Asymptotic behavior of the quadratic variation of the sum of two Hermite processes of consecutive orders

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ASYMPTOTIC BEHAVIOR OF THE QUADRATIC VARIATION OF THE SUM OF TWO HERMITE PROCESSES OF CONSECUTIVE ORDERS

M. CLAUSEL, F. ROUEFF, M. S. TAQQU, AND C. TUDOR

Abstract. Hermite processes are self–similar processes with stationary increments which appear as limits of normalized sums of random variables with long range dependence. The Hermite process of order 1 is fractional Brownian motion and the Hermite process of order 2 is the Rosenblatt process. We consider here the sum of two Hermite processes of order $q \geq 1$ and $q+1$ and of different Hurst parameters. We then study its quadratic variations at different scales. This is akin to a wavelet decomposition. We study both the cases where the Hermite processes are dependent and where they are independent. In the dependent case, we show that the quadratic variation, suitably normalized, converges either to a normal or to a Rosenblatt distribution, whatever the order of the original Hermite processes.

1. Introduction

The (centered) quadratic variation of a process $\{Z_t, t \geq 0\}$ is usually defined as

$$V_N(Z) = \sum_{i=0}^{N-1} \left[(Z_{t_{i+1}} - Z_{t_i})^2 - \mathbb{E}(Z_{t_{i+1}} - Z_{t_i})^2 \right] \ . \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_N$. The quadratic variation plays an important role in the analysis of a stochastic process, for various reasons. For example, for Brownian motion and martingales, the limit of the sequence (1) is an important element in the Itô stochastic calculus. Another field where the asymptotic behavior of (1) is important is estimation theory: for self-similar processes the quadratic variations are used to construct consistent estimators for the self-similarity parameter. The limit in distribution of the sequence V_N yields the asymptotic behavior of the associated estimators (see e.g. [11], [10],[12], [7], [19], [20], [21]). Quadratic variations (and their generalizations) are also crucial in mathematical finance (see e.g. [2]), stochastic analysis of processes related with fractional Brownian motion (see e.g. [8], [14]) or numerical schemes for stochastic differential equations (see e.g. [13]). Variations of sums of independent Brownian motion and fractional Brownian motion are considered in [9]. The asymptotic behavior of the quadratic variation of a single Hermite process has been studied in [4].

Our purpose is to study the asymptotic behavior of the quadratic variation of a sum of two dependent Hermite processes of consecutive orders. One could consider other combinations. We focus on this one because it already displays interesting features. It shows that the quadratic variation, suitably normalized, converges either to a normal or to a Rosenblatt distribution, whatever the order of the original Hermite processes. This would not be the

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case if only one Hermite process of order at least equal to two were considered, since then the limit would always be a Rosenblatt distribution. This would also not be the case if one considered the sum of two independent Hermite processes. We show indeed that in the independent case, the quadratic variation asymptotically behaves as that of a single Hermite process.

We will thus take the process Z in (1) to be

$$Z = Z^{q,H_1} + Z^{q+1,H_2} ,$$

where $Z^{q,H}$ denotes a Hermite process of order $q \geq 1$ and with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. Hermite processes are self-similar processes with stationary increments and exhibit long-range dependence. The Hermite process of order $q \geq 1$ can be written as a multiple integral of order q with respect to the Wiener process and thus belongs to the Wiener chaos of order q.

We will consider an interspacing

$$t_i - t_{i-1} = \gamma_N$$

which may depend on N. The interspacing γ_N may be fixed (as in a time series setting), grow with N (large scale asymptotics) or decrease with N (small scale asymptotics). The case $\gamma_N = 1/N$ is referred to as *in-fill* asymptotics. From now on, the expression of $V_N(Z)$ reads

$$V_N(Z) = \sum_{i=0}^{N-1} \left[(Z_{\gamma_N(i+1)} - Z_{\gamma_N i})^2 - \mathbb{E}(Z_{\gamma_N(i+1)} - Z_{\gamma_N i})^2\right] . \tag{2}$$

Such an interspacing was also considered in [19] when studying the impact of the sampling rate on the estimation of the parameters of fractional Brownian motion. Since we consider here the sum of two self-similar processes, one with self-similarity index H_1, the other with self-similarity index H_2, we expect to find several regimes depending on the growth or decay of γ_N with respect to N. It seems indeed reasonable to expect that, if $H_1 > H_2$ the first process will dominate at large scales and be negligible at small scales, and the opposite if $H_1 < H_2$. Our analysis will in fact exhibit an intermediate regime between these two. When $H_1 = H_2$, it is not clear whether one term should or should not dominate the other one.

The quadratic variation of the sum $Z = X + Y$ can obviously be decomposed into the sum of the quadratic variations of X and Y and the so-called quadratic covariation of X and Y which is defined by

$$V_N(X, Y) := \sum_{i=0}^{N-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

with $0 = t_0 < t_1 < \cdots < t_N$. The quadratic covariation shall play a central role in our analysis. The case where $X = Z^{H_1,q}$ and $Y = Z^{H_2,q+1}$ are Hermite processes of consecutive orders, exhibits an interesting situation. If the two processes are independent (that is, they are expressed as multiple integrals with respect to independent Wiener processes), then the quadratic covariation of the sum is always dominated by one of the two quadratic variations $V_N(X)$ or $V_N(Y)$. On the other hand, surprisingly, we highlight in this paper that when the two processes are dependent (they can be written as multiple integrals with respect to the same Wiener process), then it is their quadratic covariation which may determine the asymptotic behavior of $V_N(X + Y)$. We also find that there is a range of values for the interspacing γ_N where the limit is Rosenblatt in the independent case and Gaussian in the dependent case. The range includes the choice $\gamma_N = 1$ for a large set of (H_1, H_2), as illustrated by the domain $\nu_1 < 0$ in Figure 1.

A primary motivation for our work involves the analysis of wavelet estimators. Of particular interest is the case $H_2 = 2H_1 - 1$ which is related to an open problem in [6]. See Example 3 for details.

The paper is organized as follows. Section 2 contains some preliminaries on Hermite processes and their properties. The main results are stated in Section 3. The asymptotic behavior of $V_N(Z)$ is given and illustrated in Section 4. The proofs of the main theorem and propositions are given in Section 5 while Section 6 contains some technical lemmas. Basic facts about multiple Itô integrals are gathered in Appendix A.

2. Preliminaries

Recall that a process $\{X_t, t \geq 0\}$ is self–similar with index H if for any $a > 0$, $\{X_{at}, t \geq 0\}$ has the same finite-dimensional distributions as $\{a^H X_t, t \geq 0\}$. Hermite processes $\{Z_t^{H,q}, t \geq 0\}$, where $H \in (1/2, 1)$, $q = 1, 2, \cdots$ are self–similar processes with stationary increments. They appear as limits of normalized sums of random variables with long–range dependence. The parameter H is the self–similar parameter and the parameter q denotes the order of the process. The most common Hermite processes are the fractional Brownian motion $B^H = Z^{H,1}$ (Hermite process of order 1) and the Rosenblatt process $R^H = Z^{H,2}$ (Hermite process of order 2). Fractional Brownian motion (fBm) is Gaussian but all the other Hermite processes are non–Gaussian. On the other hand, because of self–similarity and stationarity of the increments, they all have zero mean and the same covariance

$$\mathbb{E}\left[Z_{t_1}^{H,q} Z_{t_2}^{H,q}\right] = \frac{1}{2}\left[|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\right] ,$$

hence $\mathbb{E}\left[(Z_1^{H,q})^2\right] = 1$. Consequently, the covariance of their increments decays slowly to zero as the lag tends to infinity, namely

$$\mathbb{E}\left[(Z_{t+1}^{H,q} - Z_t^{H,q})(Z_{s+t+1}^{H,q} - Z_{s+t}^{H,q})\right] \sim H(2H-1)s^{2H-2} \text{ as } s \to \infty .$$

Observe that the sum over $s \geq 1$ of these covariances diverges, which is an indication of "long–range" dependence.

The Hermite processes $\{Z_t^{H,q}, t \geq 0\}$ can be represented by Wiener–Itô integrals (see Appendix A for more details about stochastic integrals), namely

$$Z_t^{H,q} = c(H,q)\ I_q(L_t^{H,q}) := c(H,q) \int_{\mathbb{R}^q}' L_t^{H,q}(y_1, \cdots, y_q) \mathrm{d}B_{y_1} \cdots \mathrm{d}B_{y_q} , \tag{3}$$

where $c(H,q)$ is a positive normalizing constant, B represents standard Brownian motion and where the kernel is defined by

$$L_t^{H,q}(y_1, \cdots, y_q) = \int_0^t (u - y_1)_+^{-(\frac{1}{2} + \frac{1-H}{q})} \cdots (u - y_q)_+^{-(\frac{1}{2} + \frac{1-H}{q})} \mathrm{d}u . \tag{4}$$

The prime on the integral (3) indicates that one does not integrate over the "diagonals", where at least two entries of the vector $(y_1, \ldots, y_q)$ are equal. Observe that the kernel $L_t^{H,q}$ is symmetric and has a finite $L^2(\mathbb{R}^q)$ norm $\|L_t^{H,q}\|_2 < \infty$ because $H \in (1/2, 1)$. The Hermite process $\{Z_t^{H,q}, t \geq 0\}$ is then well–defined. It has mean zero and variance

$$\mathbb{E}\left[(Z_t^{H,q})^2\right] = c^2(H,q)\ q!\|L_t^{H,q}\|_2^2 .$$

In order to standardize the Hermite process, the positive normalizing constant $c(H,q)$ is defined by

$$c(H,q) = \left(q!\|L_1^{H,q}\|_2^2\right)^{-1/2} , \tag{5}$$

so that $\mathbb{E}\left[(Z_t^{H,q})^2\right] = t^{2H}$ for all $t \geq 0$.

The fractional Brownian motion is obtained by setting $q = 1$ and denoted by

$$B_t^H = Z_t^{H,1} := c(H,1) \int_{\mathbb{R}} \left(\int_0^t (u-y)_+^{H-3/2} \mathrm{d}u \right) \mathrm{d}B_{y_1} ,$$

while the Rosenblatt process is obtained by setting $q = 2$ and denoted by

$$R_t^H = Z_t^{H,2} := c(H,2) \int'_{\mathbb{R}^2} \left(\int_0^t (u-y_1)_+^{H-1} (u-y_2)_+^{H-1} \mathrm{d}u \right) \mathrm{d}B_{y_1} \mathrm{d}B_{y_2} .$$

The (marginal) distribution B_1^H of the standard fractional Brownian motion $B_t^H = Z_t^{H,1}$ when $t = 1$ is $\mathcal{N}(0,1)$ and the distribution R_1^H of the standard Rosenblatt process R_t^H when $t = 1$ is called the Rosenblatt distribution, see [16] and [22] for more information about that distribution. The normal distribution and the Rosenblatt distribution will appear in the limit.

The asymptotic behavior of $V_N(X)$ where X is an Hermite process, namely $X = Z^{H,q}$ was studied in [4]. The limit is either the normal distribution or the Rosenblatt distribution. The normal distribution appears in the limit when X is the fractional Brownian motion $Z^{H,1}$ with $H \in (1/2, 3/4)$. The Rosenblatt distribution appears in the limit when X is the fractional Brownian motion with $H \in (3/4, 1)$ or when $Z^{H,q}$ is a Hermite process with $q \geq 2$ and $H \in (1/2, 1)$. See Theorem 1 below, for a precise statement.

We shall focus on the simplest mixed model based on Hermite processes, that is,

$$Z_t = Z_t^{H_1,H_2} = Z_t^{H_1,q} + Z_t^{H_2,q+1} , \tag{6}$$

where $q \geq 1$ and $H_1, H_2 \in (1/2, 1)$. Processes of the type (6) appear naturally in the framework of long range dependent Gaussian subordinated processes (see [6] and Example 3 below). Observe that :

- $Z^{H,q}$ and $Z^{H,q+1}$ are defined in (3) using the same underlying Brownian motion B but different kernels L_t are involved.
- It follows from the previous point that $Z^{H,q}$ and $Z^{H,q+1}$ are uncorrelated but dependent, see [1].
- Z^{H_1,H_2} is not self–similar anymore if $H_1 \neq H_2$ but still has stationary increments.
- In the quadratic variations (2) cross–terms

$$\left(Z_{\gamma_N(i+1)}^{H_1,q} - Z_{\gamma_N i}^{H_1,q} \right) \left(Z_{\gamma_N(i+1)}^{H_2,q+1} - Z_{\gamma_N i}^{H_2,q+1} \right) ,$$

  will appear. We will show that their (renormalized) partial sum is asymptotically normal.

The notation $\xrightarrow{(d)}$ refers to the convergence in distribution and $a_N \ll b_N$ means that $a_N = o(b_N)$ as $N \to \infty$ and $a_N \sim b_N$ means $a_N/b_N \to 1$ as $N \to \infty$. The notation $X_N = o_P(1)$ means that $X_N \to 0$ in probability.

3. Main results

3.1. **Main assumptions.** Throughout the paper, we consider $H_1, H_2 \in (1/2, 1)$ and an integer $q \geq 1$,

$$V_N := V_N(Z) = \sum_{i=0}^{N-1} \left[(Z_{t_{i+1}} - Z_{t_i})^2 - \mathbb{E}(Z_{t_{i+1}} - Z_{t_i})^2 \right] \ , \tag{7}$$

where $t_i = \gamma_N i$ and Z is the sum of the two Hermite processes $Z^{H_1,q}$ and $Z^{H_2,q+1}$ as defined in (6). The sum V_N will be split into three terms as follows

$$V_N = V_N^{(1)} + V_N^{(2)} + 2V_N^{(3)} \ , \tag{8}$$

where

$$V_N^{(1)} = \sum_{i=0}^{N-1} \left[\left(Z_{t_{i+1}}^{H_1,q} - Z_{t_i}^{H_1,q} \right)^2 - \mathbb{E} \left(Z_{t_{i+1}}^{H_1,q} - Z_{t_i}^{H_1,q} \right)^2 \right] \ , \tag{9}$$

$$V_N^{(2)} = \sum_{i=0}^{N-1} \left[\left(Z_{t_{i+1}}^{H_2,q+1} - Z_{t_i}^{H_2,q+1} \right)^2 - \mathbb{E} \left(Z_{t_{i+1}}^{H_2,q+1} - Z_{t_i}^{H_2,q+1} \right)^2 \right] \ , \tag{10}$$

and

$$V_N^{(3)} = \sum_{i=0}^{N-1} \left(Z_{t_{i+1}}^{H_1,q} - Z_{t_i}^{H_1,q} \right) \left(Z_{t_{i+1}}^{H_2,q+1} - Z_{t_i}^{H_2,q+1} \right) \ . \tag{11}$$

The mean of the cross–term (11) vanishes because the terms in the product are Wiener–Itô integrals of different orders and hence are uncorrelated (see formula (53)). We further denote the corresponding standard deviations by

$$\sigma_N := \left(\mathbb{E} \left[V_N^2 \right] \right)^{1/2} \quad \text{and} \quad \sigma_N^{(i)} := \left(\mathbb{E} \left[(V_N^{(i)})^2 \right] \right)^{1/2} \quad \text{for } i = 1, 2, 3. \tag{12}$$

3.2. **Asymptotic behavior of $V_N^{(1)}$, $V_N^{(2)}$ and $V_N^{(3)}$.** To investigate the asymptotic behavior of V_N and σ_N, we shall consider the terms $V_N^{(1)}$, $V_N^{(2)}$ and $V_N^{(3)}$ separately, without any assumption on the scale sequence (γ_N).

First we recall well–known results about the asymptotic behavior of the sequences $V_N^{(1)}$ and $V_N^{(2)}$.

Theorem 1. *Denote*

$$h_1 = \max \left(\frac{1}{2}, 1 - \frac{2(1 - H_1)}{q} \right) = \begin{cases} \frac{1}{2} & \text{if } q = 1 \text{ and } H_1 \leq 3/4 \\ 1 - \frac{2(1-H_1)}{q} & \text{if } q \geq 2 \text{ or } H_1 \geq 3/4 \end{cases} \ , \tag{13}$$

and

$$\delta = 1_{\{H_1 = 3/4\} \cap \{q=1\}} \ , \tag{14}$$

that is, $\delta = 1$ if $H_1 = 3/4$ and $q = 1$, and $\delta = 0$ if $H_1 \neq 3/4$ or $q \geq 2$. Then as $N \to \infty$,

$$\sigma_N^{(1)} \sim a(q, H_1) \, \gamma_N^{2H_1} \, N^{h_1} \, (\log N)^{\delta/2} \ . \tag{15}$$

Moreover, we have the following asymptotic limits as $N \to \infty$.

(1) (a) If $q = 1$ and $H_1 \in (1/2, 3/4]$, then

$$\frac{V_N^{(1)}}{\sigma_N^{(1)}} \xrightarrow{(d)} \mathcal{N}(0, 1) \ , \tag{16}$$

(b) If $q = 1$ and $H_1 \in (3/4, 1)$ or if $q \geq 2$ and $H_1 \in (1/2, 1)$, then

$$\frac{V_N^{(1)}}{\sigma_N^{(1)}} \xrightarrow{(d)} R_1^{1-2(1-H_1)/q} \tag{17}$$

(2) If $q \geq 1$ (that is, $q+1 \geq 2$) and $H_2 \in (1/2, 1)$, then

$$\frac{V_N^{(2)}}{\sigma_N^{(2)}} \xrightarrow{(d)} R_1^{1-2(1-H_2)/(q+1)} \quad \textit{with} \quad \sigma_N^{(2)} \sim a(q+1, H_2)\, N^{[1-2(1-H_2)/(q+1)]} \gamma_N^{2H_2} \,. \tag{18}$$

Here $a(q, H_1)$ and $a(q+1, H_2)$ are positive constants.

Proof. Point (1a) goes back to [3] and Point (1b) with $q = 1$ and $H_1 \in (3/4, 1)$ goes back to [17]. Point (1b) with $q \geq 2$ and $H_1 \in (1/2, 1)$ can be deduced from [4] (see Theorem 1.1 and its proof).

For the expression of the constant $a(q, H_1)$ in (15) with $q = 1$, that is for $a(1, H_1)$, see Propositions 5.1, 5.2 and 5.3 in [20] with $H \in (1/2, 3/4)$, $H \in (3/4, 1)$ and $H = 3/4$ respectively. For the expression of $a(q, H_1)$ with $q \geq 2$ and $H_1 \in (1/2, 1)$, see Proposition 3.1 in [4]. The expression of the constant $a(q+1, H_2)$ follows from that of $a(q, H_1)$.

The exponent of γ_N in (15) and (18) results from the fact that $Z^{H_1,q}$ and $Z^{H_2,q+1}$ are self-similar with index H_1 and H_2, respectively. □

In view of the decomposition (8) and of Theorem 1, we need to investigate the asymptotic behavior of the cross–term $V_N^{(3)}$ in order to get the asymptotic behavior of V_N.

Theorem 2. *We have the following convergence and asymptotic equivalence as $N \to \infty$.*

$$\frac{V_N^{(3)}}{\sigma_N^{(3)}} \xrightarrow{(d)} \mathcal{N}(0, 1) \quad \textit{with} \quad \sigma_N^{(3)} \sim b(q, H_1, H_2)\, N^{1-(1-H_2)/(q+1)} \gamma_N^{H_1+H_2} \,,$$

where $b(q, H_1, H_2)$ is a positive constant.

Remark 1. Theorem 2 cannot be directly extended to the general case where the process Z is the sum of two Hermite processes of order q_1, q_2 with $q_2 - q_1 > 1$. This is because the proof is based on the fact that $V_N^{(3)}$ admits a Gaussian leading term. This may not happen if $q_2 - q_1 > 1$ (see the proof of Proposition 2 and Remark 5 for more details). In contrast, Theorems 2 and 5 below can be easily extended.

Proof. The proof of Theorem 2 is found in Section 5. □

3.3. **Quadratic covariation in the independent case.** Theorem 2 will imply that the term $V_N^{(3)}$, which corresponds to the quadratic covariation of $Z^{H_1,q}$ and $Z^{H_2,q+1}$, may dominate in the asymptotic behavior of the sequence V_N. On the other hand, if the two Hermite processes are independent, the quadratic covariation is always dominated by the quadratic variation of one of these processes. This is a consequence of the following theorem.

Theorem 3. *Assume that for every $t \geq 0$, $Z_t^{H_1,q}$ and $Z_t^{H_2,q+1}$ are given by (3) and define*

$$\tilde{V}_N^{(3)} = \sum_{i=0}^{N-1} \left(Z_{t_{i+1}}^{H_1,q} - Z_{t_i}^{H_1,q} \right) \left(\tilde{Z}_{t_{i+1}}^{H_2,q+1} - \tilde{Z}_{t_i}^{H_2,q+1} \right) \,,$$

where the process $\tilde{Z}^{H_2,q+1}$ is an independent copy of $Z^{H_2,q+1}$. Then, as $N \to \infty$,

$$\mathbb{E}\left[\left(\tilde{V}_N^{(3)}\right)^2\right] = o\left(\sigma_N^{(1)} \times \sigma_N^{(2)}\right) ,$$

where $\sigma_N^{(1)}$ and $\sigma_N^{(2)}$ are defined by (12).

Proof. We have, from the independence of the two Hermite processes,

$$\mathbb{E}\left[\left(\tilde{V}_N^{(3)}\right)^2\right] = \sum_{i,j=0}^{N-1} \gamma_{i,j}(Z^{H_1,q})\gamma_{i,j}(\tilde{Z}^{H_2,q+1}) ,$$

where

$$\gamma_{i,j}(X) := \mathbb{E}\left[(X_{t_{i+1}} - X_{t_i})(X_{t_{j+1}} - X_{t_j})\right] .$$

Since the covariance structure of the Hermite process $Z^{H,q}$ is the same for all $q \geq 1$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{V}_N^{(3)}\right)^2\right] &= \sum_{i,j=0}^{N-1} \gamma_{i,j}(Z^{H_1,1})\gamma_{i,j}(Z^{H_2,1}) \\
&\leq \left(\sum_{i,j=0}^{N-1} \left(\gamma_{i,j}(Z^{H_1,1})\right)^2\right)^{\frac{1}{2}} \left(\sum_{i,j=0}^{N-1} \left(\gamma_{i,j}(Z^{H_2,1})\right)^2\right)^{\frac{1}{2}} ,
\end{aligned}$$

where the last line follows from the Cauchy–Schwarz inequality. Recall that for two jointly centered Gaussian random variables X and Y, we have

$$(\mathrm{Cov}(X,Y))^2 = \frac{1}{2}\mathrm{Cov}(X^2, Y^2).$$

Hence

$$\begin{aligned}
\sum_{i,j=0}^{N-1} \left(\gamma_{i,j}(Z^{H_1,1})\right)^2 &= \sum_{i,j=0}^{N-1} \left(\mathrm{Cov}\left(Z^{H_1,1}_{t_{i+1}} - Z^{H_1,1}_{t_i}, Z^{H_1,1}_{t_{j+1}} - Z^{H_1,1}_{t_j}\right)\right)^2 \\
&= \frac{1}{2}\sum_{i,j=0}^{N-1} \mathrm{Cov}\left(\left(Z^{H_1,1}_{t_{i+1}} - Z^{H_1,1}_{t_i}\right)^2, \left(Z^{H_1,1}_{t_{j+1}} - Z^{H_1,1}_{t_j}\right)^2\right) \\
&= \frac{1}{2}\mathrm{Var}\left[\sum_{i=0}^{N-1} \left(\left(Z^{H_1,1}_{t_{i+1}} - Z^{H_1,1}_{t_i}\right)^2 - \mathbb{E}\left[\left(Z^{H_1,1}_{t_{i+1}} - Z^{H_1,1}_{t_i}\right)^2\right]\right)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left(V_N\left(Z^{1,H_1}\right)\right)^2\right]
\end{aligned}$$

by using the notation (1). Consequently,

$$\mathbb{E}\left[\left(\tilde{V}_N^{(3)}\right)^2\right] \leq \frac{1}{2}\left\{\mathbb{E}\left[\left(V_N\left(Z^{H_1,1}\right)\right)^2\right]\mathbb{E}\left[\left(V_N\left(Z^{H_2,1}\right)\right)^2\right]\right\}^{\frac{1}{2}} .$$

By Theorem 1, we know that, as $N \to \infty$, the rate of convergence of the variance of the quadratic variations of the Hermite process $Z^{H,q}$ (strictly) increases with respect to q (when H is fixed). Therefore, since $q \geq 1$ and $q+1 > 1$, we get, as $N \to \infty$,

$$\mathbb{E}\left[\left(\tilde{V}_N^{(3)}\right)^2\right] = o\left(\left\{\mathbb{E}\left[\left(V_N\left(Z^{H_1,q}\right)\right)^2\right]\mathbb{E}\left[\left(V_N\left(Z^{H_2,q+1}\right)\right)^2\right]\right\}^{\frac{1}{2}}\right) ,$$

which concludes the proof. □

4. Asymptotic behavior of the quadratic variation of the sum

4.1. **Dependent case.** It is now clear that the asymptotic behavior of V_N will depend on the relative behavior of the three sumands $V_N^{(1)}$, $V_N^{(2)}$ and $V_N^{(3)}$. More precisely we have the following result.

Theorem 4. *Let us define*

$$\nu_1 := \frac{1-H_2}{1+q} - 1 + \max\left(\frac{1}{2}, 1 - \frac{2(1-H_1)}{q}\right) \; < \; \frac{1-H_2}{1+q} =: \nu_2 \; . \tag{19}$$

Denoting δ as in (14), we have the following asymptotic equivalence as $N \to \infty$:

(1) If $\gamma_N^{H_2-H_1} \ll N^{\nu_1} (\log N)^{\delta/2}$ then

$$V_N = V_N^{(1)} \left(1 + o_P(1)\right) .$$

(2) If $N^{\nu_1} (\log N)^{\delta/2} \ll \gamma_N^{H_2-H_1} \ll N^{\nu_2}$, then

$$V_N = 2V_N^{(3)} \left(1 + o_P(1)\right) .$$

(3) If $\gamma_N^{H_2-H_1} \gg N^{\nu_2}$ then

$$V_N = V_N^{(2)} \left(1 + o_P(1)\right) .$$

Proof. Let us compare the terms $V_N^{(1)}, V_N^{(2)}$ and $V_N^{(3)}$ in each case considered in Theorem 4. By Theorem 1 and Theorem 2, we have, for some positive constants c_1, c_2 and c_3,

$$\mathbb{E}\left[\left|V_N^{(1)}\right|^2\right]^{1/2} \sim c_1 \, \gamma_N^{2H_1} \; N^{h_1} (\log N)^{\delta/2} ,$$

$$\mathbb{E}\left[\left|V_N^{(2)}\right|^2\right]^{1/2} \sim c_2 \, N^{1-2(1-H_2)/(q+1)} \gamma_N^{2H_2} ,$$

$$\mathbb{E}\left[\left|V_N^{(3)}\right|^2\right]^{1/2} \sim c_3 \, N^{1-(1-H_2)/(q+1)} \gamma_N^{H_1+H_2} ,$$

and these rates always correspond to the rate of convergence in distribution.

The different cases are obtained by using the definitions of $\nu_1 < \nu_2$ in (19) and by computing the following ratios of the above rates for $V_N^{(1)}$ versus $V_N^{(3)}$:

$$\frac{\gamma_N^{2H_1} \, N^{h_1} (\log N)^{\delta/2}}{N^{1-(1-H_2)/(q+1)} \gamma_N^{H_1+H_2}} = \frac{N^{\nu_1} (\log N)^{\delta/2}}{\gamma_N^{H_2-H_1}} , \tag{20}$$

and $V_N^{(2)}$ versus $V_N^{(3)}$:

$$\frac{N^{1-2(1-H_2)/(q+1)} \gamma_N^{2H_2}}{N^{1-(1-H_2)/(q+1)} \gamma_N^{H_1+H_2}} = \frac{\gamma_N^{H_2-H_1}}{N^{\nu_2}} . \tag{21}$$

Observing that $\nu_1 < \nu_2$ and thus $N^{\nu_1} (\log N)^{\delta/2} \ll N^{\nu_2}$, we get in the three cases :

(1) If

$$\gamma_N^{H_2-H_1} \ll N^{\nu_1} (\log N)^{\delta/2},$$

then $V_N^{(1)}$ dominates $V_N^{(3)}$ by (20). But since it implies $\gamma_N^{H_2-H_1} \ll N^{\nu_2}$, we have by (21) that $V_N^{(3)}$ dominates $V_N^{(2)}$. Hence $V_N^{(1)}$ dominates in this case.

(2) If

$$N^{\nu_1} (\log N)^{\delta/2} \ll \gamma_N^{H_2-H_1} \ll N^{\nu_2},$$

then $V_N^{(3)}$ dominates both $V_N^{(1)}$ and $V_N^{(2)}$ by (20) and (21), respectively.

(3) If

$$\gamma_N^{H_2-H_1} \gg N^{\nu_2},$$

then $V_N^{(2)}$ dominates $V_N^{(3)}$ by (21). But since it implies $\gamma_N^{H_2-H_1} \gg N^{\nu_1} (\log N)^{\delta/2}$, we have by (20) that $V_N^{(3)}$ dominates $V_N^{(1)}$. Hence $V_N^{(2)}$ dominates in this case.

This concludes the proof of Theorem 4. □

Remark 2. Note that $\nu_2 > 0$ but ν_1 can be positive, zero, or negative. In fact,

$$\nu_1 = \begin{cases} \frac{1-H_2}{1+q} - \frac{1}{2} & \text{if } q = 1 \text{ and } H_1 < 3/4 \\ \frac{1-H_2}{1+q} - \frac{2(1-H_1)}{q} & \text{otherwise.} \end{cases}$$

It follows that

$$\nu_1 \le 0 \Longleftrightarrow H_2 \ge 1 - \frac{2(q+1)(1-H_1)}{q} ,$$

with equality on the left-hand side if and only if there is equality on the right-hand side. The equality case corresponds to having (H_1, H_2) on the segment with end points $(1 - q/(4(q + 1)), 1/2)$ and $(1, 1)$, see Figure 1. The H_1 coordinate of the bottom end point is $1-q/(4(q+1))$. For $q = 1$ it equals $1 - 1/8 = 0.875$ and, as $q \to \infty$, it decreases towards $3/4$.

Let us illustrate Theorem 4 with some examples.

Example 1. In the particular case where $H_1 = H_2$, by Remark 2, we always have $\nu_1 < 0$. It follows that we are in Case (2) of Theorem 4 whatever the values of $q = 1, 2, \ldots$ and the interspacing scale γ_N. Thus, the dominant part of V_N is the summand $2V_N^{(3)}$. By Theorem 2, we conclude that the limit of the normalized quadratic variation of the sum of two Hermite processes with the same self-similarity index and successive orders is asymptotically Gaussian.

Example 2. When $\gamma_N = 1$, the asymptotic behavior of the quadratic variation depends on the sign of ν_1. If $\nu_1 < 0$, we are in Case (2) of Theorem 4, the dominant part of V_N is the summand $2V_N^{(3)}$ and the limit is asymptotically Gaussian by Theorem 2. If $\nu_1 > 0$, we are in Case (1) of Theorem 4, the dominant part of V_N is the summand $V_N^{(1)}$ and by Theorem 1, the limit is Rosenblatt. Indeed, $\nu_1 > 0$ excludes $q = 1$ and $H_1 \le 3/4$, see Figure 1.

Example 3. The case $H_2 = 2H_1 - 1$ and $q = 1$ is of special interest because it is related to an open problem in [6]. Let us recall the context. Suppose you have unit variance Gaussian stationary data $Y_i, i \ge 1$ with spectral density $f(\lambda)$ which blows up like $|\lambda|^{-2d}$ at the origin, with $1/4 < d < 1/2$. Then the partial sums behave asymptotically like fractional Brownian motion with index H_1, where

$$2H_1 = (2d - 1) + 2 = 2d + 1 . \tag{22}$$

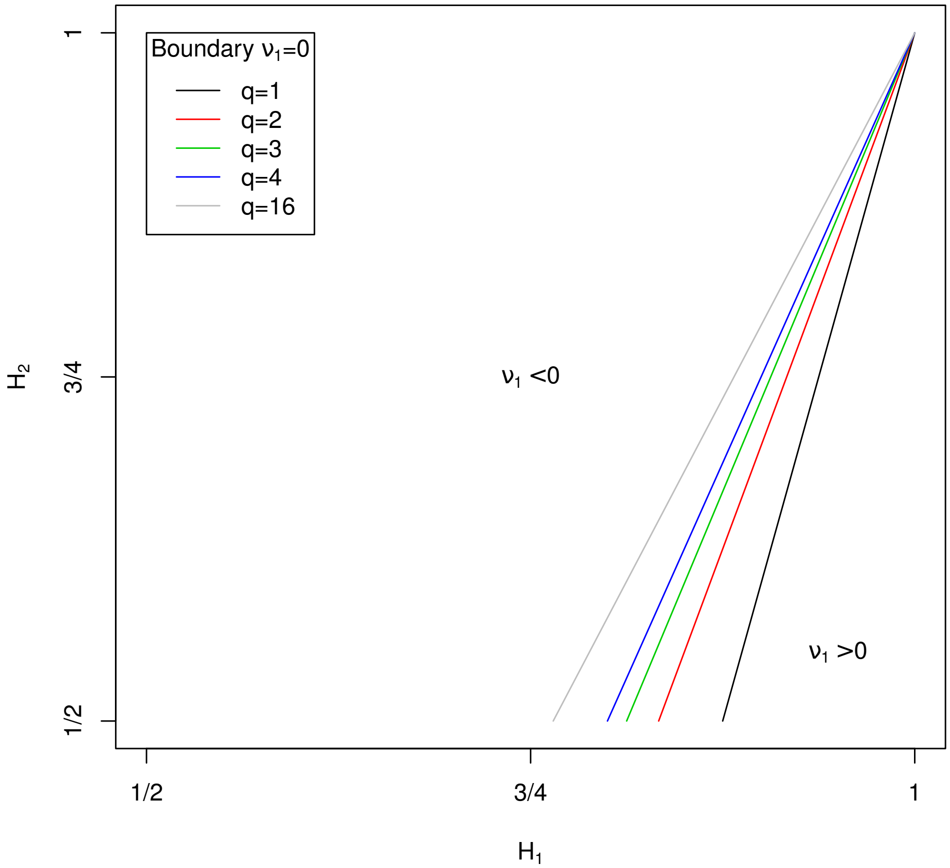

FIGURE 1. Domains of points (H_1, H_2) where the signs of ν_1 is negative or positive. The lines show the boundary between these sets. The right-hand line corresponds to $q = 1$ and the left-hand line to $q = 16$. The processes here are dependent.

On the other hand, referring to [5], the partial sums of $Y_i^2 - 1, i \geq 1$ behave like a Rosenblatt process with index H_2, where

$$2H_2 = 2(2d - 1) + 2 = 4d \; . \tag{23}$$

It follows that, conveniently normalized, for n large, $\sum_{k=1}^{[nt]}(Y_k + Y_k^2 - 1)$ can be seen as a process Z_t as defined by (6) with $H_2 = 2H_1 - 1$.

Applying Theorem 4 with $H_2 = 2H_1 - 1$ and $q = 1$, we obtain the following result.

Corollary 1. *If $H_2 = 2H_1 - 1$ and $q = 1$, we have the following asymptotic equivalence as $N \to \infty$:*

(1) If $\gamma_N \gg N$ then

$$V_N = V_N^{(1)} \left(1 + o_P(1)\right) .$$

(2) If $N^{-1} \ll \gamma_N \ll N$, then

$$V_N = 2V_N^{(3)} \left(1 + o_P(1)\right) .$$

(3) If $\gamma_N \ll N^{-1}$ then
$$V_N = V_N^{(2)} \left(1 + o_P(1)\right) .$$

Proof. Observe that if $H_2 = 2H_1 - 1$ and $q = 1$, one has $H_2 - H_1 = H_1 - 1 < 0$. In addition, the expression of the two exponents ν_1, ν_2 in (19) can be simplified as follows :
$$\nu_1 = \max\left(\frac{1}{2} - H_1, H_1 - 1\right) = -\min\left(H_1 - \frac{1}{2}, 1 - H_1\right) \quad \text{and} \quad \nu_2 = 1 - H_1.$$
Then $\nu_1/(H_2 - H_1) = \min\left(1, \frac{H_1 - 1/2}{1 - H_1}\right)$ and $\nu_2/(H_2 - H_1) = -1$. Moreover observe that $H_2 = 2H_1 - 1 > 1/2$ implies $H_1 > 3/4$ which in turns implies $\min\left(1, \frac{H_1 - 1/2}{1 - H_1}\right) = 1$ and $\delta = 0$. Thus $\nu_1/(H_2 - H_1) = 1$ and $\nu_2/(H_2 - H_1) = -1$. Corollary 1 then follows from Theorem 4. $\square$

4.2. **Independent case.** Theorem 4 should be contrasted with the following result involving independent processes.

Theorem 5. *Assume that the process $\tilde{Z}^{H_2,q+1}$ is an independent copy of $Z^{H_2,q+1}$. Let*
$$\tilde{V}_N = V_N(Z^{H_1,q} + \tilde{Z}^{H_2,q+1})$$
and
$$\tilde{V}_N^{(2)} = V_N(\tilde{Z}^{H_2,q+1}).$$
Define δ as in (14) and ν_1, ν_2 as in (19). We have the following asymptotic equivalence as $N \to \infty$:

(1) If $\gamma_N^{2(H_2 - H_1)} \ll N^{\nu_1 + \nu_2} (\log N)^{\delta/2}$ then
$$\tilde{V}_N = V_N^{(1)} \left(1 + o_P(1)\right) .$$

(2) If $\gamma_N^{2(H_2 - H_1)} \gg N^{\nu_1 + \nu_2} (\log N)^{\delta/2}$ then
$$\tilde{V}_N = \tilde{V}_N^{(2)} \left(1 + o_P(1)\right) .$$

Proof. By Theorem 3, we only need to compare $V_N^{(1)} = V_N(Z^{H_1,q})$ and $\tilde{V}_N^{(2)} = V_N(\tilde{Z}^{H_2,q+1}) \overset{d}{=} V_N(Z^{H_2,q+1}) = V_N^{(2)}$. Using Theorem 1 as in the proof of Theorem 4, the ratio between (20) and (21), gives that, as $N \to \infty$,
$$\frac{\sigma_N^{(1)}}{\sigma_N^{(2)}} \sim c \, \frac{N^{\nu_1 + \nu_2} (\log N)^{\delta/2}}{\gamma_N^{2(H_2 - H_1)}} ,$$
where c is a positive constant. This concludes the proof of Theorem 5. $\square$

Remark 3. Note that $\nu_1 + \nu_2$ can be positive, zero, or negative. In fact,
$$\nu_1 + \nu_2 = \begin{cases} \frac{2(1-H_2)}{1+q} - \frac{1}{2} & \text{if } q = 1 \text{ and } H_1 < 3/4 \\ \frac{2(1-H_2)}{1+q} - \frac{2(1-H_1)}{q} & \text{otherwise.} \end{cases}$$
It follows that
$$\nu_1 + \nu_2 \leq 0 \iff H_2 \geq 1 - \frac{(q+1)(1-H_1)}{q} ,$$
with equality on the left-hand side if and only if there is equality on the right-hand side. The equality case corresponds to having (H_1, H_2) on the segment with end points $(1 - q/(2(q+1)), 1/2)$ and $(1, 1)$, see Figure 2. The H_1 coordinate of the bottom end point is $1 - q/(2(q+1))$.

For $q = 1$ it equals $3/4$ and, as $q \to \infty$, it decreases towards $1/2$. In contrast with the dependent case described in Remark 2, the bottom of the boundary lines are pushed to the left, with half the slopes, compare Figures 1 and 2.

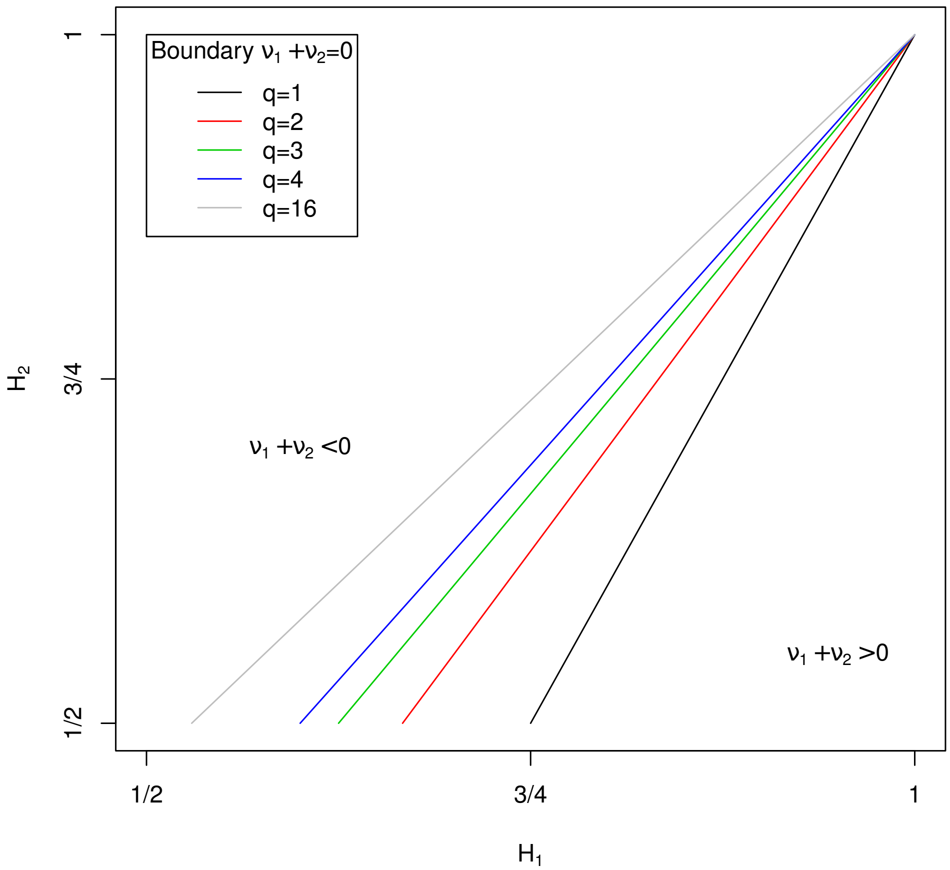

FIGURE 2. Domains of points (H_1, H_2) where the signs of $\nu_1 + \nu_2$ is negative or positive. The lines show the boundary between these sets. The right-hand line corresponds to $q = 1$ and the left-hand line to $q = 16$. The processes here are independent.

We now illustrate Theorem 5 where the sum of two independent processes is considered.

Example 4. If $H_1 = H_2$, by Remark 3, we always have $\nu_1 + \nu_2 < 0$. Thus the dominant part of $\tilde{V}_N$ is always $\tilde{V}_N^{(2)}$. By Theorem 1, we conclude that the limit of the normalized quadratic variation of the sum of two *independent* Hermite processes with the same self-similarity index and successive orders is asymptotically Rosenblatt.

Example 5. When $\gamma_N = 1$, the asymptotic behavior of the quadratic variation depends on the sign of $\nu_1 + \nu_2$. If $\nu_1 + \nu_2 < 0$, we are in Case 2 of Theorem 5, the dominant part of V_N is $\tilde{V}_N^{(2)}$ and the limit is asymptotically Rosenblatt by Theorem 1. If $\nu_1 + \nu_2 > 0$, we are in Case 1 of Theorem 5, the dominant part of V_N is $V_N^{(1)}$ and by Theorem 1, the limit is Rosenblatt. Indeed, $\nu_1 + \nu_2 > 0$ excludes the case $q = 1$ and $H_1 \leq 3/4$, see Figure 2.

Remark 4. In Examples 1 and 4, we considered the setting $H_1 = H_2$ in the dependent and independent cases. We see that the corresponding limits always differ, it is Gaussian in the dependent case and it is Rosenblatt in the independent case.

The contrast between Examples 2 and 5, which both correspond to the setting $\gamma_N = 1$ is a bit more involved. If $\nu_1 > 0$, then $\nu_1 + \nu_2 > 0$ and we have the same asymptotic behavior in both cases and the asymptotic limit is Rosenblatt. On the other hand, if $\nu_1 < 0$, then, in the dependent case we have a Gaussian limit and in the independent case we again have a Rosenblatt limit.

5. Proof of Theorem 2

The proof of Theorem 2 is based on its decomposition in Wiener chaos of the cross term $V_N^{(3)}$. We first need some notation : For any q and $(H_1, H_2) \in (1/2, 1)^2$, set

$$H_1^*(q) = \frac{1-H_1}{q} + \frac{1-H_2}{q+1}\ . \tag{24}$$

The function $\widetilde{\beta}_{a,b}$ will appear as part of the kernel involved in the Wiener chaos expansion of $V_N^{(3)}$. It is defined on $\mathbb{R}^2 \setminus \{(u,v), u = v\}$ for any $a, b > -1$ such that $a + b < -1$ as :

$$\widetilde{\beta}_{a,b}(u,v) = \begin{cases} \beta(a+1, -1-a-b) \text{ if } u < v, \\ \beta(b+1, -1-a-b) \text{ if } v < u\ , \end{cases} \tag{25}$$

where β denotes the beta function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x, y > 0\ .$$

Proposition 1. *The sum $V_N^{(3)}$ admits the following expansion into Wiener chaos :*

$$V_N^{(3)} = \sum_{k=0}^{q} V_N^{(3,k)}\ , \tag{26}$$

where for every $k = 0, \cdots, q$,

$$V_N^{(3,k)} = M(k, q, H_1, H_2) I_{2q+1-2k}\left(\sum_{i=0}^{N-1} f_{N,i}^{(k)}\right)\ , \tag{27}$$

with

$$\begin{aligned} f_{N,i}^{(k)}(y_1, \cdots, y_{2q+1-2k}) \quad &= \quad \int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} \left[\prod_{i=1}^{q-k}(u-y_i)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}\right]\left[\prod_{i=q-k+1}^{2q+1-2k}(v-y_i)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\right] \\ &\times \left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(u,v)\right]^k |u-v|^{-kH_1^*(q)}\mathrm{d}u\mathrm{d}v\ , \end{aligned}$$

where $\widetilde{\beta}_{a,b}$ has been defined in (25) and, defining $c(H,q)$ as in (5),

$$M(k, q, H_1, H_2) = c(H_1, q)c(H_2, q+1)k!\binom{q}{k}\binom{q+1}{k}\ .$$

Proof. Using the integral expression (3) of the two Hermite processes $Z^{(q,H_1)}$ and $Z^{(q+1,H_2)}$ and by definition (11) of the sum $V_N^{(3)}$, we get that

$$\bar{V}_N^{(3)} := \frac{V_N^{(3)}}{c(H_1,q)c(H_2,q+1)} = \sum_{i=0}^{N-1} I_q\left(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}\right) I_{q+1}\left(L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1}\right) ,$$

where the two kernels $L_t^{H_1,q}$, $L_t^{H_2,q+1}$ are defined in (4). We now use the product formula (55) and deduce that

$$\begin{aligned}\bar{V}_N^{(3)} &= \sum_{i=0}^{N-1}\left[\sum_{k=0}^{q} k!\binom{q}{k}\binom{q+1}{k} I_{2q+1-2k}\left((L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}) \otimes_k (L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1})\right)\right] \\ &= \sum_{k=0}^{q} k!\binom{q}{k}\binom{q+1}{k} I_{2q+1-2k}\left(\sum_{i=0}^{N-1}(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}) \otimes_k (L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1})\right) . \end{aligned} \tag{28}$$

To get an explicit expression for each term

$$\left(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}\right) \otimes_k \left(L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1}\right) ,$$

we use the definition of the $\otimes_k$ product given in the appendix :

$$\begin{aligned} &\left[\left(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}\right) \otimes_k \left(L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1}\right)\right](y_1,\cdots,y_{2q+1-2k}) \\ = &\int_{\mathbb{R}^k}\left(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}\right)(y_1,\cdots,y_{q-k},x_1,\cdots,x_k) \\ &\times\left(L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1}\right)(y_{q-k+1},\cdots,y_{2q+1-2k},x_1,\cdots,x_k)\mathrm{d}x_1\cdots\mathrm{d}x_k .\end{aligned}$$

Using the specific form (4) of the kernel $L_t^{H,q}$ and the Fubini Theorem, the last formula reads

$$\begin{aligned} &\left[\left(L_{t_{i+1}}^{H_1,q} - L_{t_i}^{H_1,q}\right) \otimes_k \left(L_{t_{i+1}}^{H_2,q+1} - L_{t_i}^{H_2,q+1}\right)\right](y_1,\cdots,y_{2q+1-2k}) \\ = &\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\left[\prod_{i=1}^{q-k}(u-y_i)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}\right]\left[\prod_{i=q-k+1}^{2q+1-2k}(v-y_i)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\right] \\ &\times\left[\prod_{i=1}^{k}\int_{\mathbb{R}}(u-x_i)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}(v-x_i)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\mathrm{d}x_i\right]\mathrm{d}u\mathrm{d}v \\ = &\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\left[\prod_{i=1}^{q-k}(u-y_i)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}\right]\left[\prod_{i=q-k+1}^{2q+1-2k}(v-y_i)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\right] \\ &\times\left[\int_{-\infty}^{u\wedge v}(u-x)^{-(\frac{1}{2}+\frac{1-H_1}{q})}(v-x)^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\mathrm{d}x\right]^k\mathrm{d}u\mathrm{d}v .\end{aligned}$$

But for any real numbers $a, b > -1$ such that $a + b < -1$, Lemma 1 implies that

$$\int_{-\infty}^{u\wedge v}(u-x)^a(v-x)^b\mathrm{d}x = \widetilde{\beta}_{a,b}(u,v)|u-v|^{a+b+1} , \tag{29}$$

where the function $\widetilde{\beta}_{a,b}$ has been defined in (25). Hence

$$\begin{aligned}&\left[\left(L^{H_1,q}_{t_{i+1}}-L^{H_1,q}_{t_i}\right)\otimes_k\left(L^{H_2,q+1}_{t_{i+1}}-L^{H_2,q+1}_{t_i}\right)\right](y_1,\cdots,y_{2q+1-2k})\\ =\ &\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\left[\prod_{i=1}^{q-k}(u-y_i)_+^{-\left(\frac12+\frac{1-H_1}{q}\right)}\right]\left[\prod_{i=q-k+1}^{2q+1-2k}(v-y_i)_+^{-\left(\frac12+\frac{1-H_2}{q+1}\right)}\right]\\ &\times\left[\widetilde{\beta}_{-(\frac12+\frac{1-H_1}{q}),-(\frac12+\frac{1-H_2}{q+1})}(u,v)\right]^k|u-v|^{-kH_1^*(q)}\mathrm{d}u\mathrm{d}v\ ,\end{aligned}$$

where we defined H_1^* in (24). Combining this equality and relation (28) then leads to the decomposition (26) of the sum $V_N^{(3)}$. This completes the proof of Proposition 1. ☐

We now bound the L^2–norm of $V_N^{(3,k)}$ for any $k=0,\cdots,q$ and deduce that the terms $V_N^{(3,k)}$ are for all $k<q$ negligible with respect to $V_N^{(3,q)}$. Proposition 2 below then directly implies Theorem 2. We set

$$\varepsilon(\alpha)=\begin{cases}1\text{ if }\alpha=1,\\ 0\text{ otherwise.}\end{cases}\tag{30}$$

Proposition 2. *For any $k=0,\cdots,q-1$,*

$$\|V_N^{(3,k)}\|^2\le\ C\ N^{[2-\min(\alpha,1)]}(\log(N))^{\varepsilon(\alpha)}\gamma_N^{2H_1+2H_2}\ ,\tag{31}$$

where ε is defined in (30),

$$\alpha=2(q-k)(1-H_1)/q+2(q+1-k)(1-H_2)/(q+1)\tag{32}$$

and as $N\to\infty$,

$$N^{-2+\frac{2(1-H_2)}{q+1}}\gamma_N^{-2(H_1+H_2)}\|V_N^{(3,q)}\|^2\to b^2(H_1,H_2,q)\ \ ,\tag{33}$$

for some $b(H_1,H_2,q)>0$. The leading term is the one with $k=q$. Moreover, $V_N^{(3,q)}$ is a Gaussian random variable, and thus

$$N^{\frac{(1-H_2)}{q+1}-1}\gamma_N^{-(H_1+H_2)}V_N^{(3,q)}\xrightarrow{(d)}b(H_1,H_2,q)\,\mathcal{N}(0,1)\ .$$

Remark 5. The proof of (33) is based on the fact that, because Z is the sum of two Hermite processes of consecutive orders, then $V_N^{(3,q)}$ has a centered Gaussian term in its decomposition (28) and this term turns out to be the leading term. We can then deduce its asymptotic behavior from that of its variance. Since the variance of a simple Wiener–Itô integral is related to the L^2–norm of the integrand, we obtain (33). Note that this proof does not extend to the case where Z is the sum of two Hermite processes of order q_1,q_2 with $q_2-q_1>1$. In that case, there is no Gaussian term in the sum $V_N^{(3)}$ and hence no Gaussian leading term. Thus, Proposition 2 cannot be extended in a simple way to more general cases.

Proof. We use the notation of Proposition 1. If $n\ge 2$, we have by (54), that $\mathbb{E}[I_n(f)^2]\le n!\|f\|_2^2$ whereas in the case $n=1$, f is trivially symmetric and this inequality becomes an equality. We first consider the case $k=0,\cdots,q-1$. By the integral definition (27) of the terms $V_N^{(3,k)}$, we get that

$$\mathbb{E}\left[\left|V_N^{(3,k)}\right|^2\right]\le M_1^2(k,q,H_1,H_2)\int_{\mathbb{R}^{2q+1-2k}}\left[\sum_{i,j=0}^{N-1}f_{N,i}^{(k)}(y)f_{N,j}^{(k)}(y)\right]\mathrm{d}y_1\cdots\mathrm{d}y_{2q+1-2k}\ ,$$

with

$$M_1^2(k,q,H_1,H_2) = (2q+1-2k)!M^2(k,q,H_1,H_2).$$

Using the explicit expression of $f_{N,\ell}^{(k)}$ given for any $\ell = 0,\cdots,N-1$ in Proposition 1, we deduce that :

$$\mathbb{E}\left[\left|V_N^{(3,k)}\right|^2\right] \leq M_1^2(k,q,H_1,H_2)\int_{\mathbb{R}^{2q+1-2k}}\left[\sum_{i,j=0}^{N-1} g_{N,i,j}^{(q,k)}(y)\right]\mathrm{d}y_1\cdots\mathrm{d}y_{2q+1-2k}\ , \tag{34}$$

with

$$\begin{aligned}
&g_{N,i,j}^{(q,k)}(y_1,\cdots,y_{2q+1-2k})\\
=\ &\int_{u=t_i}^{t_{i+1}}\int_{v=t_i}^{t_{i+1}}\int_{u'=t_j}^{t_{j+1}}\int_{v'=t_j}^{t_{j+1}}\prod_{\ell=1}^{q-k}\left[(u-y_\ell)_+(u'-y_\ell)_+\right]^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)}\\
&\times\prod_{\ell=q-k+1}^{2q-2k+1}\left[(v-y_\ell)_+(v'-y_\ell)_+\right]^{-\left(\frac{1}{2}+\frac{1-H_2}{q+1}\right)}\\
&\times\left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(u,v)\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(u',v')|u-v|^{-H_1^*(q)}|u'-v'|^{-H_1^*(q)}\right]^k \mathrm{d}u\mathrm{d}v\mathrm{d}u'\mathrm{d}v'\ .
\end{aligned}$$

On the other hand, equality (42) of Lemma 1 implies that for any $\ell = 1,\cdots,q-k$

$$\int_{y_\ell\in\mathbb{R}}(u-y_\ell)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}(u'-y_\ell)_+^{-(\frac{1}{2}+\frac{1-H_1}{q})}\mathrm{d}y_\ell = \beta(a_1+1,-2a_1-1)|u-u'|^{-\frac{(2-2H_1)}{q}}\ ,$$

with $a_1 = -(1/2+(1-H_1)/q)$ and that for any $\ell = q-k+1,\cdots,2q-2k+1$

$$\int_{y_\ell\in\mathbb{R}}(v-y_\ell)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}(v'-y_\ell)_+^{-(\frac{1}{2}+\frac{1-H_2}{q+1})}\mathrm{d}y_\ell = \beta(a_2+1,-2a_2-1)|v-v'|^{-\frac{(2-2H_2)}{q+1}}\ ,$$

with $a_2 = -(1/2+(1-H_2)/(q+1))$. Hence, combining the Fubini theorem, inequality (34) and these two last equalities implies that for some $M_2(k,q,H_1,H_2) > 0$

$$\mathbb{E}\left[\left(V_N^{(3,k)}\right)^2\right] \quad\leq\quad M_2^2(k,q,H_1,H_2)\sum_{i,j=0}^{N-1}\int_{u=t_i}^{t_{i+1}}\int_{v=t_i}^{t_{i+1}}\int_{u'=t_j}^{t_{j+1}}\int_{v'=t_j}^{t_{j+1}} h(u,u',v,v')\mathrm{d}u\mathrm{d}v\mathrm{d}u'\mathrm{d}v'\ ,$$

with

$$h(u,u',v,v') = \left[|u-u'|^{-\frac{2-2H_1}{q}}\right]^{q-k}\left[|v-v'|^{-\frac{2-2H_2}{q+1}}\right]^{q-k+1}$$

$$\times\left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(u,v)\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(u',v')|u-v|^{-H_1^*(q)}|u'-v'|^{-H_1^*(q)}\right]^k\ ,$$

and

$$M_2^2(k,q,H_1,H_2) = M_1(k,q,H_1,H_2)^2\beta(a_1+1,-2a_1-1)^{q-k}\beta(a_2+1,-2a_2-1)^{q-k+1}\ .$$

Recall that $t_i = i\gamma_N$ and use the change of variables

$$U = \gamma_N^{-1}(u-i\gamma_N),\ V = \gamma_N^{-1}(v-i\gamma_N),\ U' = \gamma_N^{-1}(u'-j\gamma_N),\ V' = \gamma_N^{-1}(v'-j\gamma_N)\ .$$

We have

$$\begin{aligned}\mathbb{E}\left[\left(V_N^{(3,k)}\right)^2\right] &\leq c\gamma_N^{-2kH_1^*(q)}\gamma_N^{-\frac{(q-k)(2-2H_1)}{q}}\gamma_N^{-\frac{(q-k+1)(2-2H_2)}{q+1}}\gamma_N^4 \\ &\quad \times\sum_{i,j=0}^{N-1}\int_{[0,1]^4} H_{i,j}(U,U',V,V')\mathrm{d}U\mathrm{d}V\mathrm{d}U'\mathrm{d}V' \;,\end{aligned}$$

with

$$\begin{aligned}H_{i,j}(U,U',V,V') &= \left[|U-U'+i-j|^{-\frac{2(1-H_1)}{q}}\right]^{q-k}\left[|V-V'+i-j|^{-\frac{2(1-H_2)}{q+1}}\right]^{q-k+1} \quad (35)\\ &\quad \times\left(\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(U,V)\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(U',V')\right)^k \quad (36)\\ &\quad \times|U-V|^{-kH_1^*(q)}\cdot|U'-V'|^{-kH_1^*(q)}\;,\end{aligned}$$

since $\widetilde{\beta}_{a,b}(u,v)$ only depends on the sign of $u-v$. Now we simplify the expression involving powers of γ_N. Since

$$\begin{aligned}&-2kH_1^*(q)-\frac{(q-k)(2-2H_1)}{q}-\frac{(q-k+1)(2-2H_2)}{q+1}+4\\ =\;& -2k\frac{1-H_1}{q}-2k\frac{1-H_2}{q+1}-\frac{(q-k)(2-2H_1)}{q}-\frac{(q-k+1)(2-2H_2)}{q+1}+4\\ =\;& \left(-\frac{2q}{q}-\frac{2(q+1)}{q+1}+4\right)+H_1\left(\frac{2k}{q}+\frac{2(q-k)}{q}\right)+H_2\left(\frac{2k}{q+1}+\frac{2(q-k+1)}{q+1}\right)\\ =\;& 2H_1+2H_2\;, \qquad (37)\end{aligned}$$

we deduce that

$$\mathbb{E}\left[\left(V_N^{(3,k)}\right)^2\right]\leq c\gamma_N^{2H_1+2H_2}\sum_{i,j=0}^{N-1}\int_{[0,1]^4}H_{i,j}(U,U',V,V')\mathrm{d}U\mathrm{d}V\mathrm{d}U'\mathrm{d}V'\;. \qquad (38)$$

To obtain (31), we check that we can apply Lemma 2 below with

$$\alpha_1=\frac{2(q-k)(1-H_1)}{q},\;\alpha_2=\frac{2(q-k+1)(1-H_2)}{q+1}\;,$$

and

$$F(U,V)=\left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(U,V)|U-V|^{-H_1^*(q)}\right]^k\;.$$

Since $kH_1^*(q)\leq k(2q+1)/(2q(q+1))<1$, Condition (45) holds.

It remains to check Condition (46). Note that $\widetilde{\beta}_{H_1^*(q),H_2^*(q)}$ is bounded and then, for some $C>0$,

$$|F(U,V)|\leq C\,|U-V|^{-kH_1^*(q)}\;.$$

We deduce the finiteness of the integrals

$$\int_{[0,1]^4}|U-U'+\ell|^{-\alpha_1}|V-V'+\ell|^{-\alpha_2}F(U,V)F(U',V')\mathrm{d}U\mathrm{d}U'\mathrm{d}V\mathrm{d}V'\;,$$

as follows :

(1) if $\ell = 0$, we observe that $\alpha_1, \alpha_2, kH_1^*(q) \in (0,1)$ and by (37),

$$\alpha_1 + \alpha_2 + 2kH_1^*(q) = 2(1-H_1) + 2(1-H_2) < 3$$

We then apply Part (1) of Lemma 3.

(2) if $\ell = 1$ or $\ell = -1$, we observe that $\alpha_1, \alpha_2, kH_1^*(q) \in (0,1)$ and apply Part (2) of Lemma 3.

(3) if $|\ell| \geq 2$, we observe that on $[0,1]^2$,

$$|U-U'+\ell|^{-\alpha_1}|V-V'+\ell|^{-\alpha_2} \leq ||\ell|-1|^{-\alpha_1-\alpha_2}\left(\int_{[0,1]}|U-V|^{-kH_1^*(q)}\mathrm{d}U\mathrm{d}V\right)^2 < \infty\,,$$

since $kH_1^*(q) < 1$.

This completes the proof of inequality (31) in the case $k \in \{0, \cdots, q-1\}$.

Now we consider the case where $k = q$. The approach is exactly the same except that Inequality (38) becomes an equality because in (27) $I_{2q+1-2k} = I_1$ becomes a Gaussian integral. One then has

$$\mathbb{E}\left[\left(V_N^{(3,q)}\right)^2\right] = c\gamma_N^{2H_1+2H_2}\sum_{i,j=0}^{N-1}\int_{[0,1]^4} H_{i,j}(U,U',V,V')\mathrm{d}U\mathrm{d}V\mathrm{d}U'\mathrm{d}V'\,, \tag{39}$$

with (see 35),

$$\begin{aligned} H_{i,j}(U,U',V,V') \quad &= \quad \left[|V-V'+i-j|^{-\frac{2(1-H_2)}{q+1}}\right]\left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(U,V)|U-V|^{-H_1^*(q)}\right]^q \\ &\times\left[\widetilde{\beta}_{-(\frac{1}{2}+\frac{1-H_1}{q}),-(\frac{1}{2}+\frac{1-H_2}{q+1})}(U',V')|U'-V'|^{-H_1^*(q)}\right]^q\,. \end{aligned}$$

To conclude, we now apply Part (2) of Lemma 2 with

$$\alpha_1 = 0,\ \alpha_2 = \frac{2(1-H_2)}{q+1}\,,$$

and

$$F(u,v) = \left[\widetilde{\beta}_{H_1^*(q),H_2^*(q)}(U,V)|U-V|^{-H_1^*(q)}\right]^q\,,$$

Since $\alpha_1 + \alpha_2 < 1$, the equality (33) follows. Observe that α in (32) decreases with k. Therefore the leading term of the sum (26) is obtained for $k = q$, that is, the summand in the first Wiener chaos. Finally, observe that since this term is Gaussian, convergence of the variance implies convergence in distribution. This completes the proof of Proposition 2 and hence of Theorem 2. □

6. Technical lemmas

Lemma 1. *Consider the special function β defined for any $x, y > 0$ as*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t\,.$$

Define on $\mathbb{R}^2 \setminus \{(u,v), u = v\}$, for any $a, b > -1$ such that $a + b < -1$ the function $\widetilde{\beta}_{a,b}$ as :

$$\widetilde{\beta}_{a,b}(u,v) = \begin{cases} \beta(a+1,-1-a-b) \text{ if } u < v, \\ \beta(b+1,-1-a-b) \text{ if } v < u\,. \end{cases} \tag{40}$$

Then
$$\int_{-\infty}^{u\wedge v}(u-s)^a(v-s)^b\mathrm{d}s=\widetilde{\beta}_{a,b}(u,v)|u-v|^{a+b+1}\ . \tag{41}$$
In particular,
$$\int_{-\infty}^{u\wedge v}(u-s)^a(v-s)^b\mathrm{d}s\leq C(a,b)|u-v|^{a+b+1}\ , \tag{42}$$
with
$$C(a,b)=\sup_{(u,v)\in\mathbb{R}^2}\left[\widetilde{\beta}_{a,b}(u,v)\right]<\infty\ . \tag{43}$$

Proof. We use the equivalent definition of function β
$$\beta(x,y)=\int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t\ . \tag{44}$$
Consider first the case where $u<v$. In the integral $\int_{-\infty}^{u}(u-s)^a(v-s)^b\mathrm{d}s$, we set $s'=(u-s)/(v-u)$. We get
$$\begin{aligned}\int_{-\infty}^{u}(u-s)^a(v-s)^b\mathrm{d}s &= \int_0^{\infty}\left[(v-u)^a(s')^a\right]\left[(v-u)^b(1+s')^b\right](v-u)\mathrm{d}s'\\ &= (v-u)^{a+b+1}\int_0^{\infty}(s')^a(1+s')^b\mathrm{d}s'\ .\end{aligned}$$
Hence, in view of (44), we deduce that
$$\int_{-\infty}^{u\wedge v}(u-s)^a(v-s)^b\mathrm{d}s=(v-u)^{a+b+1}\beta(x,y)$$
with $x-1=a$ and $x+y=-b$. This implies (41) in the case $u<v$. The other case $v<u$ is obtained by symmetry.

The finiteness of the constant $C(a,b)$ results from the fact that by definition of $\widetilde{\beta}$,
$$\sup_{(u,v)\in\mathbb{R}^2}\widetilde{\beta}(u,v)=\max(\beta(a+1,-1-a-b),\beta(b+1,-1-a-b))\ ,$$
which is finite (since $\beta(x,y)$ is finite for each $x,y>-1$). □

Lemma 2. *Let $\alpha_1,\alpha_2\in[0,1)$, F a function defined from $[0,1]^2$ to $\mathbb{R}_+^*$ such that*
$$\gamma=\int_{[0,1]^2}F(U,V)\mathrm{d}U\mathrm{d}V<\infty\ . \tag{45}$$
and for any $\ell\in\mathbb{Z}$
$$\Delta(\ell)=\int_{[0,1]^4}\left[\left|U-U'+\ell\right|^{-\alpha_1}\left|V-V'+\ell\right|^{-\alpha_2}\right]F(U,V)F(U',V')\mathrm{d}U\mathrm{d}V\mathrm{d}U'\mathrm{d}V'<\infty\ . \tag{46}$$
Then

(1) if $\alpha_1+\alpha_2\geq 1$, there exists some $C>0$ such that
$$\sum_{i,j=0}^{N-1}\Delta(i-j)\leq CN\log(N)^{\varepsilon(\alpha_1+\alpha_2)}\ , \tag{47}$$
where ε has been defined in (30).

(2) if $\alpha_1+\alpha_2<1$, we have

$$\lim_{N\to\infty}\left(\frac{N^{\alpha_1+\alpha_2}}{N^2}\left[\sum_{i,j=0}^{N-1}\Delta(i-j)\right]\right)=\frac{2\gamma^2}{(1-\alpha_1-\alpha_2)(2-\alpha_1-\alpha_2)}\;. \tag{48}$$

Proof. We first observe that $\Delta(\ell)=\Delta(-\ell)$ for all $\ell\in\mathbb{Z}$ and thus

$$\sum_{i,j=0}^{N-1}\Delta(i-j)=N\left[\Delta(0)+2\sum_{\ell=1}^{N-1}(1-\ell/N)\;\Delta(\ell)\right]\;. \tag{49}$$

Note that for all $\ell\geq 2$ and $U,U',V,V'\in[0,1]^4$, we have

$$\ell-1\leq|U-U'+\ell|\leq\ell+1\;.$$

Hence, for all $\ell\geq 2$,

$$\gamma^2\;(\ell+1)^{-\alpha_1-\alpha_2}\leq\Delta(\ell)\leq\gamma^2\;(\ell-1)^{-\alpha_1-\alpha_2}\;. \tag{50}$$

We now consider two cases.

Case (1) : Suppose $\alpha_1+\alpha_2\geq 1$. We get from (49) and (50) that

$$\sum_{i,j=0}^{N-1}\Delta(i-j)\leq N\left[\Delta(0)+2\Delta(1)+2\gamma^2\sum_{\ell=2}^{N-1}(\ell-1)^{-\alpha_1-\alpha_2}\right]=O\left(N\log(N)^{\varepsilon(\alpha_1+\alpha_2)}\right)\;.$$

The bound (47) follows.

Case (2) : We now assume that $\alpha_1+\alpha_2<1$. In this case, using that, as $N\to\infty$,

$$\sum_{\ell=2}^{N-1}(\ell+1)^{-\alpha_1-\alpha_2}=\int_1^N u^{-\alpha_1-\alpha_2}\mathrm{d}u+O(1)\;,$$
$$\sum_{\ell=2}^{N-1}\ell(\ell+1)^{-\alpha_1-\alpha_2}=\int_1^N (u-1)u^{-\alpha_1-\alpha_2}\mathrm{d}u+O(1)\;,$$

we get

$$\begin{aligned}\sum_{\ell=2}^{N-1}(1-\ell/N)\;(\ell+1)^{-\alpha_1-\alpha_2}&=\int_1^N u^{-\alpha_1-\alpha_2}\mathrm{d}u-\frac{1}{N}\int_1^N(u-1)u^{-\alpha_1-\alpha_2}\mathrm{d}u+O(1)\\&=\frac{N^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}-\frac{N^{1-\alpha_1-\alpha_2}}{2-\alpha_1-\alpha_2}+O(1)\\&\sim\frac{N^{1-\alpha_1-\alpha_2}}{(1-\alpha_1-\alpha_2)(2-\alpha_1-\alpha_2)}\;.\end{aligned}$$

Similarly, using instead that, as $N\to\infty$,

$$\sum_{\ell=2}^{N-1}(\ell-1)^{-\alpha_1-\alpha_2}=\int_1^N u^{-\alpha_1-\alpha_2}\mathrm{d}u+O(1)\;,$$
$$\sum_{\ell=2}^{N-1}\ell(\ell-1)^{-\alpha_1-\alpha_2}=\int_1^N (u+1)u^{-\alpha_1-\alpha_2}\mathrm{d}u+O(1)\;,$$

we get the same asymptotic equivalence, namely,

$$\sum_{\ell=2}^{N-1}(1-\ell/N)\,(\ell-1)^{-\alpha_1-\alpha_2}\sim\frac{N^{1-\alpha_1-\alpha_2}}{(1-\alpha_1-\alpha_2)(2-\alpha_1-\alpha_2)}\,.$$

Hence, with (49) and (50), we get (48). □

Lemma 3. *Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in(0,1)$.*

(1) Assume that

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4<3\,.$$

Then

$$\int_{[0,1]^4}|u_1-u_2|^{-\alpha_1}|u_2-u_3|^{-\alpha_2}|u_3-u_4|^{-\alpha_3}|u_4-u_1|^{-\alpha_i}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\,, \tag{51}$$

is finite.

(2) Let $\varepsilon\in\{-1,1\}$, then,

$$\int_{[0,1]^4}|u_1-u_2+\varepsilon|^{-\alpha_1}|u_2-u_3+\varepsilon|^{-\alpha_2}|u_3-u_4|^{-\alpha_3}|u_4-u_1|^{-\alpha_i}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\,, \tag{52}$$

is finite.

Proof. We shall apply the power counting theorem in [18], in particular Corollary 1 of this paper. Since the exponents are $-\alpha_i>-1$, $i=1,\cdots,4$, we need only to consider non–empty padded subsets of the set

$$T=\{u_1-u_2,\,u_2-u_3,\,u_3-u_4,u_4-u_1\}\,.$$

A set $W\subset T$ is said to be "padded" if for every element M in W, M is also a linear combination of elements in $W\setminus\{M\}$. That is, M can be obtained as linear combination of other elements in W. Since T above is the only non–empty padded set and since

$$d_0(T)=\mathrm{rank}(T)+\sum_T(-\alpha_i)=3-\sum_{i=1}^4\alpha_i>0\,,$$

we conclude that the integral (51) converges. This completes the proof of Part (1) of Lemma 3.

The proof of Part (2) of Lemma 3 is even simpler since there is no padded subsets of T and thus the integral (52) always converges. □

Appendix A. Multiple Wiener-Itô Integrals

Let $B=(B_t)_{t\in\mathbb{R}}$ be a classical Wiener process on a probability space $(\Omega,\mathcal{F},\mathbf{P})$. If $f\in L^2(\mathbb{R}^n)$ with $n\geq1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to B. The basic reference is the monograph [15]. Let $f\in\mathcal{S}_n$ be an elementary symmetric function with n variables that can be written as $f=\sum_{i_1,\ldots,i_n}c_{i_1,\ldots,i_n}1_{A_{i_1}\times\ldots\times A_{i_n}}$, where the coefficients satisfy $c_{i_1,\ldots,i_n}=0$ if two indexes i_k and i_l are equal and the sets $A_i\in\mathcal{B}(\mathbb{R})$ are pairwise disjoint. For such a step function f we define

$$I_n(f)=\sum_{i_1,\ldots,i_n}c_{i_1,\ldots,i_n}B(A_{i_1})\ldots B(A_{i_n})$$

where we put $B(A) = \int_{\mathbb{R}} 1_A(s) dB_s$. It can be seen that the application I_n constructed above from $\mathcal{S}_n$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_n$ in the sense

$$\mathbb{E}\left[I_n(f) I_m(g)\right] = n! \langle f, g \rangle_{L^2(T^n)} \text{ if } m = n \tag{53}$$

and

$$\mathbb{E}\left[I_n(f) I_m(g)\right] = 0 \text{ if } m \neq n.$$

Since the set $\mathcal{S}_n$ is dense in $L^2(\mathbb{R}^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.

One has $I_n(f) = I_n\left(\tilde{f}\right)$, where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

σ running over all permutations of $\{1, \cdots, n\}$. Thus

$$\mathbb{E}\left[I_n(f)^2\right] = \mathbb{E}\left[I_n(\tilde{f})^2\right] = n! \|\tilde{f}\|_2^2 \leq n! \|f\|_2^2 . \tag{54}$$

We will need the general formula for calculating products of Wiener chaos integrals of any orders m, n for any symmetric integrands $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, which is

$$I_m(f) I_n(g) = \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} I_{m+n-2k}(f \otimes_k g), \tag{55}$$

where the contraction $f \otimes_k g$ is defined by

$$\begin{aligned} &(f \otimes_k g)(s_1, \ldots, s_{m-k}, t_1, \ldots, t_{n-k}) \\ &= \int_{\mathbb{R}^k} f(s_1, \ldots, s_{m-k}, u_1, \ldots, u_k) g(t_1, \ldots, t_{n-k}, u_1, \ldots, u_k) \mathrm{d}u_1 \ldots \mathrm{d}u_k . \end{aligned} \tag{56}$$

Note that the contraction $(f \otimes_k g)$ is an element of $L^2(\mathbb{R}^{m+n-2k})$ but it is not necessarily symmetric. We will denote its symmetrization by $(f \tilde{\otimes}_k g)$.

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