tilde{u}}(\tilde{u}_j) \ . \tag{8.1}$$

Next we take advantage of the identity

$$U^*V = \overline{F}(H^2_u)F(K^2_u) \ ,$$

by evaluating $U^*S^*U$ on the closed range of $H_u$. We compute

$$\begin{aligned} U^*S^*UH_u &= U^*S^*H_{\tilde{u}}U = U^*K_{\tilde{u}}U = U^*VK_uV^*U \\ &= \overline{F}(H_u^2)F(K_u^2)K_u\overline{F}(K_u^2)F(H_u^2) = \overline{F}(H_u^2)F(K_u^2)^2K_uF(H_u^2) \\ &= \overline{F}(H_u^2)F(K_u^2)^2S^*\overline{F}(H_u^2)H_u \ , \end{aligned}$$

and we conclude, on $\overline{\mathrm{Ran}(H_u)}$,

$$U^*S^*U = \overline{F}(H_u^2)F(K_u^2)^2S^*\overline{F}(H_u^2) \ . \tag{8.2}$$

For simplicity, set $D := D_{2j-1}$ and $d := d_{2j-1}$. Recall from Proposition 1 that a basis of $E_u(\rho_j)$ is

$$\left(\frac{z^a}{D}H_u(u_j) \ , \ a = 0, \dots, d\right) ,$$

and a basis of $F_u(\rho_j) = E_u(\rho_j) \cap u^\perp$ is

$$\left(\frac{z^b}{D}H_u(u_j) \ , \ b = 0, \dots, d-1\right) .$$

For $a = 1, \dots, d_{2j-1}$, we infer

$$U^*S^*U\left(\frac{z^a}{D}H_u(u_j)\right) = \frac{z^{a-1}}{D}H_u(u_j) \ ,$$

or

$$U\left(\frac{z^a}{D}H_u(u_j)\right) = (S^*)^{d-a}U\left(\frac{z^d}{D}H_u(u_j)\right) \ , \ a = 0, \dots, d \ .$$

This implies, for $a = 0, \dots, d-1$, that the right hand side belongs to $F_{\tilde{u}}(\rho_j)$. On the other hand, if $P \in \mathbb{C}[z]$ has degree at most $d$, one easily checks that

$$S^*\left(\frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\right) = P(0)K_{\tilde{u}}(\tilde{u}_j) + R \ , \ R \in F_{\tilde{u}}(\rho_j) \ .$$

Notice that the right hand side belongs to $F_{\tilde{u}}(\rho_j)$ if and only if $K_{\tilde{u}}(\tilde{u}_j) \in F_{\tilde{u}}(\rho_j)$ or $P(0) = 0$. Assume for a while that $K_{\tilde{u}}(\tilde{u}_j)$ does not belong to $F_{\tilde{u}}(\rho_j)$. Then, writing

$$U\left(\frac{z^d}{D}H_u(u_j)\right) = \frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j) \ ,$$

and using that, for $a = 0, \dots, d-1$,

$$(S^*)^{d-a}\left(\frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\right) \in F_{\tilde{u}}(\rho_j) \ ,$$

we infer $P(0) = 0$, and, by iterating this argument, that $P$ is divisible by $z^d$, in other words,

$$U\left(\frac{z^d}{D}H_u(u_j)\right) = c\frac{z^d}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j) \ ,$$

for some $c \in \mathbb{C}$, and conclude

$$U\left(\frac{z^a}{D}H_u(u_j)\right) = c\frac{z^a}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\ ,\ a = 0, \dots, d\ .$$

Comparing to formula (8.1) for $U(H_u(u_j))$, we obtain

$$cD(z) = \overline{F}(\rho_j^2)\tilde{D}(z)\ .$$

Since $D(0) = 1 = \tilde{D}(0)$, we conclude $c = \overline{F}(\rho_j^2)$, $D = \tilde{D}$, and finally

$$\Psi_{2j-1}(\tilde{u}) = \overline{F}(\rho_j^2)^2\Psi_{2j-1}(u)\ .$$

We now turn to study the special case $K_{\tilde{u}}(\tilde{u}_j) \in F_{\tilde{u}}(\rho_j)$. This reads

$$0 = (K_{\tilde{u}}^2 - \rho_j^2 I)K_{\tilde{u}}(\tilde{u}_j) = K_{\tilde{u}}((H_{\tilde{u}}^2 - \rho_j^2 I)\tilde{u}_j - \|\tilde{u}_j\|^2\tilde{u}) = -\|\tilde{u}_j\|^2 K_{\tilde{u}}(\tilde{u})\ .$$

In other words, this imposes $K_{\tilde{u}}(\tilde{u}) = 0$, or $\tilde{u} = \rho\tilde{\Psi}$, where $\tilde{\Psi}$ is a Blaschke product of degree $d$. Making $V^*$ act on the identity $K_{\tilde{u}}(\tilde{u}) = 0$, we similarly conclude $u = \rho\Psi$, where $\Psi$ is a Blaschke product of degree $d$, so what we have to check is that $\Psi$ and $\tilde{\Psi}$ only differ by a phase factor. In this case, $S^*$ sends $E_u(\rho) = \mathrm{Ran}(H_u)$ into $F_u(\rho)$, so that (8.2) becomes, on $\mathrm{Ran}(H_u)$,

$$U^*S^*U = S^*\ .$$

In other words, the actions of $S^*$ on $W := \mathrm{span}\left(\frac{z^a}{D}, a = 0, \dots, d\right)$ and on $\tilde{W} := \mathrm{span}\left(\frac{z^a}{\tilde{D}}, a = 0, \dots, d\right)$ are conjugated. Writing

$$D(z) = \prod_{p\in\mathcal{P}}(1 - \overline{p}z)^{m_p}\ ,$$

where $\mathcal{P}$ is a finite subset of $\mathbb{D}\setminus\{0\}$, and $m_p$ are positive integers, and using the elementary identities

$$(S^* - \overline{p}I)\left(\frac{1}{(1-\overline{p}z)^k}\right) = S^*\left(\frac{1}{(1-\overline{p}z)^{k-1}}\right)\ ,$$

one easily checks that the eigenvalues of $S^*$ on $W$ are precisely the $\overline{p}$'s, for $p \in \mathcal{P}$, and 0, with the corresponding algebraic multiplicities $m_p$ and

$$m_0 = 1 + d - \sum_{p\in\mathcal{P}} m_p\ .$$

We conclude that $D = \tilde{D}$, whence the claim.

Next, we study the case $r = 2k$. Then

$$V(u_k') = F(\sigma_k^2)\tilde{u}_k'\ ,\ V(K_u(u_k')) = \overline{F}(\sigma_k^2)K_{\tilde{u}}(\tilde{u}_k')\ .$$

Denote by $P_u$ the orthogonal projector onto $\overline{\mathrm{Ran}(H_u)}$, and compute

$$\begin{aligned} V^*P_{\tilde{u}}SVK_u &= V^*P_{\tilde{u}}SK_{\tilde{u}}V = V^*(H_{\tilde{u}} - (\tilde{u}|\,.\,)P_{\tilde{u}}(1))V \\ &= V^*U(H_u - (U^*(\tilde{u})|\,.\,)U^*(P_{\tilde{u}}(1)))U^*V \\ &= \overline{F}(K_u^2)F(H_u^2)(H_u - (\overline{F}(H_u^2)u|\,.\,)F(H_u^2)P_u(1))\overline{F}(H_u^2)F(K_u^2) \\ &= \overline{F}(K_u^2)F(H_u^2)^2(H_u - (u|\,.\,)P_u(1))F(K_u^2) \\ &= \overline{F}(K_u^2)F(H_u^2)^2P_uSK_uF(K_u^2) = \overline{F}(K_u^2)F(H_u^2)^2P_uS\overline{F}(K_u^2)K_u \ , \end{aligned}$$

so that, on $\overline{\mathrm{Ran}(K_u)}$,

$$V^*P_{\tilde{u}}SV = \overline{F}(K_u^2)F(H_u^2)^2P_uS\overline{F}(K_u^2) \ . \tag{8.3}$$

For simplicity again, set $D := D_{2k}$ and $d := d_{2k}$. Recall from proposition 1 that a basis of $F_u(\sigma_k)$ is

$$\left(\frac{z^a}{D}u_k' \ , \ a = 0, \dots, d\right),$$

and a basis of $E_u(\sigma_k) = F_u(\sigma_k) \cap u^\perp$ is

$$\left(\frac{z^a}{D}u_k' \ , \ a = 1, \dots, d\right) \ .$$

Applying identity (8.3) to $\frac{z^a}{D}u_k'$ for $a = 0, \dots, d-1$, we infer

$$V\left(\frac{z^a}{D}u_k'\right) = (P_{\tilde{u}}S)^aV\left(\frac{1}{D}u_k'\right) \ , \ a = 0, \dots, d \ .$$

In particular, the right hand side belongs to $E_{\tilde{u}}(\sigma_k)$ for $a = 1, \dots, d$. On the other hand, if $Q \in \mathbb{C}[z]$ has degree at most $d$,

$$P_{\tilde{u}}S\left(\frac{Q}{\tilde{D}}\tilde{u}_k'\right) = \gamma P_{\tilde{u}}SK_{\tilde{u}}(\tilde{u}_k') + R \ , \ R \in E_{\tilde{u}}(\sigma_k) \ ,$$

where $\gamma\sigma_k\mathrm{e}^{-i\tilde{\psi}_{2k}}$ is the coefficient of $z^d$ in $Q$. Therefore the left hand side belongs to $E_{\tilde{u}}(\sigma_k)$ if and only if

$$\begin{aligned} 0 &= \gamma(H_{\tilde{u}}^2 - \sigma_k^2 I)P_{\tilde{u}}SK_{\tilde{u}}(\tilde{u}_k') \\ &= \gamma(H_{\tilde{u}}^2 - \sigma_k^2 I)(H_{\tilde{u}}(\tilde{u}_k') - \|\tilde{u}_k'\|^2P_{\tilde{u}}(1)) \\ &= \gamma\sigma_k^2\|\tilde{u}_k'\|^2P_{\tilde{u}}(1) \ , \end{aligned}$$

which is impossible. We conclude that $\gamma = 0$, which means that the degree of $Q$ is at most $d-1$. Iterating this argument, we infer

$$V\left(\frac{1}{D}u_k'\right) = c\frac{1}{\tilde{D}}\tilde{u}_k' \ ,$$

for some $c \in \mathbb{C}$, and finally

$$V\left(\frac{z^a}{D}u_k'\right) = c\frac{z^a}{\tilde{D}}\tilde{u}_k' \ , \ a = 0, \dots d \ .$$

Comparing to the above formula for $V(u_k')$, we obtain

$$cD = F(\sigma_k^2)\tilde{D} \ ,$$

thus, since $D(0) = 1 = \tilde{D}(0)$, we have $D = \tilde{D}$, $c = F(\sigma_k^2)$, and finally $\Psi_{2k}(\tilde{u}) = F(\sigma_k^2)^2 \Psi_{2k}(u)$ .

Assume that (2) holds. Define $F : \Sigma_H(u) \cup \Sigma_K(u) \to \mathbb{S}^1$ by

$$F(\rho_j^2) = \mathrm{e}^{-i\frac{\gamma_{2j-1}}{2}} \ ; \ F(\sigma_k^2) = \mathrm{e}^{i\frac{\gamma_{2k}}{2}} \ ,$$

and, if necessary, we define $F(0)$ to be any complex number of modulus 1. Next we define $U$ on the closed range of $H_u$, which is the closed orthogonal sum of $E_u(s_r)$. Thus we just have to define

$$U : E_u(s_r) \to E_{\tilde{u}}(s_r).$$

If $r = 2j-1$, we set

$$U\left(\frac{z^a}{D_{2j-1}} H_u(u_j)\right) = \overline{F}(\rho_j^2) \frac{z^a}{D_{2j-1}} H_{\tilde{u}}(\tilde{u}_j) \ , \ a = 0, \dots, d_{2j-1} \ . \tag{8.4}$$

If $r = 2k$ and $d_{2k} \geq 1$, we set

$$U\left(\frac{z^b}{D_{2k}} u_k'\right) = F(\sigma_k^2) \frac{z^b}{D_{2k}} \tilde{u}_k' \ , \ b = 1, \dots, d_{2k} \ . \tag{8.5}$$

Using (8.4) we obtain

$$\begin{aligned} U(u_j) &= \frac{1}{\rho_j} U(\Psi_{2j-1}(u) H_u(u_j)) = \frac{1}{\rho_j} \overline{F}(\rho_j^2) \Psi_{2j-1}(u) H_{\tilde{u}}(\tilde{u}_j) \\ &= \frac{1}{\rho_j} F(\rho_j^2) \Psi_{2j-1}(\tilde{u}) H_{\tilde{u}}(\tilde{u}_j) = F(\rho_j^2) \tilde{u}_j \ . \end{aligned}$$

Consequently, we get

$$U(u) = \sum_j U(u_j) = \sum_j F(\rho_j^2) \tilde{u}_j = F(H_{\tilde{u}}^2) \tilde{u} \ .$$

A similar argument combined to Proposition 1 leads to

$$U H_u = H_{\tilde{u}} U.$$

Next, we prove that $U$ is unitary. It is enough to prove that every map $U : E_u(s_r) \to E_{\tilde{u}}(s_r)$ is unitary, or that the Gram matrix of a basis of $E_u(s_r)$ is equal to the Gram matrix of its image. We first deal with $r = 2j-1$. Equivalently, we prove that, for $a, b = 0, \dots, d_{2j-1} - 1$,

$$\left(\frac{z^a}{D_{2j-1}} H_u(u_j) | \frac{z^b}{D_{2j-1}} H_u(u_j)\right) = \left(\frac{z^a}{D_{2j-1}} H_{\tilde{u}}(\tilde{u}_j) | \frac{z^b}{D_{2j-1}} H_{\tilde{u}}(\tilde{u}_j)\right) \ .$$

We set

$$\zeta_{a-b} := \left(\frac{z^a}{D_{2j-1}} H_u(u_j) | \frac{z^b}{D_{2j-1}} H_u(u_j)\right) , \ a, b = 0, \dots, d_{2j-1} - 1 \ ,$$

and we notice that $\zeta_{-k} = \overline{\zeta}_k$ , $k = -d_{2j-1}, \dots, d_{2j-1}$ . We drop the subscript $2j-1$ for simplicity and we set

$$D(z) := 1 + \overline{a}_1 z + \cdots + \overline{a}_d z^d.$$

As $\Psi H_u(u_j)$ is orthogonal to $\frac{z^a}{D} H_u(u_j)$ for $a = 0, \dots, d-1$, and $\|H_u(u_j)\|^2 = \rho_j^2 \tau_j^2$, we obtain the system

$$\left\{ \begin{array}{l} \zeta_{d-b} + a_1 \zeta_{d-b-1} + \cdots + a_d \zeta_{-b} = 0 \ , \ b = 0, \dots, d-1 \ , \\ \zeta_0 + a_1 \zeta_{-1} + \cdots + a_d \zeta_{-d} = \rho_j^2 \tau_j^2 \ . \end{array} \right. \tag{8.6}$$

**Lemma 9.** *Let $a_1, \dots a_d$ be complex numbers such that the polynomial $z^d + a_1 z^{d-1} + \cdots + a_d$ has all its roots in $\mathbb{D}$. Then the system (8.6) has at most one solution $\zeta_k$, $k = -d \dots, d$ with $\overline{\zeta}_k = \zeta_{-k}$.*

Assume for a while that this lemma is proved. Since $\tau_j^2$ can be expressed in terms of the $(s_r)$'s — see (A.5), we infer that $U : E_u(\rho_j) \to E_{\tilde{u}}(\rho_j)$ is unitary. Similarly, one proves that the Gram matrix of the basis

$$\frac{z^a}{D_{2k}} u'_k, \ a = 0, \dots, d_{2k}$$

of $F_u(\sigma_k)$ only depends on the $(s_r)$'s and on $D_{2k}$. In particular,

$$U : E_u(\sigma_k) \to E_{\tilde{u}}(\sigma_k)$$

is unitary and finally is unitary from the closed range of $H_u$ onto the closed range of $H_{\tilde{u}}$.

Next, we construct $V$ on the closed range of $H_u$ which is the orthogonal sum of the $F_u(\sigma)$ for $\sigma \in \Sigma_H \cup \Sigma_K$. Thus we just have to define $V : F_u(\sigma) \to E_{\tilde{u}}(\sigma)$ for $\sigma \in \Sigma_H \cup \Sigma_K$.

If $r = 2j - 1$, we set

$$V\left( \frac{z^a}{D_{2j-1}} H_u(u_j) \right) = \overline{F}(\rho_j^2) \frac{z^a}{D_{2j-1}} H_{\tilde{u}}(\tilde{u}_j) \ , \ a = 1, \dots, d_{2j-1} \ . \tag{8.7}$$

If $r = 2k$ and $d_{2k} \geq 1$, we set

$$V\left( \frac{z^b}{D_{2k}} u'_k \right) = F(\sigma_k^2) \frac{z^b}{D_{2k}} \tilde{u}'_k \ , \ b = 0, \dots, d_{2k} \ . \tag{8.8}$$

Similarly, if $0 \in \Sigma_K$, we define $V(u'_0) = F(0)\tilde{u}'_0$. Using (8.8) we get $V(u'_k) = F(\sigma_k^2)\tilde{u}'_k$. Consequently,

$$V(u) = V(u'_0) + \sum_k V(u'_k) = F(K_{\tilde{u}}^2)\tilde{u} \ .$$

A similar argument combined with Proposition 1 leads to

$$V K_u = K_{\tilde{u}} V.$$

Using again Lemma 9, $V$ is unitary from the closed range of $H_u$ onto the closed range of $H_{\tilde{u}}$.

Now we define $U$ and $V$ on the kernel of $H_u$ which is either $\{0\}$ or an infinite dimensional separable Hilbert space. From Corollary 5 of Appendix A, the cancellation of $\ker H_u$ only depends on the $s_r$'s. Therefore, $\ker H_u$ and $\ker H_{\tilde{u}}$ are isometric. We then define $U = V$ from $\ker H_u$ onto $\ker H_{\tilde{u}}$ to be any unitary operator.

It remains to prove that $U^*V = \overline{F}(H_u^2)F(K_u^2)$. On $\ker H_u$, it is trivial since $U^*V = I = \overline{F}(0)F(0)$. Similarly, it is trivial on vectors

$$\frac{z^a}{D_{2k}} u'_k \ , a = 1, \dots, d_{2k} \ .$$

It remains to prove the equality for $u'_0$, $u'_k$. We write

$$\begin{aligned} U^*V(u'_k) &= F(\sigma_k^2)U^*(\tilde{u}'_k) = F(\sigma_k^2)U^* \left( \kappa_k^2 \sum_j \frac{\tilde{u}_j}{\rho_j^2 - \sigma_k^2} \right) \\ &= F(\sigma_k^2) \sum_j \overline{F}(\rho_j^2)\kappa_k^2 \frac{u_j}{\rho_j^2 - \sigma_k^2} = F(\sigma_k^2)\overline{F}(H_u^2)u'_k \\ &= \overline{F}(H_u^2)F(K_u^2)(u'_k) \ . \end{aligned}$$

A similar arguments holds for $U^*V(u'_0)$.

It remains to prove Lemma 9. It is sufficient to prove that the only solution of the homogeneous system

$$\left\{ \begin{array}{l} \zeta_{d-b} + a_1\zeta_{d-b-1} + \cdots + a_d\zeta_{-b} = 0 \ , \ b = 0, \dots, d-1 \ , \\ \zeta_0 + a_1\zeta_{-1} + \cdots + a_d\zeta_{-d} = 0 \ , \end{array} \right. \tag{8.9}$$

with $\overline{\zeta}_k = \zeta_{-k}$ , $k = 0, \dots, d$, is the trivial solution $\zeta = 0$.

We proceed by induction on $d$. For $d = 1$, the system reads

$$\left\{ \begin{array}{l} \zeta_1 + a_1\zeta_0 = 0 \ , \ , \\ \zeta_0 + a_1\overline{\zeta}_1 = 0 \ . \end{array} \right.$$

Since $|a_1| < 1$, this trivially implies $\zeta_0 = \zeta_1 = 0$.

For a general $d$, we plug the expression

$$\zeta_d = -(a_1\zeta_{d-1} + \cdots + a_d\zeta_0)$$

into the last equation. We get

$$\zeta_0 + b_1\overline{\zeta}_1 + \cdots + b_{d-1}\overline{\zeta}_{d-1} = 0 \tag{8.10}$$

with

$$b_k = \frac{a_k - a_d\overline{a}_{d-k}}{1 - |a_d|^2} \ , k = 1, \dots, d-1 \ .$$

Notice that from Proposition 4 of Appendix B, $|a_d| < 1$ and the polynomial $z^{d-1} + b_1 z^{d-2} + \cdots + b_{d-1}$ has all its roots in $\mathbb{D}$. For $b = 1, \dots, d-1$, we multiply by $a_d$ the conjugate of equation

$$\zeta_b + a_1\zeta_{b-1} + \cdots + a_d\zeta_{b-d} = 0$$

and substract the result from equation

$$\zeta_{d-b} + a_1\zeta_{d-b-1} + \cdots + a_d\zeta_{-b} = 0 \ .$$

This yields

$$\zeta_{d-b} + b_1\zeta_{d-b-1} + \cdots + b_{d-1}\zeta_{1-b} = 0 \ .$$

Together with Equation (8.10), this is exactly the system at order $d-1$ with coefficients $b_1, \dots, b_{d-1}$. By induction, we obtain

$$\zeta_0 = \zeta_1 = \cdots = \zeta_{d-1} = 0$$

and finally $\zeta_d = 0$.

This completes the proof.

□

## 9. The symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$

In this section, we prove the last part of Theorem 4, namely that the symplectic form $\omega$ restricted to $\mathcal{V}_{(d_1,\dots,d_n)}$ is given by

$$\omega = \sum_{r=1}^{n} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r \ . \tag{9.1}$$

Recall that the variable $\psi_r$ is connected to the Blaschke product $\Psi_r$ through the identity

$$\Psi_r = \mathrm{e}^{-i\psi_r}\chi_r \ ,$$

where $\chi_r$ is a Blaschke product built with a monic polynomial. Given an integer $k$, we denote by $\mathcal{B}_k^\sharp$ the submanifold of $\mathcal{B}_k$ made with Blaschke products built with monic polynomials of degree $k$.

Let us first point out that we get the following result as a corollary.

**Corollary 3.** *The manifold $\mathcal{V}_{(d_1,\dots,d_n)}$ is an involutive submanifold of $\mathcal{V}(d)$, where*

$$d = 2\sum_{r=1}^{n} d_r + n.$$

*Moreover, $\mathcal{V}_{(d_1,\dots,d_n)}$ is the disjoint union of the symplectic manifolds*

$$\mathcal{W}(\chi_1, \dots, \chi_n) := \Phi^{-1}(\Omega_n \times (\mathbb{S}^1\chi_1 \times \cdots \times \mathbb{S}^1\chi_n)) \ ,$$

*on which*

$$\left(\frac{s_r^2}{2}, \psi_r\right)_{1\le r\le n}$$

*are action angle variables for the cubic Szegő flow.*

*Proof.* From the definition of an involutive submanifold, one has to prove that, at every point $u$ of $\mathcal{V}_{(d_1,\dots,d_n)}$, the tangent space $T_u\mathcal{V}_{(d_1,\dots,d_n)}$ contains its orthogonal relatively to $\omega$. We use an argument of dimension. Namely, one has

$$\begin{aligned} \dim_{\mathbb{R}}(T_u\mathcal{V}_{(d_1,\dots,d_n)})^\perp &= \dim_{\mathbb{R}} T_u\mathcal{V}(d) - \dim_{\mathbb{R}} T_u\mathcal{V}_{(d_1,\dots,d_n)} \\ &= 2d - (2n + 2\sum_{r=1}^{n} d_r) = 2\sum_{r=1}^{n} d_r \ . \end{aligned}$$

One the other hand, from equation (9.1), the tangent space to the manifold

$$\mathcal{F}(u) := \Phi^{-1}\left(\{(s_r(u))\} \times \prod_{r=1}^{n} \mathrm{e}^{-i\psi_r(u)}\mathcal{B}_{d_r}^{\sharp}\right)$$

is clearly a subset of $(T_u\mathcal{V}_{(d_1,\ldots,d_n)})^{\perp}$. Since its dimension equals $2\sum d_r$, we get the equality and hence the first result. The second result is an immediate consequence of the previous sections. □

**Remark 2.**

- *As this is the case for any involutive submanifold of a symplectic manifold, the subbundle $(T\mathcal{V}_{(d_1,\ldots,d_n)})^{\perp}$ of $T\mathcal{V}_{(d_1,\ldots,d_n)}$ is integrable. The leaves of the corresponding isotropic foliation are the manifolds $\mathcal{F}(u)$ above.*
- *The Lagrangian tori of $\mathcal{W}(\chi_1,\ldots,\chi_n)$ corresponding to the above action angle variables are precisely the tori studied in section 8.*

Now, we prove equality (9.1). We first establish the following lemma, as a consequence of Theorem 10.

**Lemma 10.** *On $\mathcal{V}_{(d_1,\ldots,d_n)}$,*

$$\omega = \sum_{r=1}^{n} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r + \tilde{\omega} \ .$$

*where, for any $1 \leq r \leq n$,*

$$i_{\frac{\partial}{\partial\psi_r}}\tilde{\omega} = 0 \ .$$

*Proof.* Taking the interior product of both sides of identity (7.7) with the restriction of $\omega$ to $\mathcal{V}_{(d_1,\ldots,d_n)}$, we obtain

$$-d(\log J^y) = \sum_{r=1}^{n}(-1)^r \frac{2y}{1+ys_r^2} i_{\frac{\partial}{\partial\psi_r}}\omega \ .$$

On the other hand, from formula (A.7) in Appendix A,

$$d(\log(J^y)) = \sum_{r=1}^{n}(-1)^r \frac{2y}{1+ys_r^2} d\left(\frac{s_r^2}{2}\right) \ .$$

Identification of residues in the $y$ variables yields

$$d\left(\frac{s_r^2}{2}\right) = -i_{\frac{\partial}{\partial\psi_r}}\omega \ , \ r = 1,\ldots,n \ .$$

Since

$$i_{\frac{\partial}{\partial\psi_r}}\left(\sum_{r'=1}^{n} d\left(\frac{s_{r'}^2}{2}\right) \wedge d\psi_{r'}\right) = -d\left(\frac{s_r^2}{2}\right) \ ,$$

this completes the proof. □

Since $d\omega = 0$, we have $d\tilde{\omega} = 0$. Combining this information with $i\frac{\partial}{\partial\psi_r}\tilde{\omega} = 0$ , we conclude that

$$\tilde{\omega} = \pi^*\beta \ ,$$

where $\beta$ is a closed 2-form on $\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}^\sharp$, and

$$\pi(u) := ((s_r(u))_{1\leq r\leq n}, (\chi_r(u)_{1\leq r\leq n}) \ .$$

In order to prove that $\tilde{\omega} = 0$, it is therefore sufficient to prove that $\tilde{\omega} = 0$ on the submanifold

$$\mathcal{V}_{(d_1,\ldots,d_n),\mathrm{red}} := \Phi^{-1}\left(\Omega_n \times \prod_{r=1}^n \mathcal{B}_{d_r}^\sharp\right)$$

given by the equations $\psi_r = 0$ , $r = 1, \ldots, n$.

**Lemma 11.** *The restriction of $\omega$ to $\mathcal{V}_{(d_1,\ldots,d_n),\mathrm{red}}$ is 0.*

*Proof.* Consider the differential form $\alpha$ of of degree 1 defined

$$\langle\alpha(u), h\rangle := \mathrm{Im}(u|h) \ .$$

It is elementary to check that

$$\frac{1}{2}d\alpha = \omega \ ,$$

hence the statement is consequence of the fact that the restriction of $\alpha$ to $\mathcal{V}_{(d_1,\ldots,d_n),\mathrm{red}}$ is 0. Let us prove this stronger fact. By a density argument, we may assume that $n = 2q$ is even, and that the Blaschke products $\chi_r(u)$ have only simple zeroes. Firstly, we describe the tangent space of $\mathcal{V}_{(d_1,\ldots,d_n),\mathrm{red}}$ at a generic point. We use the notation of section 4.2.

**Lemma 12.** *The tangent vectors to $\mathcal{V}_{(d_1,\ldots,d_n),\mathrm{red}}$ at a generic point $u$ where every $\chi_r$ has only simple zeroes are linear combinations with real coefficients of $u_j, u_jH_u(u_\ell), 1 \leq j, \ell \leq q$, and of the following functions, for $\zeta \in \mathbb{C}$ and $1 \leq j, k \leq q$,*

$$\begin{aligned}
\dot{u}_{\chi_{2j-1},\zeta}(z) &:= \left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right) u_j(z)H_u(u_j)(z) \ , \ \chi_{2j-1}(p) = 0 \ , \\
\dot{u}_{\chi_{2k},\zeta}(z) &:= \left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right) zu_k'(z)K_u(u_k')(z) \ , \ \chi_{2k}(p) = 0 \ .
\end{aligned}$$

We assume Lemma 12 and show how it implies Lemma 11. Notice that

$$\begin{aligned}
(u|u_j) &= \|u_j\|^2 \ , \ (u|u_jH_u(u_j)) = \|H_u(u_j)\|^2 \ , \\
(u|H_u(u_\ell)u_j) &= (H_u(u_j)|H_u(u_\ell)) = 0 \ , \ j \neq \ell \ ,
\end{aligned}$$

and therefore $\alpha(u)$ cancels on $u_j, u_j H_u(u_\ell), 1 \le j, \ell \le q$. We now deal with vectors $\dot{u}_{\chi_r,\zeta}$.

$$\begin{aligned}(u|\dot{u}_{\chi_{2j-1},\zeta}) &= \zeta\left(u\Big|\frac{z}{1-\overline{p}z}u_j H_u(u_j)\right) - \overline{\zeta}\left(u\Big|\frac{u_j}{z-p}H_u(u_j)\right)\\ &= \zeta\left(H_u(u_j)\Big|\frac{z}{1-\overline{p}z}H_u(u_j)\right) - \overline{\zeta}\left(H_u^2(u_j)\Big|\frac{u_j}{z-p}\right)\ ,\end{aligned}$$

where we used that $u_j/(z-p)$ belongs to $L^2_+$. Since $H_u^2(u_j) = \rho_j^2 u_j$ and $\rho_j^2|u_j|^2 = |H_u(u_j)|^2$ on the unit circle, we infer

$$(u|\dot{u}_{\chi_{2j-1},\zeta}) = \zeta\left(H_u(u_j)\Big|\frac{z}{1-\overline{p}z}H_u(u_j)\right) - \overline{\zeta}\left(\frac{z}{1-\overline{p}z}H_u(u_j)\Big|H_u(u_j)\right)$$

and consequently

$$\langle \alpha(u), \dot{u}_{\chi_{2j-1},\zeta}\rangle = 2\mathrm{Im}\ \zeta\left(H_u(u_j)\Big|\frac{z}{1-\overline{p}z}H_u(u_j)\right)\ ,$$

which is 0 for every $\zeta \in \mathbb{C}$ if and only if

$$\left(H_u(u_j)\Big|\frac{z}{1-\overline{p}z}H_u(u_j)\right) = 0\ .$$

Let us prove this identity. Set

$$v := \frac{z}{1-\overline{p}z}H_u(u_j)\ .$$

Notice that, since $\chi_{2j-1}(p) = 0$, $v \in E_u(\rho_j)$, and moreover

$$(v|1) = v(0) = 0\ .$$

Therefore

$$(H_u(u_j)|v) = (H_u(v)|u_j) = (H_u(v)|u) = (1|H_u^2(v)) = \rho_j^2(1|v) = 0\ .$$

We conclude that

$$\langle \alpha(u), \dot{u}_{\chi_{2j-1},\zeta}\rangle = 0\ .$$

Similarly, we calculate

$$\begin{aligned}(u|\dot{u}_{\chi_{2k},\zeta}) &= \zeta\left(u\Big|\frac{z}{1-\overline{p}z}zu_k' K_u(u_k')\right) - \overline{\zeta}\left(u\Big|\frac{K_u(u_k')}{z-p}zu_k'\right)\\ &= \zeta\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right) - \overline{\zeta}\left(K_u(u_k')\Big|\frac{K_u(u_k')}{z-p}\right)\ ,\end{aligned}$$

where we have used that $K_u(u_k')/(z-p)$ belongs to $L^2_+$. We conclude that

$$\langle \alpha(u), \dot{u}_{\chi_{2k},\zeta}\rangle = 2\mathrm{Im}\ \zeta\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right)\ ,$$

which is 0 for every $\zeta \in \mathbb{C}$ if and only if

$$\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right) = 0\ .$$

Since $|K_u(u'_k)|^2 = \sigma_k^2 |u'_k|^2$ on the unit circle, we are left to prove

$$\left( u'_k \middle| \frac{z}{1-\overline{p}z} u'_k \right) = 0 \ .$$

Set

$$w := \frac{1}{1-\overline{p}z} u'_k \ .$$

We notice that $w \in F_u(\sigma_k)$, and that $zw \in F_u(\sigma_k)$. Moreover,

$$w = \frac{1}{1-|p|^2} (\overline{p}\chi_p + 1) u'_k \ ,$$

therefore, setting $\chi_{2k} := g_k \chi_p$,

$$K_u(w) = \frac{\sigma_k}{1-|p|^2} (p g_k u'_k + \chi_{2k} u'_k) = \frac{\sigma_k g_k}{1-|p|^2} (p + \chi_p) u'_k = \sigma_k g_k z w \ .$$

In particular,

$$(K_u(w)|1) = K_u(w)(0) = 0 \ .$$

We now conclude as follows,

$$\left( u'_k \middle| \frac{z}{1-\overline{p}z} u'_k \right) = (u'_k | zw) = (u|zw) = (K_u(w)|1) = 0 \ .$$

This completes the proof up to the proof of lemma 12. □

Let us prove lemma 12. We are going to use formulae from section 4.2, namely

$$u(z) = \sum_{j=1}^{q} \chi_{2j-1}(z) h_j(z) \ ,$$

where $\mathcal{H}(z) = (h_\ell(z))_{1 \le \ell \le q}$ satisfies

$$\mathcal{C}(z)\mathcal{H}(z) = \mathbf{1} \ , \ \mathcal{C}(z) = \left( \frac{\rho_\ell - \sigma_k z \chi_{2k}(z) \chi_{2\ell-1}(z)}{\rho_\ell^2 - \sigma_k^2} \right)_{1 \le k, \ell \le q} \ .$$

If we denote by $\dot{}$ the derivative with respect to one of the parameters $\rho_j, \sigma_k$ or one of the coefficients of the $\chi_r$, we have

$$\dot{u}(z) = \sum_{\ell=1}^{q} (\dot{\chi}_{2\ell-1}(z) h_\ell(z) + \chi_{2\ell-1}(z) \dot{h}_\ell(z)) \ ,$$

with

$$\dot{\mathcal{H}}(z) = -\mathcal{C}(z)^{-1} \dot{\mathcal{C}}(z) \mathcal{H}(z) \ .$$

In the case of the derivative with respect to $\rho_j$, one gets

$$\dot{h}_\ell(z) = h_j(z) \sum_{k=1}^{q} \frac{(-1)^{k+\ell} \Delta_{k\ell}(z) (\rho_j^2 + \sigma_k^2 - 2\sigma_k \rho_j z \chi_{2j-1}(z) \chi_{2k}(z))}{\det \mathcal{C}(z) (\rho_j^2 - \sigma_k^2)^2} \ ,$$

and therefore, from formula (4.16),

$$\begin{aligned}\dot{u}(z) &= h_j(z)\sum_{k=1}^{q} u_k'(z)\frac{(\rho_j^2+\sigma_k^2-2\sigma_k\rho_j z\chi_{2j-1}(z)\chi_{2k}(z))}{(\rho_j^2-\sigma_k^2)^2}\\ &= \frac{H_u(u_j)(z)}{\rho_j}\sum_{k=1}^{q}\frac{\rho_j^2+\sigma_k^2}{(\rho_j^2-\sigma_k^2)^2}u_k'(z)-2\rho_j u_j(z)\sum_{k=1}^{q}\frac{zK_u(u_k')(z)}{(\rho_j^2-\sigma_k^2)^2}\ .\end{aligned}$$

Observe that

$$zK_u(u_k')(z)=[SS^*H_u(u_k')](z)=H_u(u_k')(z)-\kappa_k^2\ ,$$

and that $u_k'$ is a linear combination with real coefficients of $u_\ell$ in view of (2.1). We infer that, in this case, $\dot{u}$ is a linear combination with real coefficients of $u_j$ and $u_m H_u(u_\ell)$.

In the case of the derivative with respect to $\sigma_k$, one similarly gets

$$\dot{h}_\ell(z)=\sum_{j=1}^{q}h_j(z)\frac{(-1)^{k+\ell}\Delta_{k\ell}(z)\left(z\chi_{2j-1}(z)\chi_{2k}(z)(\rho_j^2+\sigma_k^2)-2\sigma_k\rho_j\right)}{\det\mathcal{C}(z)(\rho_j^2-\sigma_k^2)^2}\ ,$$

and therefore, from formula (4.16),

$$\begin{aligned}\dot{u}(z) &= u_k'(z)\sum_{j=1}^{q}h_j(z)\frac{z\chi_{2j-1}(z)\chi_{2k}(z)(\rho_j^2+\sigma_k^2)-2\sigma_k\rho_j}{(\rho_j^2-\sigma_k^2)^2}\\ &= \frac{zK_u(u_k')(z)}{\sigma_k}\sum_{j=1}^{q}\frac{(\rho_j^2+\sigma_k^2)u_j(z)}{(\rho_j^2-\sigma_k^2)^2}-2\sigma_k u_k'(z)\sum_{j=1}^{q}\frac{H_u(u_j)(z)}{(\rho_j^2-\sigma_k^2)^2}\ ,\end{aligned}$$

which is a linear combination with real coefficients of $u_j$ and $u_j H_u(u_\ell)$.

In the case of a derivative with respect to one of the zeroes of $\chi_{2j-1}$, we obtain a simpler identity,

$$\dot{h}_\ell(z)=\frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)}u_j(z)\sum_{k=1}^{q}\frac{(-1)^{k+\ell}\Delta_{k\ell}(z)z\sigma_k\chi_{2k}(z)}{\det\mathcal{C}(z)(\rho_j^2-\sigma_k^2)}\ ,$$

and therefore, from formulae (4.16), (2.1) and (A.11),

$$\begin{aligned}\dot{u}(z) &= \frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)}u_j(z)+\sum_{\ell=1}^{q}\chi_{2\ell-1}(z)\dot{h}_\ell(z)\\ &= \frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)}\frac{u_j(z)H_u(u_j)(z)}{\tau_j^2}\ .\end{aligned}$$

In the case of a derivative with respect to one of the zeroes of $\chi_{2k}$, we obtain similarly,

$$\dot{h}_\ell(z)=\frac{\dot{\chi}_{2k}(z)}{\chi_{2k}(z)}\frac{(-1)^{k+\ell}\Delta_{k\ell}(z)zK_u(u_k')(z)}{\det\mathcal{C}(z)}\ ,$$

and therefore

$$\begin{aligned}\dot{u}(z) &= \sum_{\ell=1}^{q} \chi_{2\ell-1}(z)\dot{h}_\ell(z) \\ &= \frac{\dot{\chi}_{2k}(z)}{\chi_{2k}(z)} \frac{u'_k(z) z K_u(u'_k)(z)}{\kappa_k^2} \ .\end{aligned}$$

The proof of Lemma 12 is completed by observing that, since $\chi_r$ is a product of functions $\chi_p$ for $|p| < 1$, $\dot{\chi}_r/\chi_r$ is a sum of terms of the form

$$\left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right)$$

where $\zeta := \dot{p}$.

## Appendix A. Some Bateman-type formulae

Let $u \in VMO_+$. Denote by $(\rho_j)$ the decreasing sequence of elements of $\Sigma_H(u)$, and by $(\sigma_k^2)$ the decreasing sequence of elements of $\Sigma_K(u)$. Recall that both sequences are either finite or infinite, with the same number of elements, and we have

$$\rho_1^2 > \sigma_1^2 > \rho_2^2 > \dots$$

Denote by $u_j$ the orthogonal projection of $u$ onto $E_u(\rho_j) = \ker(H_u^2 - \rho_j^2 I)$, and by $\tau_j$ the norm of $u_j$. Similarly, denote by $u'_k$, the orthogonal projection of $u$ onto $F_u(\sigma_k) := \ker(K_u^2 - \sigma_k^2 I)$, and by $\kappa_k$ the norm of $u'_k$. In this section, we state and prove several formulae connecting these sequences. These formulae are based on the special case of a general formula for the resolvent of a finite rank perturbation of an operator, which seems to be due to Bateman [4] in the framework of Fredholm integral equations. Further references can be found in Chap. II, sect. 4.6 of [16], section 106 of [2] and [20], from which we borrowed this information.

**Proposition 3.** *The following functions coincide respectively for $x$ outside the set $\{\frac{1}{\rho_j^2}\}$ and outside the set $\{\frac{1}{\sigma_k^2}\}$.*

$$\prod_j \frac{1-x\sigma_j^2}{1-x\rho_j^2} = 1 + x\sum_j \frac{\tau_j^2}{1-x\rho_j^2} \tag{A.1}$$

$$\prod_j \frac{1-x\rho_j^2}{1-x\sigma_j^2} = 1 - x\left(\sum_j \frac{\kappa_j^2}{1-x\sigma_j^2}\right) \ . \tag{A.2}$$

*Furthermore,*

$$1 - \sum_j \frac{\tau_j^2}{\rho_j^2} = \prod_j \frac{\sigma_j^2}{\rho_j^2} \ , \tag{A.3}$$

*and, if* $\prod_j \frac{\sigma_j^2}{\rho_j^2} = 0$,

$$\sum_j \frac{\tau_j^2}{\rho_j^4} = \frac{1}{\rho_1^2} \prod_j \frac{\sigma_j^2}{\rho_{j+1}^2} \ . \tag{A.4}$$

*The* $\tau_j^2$*'s are given by*

$$\tau_j^2 = \left(\rho_j^2 - \sigma_j^2\right) \prod_{k \neq j} \frac{\rho_j^2 - \sigma_k^2}{\rho_j^2 - \rho_k^2} \ , \tag{A.5}$$

*and the* $\kappa_j^2$*'s by*

$$\kappa_j^2 = (\rho_j^2 - \sigma_j^2) \prod_{k \neq j} \frac{\sigma_j^2 - \rho_k^2}{\sigma_j^2 - \sigma_k^2}. \tag{A.6}$$

*Proof.* For $x \notin \{\frac{1}{\rho_j^2}\}$, we set

$$J(x) := ((I - xH_u^2)^{-1}(1)|1).$$

We claim that

$$J(x) = \prod_j \frac{1 - x\sigma_j^2}{1 - x\rho_j^2}. \tag{A.7}$$

Indeed, let us first assume that $H_u^2$ and $K_u^2$ are of trace class and compute the trace of $(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}$. We write

$$[(I-xH_u^2)^{-1} - (I-xK_u^2)^{-1}](f) = \frac{x}{J(x)}(f|(I-xH_u^2)^{-1}u) \cdot (I-xH_u^2)^{-1}u.$$

Consequently, taking the trace, we get

$$\mathrm{Tr}[(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}] = \frac{x}{J(x)} \|(I - xH_u^2)^{-1}u\|^2.$$

Since, on the one hand

$$\|(I - xH_u^2)^{-1}u\|^2 = ((I - xH_u^2)^{-1}H_u^2(1)|1) = J'(x)$$

and on the other hand

$$\begin{aligned} \mathrm{Tr}[(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}] &= x\mathrm{Tr}[H_u^2(I - xH_u^2)^{-1} - K_u^2(I - xK_u^2)^{-1}] \\ &= x \sum_j \left( \frac{\rho_j^2}{1 - \rho_j^2 x} - \frac{\sigma_j^2}{1 - \sigma_j^2 x} \right) \ , \end{aligned}$$

where we used proposition 2. On the other hand,

$$\sum_j \left( \frac{\rho_j^2}{1 - \rho_j^2 x} - \frac{\sigma_j^2}{1 - \sigma_j^2 x} \right) = \frac{J'(x)}{J(x)} \ , \ x \notin \left\{ \frac{1}{\rho_j^2}, \frac{1}{\sigma_j^2} \right\} \ . \tag{A.8}$$

This gives equality (A.7) for $H_u^2$ and $K_u^2$ of trace class. To extend this formula to compact operators, we remark that $\sum_j (\rho_j^2 - \sigma_j^2)$ converges since $0 \leq \rho_j^2 - \sigma_j^2 \leq \rho_j^2 - \rho_{j+1}^2$ and $(\rho_j^2)$ tends to zero from the

compactness of $H_u^2$. Hence the infinite product in formula (A.7) and the above computation makes sense for compact operators.

On the other hand, for $x \notin \{\frac{1}{\rho_j^2}\}$, if $\tau_j$ denotes the norm of $u_j$,

$$\begin{aligned} J(x) &= ((I - xH_u^2)^{-1}(1)|1) = 1 + x((I - xH_u^2)^{-1}(u)|u) \\ &= 1 + x(\sum_j (I - xH_u^2)^{-1}(u_j)|u)) = 1 + x \sum_j \frac{\tau_j^2}{1 - x\rho_j^2} \end{aligned}$$

hence

$$\prod_j \frac{1 - x\sigma_j^2}{1 - x\rho_j^2} = 1 + x \sum_j \frac{\tau_j^2}{1 - x\rho_j^2} \,. \tag{A.9}$$

Passing to the limit as $x$ goes to $-\infty$ in (A.1), we obtain (A.3). If we assume that the left hand side of (A.3) cancels, then (A.1) can be rewritten as

$$\prod_j \frac{1 - x\sigma_j^2}{1 - x\rho_j^2} = \sum_j \frac{\tau_j^2}{\rho_j^2(1 - x\rho_j^2)} \,.$$

Multiplying by $x$ and passing to the limit as $x$ goes to $-\infty$ in this new identity, we obtain (A.4). Furthermore, we multiply both terms of (A.1) by $(1 - x\rho_j^2)$ and we let $x$ go to $1/\rho_j^2$. We get

$$\tau_j^2 = \left(\rho_j^2 - \sigma_j^2\right) \prod_{k \neq j} \frac{\rho_j^2 - \sigma_k^2}{\rho_j^2 - \rho_k^2} \,.$$

For Equality (4.5), we do almost the same analysis. First, we establish as above that

$$\frac{1}{J(x)} = 1 - x((I - xK_u^2)^{-1}(u)|u) = 1 - x \sum_k \frac{\kappa_k^2}{1 - x\sigma_k^2}$$

where $\kappa_k^2 = \|u_k'\|^2$.

Identifying this expression with

$$\frac{1}{J(x)} = \prod_j \frac{1 - x\rho_j^2}{1 - x\sigma_j^2}$$

we get

$$\kappa_j^2 = (\rho_j^2 - \sigma_j^2) \prod_{k \neq j} \frac{\sigma_j^2 - \rho_k^2}{\sigma_j^2 - \sigma_k^2}.$$

□

As a consequence of the previous lemma, we get the following couple of corollaries.

**Corollary 4.** *For any $k, r \geq 1$, we have*

$$\sum_j \frac{\tau_j^2}{\rho_j^2 - \sigma_k^2} = 1 \tag{A.10}$$

$$\sum_j \frac{\kappa_j^2}{\rho_k^2 - \sigma_j^2} = 1 \tag{A.11}$$

$$\sum_j \frac{\tau_j^2}{(\rho_j^2 - \sigma_k^2)(\rho_j^2 - \sigma_r^2)} = \frac{1}{\kappa_k^2}\delta_{kr} \tag{A.12}$$

$$\sum_j \frac{\kappa_j^2}{(\sigma_j^2 - \rho_k^2)(\sigma_j^2 - \rho_r^2)} = \frac{1}{\tau_k^2}\delta_{kr} \tag{A.13}$$

*Proof.* The first two equalities (A.10) and (A.11) are obtained by making $x = \frac{1}{\sigma_k^2}$ and $x = \frac{1}{\rho_k^2}$ respectively in formula (A.1) and formula (A.2). For equality (A.12) in the case $k = r$, we first make the change of variable $y = 1/x$ in formula (A.1) then differentiate both sides with respect to $y$ and make $y = \sigma_r^2$. Equality (A.13) in the case $k = r$ follows by differentiating equation (A.2) and making $x = \frac{1}{\rho_m^2}$. Both equalities in the case $m \neq p$ follow directly respectively from equality (A.10) and equality (A.11). □

**Corollary 5.** *The kernel of $H_u$ is $\{0\}$ if and only if*

$$\prod_j \frac{\sigma_j^2}{\rho_j^2} = 0 \ , \ \prod_j \frac{\sigma_j^2}{\rho_{j+1}^2} = \infty \ .$$

*Proof.* By the first part of theorem 4 in [10] — which is independent of multiplicity assumptions— , the kernel of $H_u$ is $\{0\}$ if and only if $1 \in \overline{R} \setminus R$, where $R = \mathrm{Ran}(H_u)$ denotes the range of $H_u$. On the other hand,

$$u = \sum_j u_j = \sum_j \frac{H_u(H_u(u_j))}{\rho_j^2} \ ,$$

hence the orthogonal projection of 1 onto $\overline{R}$ is

$$\sum_j \frac{H_u(u_j)}{\rho_j^2} \ .$$

Consequently, $1 \in \overline{R}$ if and only if

$$1 = \sum_j \left\| \frac{H_u(u_j)}{\rho_j^2} \right\|^2 = \sum_j \frac{\tau_j^2}{\rho_j^2} \ .$$

Moreover, if this is the case,

$$1 = \sum_j \frac{H_u(u_j)}{\rho_j^2}$$

and $1 \in R$ if and only if the series $\sum_j u_j/\rho_j^2$ converges, which is equivalent to

$$\sum_j \frac{\tau_j^2}{\rho_j^4} < \infty \ .$$

Hence $1 \in \overline{R} \setminus R$ if and only if

$$\sum_j \frac{\tau_j^2}{\rho_j^2} = 1 \ , \ \sum_j \frac{\tau_j^2}{\rho_j^4} = \infty \ ,$$

which is the claim, in view of identities (A.3) and (A.4). □

## Appendix B. The structure of finite Blaschke products

In this appendix, we describe the set $\mathcal{B}_d$ of Blaschke products of degree $d$. Every element of $\mathcal{B}_d$ can be written

$$\Psi = \mathrm{e}^{-i\psi}\chi \ ,$$

where $\psi \in \mathbb{T}$ and $\chi \in \mathcal{B}_d^\sharp$ is a Blaschke product of the form

$$\chi(z) = \frac{P(z)}{z^d \overline{P}\left(\frac{1}{z}\right)} \ ,$$

where $P(z) = z^d + a_1 z^{d-1} + \cdots + a_d$ is a monic polynomial of degree $d$ with all its zeroes in the open unit disc $\mathbb{D}$. Conversely, if $P$ is such a polynomial, then $\chi$ is a Blaschke product of degree $d$. We denote by $\mathcal{O}_d$ the open subset of $\mathbb{C}^d$ made of such $(a_1, \ldots, a_d)$. The following result is connected to the Schur–Cohn criterion [26], [6], and is classical in control theory, see *e.g.* [15] and references therein. For the sake of completeness, we give a self contained proof.

**Proposition 4.** *For every $d \geq 1$ and $(a_1, \ldots, a_d) \in \mathbb{C}^d$, the following two assertions are equivalent.*

(1) $(a_1, \ldots, a_d) \in \mathcal{O}_d$ .
(2) $|a_d| < 1$ *and*

$$\left( \frac{a_k - a_d \overline{a}_{d-k}}{1 - |a_d|^2} \right)_{1 \leq k \leq d-1} \in \mathcal{O}_{d-1} \ .$$

*In particular, for every $d \geq 0$, $\mathcal{O}_d$ is diffeomorphic to $\mathbb{R}^{2d}$.*

*Proof.* Consider the rational functions

$$\chi(z) = \frac{z^d + a_1 z^{d-1} + \cdots + a_d}{1 + \overline{a}_1 z + \cdots + \overline{a}_d z^d} \ ,$$

and

$$\tilde{\chi}(z) = \frac{\chi(z) - \chi(0)}{1 - \overline{\chi(0)}\chi(z)} = z \frac{z^{d-1} + b_1 z^{d-2} + \cdots + b_{d-1}}{1 + \overline{b}_1 z + \cdots + \overline{b}_{d-1} z^{d-1}} \ , \ b_k := \frac{a_k - a_d \overline{a}_{d-k}}{1 - |a_d|^2} \ .$$

If (1) holds true, then $\chi \in \mathcal{B}_d$, which implies

$$\forall z \in \mathbb{D}, |\chi(z)| < 1 \ , \ |\chi(\mathrm{e}^{ix})| = 1 \ . \tag{B.1}$$

In particular, $\chi(0) = a_d \in \mathbb{D}$, and therefore the numerator and the denominator of $\tilde{\chi}$ have no common root. Moreover,

$$\forall z \in \mathbb{D}, |\tilde{\chi}(z)| < 1 \ , \ |\tilde{\chi}(\mathrm{e}^{ix})| = 1 \ . \tag{B.2}$$

This implies $\tilde{\chi} \in \mathcal{B}_d$, hence (2). Conversely, if (2) holds, then $\tilde{\chi}$ satisfies (B.2) and has degree $d$, hence

$$\chi(z) = \frac{\tilde{\chi}(z) + a_d}{1 + \overline{a}_d \tilde{\chi}(z)}$$

satisfies (B.1) and has degree $d$, whence (1).

The second statement follows from an easy induction argument on $d$, since $\mathcal{O}_1 = \mathbb{D}$ is diffeomorphic to $\mathbb{R}^2$. □

## Appendix C. Two results by Adamyan–Arov–Krein

In this appendix, we recall the proof of two important results by Adamyan–Arov–Krein, which have been used throughout our paper. The proof is translated from [1] into our representation of Hankel operators, and is given for the convenience of the reader.

**Theorem** (Adamyan, Arov, Krein [1])**.** *Let $u \in VMO_+ \setminus \{0\}$. Denote by $(\underline{s}_k(u))_{k\geq 0}$ the sequence of singular values of $H_u$, namely the eigenvalues of $|H_u| := \sqrt{H_u^2}$, in decreasing order, and repeated according to their multiplicity.Let $k \geq 0, m \geq 1$, such that*

$$\underline{s}_{k-1}(u) > \underline{s}_k(u) = \cdots = \underline{s}_{k+m-1}(u) = s > \underline{s}_{k+m}(u) \ ,$$

*with the convention $\underline{s}_{-1}(u) := +\infty$.*

(1) *For every $h \in E_u(s)\setminus\{0\}$, there exists a polynomial $P \in \mathbb{C}_{m-1}[z]$ such that*

$$\forall z \in \mathbb{D} \ , \ \frac{sh(z)}{H_u(h)(z)} = \frac{P(z)}{z^{m-1}\overline{P}\left(\frac{1}{z}\right)} \ .$$

(2) *There exists a rational function $r$ with no pole on $\overline{\mathbb{D}}$ such that $\mathrm{rk}(H_r) = k$ and*

$$\|H_u - H_r\| = s \ .$$

*Proof.* We start with the case $k = 0$. In this case the statement (2) is trivial, so we just have to prove (1). This is a consequence of the following lemma.

**Lemma 13.** *Assume $s = \|H_u\|$. For every $h \in E_u(s) \setminus \{0\}$, consider the following inner outer decompositions,*

$$h = ah_0 \ , \ s^{-1}H_u(h) = bf_0 \ .$$

*If $c$ is an arbitrary inner divisor of $ab$, $ab = cc'$, then $ch_0 \in E_u(s)$, with*

$$H_u(ch_0) = sc'f_0 \ , \ H_u(c'f_0) = sch_0 \ . \tag{C.1}$$

*In particular, $a, b$ are finite Blaschke products and*

$$\deg(a) + \deg(b) + 1 \leq \dim E_u(s) \ . \tag{C.2}$$

*Furthermore, there exists an outer function $h_0$ such that, if $m := \dim E_u(s)$,*

$$E_u(s) = \mathbb{C}_{m-1}[z]h_0 \ , \tag{C.3}$$

*and there exists $\varphi \in \mathbb{T}$ such that, for every $P \in \mathbb{C}_{m-1}[z]$,*

$$H_u(Ph_0)(z) = s\mathrm{e}^{i\varphi} z^{m-1} \overline{P}\left(\frac{1}{z}\right) h_0(z) \ . \tag{C.4}$$

Let us prove this lemma. We will need a number of elementary properties of Toeplitz operators $T_b$ defined by equation (6.2), where $b$ is a function in $L^\infty_+ := L^2_+ \cap L^\infty$, which we recall below. In what follows, $b$ denotes a function in $L^\infty_+$ and $u \in BMO_+$.

(1)
$$H_u T_b = T_{\overline{b}} H_u = H_{T_{\overline{b}} u} \ .$$

(2) If $|b| \leq 1$ on $\mathbb{S}^1$,
$$H_u^2 \geq T_{\overline{b}} H_u^2 T_b \ .$$

(3) If $|b| = 1$ on $\mathbb{S}^1$, namely $b$ is an inner function,
$$\forall f \in L^2_+ \ , \ f = T_b T_{\overline{b}} f \Longleftrightarrow \|f\| = \|T_{\overline{b}} f\| \ .$$

Indeed, (1) is just equivalent to the elementary identities

$$\Pi(u\overline{bh}) = \Pi(\overline{b}\Pi(u\overline{h})) = \Pi((\Pi(\overline{b}u)\overline{h}) \ .$$

As for (2), we observe that $T_b^* = T_{\overline{b}}$ and

$$\|T_{\overline{b}} h\| \leq \|\overline{b}h\| \leq \|h\| \ .$$

Hence, using (1),

$$(H_u^2 h|h) - (T_{\overline{b}} H_u^2 T_b h|h) = \|H_u(h)\|^2 - \|T_{\overline{b}} H_u(h)\|^2 \geq 0 \ .$$

Finally, for (3) we remark that, if $b$ is inner, $T_{\overline{b}} T_b = I$ and $T_b T_{\overline{b}}$ is the orthogonal projector onto the range of $T_b$, namely $bL^2_+$. Since $\|T_{\overline{b}} f\| = \|T_b T_{\overline{b}} f\|$, (3) follows.

Let us come back to the proof of Lemma 13. Starting from

$$H_u(h) = sf \ , \ H_u(f) = sh \ , \ h = ah_0 \ , \ f = bf_0 \ , \ ab = cc' \ ,$$

we obtain, using property (1),

$$T_{\overline{c'}} H_u(ch_0) = H_u(cc'h_0) = T_{\overline{b}} H_u(h) = sf_0 \ .$$

In particular,

$$\|H_u(ch_0)\| \geq \|T_{\overline{c'}} H_u(ch_0)\| = s\|f_0\| = s\|f\| = s\|h\| = s\|ch_0\| \ .$$

Since $s = \|H_u\|$, all the above inequalities are equalities, hence $ch_0 \in E_u(s)$, and, using (3),

$$H_u(ch_0) = T_{c'} T_{\overline{c'}} H_u(ch_0) = sc'f_0 \ .$$

The second identity in (C.1) immediately follows. Remark that, if $\Psi$ is an inner function of degree at least $d$, there exist $d+1$ linearly independent inner divisors of $\Psi$ in $L^\infty_+$. Then inequality (C.2) follows. Let us come to the last part. Since $\dim E_u(s) = m$, there exists $h \in E_u(s) \setminus \{0\}$ such that the first $m-1$ Fourier coefficients of $h$ cancel, namely

$$h = z^{m-1}\tilde{h} \ .$$

Considering the inner outer decompositions

$$\tilde{h} = ah_0 \ , \ H_u(h) = sbf_0 \ ,$$

and applying the first part of the lemma, we conclude that $\deg(a) + \deg(b) = 0$, hence, up to a slight change of notation, $a = b = 1$, and, for $\ell = 0, 1, \dots, m-1$,

$$H_u(z^\ell h_0) = sz^{m-1-\ell} f_0 \ , \ H_u(z^{m-\ell-1} f_0) = sz^\ell h_0 \ .$$

This implies

$$E_u(s) = \mathbb{C}_{m-1}[z]h_0 = \mathbb{C}_{m-1}[z]f_0 \ .$$

Since $\|h_0\| = \|h\| = \|f\| = \|f_0\|$, it follows that $f_0 = \mathrm{e}^{i\varphi}h_0$, and (C.4) follows from the antilinearity of $H_u$. The proof of Lemma 13 is complete.

Let us complete the proof of the theorem by proving the case $s < \|H_u\|$. The crucial new observation is the following.

**Lemma 14.** *There exists a function $\phi \in L^\infty$ such that $|\phi| = 1$ on $\mathbb{S}^1$, and such that the operators $H_u$ and $H_{s\Pi(\phi)}$ coincide on $E_u(s)$.*

Let us prove this lemma. For every pair $(h, f)$ of elements of $E_u(s)$ such that $H_u(h) = sf, H_u(f) = sh$, we claim that the function

$$\phi := \frac{f}{\overline{h}} \ ,$$

does not depend on the choice of the pair $(h, f)$. Indeed, it is enough to check that, if $(h', f')$ is another such pair,

$$f\overline{h}' = f'\overline{h} \ .$$

In fact, for every $n \geq 0$,

$$\begin{aligned} s(f\overline{h}'|z^n) &= (H_u(h)|S^n h') = ((S^*)^n H_u(h)|h') \\ &= (H_u(S^n h)|h') = (H_u(h')|S^n h) = s(f'\overline{h}|z^n) \ . \end{aligned}$$

Changing the role of $(h, h')$ and $(f, f')$, we get the claim. Finally the fact $|\phi| = 1$ comes from applying the above identity to the pairs $(h, f)$ and $(f, h)$. Then we just have to check that, for every such pair,

$$H_{s\Pi(\phi)}(h) = s\Pi(\Pi(\phi)\overline{h}) = s\Pi(\phi\overline{h}) = sf \ .$$

This completes the proof of Lemma 14.

Let us come to part (2) of the Theorem. Introduce

$$v := s\Pi(\phi) \ .$$

We are going to show that $r := u - v$ is a rational function with no pole on $\overline{\mathbb{D}}$, $\mathrm{rk}(H_r) = k$ and

$$\|H_u - H_r\| = s \ .$$

Since, for every $h \in L^2_+$,

$$H_v(h) = s\Pi(\phi\overline{h}) \ ,$$

we infer $\|H_v\| \leq s$, and from $E_u(s) \subset E_v(s)$, we conclude

$$\|H_v\| = s \ .$$

Because of (1.6), $H_u$ and $H_v$ coincide on the smallest shift invariant closed subspace of $L^2_+$ containing $E_u(s)$. By Beurling's theorem [5], this subspace is $aL^2_+$ for some inner function $a$. Then $H_r = 0$ on $aL^2_+$, hence the rank of $H_r$ is at most the dimension of $(aL^2_+)^\perp$. Since

$$\|H_u - H_r\| = \|H_v\| = s < \underline{s}_{k-1}(u) \ ,$$

the rank of $H_r$ cannot be smaller than $k$, and the result will follow by proving that the dimension of $(aL^2_+)^\perp$ is $k$.

We can summarize the above construction as

$$H_{T_{\overline{a}}u} = H_u T_a = H_v T_a = H_{T_{\overline{a}}v} \ .$$

The above Hankel operator is compact and its norm is at most $s$. In fact, if $H_u(h) = sf$ , $H_u(f) = sh$, with $f = a\tilde{f}$, it is clear from property (1) above that

$$H_{T_{\overline{a}}u}(h) = s\tilde{f} \ , \ H_{T_{\overline{a}}u}(\tilde{f}) = sh \ .$$

In particular,

$$\|H_{T_{\overline{a}}u}\| = s \ .$$

Applying property (C.1) from Lemma 13, we conclude that there exists an outer function $h_0$ such that

$$ch_0 \in E_{T_{\overline{a}}u}(s)$$

for every inner divisor $c$ of $a$. Moreover, $a$ is a Blaschke product of finite degree $d$. Since $h_0$ is outer, it does not vanish at any point of $\mathbb{D}$, therefore, it is easy to find $d$ inner divisors $c_1, \ldots, c_d$ of $a$ such that $c_1h_0, \ldots, c_dh_0$ are linearly independent and generate a vector subspace $\tilde{E}$ satisfying

$$\tilde{E} \cap aL^2_+ = \{0\} \ .$$

Consequently, we obtain

$$\tilde{E} \oplus E_u(s) \subset E_{T_{\overline{a}}u}(s),$$

whence

$$d' := \dim E_{T_{\overline{a}}u}(s) \geq d + m \ . \tag{C.5}$$

On the other hand, by property (2) above, we have

$$H^2_{T_{\overline{a}}u} = T_{\overline{a}} H^2_u T_a \leq H^2_u \ ,$$

therefore, from the min-max formula,

$$\forall n, \underline{s}_n(T_{\overline{a}}u) \leq \underline{s}_n(u) \ .$$

In particular, from the definition of $d'$,

$$s = \underline{s}_{d'-1}(T_{\overline{a}}u) \leq \underline{s}_{d'-1}(u) \ ,$$

which imposes, in view of the assumption, $d' - 1 \leq k + m - 1$, in particular, in view of (C.5),

$$d \leq k \ .$$

Finally, notice that, since $a$ has degree $d$, the dimension of $(aL^2_+)^\perp$ is $d$. Hence, by the min-max formula again,

$$\underline{s}_d(u) \leq \sup_{h \in aL^2_+ \setminus \{0\}} \frac{\|H_u(h)\|}{\|h\|} \leq s < \underline{s}_{k-1}(u) \ .$$

This imposes $d \geq k$, and finally

$$d = k \ , \ d' = k + m \ ,$$

and part (2) of the theorem is proved.

In order to prove part (1), we apply properties (C.3) and (C.4) of Lemma 13. We describe elements of $E_{T_{\overline{a}}u}(s)$ as

$$h(z) = Q(z) h_0(z) \ ,$$

where $h_0$ is outer and $Q \in \mathbb{C}_{k+m-1}[z]$. Moreover, if $h \in E_u(s)$, then $h = a\tilde{h}$, $H_u(h) = sa\tilde{f}$, where $\tilde{h}, \tilde{f} \in E_{T_{\overline{a}}u}(s)$. This reads

$$Q(z) = a(z)\tilde{Q}(z) \ ,$$

If we set

$$a(z) = \frac{z^k \overline{D}\left(\frac{1}{z}\right)}{D(z)} \ ,$$

where $D \in \mathbb{C}_k[z]$ and $D(0) = 1$, and $D$ has no zeroes in $\mathbb{D}$, this implies

$$Q(z) = z^k \overline{D}\left(\frac{1}{z}\right) P(z) \ , \ P \in \mathbb{C}_{m-1}[z] \ .$$

Moreover,

$$H_{T_{\overline{a}}u}(h)(z) = s\mathrm{e}^{i\varphi} z^{m+k-1} \overline{Q}\left(\frac{1}{z}\right) h_0(z) = s\mathrm{e}^{i\varphi} D(z) z^{m-1} \overline{P}\left(\frac{1}{z}\right) h_0(z) \ ,$$

and

$$H_u(h)(z) = a(z) H_{T_{\overline{a}}u}(h)(z) = s\mathrm{e}^{i\varphi} z^k \overline{D}\left(\frac{1}{z}\right) z^{m-1} \overline{P}\left(\frac{1}{z}\right) h_0(z) \ .$$

Changing $P$ into $P\mathrm{e}^{-i\varphi/2}$, this proves part (1) of the theorem. □

## References


[1] Adamyan, V. M., Arov, D. Z., Krein, M. G., *Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem.* (Russian) Mat. Sb. (N.S.) 86(128) (1971), 34–75; English transl. Math USSR. Sb. 15 (1971), 31–73.

[2] Akhiezer, N.I., Glazman, I.M., *Theory of Linear Operators in Hilbert Space*, 2nd ed., Nauka Moscow, 1966 (Russian); English transl. of the 1st ed., Frederick Ungar Publishing, New York, vol.1, 1961; vol. 2, 1963.

[3] Baratchart, L., Seyfert, F., *An $L^p$ analog to AAK theory for $p \geq 2$*, J. Funct. Analysis 191 (2002), 52–122.

[4] Bateman, H., *A formula for solving function of a certain integral equation of the second kind*, Messenger Math. 37 (1908), 179–187.

[5] Beurling, A., *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949), 239–255.

[6] Cohn, A., *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. 14 (1922), 110–148.

[7] Fefferman, C., *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[8] Gérard, P., Grellier, S., *The cubic Szegő equation* , Ann. Scient. Éc. Norm. Sup. 43 (2010), 761–810.

[9] Gérard, P., Grellier, S., *Invariant Tori for the cubic Szegő equation*, Invent. Math. 187 (2012), 707–754.

[10] Gérard, P., Grellier, S., *Inverse spectral problems for compact Hankel operators*, Journal of the Institute of Mathematics of Jussieu, available on CJO2013. doi:10.1017/S1474748013000121, published online on 18 April 2013.

[11] Gérard, P., Grellier, S., *An explicit formula for the cubic Szegő equation*, to appear in Trans. A.M.S.

[12] Gérard, P., Koch, H., *The cubic Szegő equation on $BMO$*, in preparation.

[13] Gérard, P., Pushnitski, A., *An inverse problem for self-adjoint positive Hankel operators*, preprint, arXiv:1401.2042[math.SP].

[14] Hartman, P., *On completely continuous Hankel matrices ,* Proc. Amer. Math. Soc. 9 (1958), 862–866.

[15] Hernández B. A., Frías-Armenta M. E., Verduzco F., *On differential structures of polynomial spaces in control theory*, Journal of Systems Science and Systems Engineering 21 (2012), 372–382.

[16] Kantorovitch, L.V., Krylov, V.I., *Approximate Methods of Higher Analysis*, 5th ed.,FML, Moscow, 1962 (Russian); English transl. of the 3rd ed. Benter Interscience Publishers, New York, 1958.

[17] Kappeler, T., Pöschel, J., *KdV & KAM*, A Series of Modern Surveys in Mathematics, vol. 45, Springer-Verlag, 2003.

[18] Levitan, B.M., Zhikov, V.V., *Almost periodic functions and differential equations*, Cambridge University Press, 1982.

[19] Megretskii, A V., Peller, V. V., and Treil, S. R., *The inverse problem for selfadjoint Hankel operators*, Acta Math. 174 (1995), 241–309.

[20] Nazarov, A.I., *On a set of Gaussian random functions*, translated from Russian, Theory Probab. Appl., 54 (2010), 203–216.

[21] Nehari, Z., *On bounded bilinear forms.* Ann. Math. 65, 153–162 (1957).

[22] Nikolskii, N. K., *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz.* Translated from the French by Andreas Hartmann. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[23] Peller, V.V., *Smoothness of Schmidt functions of smooth Hankel operators* In Lect. Notes. Math. 1302, Springer-Verlag, Berlin, 1988, 237–246.
[24] Peller, V.V., *Hankel Operators and their applications* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.
[25] Rudin, W., *Real and complex analysis.* Tata Mc Graw Hill, New Hill, 1974.
[26] Schur, I., *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind,* J. Reine Angew. Math., 148 (1918), 122–145.

Université Paris-Sud XI, Laboratoire de Mathématiques d'Orsay, CNRS, UMR 8628, et Institut Universitaire de France
*E-mail address*: Patrick.Gerard@math.u-psud.fr

Fédération Denis Poisson, MAPMO-UMR 6628, Département de Mathématiques, Université d'Orleans, 45067 Orléans Cedex 2, France
*E-mail address*: Sandrine.Grellier@univ-orleans.fr