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A CENTRAL LIMIT THEOREM FOR THE EULER CHARACTERISTIC OF A GAUSSIAN EXCURSION SET

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We study the Euler characteristic of an excursion set of a stationary isotropic Gaussian random field $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$. Let consider a fix level $u \in \mathbb{R}$ and also the excursion set above u, $\{t \in \mathbb{R}^d : X(t) \geq u\}$. We take the restriction to a compact domain considering for any bounded rectangle $T \subset \mathbb{R}^d$, $A(T,u) = \{t \in T : X(t) \geq u\}$. The aim of this paper is to establish a central limit theorem for the Euler characteristic of $A(T,u)$ as T grows to $\mathbb{R}^d$, as conjectured by R. Adler more than ten years ago [3].

The required assumption on X is having trajectories in $C^3(\mathbb{R}^d)$. It is stronger than Geman's assumption traditionally used in dimension one. Nevertheless, our result extends to higher dimension what is known in dimension one. In that case the Euler characteristic of $A(T,u)$ equals the number of up-crossings of X at level u.

Introduction

The Euler characteristic, also called Euler-Poincaré index, is one of the additive functionals that can be defined on the collection of all compact sets of $\mathbb{R}^d$. It contains information that describes (a part of) the topology of a compact set in $\mathbb{R}^d$. In dimension one, the Euler characteristic is the number of disjoint intervals constituting the compact set. Intuitively, in dimension two, the Euler characteristic equals the number of connected components minus the number of "holes" in the compact set. In dimension three, it equals the number of connected components minus the number of "handles" plus the number of "interior hollows".

We are interested in the Euler characteristic of an excursion set $A(T,u) = \{t \in T : X(t) \geq u\}$ for a real valued smooth stationary isotropic Gaussian field $X = \{X(t) : t \in \mathbb{R}^d\}$, a bounded closed rectangle $T \subset \mathbb{R}^d$ and a level u. We denote it as $\chi(X,T,u)$ for a while. One should consider $\chi(X,T,u)$ as an extension in dimension greater than one of the very precious tool in dimension $d = 1$ that is $U^X(T,u)$ the number of up-crossings at the level u of X on the interval T. In 2000, Adler [3] conjectured that $\chi(X,T,u)$ satisfies a central limit theorem (CLT) as T grows to $\mathbb{R}^d$. We prove it in the present paper. In dimension one, a CLT result for $U^X(T,u)$ can be found in [10] Chapter 10.

Actually, twenty years ago, Worsley [31] discovered that the expectation of $\chi(X,T,u)$ can be explicitly computed as a function of u depending on the covariance structure of X. When one wants to establish a CLT, the first step consists in estimating the asymptotic variance. Unfortunately the tricky method that yields the expectation formula cannot operate to compute the variance. In [29], formulae for the higher moments of the Differential Topology characteristic of $A(T,u)$, which

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is not so far from $\chi(X,T,u)$, are obtained for fixed T under convenient assumptions. Our method allows us to obtain an L^2 approximation of $\chi(X,T,u)$. In order to prove that the asymptotic variance is finite, we use an Hermite decomposition. The second step towards a CLT is to get the asymptotic normality. At this stage, the expansion of $\chi(X,T,u)$ into stochastic integrals with respect to Hermite polynomials is used again as well as a continuous parameter version of the celebrated Breuer-Major theorem (see [8] for instance). Precisely, we follow Nourdin et al. [22] but with some modifications motivated by the fact that our process has parameter in $\mathbb{R}^d$ instead of $\mathbb{Z}$ as in the cited article.

In the present paper, two types of tools are mixed. One is concerned with level functionals and its Hermite expansion. This point of view allows us to use the revisited Stein method presented for instance in [25, 24, 22]. A CLT result for $\chi(X,T,u)$ appears then as a consequence of the asymptotic normality of each stochastic integral with respect to Hermite polynomials. The second tool deals with the geometrical aspect of the work. Actually we have to consider the random vectorial field $\mathbf{X} = (X, \nabla X, \nabla^2 X)$ and it is convenient to see it as a random function from $\mathbb{R}^d$ to $\mathbb{R}^D$ with $D = 1 + d + d(d+1)/2$. It implies tedious differential calculus in high dimension.

Beyond our work, the study of the excursion sets of a stationary field is a very popular theme. Many authors were and still are interested in this domain as proved by the successful books of Adler & Taylor [5] and Azas & Wshebor [10], or the recent papers [9, 27]. On the one hand, the description of the excursion sets appears very powerful to characterize the field X. For instance since the first Adler's book [2] one knows that the expectation of $\chi(X,T,u)$ is a good approximation for the probability of the maximum of X on T to be greater than u. Also the line integral with respect of the level curve at any level u provides information on the anisotropy property of X (see the works of Cabaa [12] and Iribarren [16]). On the other hand, at least in the Gaussian case, accurate methods such as the theory of crossings can be used to get explicit values for level functionals (see the seminal work of Slud [26] and also the paper of Kratz and Len [17]).

Our study for establishing a CLT for level functionals has many precursors in the literature. These functionals are usually used to build statistical estimators or to construct statistical tests. The first one that we can cite is Adler's work [1] that uses the Euler characteristic of an excursion set to build a spectral moment estimator for two dimensional Gaussian fields. Moreover, this estimator was proved to satisfy asymptotical normality. Afterwards, the two cited works [12] and [16] established a CLT for studying the asymptotic behavior of estimators based on the level sets, actually the line integral with respect to a level curve. Following this direction, we have in mind statistical outcomes of our result. They could serve various fields of application such as brain exploration or representation of the universe following [30] or the nice introduction of the forthcoming book [6], as well as worn surfaces or more generally rough surfaces as proposed in [7, 28]. Our result could be used to get the asymptotic distribution of the statistic under the null hypothesis in a test of normality. Furthermore, it should also give a functional CLT for $u \mapsto \chi(X,T,u)$ as in [21, 28] where similar questions are studied. We also have in mind extensions to non-Gaussian fields or to Gaussian non stationary fields, starting from the recent results concerning the volume ([21, 11]) or the expected Euler characteristic ([13, 7]) of excursion sets in these cases.

Hypothesis on X.

All over the paper we deal with a centered stationary isotropic Gaussian field $X = \{X(t) : t \in \mathbb{R}^d\}$ such that $\mathrm{Var}(X(0)) = 1$. We also assume that almost every realization of X is of class C^3 on $\mathbb{R}^d$. This last hypothesis should certainly be weakened, but we use it in this form to make the computations as fluent as possible.
We write X_i and X_{ij} the derivatives of X of first and second order: $\frac{\partial X}{\partial t_i}(t) = X_i(t)$ and $\frac{\partial^2 X}{\partial t_i \partial t_j}(t) = X_{ij}(t)$ as well as $\nabla X(t) = (X_i(t))_{1\le i\le d}$ and $\nabla^2 X(t) = (X_{ij}(t))_{1\le i,j\le d}$. Denoting by r the covariance function of X,

$$r(t) = \mathrm{Cov}(X(0), X(t))\,,$$

the assumption on X implies that $r \in C^6(\mathbb{R}^d)$ and for any multidimensional index $\mathbf{m} = (i_1, \dots, i_k)$ with $1 \le k \le 6$ and $1 \le i_j \le d$, we write

$$\frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t) = \frac{\partial^k r}{\partial t_{i_1} \dots \partial t_{i_k}}(t) = r^{(k)}_{i_1 \cdots i_k}(t)\,.$$

It is well know that for any fixed t, $X(t)$ and $\nabla X(t)$ are independent, as well as $\nabla X(t)$ and $\nabla^2 X(t)$. Moreover, since X is isotropic, for any fixed t and any $1 \le i \ne j \le d$, $X_i(t)$ and $X_j(t)$ are independent and there exists a real number $\lambda \ge 0$ such that for any i, $r^{(2)}_{ii}(0) = -\lambda$. In order to avoid working with a degenerate field X, we assume that $\lambda > 0$.

Outline of the article.

Since we are interested in the asymptotic as the rectangle T tends to $\mathbb{R}^d$, we start our study without taking into account what happens on the boundary of T. Hence, in Section 1 instead of considering $\chi(X, T, u)$, the Euler characteristic of the excursion above u, we consider $\varphi(X, T, u)$, a modified quantity inspired by [5] Lemma 11.7.1, which we call modified Euler characteristic of the excursion above u. Roughly speaking, by applying Morse's theorem, both notions coincides on the interior of T. The precise definition of $\varphi(X, T, u)$ is given in Section 1.2 whereas the definition of $\chi(X, T, u)$ stands in Section 2.3.

Section 1 is devoted to the study of the L^2 properties of $\varphi(X, T, u)$. In Section 1.1, we first establish a lemma that gives an upper bound for the conditional expectation of $[\det(\nabla X(0))]^2$ given that $\nabla X(t) = \nabla X(0) = x$ for t in a neighborhood of 0. As a by product, we obtain some results on the second moment of $N^{\nabla X}(T, v) = \{t \in T\,;\, \nabla X(t) = v\}$ which are of interest for their own (see Proposition 1.2). In Section 1.2 we state that the usual approximation of the number of roots of a vector field by the area formula (named as "Metatheorem" in [5], Th.11.2.3) not only holds almost surely, but also in $L^2(\Omega)$. Taking the limit of the Hermite expansion of this approximation yields the expansion of $\varphi(X, T, u)$ in Section 1.3.

Section 2 deals with the main result, namely Theorem 2.6 that gives a CLT for $\chi(X, T, u)$ as $T \nearrow \mathbb{R}^d$. We first solve this question for $\varphi(X, T, u)$ in Section 2.2, after establishing in Section 2.1 that the asymptotic variance of $|T|^{-1/2}\varphi(X, T, u)$ is finite. The asymptotic normality of $\varphi(X, T, u)$ is obtained through a Breuer-Major type argument. We prove it in our setting, in other words for a Gaussian process indexed by a d-dimensional continuous parameter. This last result (Proposition 2.4) is interesting for its own and should be used separately in further works. In Section 2.3 we use the theory of Morse to transfer the CLT from $\varphi(X, T, u)$ to

$\chi(X,T,u)$. Actually, it allows us to prove that the difference between these two quantities involves the l-dimensional faces of T with $l < d$, and hence is negligible with respect to $|T|^{1/2}$.

Two technical proofs have been postponed in Section 3. The proof of Lemma 1.1 includes differential calculus and sharp estimates. The proof of Lemma 2.2 deals with specific Gaussian calculus. It shows how tricky must be the computations as soon as one wants to obtain explicit formula in this domain.

Since the field X and the level u will be fixed almost everywhere in the rest of the paper, we drop the dependence in our notations and, from now on, we write $\varphi(T)$ instead of $\varphi(X,T,u)$, as well as $\chi(T)$ instead of $\chi(X,T,u)$.

1. L^2 properties of $\varphi(T)$

Let $T = \Pi_{1\leq i\leq d}[a_i, b_i]$ be a bounded rectangle in $\mathbb{R}^d$ and let consider the excursion set $A(T,u) = \{t \in T;\, X(t) \geq u\}$.

1.1. Variance of the number of crossings of ∇X.

In this section we introduce the number of crossings of the vectorial random field ∇X at level v in T. This random variable will be used as an auxiliary tool to obtain an approximation in $L^2(\Omega)$ of the Euler characteristic of $A(T,u)$. For any $v \in \mathbb{R}^d$, we denote as $N^{\nabla X}(T,v)$ the aforementioned random variable

$$N^{\nabla X}(T,v) = \#\{t \in T :\, \nabla X(t) = v\}.$$

The two following Rice's formulas are shown in [10] or [5].

$$\begin{aligned} \mathbb{E}[N^{\nabla X}(T,v)] &= |T|\,\mathbb{E}[|\det \nabla X(0)|]\, p_0(v), \\ \mathbb{E}[N^{\nabla X}(T,v)(N^{\nabla X}(T,v)-1)] &= \int_T \int_T \mathbb{E}[|\det \nabla^2 X(t)\ \det \nabla^2 X(s)| \\ &\quad /\ \nabla X(t) = \nabla X(s) = v]\, p_{t,s}(v,v)\, dtds, \end{aligned} \tag{1}$$

where $p_t(.)$ and $p_{t,s}(.,.)$ are the probability density functions of $\nabla X(t)$ and $(\nabla X(t), \nabla X(s))$ respectively. Using the stationarity of X, we get

$$\begin{aligned} \mathbb{E}[N^{\nabla X}(T,v)(N^{\nabla X}(T,v)-1)] &= \int_{T_0} |T \cap (T-s)|\, \mathbb{E}[|\det \nabla^2 X(0)\ \det \nabla^2 X(s)| \\ &\quad /\ \nabla X(0) = \nabla X(s) = v]\, p_{0,s}(v,v) ds \end{aligned}$$

where T_0 denotes the rectangle around 0 obtained from $T = \Pi_{1\leq j\leq d}[a_j, b_j]$ by prescribing $T_0 = \Pi_{1\leq j\leq d}[a_j - b_j, b_j - a_j]$. In the following lemma we give an upper bound for the above integrand. Its proof can be found in Section 3.

Lemma 1.1. *Let X be a stationary isotropic Gaussian field with trajectories of class C^3. For all $v, s \in \mathbb{R}^d$, let*

$$F(v,s) = \mathbb{E}[|\det \nabla^2 X(0)\ \det \nabla^2 X(s)| \,/\, \nabla X(0) = \nabla X(s) = v]\, p_{0,s}(v,v).$$

Then, for any compact set $K \subset \mathbb{R}^d$, $\exists\, C_K$ and δ_K such that $\forall v \in K$, $\forall s \in \mathbb{R}^d$,

- $|F(v,s)| \leq C_K$
- $||s|| \leq \delta_K \Rightarrow |F(v,s)| \leq C_K ||s||$.

In this form the second factorial moment of $N^{\nabla X}(T,v)$ results finite.

The lemma allows us to obtain two useful results. Let us define the following approximation sequence

$$N_\varepsilon^{\nabla X}(T,v) = \int_T |\det(\nabla^2 X(t))|\, \delta_\varepsilon(\nabla X(t) - v)dt\,, \tag{2}$$

where $\delta_\varepsilon = (2\varepsilon)^{-d} 1_{[-\varepsilon,\varepsilon]^d}$. It is well known (see [5] Th.11.2.3 for instance) that $N_\varepsilon^{\nabla X}(T,0) \underset{\varepsilon\to 0}{\to} N^{\nabla X}(T,0)$ almost surely.

Proposition 1.2. *Under the hypothesis of Lemma 1.1 we have*

1. *$v \mapsto \mathbb{E}[(N^{\nabla X}(v))^2]$ is a continuous function on $\mathbb{R}^d$*
2. *$N_\varepsilon^{\nabla X}(T,0) \underset{\varepsilon\to 0}{\to} N^{\nabla X}(T,0)$ in $L^2(\Omega)$.*

Proof. The first assertion holds true if we show the continuity of the second factorial moment of $N^{\nabla X}(T,.)$ since, according to (1), $v \mapsto \mathbb{E}(N^{\nabla X}(T,v))$ clearly appears as a continuous function. From Lemma 1.1 we know that for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_{|s|<\delta} F(v,s)ds < \varepsilon$ uniformly in v. Then,

$$\begin{aligned}
&|\mathbb{E}[N^{\nabla X}(T,v)(N^{\nabla X}(T,v)-1)] - \mathbb{E}[N^{\nabla X}(T,v')(N^{\nabla X}(T,v')-1)]| \\
&\qquad \le \int_{|s|<\delta} F(v,s)ds + \int_{|s|<\delta} F(v',s)ds + |\int_{|s|>\delta} (F(v,s) - F(v',s))ds| \\
&\qquad\qquad \le 2\varepsilon + |\int_{|s|>\delta} (F(v,s) - F(v',s))ds|.
\end{aligned}$$

The continuity follows from the continuity of $v \mapsto F(v,s)$ for any fixed s, by taking the lim sup as $v' \to v$ in the left hand side of the inequality.

To prove the second assertion we use the area formula ([10] Prop. 6.1 for instance)

$$N_\varepsilon^{\nabla X}(T,0) = \int_T |\det(\nabla^2 X(t))|\, \delta_\varepsilon(\nabla X(t))dt = \int_{\mathbb{R}^d} N^{\nabla X}(T,u)\, \delta_\varepsilon(u)du.$$

Thus

$$\begin{aligned}
\mathbb{E}[(N^{\nabla X}(T,0))^2] &\le \limsup_{\varepsilon\to 0} \mathbb{E}[(\int_T |\det(\nabla^2 X(t))|\, \delta_\varepsilon(\nabla X(t))dt)^2] \\
&= \limsup_{\varepsilon\to 0} \mathbb{E}[(\int_{\mathbb{R}^d} N^{\nabla X}(T,v)\delta_\varepsilon(v)dv)^2] \\
&\le \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbb{E}[(N^{\nabla X}(T,v))^2]\delta_\varepsilon(v)dv = \mathbb{E}[(N^{\nabla X}(T,0))^2].
\end{aligned}$$

In the first line we have used Fatou's Lemma and the a.s. convergence of $N_\varepsilon^{\nabla X}(T,0)$ to $N^{\nabla X}(T,0)$, in the second one the area formula and finally in the third one Jessen inequality and the continuity proved before. This implies that

$$\mathbb{E}[(N_\varepsilon^{\nabla X}(T,0))^2] \to \mathbb{E}[(N^{\nabla X}(T,0))^2],$$

and this fact combined with the a.s. convergence gives the point (2). □

Remark. In dimension $d = 1$, the fact that the number of up-crossings of X at level u, namely $U^X(T,u)$, can be approximated in $L^2(\Omega)$ has already been established (see for instance Th.10.10 in [10]). The usual condition for this result to

hold is $\int_{|t|\leq 1} \Theta(t)/|t|\, dt < +\infty$ where $\Theta(t) = r''(t) + \lambda$, known as the Geman's assumption. It is weaker than assumption **(H)** and results to be a necessary condition for $U^X(T,u)$ to belong to $L^2(\Omega)$ (see [18]).

1.2. L^2 approximation of $\varphi(T)$ via the area formula.

It is time to give a precise definition of $\varphi(T)$, the modified Euler characteristic of the excursion set $A(T,u)$. We introduce

$$\varphi(T) = \sum_{k=0}^{d} (-1)^k \mu_k(T)$$

with

$$\mu_k(T) = \#\{t \in T :\, X(t) \geq u,\, \nabla X(t) = 0,\, \text{index}(\nabla^2 X(t)) = d-k\}.$$

Here the "index" stands for the number of negative eigenvalues.

Proposition 1.3. *For X a stationary isotropic Gaussian field with trajectories of class C^3, we have $\varphi(T) \in L^2(\Omega)$ and the following convergence holds almost surely and in $L^2(\Omega)$*

$$\varphi(T) = \lim_{\varepsilon \to 0^+} (-1)^d \int_T \det(\nabla^2 X(t)) 1_{[u,\infty)}(X(t))\, \delta_\varepsilon(\nabla X(t)) dt\,. \tag{3}$$

Proof. The almost sure convergence in (3) is contained in Theorem 11.2.3 of [5], so we only prove the convergence in $L^2(\Omega)$.
Let us denote by $\varphi(\varepsilon, T)$ the right hand side of (3). We have the following trivial bound

$$\varphi(\varepsilon, T) \leq N_\varepsilon^{\nabla X}(T,0).$$

But in Proposition 1.2 we have shown the convergence of $\mathbb{E}[(N_\varepsilon^{\nabla X}(T,0))^2]$ towards $\mathbb{E}[(N^{\nabla X}(T,0))^2]$, then the dominated convergence theorem allows us to conclude that

$$\mathbb{E}[\varphi^2(\varepsilon,T)] \to \mathbb{E}[\varphi^2(T)] \leq \mathbb{E}[(N^{\nabla X}(T,0))^2] < +\infty.$$

We obtain as a bonus that $\varphi(T) \in L^2(\Omega)$. Furthermore, the above convergence of the $L^2(\Omega)$ norms and the almost sure convergence of $\varphi(\varepsilon,T)$ imply

$$\mathbb{E}[(\varphi(\varepsilon,T) - \varphi(T))^2] \to 0. \ \square$$

Remark: The first study in dimension $d > 1$ on the finiteness of the two order moment of the Euler characteristic was the article of Adler and Hasofer [4]. More recently the fact that $\varphi(T)$ belongs to $L^2(\Omega)$ has been implicitly established in [29] under convenient assumptions. In the eighties of the past century some articles were devoted to establish the finiteness of the second moment for the number of roots of a stationary Gaussian random field (see [14], [20]). Our proof is somewhat inspired by Elizarov's work [14].

1.3. **Hermite type expansion of $\varphi(T)$.**
In what follows, we use the Hermite polynomials $(H_n)_{n\in\mathbb{N}}$ defined by $H_n(x) = (-1)^n\, e^{x^2/2}\, \frac{\partial^n}{\partial x^n}(e^{-x^2/2})$. They provide an orthonormal basis of $L^2(\mathbb{R}, \phi(x)dx)$ where ϕ denotes the standard Gaussian density on $\mathbb{R}$. We also denote by ϕ_m the standard Gaussian density on $\mathbb{R}^m$.

In order to get an expansion of $\varphi(T)$ as stochastic integrals with respect to Hermite polynomials, as a first step, we establish the expansion of the right hand side of (3).
In the following we identify any symmetric matrix of size $d\times d$ with the $d(d+1)/2$-dimensional vector containing the coefficients on and above the diagonal and write $\tilde{\det}$ the associated determinant map. Let us also define

$$D = d + d(d+1)/2 + 1\,.$$

Hence, if we consider the map G_ε defined on $\mathbb{R}^D$ by

$$\begin{aligned} G_\varepsilon : \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2} \times \mathbb{R} &\to \mathbb{R} \\ (x, \mathbf{y}, z) &\mapsto \delta_\varepsilon(x)\, \tilde{\det}(\mathbf{y})\, 1_{[u,\infty)}(z)\,, \end{aligned}$$

one can remark that $G_\varepsilon(z)$ can be factorized as

$$G_\varepsilon(z) = \delta_\varepsilon(\underline{z}) f(\bar{z}) \text{ where } z = (\underline{z}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2+1}\,. \tag{4}$$

On the other hand, let us denote by $\Sigma^{\mathbf{X}}$ the covariance matrix of the D-dimensional Gaussian vector field

$$\mathbf{X}(t) = (\nabla X(t), \nabla^2 X(t), X(t))\,.$$

We choose Λ a $D\times D$ matrix such that $\Lambda\, {}^t\Lambda = \Sigma^{\mathbf{X}}$, so that we can write for any fixed $t\in\mathbb{R}^d$, $\mathbf{X}(t) = \Lambda Y(t)$ with $Y(t)$ a D-dimensional standard Gaussian vector. Given that the field X is stationary, it holds that $\nabla X(t)$ is independent from $(\nabla^2 X(t), X(t))$ for each fixed t. Hence, the matrix $\Sigma^{\mathbf{X}}$ is diagonal by blocs of respective dimensions d and $d(d+1)/2+1$ and Λ factorizes into $\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$. Furthermore, since X is isotropic, $\Lambda_1 = \sqrt{\lambda}\, I_d$ where $\lambda = -r_{ii}^{(2)}(0)$ for any $i = 1,\dots,d$. We define

$$\widetilde{G}_\varepsilon(y) = G_\varepsilon(\Lambda y) = \delta_\varepsilon(\Lambda_1 \underline{y}) f(\Lambda_2 \bar{y}) = \delta_\varepsilon \circ \Lambda_1(\underline{y})\, f\circ\Lambda_2(\bar{y}).$$

Since the map $\widetilde{G}_\varepsilon$ clearly belongs to $\mathbb{L}^2(\mathbb{R}^D, \phi_D(y)dy)$ the following expansion converges in this space

$$\widetilde{G}_\varepsilon(y) \;=\; \sum_{q=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D;\, |\mathbf{n}|=q} c(\widetilde{G}_\varepsilon, \mathbf{n})\widetilde{H_{\mathbf{n}}}(y)$$

where $\mathbf{n} = (n_1, n_2, \dots, n_D)$, $|\mathbf{n}| = n_1 + n_2 + \cdots + n_D$ and $\widetilde{H_{\mathbf{n}}}(y) = \prod\limits_{1\le j\le D} H_{n_j}(y_j)$. The $\mathbf{n}$-th Hermite coefficient of $\widetilde{G}_\varepsilon$ is given by

$$c(\widetilde{G}_\varepsilon, \mathbf{n}) = \frac{1}{\mathbf{n}!}\int_{\mathbb{R}^D} \widetilde{G}_\varepsilon(y)\widetilde{H_{\mathbf{n}}}(y)\phi_D(y)\, dy$$

with $\mathbf{n}! = n_1!n_2!\dots n_D!$. The factorization (4) induces a factorization of the Hermite coefficient into

$$c(\widetilde{G}_\varepsilon, \mathbf{n}) = c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f \circ \Lambda_2, \bar{n})$$

with self understanding notations concerning $\mathbf{n} = (\underline{n}, \bar{n})$ and the Hermite coefficients of the maps $\delta_\varepsilon \circ \Lambda_1$ and $f \circ \Lambda_2$.
Writing $\varphi(\varepsilon, T)$ as $\varphi(\varepsilon, T) = \int_T \widetilde{G}_\varepsilon(Y(t))dt$ yields the following expansion

$$\varphi(\varepsilon, T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n},\bar{n}) \\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt. \tag{5}$$

We will take the limit as ε goes to 0 in (5) to obtain the expansion of $\varphi(T)$. We first compute the limit of the coefficient $c(\delta_\varepsilon \circ \Lambda_1, \underline{n})$:

$$\begin{aligned} c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) \quad &= \quad \frac{1}{\underline{n}!} \int_{\mathbb{R}^d} \delta_\varepsilon(\lambda \underline{y}) \widetilde{H_{\underline{n}}}(\underline{y}) \phi_d(\underline{y})\, d\underline{y} \\ &\xrightarrow[\varepsilon \to 0]{} \quad \frac{1}{\underline{n}!} (2\pi\lambda)^{-d/2}\, \widetilde{H_{\underline{n}}}(\underline{0}) := d(\underline{n})\,. \end{aligned} \tag{6}$$

Let us point out that this coefficient is zero if at least one of the indices n_j is odd. In what follows, we introduce

$$a(\mathbf{n}) = d(\underline{n})\, c(f \circ \Lambda_2, \bar{n}) \quad \text{for} \quad \mathbf{n} = (\underline{n}, \bar{n}) \in \mathbb{N}^d \times \mathbb{N}^{D-d}\,. \tag{7}$$

Proposition 1.4. *Let X be a stationary isotropic Gaussian field with C^3 trajectories. The following expansion holds in $L^2(\Omega)$*

$$\varphi(T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\mathbf{n} \in \mathbb{N}^D;\, |\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt.$$

Note that, according to Mehler's Formula (see [10] Lemma 10.7), if $|\mathbf{n}| \neq |\mathbf{m}|$ then $\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s)), \widetilde{H_{\mathbf{m}}}(Y(t))) = 0$. The above expansion results orthogonal in $L^2(\Omega)$.

Proof. Let us take the formal limit of the rhs of (5) and define the random variable

$$\eta(T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\mathbf{n} \in \mathbb{N}^D;\, |\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt.$$

The first step consists in proving that $\eta(T)$ belongs to $L^2(\Omega)$. Let Q be a positive integer and let us denote by $\eta_Q(T)$ the projection of $\eta(T)$ obtained with the finite expansion for $q \leq Q$. The second moment can be bounded by using the orthogonality relations between the Hermite polynomials. Hence,

$$\begin{aligned} \mathbb{E}[\eta_Q(T)^2] \quad &= \quad \sum_{q=0}^{Q} \mathbb{E}[\,(\sum_{|\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt)^2\,] \\ &\leq \quad \sum_{q=0}^{Q} \lim_{\varepsilon \to 0} \mathbb{E}[\,(\sum_{\substack{\mathbf{n}=(\underline{n},\bar{n}) \\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt)^2] \\ &\leq \quad \lim_{\varepsilon \to 0} \mathbb{E}[\varphi(\varepsilon, T)^2] = \mathbb{E}[\varphi(T)^2] \;<\; \infty, \end{aligned}$$

where we have used Fatou's Lemma in the second line, and (5) and Proposition 1.4 in the last one. Thus $\mathbb{E}[\eta(T)^2] < \infty$.
It is now easy to conclude that $\varphi(T) = \eta(T)$ in $L^2(\Omega)$. □

2. Central limit theorem for $\chi(T)$

In this section we will prove our main result which consists in a central limit theorem for the Euler characteristic $\chi(T)$ of the excursion set $A(T,u)$ when T grows to $\mathbb{R}^d$. We will first concentrate on $\varphi(T)$ and will be interested in the asymptotics of $\zeta(T) = \frac{\varphi(T)-\mathbb{E}\varphi(T)}{|T|^{1/2}}$. To make it precise, we assume that the compact rectangle T has the following shape $T = [-N,N]^d$ with N a positive integer, and we let N go to infinity. We will prove that the random variable

$$\zeta([-N,N]^d)) = \frac{\varphi([-N,N]^d) - \mathbb{E}\varphi([-N,N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable.

We need to introduce the following assumption.
Assumption (H): Denoting $\psi(t) = \max\{\left|\frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t)\right|;\ \mathbf{m} \in \{1,\dots,d\}^k,\ 0 \le k \le 4\}$,

$$\psi \in L^1(\mathbb{R}^d) \ \text{ and } \ \psi(t) \to 0 \text{ when } ||t|| \to +\infty$$

Note that **(H)** implies that $r \in L^q(\mathbb{R}^d)$ for all $q \ge 1$ and hence that X admits a spectral density f_X which is continuous.

2.1. Asymptotic variance of $\varphi(T)$.

We start with a crucial result which states that the asymptotic variance of $\zeta([-N,N]^d))$ does not degenerate as N goes to infinity. As expected, the asymptotic variance depends on the level u.

Proposition 2.1. *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories satisfying Assumption* **(H)**. *For any level u,*

$$\mathrm{Var}\left(\zeta([-N,N]^d)\right) \underset{N\to+\infty}{\to} V(u) \text{ with } V(u) < +\infty.$$

Moreover, $V(u) \ge f_X(0)\,\lambda^d\, H_d(u)^2\,\phi(u)^2$.

Proof.
Starting from the Hermite type expansion of $\varphi([-N,N]^d)$ in $L^2(\Omega)$ as it is given in Proposition 1.4, and using the orthogonality, we obtain

$$\mathrm{Var}\left(\zeta([-N,N]^d)\right) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\, R^N(\mathbf{n},\mathbf{m}) \tag{8}$$

with

$$\begin{aligned} R^N(\mathbf{n},\mathbf{m}) &= (2N)^{-d} \int_{[-N,N]^d}\int_{[-N,N]^d} \mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s)), \widetilde{H_{\mathbf{m}}}(Y(t)))\, dsdt \\ &= \int_{[-2N,2N]^d} \mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)), \widetilde{H_{\mathbf{m}}}(Y(v)))\, \prod_{1\le k\le d}(1-\frac{|v_k|}{2N})\, dv. \end{aligned}$$

A slight generalization of Mehler's formula (see Lemma 10.7 in [10]) allows us to write for any $\mathbf{n}, \mathbf{m} \in \mathbb{N}^D$ such that $|\mathbf{n}| = |\mathbf{m}|$,

$$\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)), \widetilde{H_{\mathbf{m}}}(Y(v))) = \sum_{\substack{d_{ij}\geq 0 \\ \sum_i d_{ij}=n_j\,;\,\sum_j d_{ij}=m_i}} \mathbf{n}!\,\mathbf{m}! \prod_{1\leq i,j\leq D} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{(d_{ij})!},$$

where

$$\Gamma_{ij}^Y(v) = \mathrm{Cov}(Y_i(0), Y_j(v)) \quad \text{for } i,j = 1,\dots,D.$$

Since $\Gamma^Y(v) = \Lambda^{-1}\Gamma^{\mathbf{X}}(v)^t(\Lambda^{-1})$ with $\Gamma^{\mathbf{X}}$ the covariance function of $(\nabla X, \nabla^2 X, X)$, we have for any $v \in \mathbb{R}^d$,

$$\sup_{1\leq i,j\leq D} |\Gamma_{ij}^Y(v)| \leq K\,\psi(v),$$

where ψ has been introduced in Assumption **(H)** and K is some positive constant. Hence, for $|\mathbf{n}| = |\mathbf{m}| = q$,

$$\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)), \widetilde{H_{\mathbf{m}}}(Y(v))) \;\leq\; K'\,\psi^q(v),$$

with some positive constant K'. By Assumption **(H)**, $\psi \in L^q(\mathbb{R}^d)$, so we can apply the dominated convergence theorem and get

$$R^N(\mathbf{n},\mathbf{m}) \underset{N\to\infty}{\longrightarrow} R(\mathbf{n},\mathbf{m}) = \int_{\mathbb{R}^d} \mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)), \widetilde{H_{\mathbf{m}}}(Y(v)))\,dv. \tag{9}$$

According to (8), we have $\mathrm{Var}\,(\zeta([-N,N]^d)) = \sum_{q=1}^{\infty} V_q^N$ with

$$V_q^N \underset{N\to\infty}{\longrightarrow} V_q := \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\,R(\mathbf{n},\mathbf{m}). \tag{10}$$

Note that for any q, $V_q^N \geq 0$ and so $V_q \geq 0$. We will establish that

$$\sup_N \sum_{q=Q+1}^{\infty} V_q^N \underset{Q\to\infty}{\longrightarrow} 0. \tag{11}$$

Using Fatou's Lemma, it will prove that the series $V = \sum_{q=1}^{\infty} V_q$ is convergent and that $\mathrm{Var}\,(\zeta([-N,N]^d))$ tends to V. And the first step of Proposition 2.1 will be achieved.
Let us remark that (11) is equivalent to

$$\mathrm{Var}\left(\pi_Q(\zeta([-N,N]^d))\right) \underset{Q\to\infty}{\longrightarrow} 0 \text{ uniformely w.r.t } N,$$

where π_Q is the projection onto the terms of order $> Q$.

Let $\mathbf{s} \in \mathbb{R}^d$ and set $\theta_{\mathbf{s}}$ the shift operator associated with the field X, i.e. $\theta_{\mathbf{s}}X_\cdot = X_{\mathbf{s}+\cdot}$. Introducing the set of indices

$$\mathcal{I}_N = [-N,N)^d \cap \mathbb{Z}^d$$

we can write

$$\zeta([-N,N]^d) = \zeta([-N,N)^d) = (2N)^{-d/2} \sum_{\mathbf{s}\in\mathcal{I}_N} \theta_{\mathbf{s}} \circ \zeta([0,1)^d).$$

Then, denoting by $V_{N,Q}$ the variance $\mathrm{Var}\left(\pi_Q(\zeta([-N,N]^d))\right)$ and using the stationarity of X, we obtain

$$V_{N,Q} = (2N)^{-d} \sum_{\mathbf{s}\in\mathcal{I}_{2N}} \alpha_{\mathbf{s}}(N)\, \mathbb{E}\left(\pi_Q(\zeta([0,1)^d))\, \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0,1)^d)))\right),$$

where $\alpha_{\mathbf{s}}(N)$ denotes the cardinal of $\{\mathbf{t} \in \mathcal{I}_N : \mathbf{t}-\mathbf{s} \in \mathcal{I}_N\}$ which is certainly less than $(2N)^d$.

Let chose a such that $\psi(\mathbf{s}) \le \rho < 1/K$ for $||\mathbf{s}||_\infty \ge a$. We split $V_{N,Q}$ into $V^1_{N,Q} + V^2_{N,Q}$ where in $V^1_{N,Q}$ the sum runs for the indices $\mathbf{s} \in \{\mathbf{s} \in \mathcal{I}_{2N} : ||\mathbf{s}||_\infty < a+1\}$ and in $V^2_{N,Q}$ for $\{\mathbf{s} \in \mathcal{I}_{2N} : ||\mathbf{s}||_\infty \ge a+1\}$.

At first, it holds for $2N > a+1$,

$$|V^1_{N,Q}| \le (2N)^{-d}\,(2a+2)^d\,(2N)^d\,\mathbb{E}\left(\pi_Q(\zeta([0,1)^d))^2\right)$$

which goes to 0 as Q goes to ∞ uniformly with respect to N.

Next, for any $\mathbf{s} \in \mathcal{I}_{2N}$ such that $||\mathbf{s}||_\infty \ge a+1$, we write

$$\mathbb{E}\left(\pi_Q(\zeta([0,1)^d))\, \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0,1)^d)))\right)$$

$$= \sum_{q=Q+1}^{\infty} \int_{[0,1)^d} \int_{[0,1)^d} \mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))]dtdu, \tag{12}$$

where

$$\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))] = \mathbb{E}[\sum_{\mathbf{n}\in\mathbb{N}^D;|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H}_{\mathbf{n}}(Y(t)) \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H}_{\mathbf{n}}(Y(\mathbf{s}+u)].$$

Arcones inequality ([8] Lemma 1), implies that

$$|\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))]| \le K^q\,\psi^q(\mathbf{s}-(u-t)) \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})^2\mathbf{n}!.$$

Let us remark that the series $\sum_{\mathbf{n}\in\mathbb{N}^D} a(\mathbf{n})^2\mathbf{n}!$ diverges so that we have to handle carefully in what follows. Recall that equation (7) writes as $a(\mathbf{n}) = d(\underline{n})\, c(f\circ\Lambda_2, \overline{n})$ with $d(\underline{n})$ given by (6). In Imkeller et al. [15] it is shown that $\sup_x |\frac{H_l(x)}{\sqrt{l!}}\,\varphi(x)| \le C$ for a universal constant C. This yields $d^2(\underline{n})\underline{n}! \le C^d$ and hence

$$\sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})^2\mathbf{n}! \le C^d\,q^d \sum_{|\overline{n}|\le q} c(f\circ\Lambda_2,\overline{n})^2\overline{n}! \le C^d\,q^d\,||f\circ\Lambda_2||^2.$$

Therefore the absolute value of (12) can be bounded by

$$C^d\,||f\circ\Lambda_2||^2 \sum_{q=Q+1}^{\infty} q^d\,K^q \int_{[0,1)^d}\int_{[0,1)^d} \psi^q(\mathbf{s}-(u-t))\,dudt.$$

Hence

$$\begin{aligned}
|V^2_{N,Q}| &\le \sum_{\mathbf{s}\in\mathcal{I}_{2N};\,||\mathbf{s}||_\infty\ge a+1} \left|\mathbb{E}\left(\pi_Q(\zeta([0,1)^d))\,\pi_Q(\theta_{\mathbf{s}}\circ(\zeta([0,1)^d)))\right)\right| \\
&\le C^d\,||f\circ\Lambda_2||^2 \sum_{q=Q+1}^{\infty} q^d\,K^q\,\rho^{q-1} \\
&\quad\times \sum_{\mathbf{s}\in\mathcal{I}_{2N};\,||\mathbf{s}||_\infty\ge a+1} \int_{[0,1)^d}\int_{[0,1)^d} \psi(\mathbf{s}-(u-t))\,dudt,
\end{aligned}$$

where we have used that for any $||\mathbf{s}||_\infty \geq a+1$ and $u,t \in [0,1)^d$, $\psi(\mathbf{s}-(u-t)) \leq \rho$. On the one hand, since $\rho < 1/K$, $\sum_{q=Q+1}^{\infty} q^d K^q \rho^{q-1}$ is the tail of a convergent series. On the other hand,

$$\sum_{\mathbf{s}\in\mathcal{I}_{2N};\, ||\mathbf{s}||_\infty \geq a+1} \int_{[0,1)^d}\int_{[0,1)^d} \psi(\mathbf{s}-(u-t))\, du dt$$
$$\leq \sum_{\mathbf{s}\in\mathcal{I}_{2N}} \int_{[0,2)^d} \psi(\mathbf{s}+u)\, du \leq 2\int_{\mathbb{R}^d} \psi(u)\, du < +\infty.$$

Hence $\sup_N |V_{N,Q}^2|$ goes to 0 as Q goes to infinity and we have proved that $\mathrm{Var}\zeta([-N,N]^d)$ tends to

$$V = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\, R(\mathbf{n},\mathbf{m}) < +\infty. \tag{13}$$

The first assertion of Proposition 2.1 being established, it remains to prove that $V \geq f_X(0)\,\lambda^d\, H_d(u)^2\, \phi(u)^2$.
Actually, in the sum (13), each q-term is non-negative so that V is greater than the $q=1$ term. The next lemma, which is proved in the Appendix allows us to conclude for the lower bound of V. □

Lemma 2.2. *Let us denote by V_1 the term corresponding to $q=1$ in the sum* (13). *Then*

$$V_1 = V_1(u) = f_X(0)\,\lambda^d\, H_d(u)^2\, \phi(u)^2$$

2.2. Central Limit Theorem for $\varphi(T)$.

Theorem 2.3. *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories and satisfying Assumption* **(H)**. *As $N \nearrow +\infty$,*

$$\zeta([-N,N]^d) = \frac{\varphi([-N,N]^d) - \mathbb{E}\varphi([-N,N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable with finite variance V given by (13).

Proof. By the proof of Proposition 2.1 (see (11)), we already know that

$$\sup_N \mathrm{Var}\left(\pi_Q(\zeta([-N,N]^d))\right) \underset{Q\to\infty}{\longrightarrow} 0\ .$$

So $\pi_Q(\zeta([-N,N]^d)) \to 0$ in $L^2(\Omega)$ when $N\to\infty$ and $Q\to\infty$ in this order. Hence in order to establish the CLT for $\zeta([-N,N]^d)$, it is enough to show the asymptotic normality, for a fixed Q as N goes to infinity, of the sequence

$$\begin{aligned} \pi^Q(\zeta([-N,N]^d)) &= \zeta([-N,N]^d) - \pi_Q(\zeta([-N,N]^d)) \\ &= \frac{1}{(2N)^{d/2}} \int_{[-N,N]^d} \sum_{q=1}^{Q} G_q(Y(t))\, dt, \end{aligned} \tag{14}$$

where we have defined $G_q(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H_{\mathbf{n}}}(\mathbf{x})$. Note that Proposition 2.1 states that the asymptotic variance of $\pi^Q(\zeta([-N,N]^d))$ is finite. Then, the result follows from the classical Breuer-Major Theorem. A reference for this theorem

can be found for instance in the paper of Arcones [8]. However, in this result the parameter set is the set of integers whereas in our setting the parameter set is $\mathbb{R}^d$. Therefore, for completeness, we give a statement and a proof of this famous theorem, both adapted to our setting, namely Proposition 2.4 below. □

Proposition 2.4. *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories that satisfies Assumption* **(H)**. *For any fixed positive integer Q, as $N \nearrow +\infty$, $\pi^Q(\zeta([-N,N]^d))$ converges in distribution to a centered Gaussian variable with finite variance $\sigma_Q^2 = \sum_{q=1}^Q V_q$, where the V_q's are introduced in* (10).

Proof.
Our proof follows very closely the proof of the CLT in Nourdin et al. [22], considering $\mathbb{R}^d$ as parameter set instead of $\mathbb{Z}$.
To set up our framework we must consider the machinery of the isonormal Hilbert space associated with the zero mean stationary Gaussian vector field $Y : \Omega \times \mathbb{R}^d \to \mathbb{R}^D$, we denote it by $\mathcal{H}$. We consider $F(\lambda) := (f_{jl}^Y(\lambda))_{1\le j,l\le D}$ the definite positive self-adjoint matrix of the spectral density functions of Y and let $B(\lambda) = (b_{jl}(\lambda))_{1\le j,l\le D}$ be a square root of $F(\lambda)$. For $t \in \mathbb{R}^d$ and $j = 1,\dots,D$, let us introduce the function

$$\varphi_{t,j} : (\lambda, l) \mapsto e^{i<t,\lambda>} b_{jl}(\lambda)\mathbf{e}_l\,,$$

where $(\mathbf{e}_l)_{1\le l\le D}$ is the canonical basis of $\mathbb{R}^D$.
The Itô formula for multiple Wiener integrals (see [23] page 13 or [19] page 37) allows us to write the next formula. For any $k = 1,\dots,D$ and any integer $l \ge 0$,

$$H_l(Y_k(s)) = I_l(\varphi_{s,k} \otimes \dots \otimes \varphi_{s,k}) = I_l(\varphi_{s,k}^{\otimes l}),$$

where the tensorial product has l terms. We use it to compute $G_q(Y(s))$ as follows

$$\begin{aligned} G_q(Y(s)) &= \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n}) \sum_{\mathbf{m}\in\mathcal{A}_\mathbf{n}} I_q(\varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q}) \\ &= \sum_{\mathbf{m}\in\{1,2,\dots,D\}^q} b_\mathbf{m} \sum_{\mathbf{m}\in\mathcal{A}_\mathbf{n}} I_q(\varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q}), \end{aligned}$$

where for $\mathbf{n} = (n_1,\dots,n_D) \in \mathbb{N}^D$ such that $|\mathbf{n}| = q$, we define

$$\mathcal{A}_\mathbf{n} = \{\mathbf{m} \in \{1,\dots,D\}^q : \sum_{j=1}^q 1_{\{i\}}(m_j) = n_i\,,\ \forall i\},$$

and remark that $\{1,2,\dots,D\}^q = \cup_{\mathbf{n}\in\mathbb{N}^D;|\mathbf{n}|=q}\mathcal{A}_\mathbf{n}$. It allows the following notation

$$b_\mathbf{m} = a(\mathbf{n}) \text{ for } \mathbf{m} \in \mathcal{A}_\mathbf{n} \text{ and so } \mathbf{m} \mapsto b_\mathbf{m} \text{ is symmetric on } \{1,2,\dots,D\}^q.$$

Fubini's theorem for multiple Wiener integrals applied to formula (14) yields

$$\pi^Q(\zeta([-N,N]^d)) = \sum_{q=1}^Q I_q(g_q^N),$$

where

$$g_q^N = \frac{1}{(2N)^{d/2}} \int_{[-N,N]^d} \sum_{\mathbf{m}\in\{1,2,\dots,D\}^q} b_\mathbf{m}\ \varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q}\, ds\,.$$

This expression corresponds with equation (4.43) in [22]. Hence, according to (4.47) of [22], if h is a two times differentiable and bounded map with bounded derivatives and if Z_Q is a centered Gaussian random variable with variance equal to $\sigma^2 = \sum_{q=1}^{Q} V_q$, then

$$| \mathbb{E}[h(Z_Q)] - \mathbb{E}[h(\pi^Q(\zeta([-N,N]^d)))] |$$

$$\leq \frac{||h''||_\infty}{2} \sum_{p,q=1}^{Q} ||\delta_{pq} V_p - \frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||_{L^2(\Omega)}, \tag{15}$$

where $\mathbf{D}$ denotes the Malliavin derivative (see [22] for its definition). We are now in position of proving the CLT for $\pi^Q(\zeta([-N,N]^d))$: it is sufficient to prove that the right hand side of (15) tends to 0.

First we consider the terms corresponding to $p = q$.

$$\begin{aligned}
& \mathbb{E}\left(\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} \right) \\
= \; & q! ||g_q^N||^2_{\mathcal{H}^{\otimes q}} \\
= \; & \frac{q!}{(2N)^d} \int_{[-N,N]^d} \int_{[-N,N]^d} \sum_{\mathbf{m},\mathbf{l} \in \{1,2,\ldots,D\}^q} b_{\mathbf{m}} b_{\mathbf{l}} \prod_{j=1}^{q} \Gamma^Y_{m_j l_j}(s_1 - s_2) ds_1 ds_2 \\
= \; & \frac{1}{(2N)^d} \int_{[-N,N]^d} \int_{[-N,N]^d} \sum_{\substack{\mathbf{n},\mathbf{n}' \in \mathbb{N}^D \\ |\mathbf{n}| = |\mathbf{n}'| = q}} a(\mathbf{n}) a(\mathbf{n}') \operatorname{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s_1)), \widetilde{H_{\mathbf{n}'}}(Y(s_2))) ds_1 ds_2.
\end{aligned}$$

The last line is equal to V_q^N and we already proved that $V_q^N \to V_q$. So

$$\mathbb{E}\left(\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} \right) \; \to \; V_q \; ,$$

in other words $\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}}$ converges to V_q in $L^1(\Omega)$. We have to prove that the convergence holds in $L^2(\Omega)$ in order to get

$$\sum_{q=1}^{Q} ||V_q - \frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||_{L^2(\Omega)} \to 0. \tag{16}$$

It will be done if we show that its variance tends to 0. Actually this last point is included in the next step ($p = q$ case).

Next step consists in showing that for $q \geq p$

$$|| \frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)} \to 0. \tag{17}$$

Formula (3.36) of [22] implies

$$\begin{aligned} & ||\frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)} \\ \leq \quad & p! \binom{q-1}{p-1}^2 (q-p)! \mathbb{E}[I_p(g_p^N)]^2 ||g_q^N \otimes_{q-p} g_q^N||_{\mathcal{H}^{\otimes 2p}} \\ + \quad & \frac{p^2}{2} \sum_{l=1}^{p-1} (l-1!^2 \binom{p-1}{l-1}^2 \binom{q-1}{l-1}^2 (p+q-2l)! \\ & (||g_p^N \otimes_{p-l} g_p^N||_{\mathcal{H}^{\otimes 2l}} + ||g_q^N \otimes_{q-l} g_q^N||_{\mathcal{H}^{\otimes 2l}}), \end{aligned}$$

where for $e < p$,

$$\begin{aligned} g_p^N \otimes_e g_p^N \quad = \quad & \frac{1}{(2N)^d} \int_{[-N,N]^d \times [-N,N]^d} \sum_{\mathbf{m}, l \in \{1,\dots,D\}^p} b_{\mathbf{m}} b_{\mathbf{l}} \prod_{j=1}^{e} \Gamma^Y_{m_j l_j}((s_1 - s_2) \\ & \times u_{s_1, m_{e+1}} \otimes \dots \otimes u_{s_1, m_p} \otimes u_{s_2, l_{e+1}} \otimes \dots \otimes u_{s_2, l_p} ds_1 ds_2 \end{aligned}$$

In this form defining $I(N) = [-N,N]^d \times [-N,N]^d \times [-N,N]^d \times [-N,N]^d$ we get

$$||g_p^N \otimes_e g_p^N||^2_{\mathcal{H}^{\otimes 2(p-e)}} \leq \big(D^p \sum_{\mathbf{m} \in \{1,\dots,D\}^p} |b_{\mathbf{m}}|^2\big)^2 \mathcal{Z}(N),$$

with

$$\mathcal{Z}(N) = \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1 - s_2) \psi^e(s_3 - s_4) \psi^{p-e}(s_1 - s_3) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_3 ds_4 \,.$$

Moreover we have $\psi^e(s_3 - s_4)\psi^{p-e}(s_1 - s_3) \leq \psi^p(s_3 - s_4) + \psi^p(s_1 - s_3)$. Thus we can write $\mathcal{Z}(N) \leq \mathcal{Z}_1(N) + \mathcal{Z}_2(N)$ where

$$\mathcal{Z}_1(N) \leq \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1 - s_2) \psi^p(s_3 - s_1) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_3 ds_4.$$

Let us look at the integral

$$\int_{[-N,N]^d} \psi^p(s_3 - s_1) ds_3 \leq \int_{\mathbb{R}^d} \psi^p(v) dv < \infty,$$

and for the remaining terms

$$\frac{1}{(2N)^{2d}} \int_{[-N,N]^d [\times -N,N]^d \times [-N,N]^d} \psi^e(s_1 - s_2) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_4$$

$$\leq \frac{1}{(2N)^{2d}} \int_{[-N,N]^d [\times [-N,N]^d} \psi^e * \psi^{p-e}(s_1 - s_4) ds_1 ds_4 \leq \frac{1}{(2N)^d} \int_{\mathbb{R}^d} \psi^e * \psi^{p-e}(s) ds \to 0.$$

The term $\mathcal{Z}_2(N)$ can be treated similarly obtaining

$$||g_p^N \otimes_e g_p^N||^2_{\mathcal{H}^{\otimes 2(p-e)}} \to 0.$$

Hence (17) holds in force. Together with (16), it implies that (15) tends to zero. □

The same proof can be used to get the next result which deals with a collection of various levels. Let us emphasize that the coefficients $a(\mathbf{n})$ appearing in (13) do depend on the level u (see (7)). We denote them as $a(\mathbf{n}, u)$ in the next theorem.

Theorem 2.5. *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories that satisfies Assumption* **(H)**. *For any level u, we denote*

$$\zeta([-N,N]^d,u)=\frac{\varphi([-N,N]^d,u)-\mathbb{E}\varphi([-N,N]^d,u)}{(2N)^{d/2}}\,.$$

Let $u_1,\dots,u_K$ be K fixed levels in $\mathbb{R}$. As $N\nearrow+\infty$, the random vector

$$\big(\zeta([-N,N]^d,u_1),\dots,\zeta([-N,N]^d,u_K)\big)$$

converges in distribution to a centered Gaussian vector with covariance matrix $(C(u_i,u_j))_{1\le i,j\le K}$ given by

$$C(u,v)=\sum_{q=1}^{\infty}\sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D\\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n},u)a(\mathbf{m},v)\,R(\mathbf{n},\mathbf{m})\,. \tag{18}$$

2.3. **Morse's theory and Central Limit Theorem for $\chi(T)$.**
We follow the presentation of Adler & Taylor's book [5] Section 9.4, inspired by Morse's theorem, to give a precise definition of $\chi(T)$, the Euler characteristic of the excursion set $A(T,u)$.

We still work with $T=[-N,N]^d$ and for $\ell=0,1,\dots,d$, we denote by $\partial_\ell T$ the collection of all the ℓ-dimensional faces of T. In particular, $\partial_d T$ only contains the interior $\overset{\circ}{T}=]-N,N[^d$ of T and $\partial_0 T=\{(\varepsilon_j N)_{1\le j\le d};\ \varepsilon_j=\pm1\}$ is the set of all the vertices of T. Each ℓ-dimensional face J of T is associated with a cardinal ℓ subset $\sigma(J)$ of $\{1,\dots,d\}$ and a sequence $(\varepsilon_j)_{j\in\{1,\dots,d\}\setminus\sigma(J)}$ in $\{-1,+1\}^{d-\ell}$ such that

$$J=\{v\in T:\ -N<v_j<N \text{ for } j\in\sigma(J),\ v_j=\varepsilon_j N \text{ for } j\notin\sigma(J)\}. \tag{19}$$

The Euler characteristic of $A(T,u)$ can be computed as

$$\chi(T)=\sum_{0\le\ell\le d}\ \sum_{J\in\partial_\ell T}\varphi(J) \tag{20}$$

where for any ℓ-dimensional face J of T

$$\begin{aligned}\varphi(J) &= \sum_{k=0}^{\ell}(-1)^k\mu_k(J) \ \text{ with}\\ \mu_k(J) &= \#\{v\in J:\ X(v)\ge u,\ X(v)=0 \text{ for } j\in\sigma(J),\\ &\quad \varepsilon_j X_j(v)>0 \text{ for } j\notin\sigma(J),\ \operatorname{index}((X_{ij}(v))_{i,j\in\sigma(J)})=\ell-k\}.\end{aligned} \tag{21}$$

Let us remark that (20) can be written $\chi(T)=\sum_{0\le\ell<d}\ \sum_{J\in\partial_\ell T}\varphi(J)\ +\ \varphi(\overset{\circ}{T})$. Moreover, Bulinskaya Lemma (Lemma 11.2.10 of [5]) entails that with probability one there is no point t in the boundary set ∂T satisfying $\nabla X(t)=0$, then $\mu_k(\overset{\circ}{T})=\mu_k(T)$. Hence, comparing the definition of $\varphi(T)$, that is given in Section 1.2, and (21), we obtain $\varphi(\overset{\circ}{T})=\varphi(T)$. Therefore, from now on we will work with this last formula

$$\chi(T)=\sum_{0\le\ell<d}\ \sum_{J\in\partial_\ell T}\varphi(J)\ +\ \varphi(T). \tag{22}$$

Recall that we want to establish that $\chi(T)$ satisfies a central limit theorem. More precisely we will prove that

$$\frac{\chi(T)-\mathbb{E}\chi(T)}{|T|^{1/2}}$$

converges in distribution to a centered Gaussian random variable as T grows to $\mathbb{R}^d$. Recall also that Proposition 2.3 already provides a CLT for $\varphi(T)$. So, according to (22), we only have to prove that for any $\ell = 0, 1, \dots, d-1$ and any face J in $\partial_\ell T$, the variance of $|T|^{-1/2}\varphi(J)$ tends to 0 as T grows to $\mathbb{R}^d$.

For $\ell = 0$, the previous statement is obvious since $\varphi(\{v\})$ is either 0 or 1 for any vertex v of T.

Let us now be concerned with $\ell \in \{1, \dots, d-1\}$. We deal with a fixed face $J \in \partial_\ell T$. We use (19) to introduce the following notations:

- With any $v \in \mathbb{R}^\ell$, we associate $v^{(J)} = (v_1^{(J)}, \dots, v_d^{(J)}) \in \mathbb{R}^d$ defined by
$$v_j^{(J)} = v_j \ \text{ if } \ j \in \sigma(J)\ ;\ v_j^{(J)} = \varepsilon_j N \ \text{ if } \ j \notin \sigma(J)\,.$$
- A random field $X^{(J)}$ is defined on $\mathbb{R}^\ell$ by
$$X^{(J)}(v) = X(v^{(J)}) \ \text{ for any } v \in \mathbb{R}^\ell\,.$$
It clearly inherits the properties of X so that $X^{(J)}$ is Gaussian, stationary, isotropic, centered and its trajectories are a.s. of class C^3.

With these notations, $\mu_k(J)$ can be written as

$$\begin{aligned}\mu_k(J) \quad = \quad & \#\{v \in [-N,N]^\ell \,:\, X^{(J)}(v) \geq u,\, \nabla X^{(J)}(v) = 0,\\ & \varepsilon_j X_j(v^{(J)}) > 0 \text{ for } j \notin \sigma(J),\ \operatorname{index}(\nabla^2 X^{(J)}(v)) = \ell - k\}.\end{aligned}$$

Following the same arguments as in Section 2, one can get an analogous proposition of Proposition 1.3

$$\begin{aligned}\varphi(J) \overset{L^2(\Omega)}{=} \lim_{\varepsilon\to 0}(-1)^\ell \int_{[-N,N]^\ell} \quad & \det(\nabla^2 X^{(J)}(v)) 1_{[u,\infty)}(X^{(J)}(v)) \delta_\varepsilon(\nabla X^{(J)}(v))\\ & \times \Pi_{j\notin\sigma(J)} 1_{(0,\infty)}(\varepsilon_j X_j(v^{(J)}))\, dv\,.\end{aligned}$$

Trivially we have

$$\begin{aligned}|(-1)^\ell \int_{[-N,N]^\ell} & \det(\nabla^2 X^{(J)}(v)) 1_{[u,\infty)}(X^{(J)}(v)) \delta_\varepsilon(\nabla X^{(J)}(v))\\ & \times \Pi_{j\notin\sigma(J)} 1_{(0,\infty)}(\varepsilon_j X_j(v^{(J)}))\, dv| \leq N_\varepsilon^{\nabla X^J}(J,0),\end{aligned}$$

where $N_\varepsilon^{\nabla X^J}(J,0)$ is an approximation of the number of zeros of $\nabla X^{(J)}$, similar to the one defined in (2). One can get as well an analogous proposition of Proposition 1.4 and at last, a similar proposition as Proposition 2.1 can be formulated: $\operatorname{Var}((2N)^{-\ell/2}(\varphi(J) - \mathbb{E}\varphi(J))) \underset{N\to+\infty}{\to} V^{(J)} < +\infty$. Hence

$$\operatorname{Var}((2N)^{-d/2}\,\varphi(J)) \underset{N\to+\infty}{\to} 0\ .$$

Finally, we are able to state our main result.

Theorem 2.6. *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories that satisfies Assumption* **(H)**. *As $N \nearrow +\infty$,*

$$\frac{\chi([-N,N]^d) - \mathbb{E}\chi([-N,N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable with finite variance V given by (13).

3. Appendix

3.1. Proof of Lemma 1.1.

We start with writing Taylor formulae around 0 for the covariance function r which is supposed to be of class C^6 as soon as X is supposed to have trajectories of class C^3. The isotropy of X allows us to write $r(t) = R(||t||)$ with $R : \mathbb{R}^+ \to \mathbb{R}$ of class C^6. We know that $R(0) = 1$, $R'(0) = R^{(3)}(0) = 0$, $R"(0) = -\lambda$ and we introduce $R^{(4)}(0) = \mu$. So

$$r(t) = 1 - \frac{\lambda}{2}||t||^2 + \frac{\mu}{4!}||t||^4 + \circ(||t||^5) \text{ as } ||t|| \to 0\,.$$

Taking the second derivatives in the above formula yields the Hessian matrix of r, namely $\nabla^2 r(t) = (r^{(2)}_{ij}(t))_{1\le i,j\le d}$ with

$$\nabla^2 r(t) = -\lambda I_d + \Theta(t) \text{ with } \Theta(t) = \frac{\mu}{3!}(||t||^2 I_d + (2t_i t_j)_{1\le i,j\le d}) + \circ(||t||^3)\,.$$

With one more derivative, we get

$$r^{(3)}_{ijm}(t) = \left\{ \begin{array}{ll} \mu t_i & \text{if } i=j=m \\ 0 & \text{if all the indices } i,j,m \text{ are different} \\ \frac{\mu}{3} t_p & \text{if two indices are equal and the third one is equal to } p \end{array} \right\} + \circ(||t||^2)\,.$$

At last,

$$r^{(4)}_{iiii}(0) = \mu\,;\, r^{(4)}_{iijj}(0) = \frac{\mu}{3} \text{ for } i \neq j$$

and all the other derivatives of order 4 at 0 equal zero.

Let us come back to the proof of Lemma 1.1 and start with getting an upper bound for $p_{0,t}(v,v)$ for t in a neighborhood of 0. The vector $(\nabla X(0), \nabla X(t))$ is a $2d$ centered Gaussian vector. Let us denote by $\Gamma^{\nabla X}(t)$ its covariance matrix. Since X is stationary and isotropic, we can write

$$\Gamma^{\nabla X}(t) = \begin{pmatrix} \lambda\, I_d & -\nabla^2 r(t) \\ -\nabla^2 r(t) & \lambda\, I_d \end{pmatrix}.$$

Computing the product of $\Gamma^{\nabla X}(t)$ with the matrix $\begin{pmatrix} I_d & 0 \\ \lambda^{-1}\nabla^2 r(t) & I_d \end{pmatrix}$, we get

$$\begin{array}{rcl} \det(\Gamma^{\nabla X}(t)) & = & \det(\lambda^2 I_d - (\nabla^2 r(t))^2) = \det(2\lambda I_d - \Theta(t))\, \det(\Theta(t)) \\ & \sim & (2\lambda)^d\, ||t||^{2d} \text{ as } ||t|| \to 0. \end{array}$$

Therefore, for a certain constant C,

$$\forall v \in \mathbb{R}^d\,,\, p_{0,t}(v,v) \le C\, ||t||^{-1}\,.$$

We turn now to the search of an upper bound for

$$g(v,t) := \mathbb{E}\left(|\det(\nabla^2 X(0))\det(\nabla^2 X(t))| \,/\, \nabla X(0) = \nabla X(t) = v\right),$$

for t in a neighborhood of 0. Using Cauchy-Schwarz inequality and the stationarity of X, we obtain

$$\begin{aligned} g(v,t)^2 &\leq \mathbb{E}\left([\det(\nabla^2 X(0))]^2 \,/\, \nabla X(0) = \nabla X(t) = v\right) \\ &\quad \times \mathbb{E}\left([\det(\nabla^2 X(t))]^2 \,/\, \nabla X(0) = \nabla X(t) = v\right) \\ &:= h(v,t)\, h(v,-t) = h(v,t)^2 \ . \end{aligned}$$

Since X is isotropic, without loss of generality we can choose to work with t in a neighborhood of 0 along a specific direction. So from now on, we consider $t = \tau e_1$ where e_1 is the first canonical vector of $\mathbb{R}^d$ and $\tau \in \mathbb{R}$, $|\tau| \sim 0$. Then

$$\Theta(\tau e_1) = \frac{\mu}{3!}\,\tau^2\, J + \circ(|\tau|^3) \text{ with } J \text{ the diagonal matrix } diag(3,1,\ldots,1)\,. \tag{23}$$

Let $K = d(d+1)/2$. In order to compute $h(v,\tau e_1)$, we consider $\nabla^2 X(0)$ as a K-dimensional Gaussian vector and we write the following K-dimensional regression system

$$\nabla^2 X(0) = A(\tau)\,\nabla X(0) + B(\tau)\,\nabla X(\tau e_1) + Z(\tau) \tag{24}$$

where $A(\tau)$ and $B(\tau)$ are two matrices of size $K \times d$ and $Z(\tau)$ is a K-dimensional centered Gaussian vector which is independant from $\nabla X(0)$ and $\nabla X(\tau e_1)$. In that form, we have

$$h(v,\tau e_1) = \mathbb{E}\left([\tilde{\det}((A(\tau)+B(\tau))\,v + Z(\tau))]^2\right) .$$

We now compute the regression coefficients. We write the covariances between the coordinates of $\nabla^2 X(0)$ and $\nabla X(0)$ in (24) and we use that $\nabla^2 X(0)$ and $\nabla X(0)$ are independant. Together with the relation (23), it allows us to write
for $1 \leq k \leq K$ and $1 \leq j \leq d$,

$$0 \;=\; \lambda A(\tau)_{kj} \;-\; B(\tau)_{kj}(-\lambda + \frac{\mu J_j}{3!}\,\tau^2 + \circ(|\tau|^3))\,.$$

In the same way, computing the covariances between the coordinates of $\nabla^2 X(0)$ and $\nabla X(\tau e_1)$ yields

$$r^{(3)}_{kj}(\tau e_1) \;=\; A(\tau)_{kj}(-\lambda + \frac{\mu J_j}{3!}\,\tau^2 + \circ(|\tau|^3)) \;-\; \lambda B(\tau)_{kj}\,.$$

Combining these last two equations gives

$$\begin{aligned} A(\tau)_{kj} &= \frac{3\alpha_{kj}}{J_j}\,\tau^{-1}\,(1+\circ(|\tau|)) \\ B(\tau)_{kj} &= -\frac{3\alpha_{kj}}{J_j}\,\tau^{-1}\,(1+\circ(|\tau|)) \\ A(\tau)_{kj} + B(\tau)_{kj} &= \frac{\mu\alpha_{kj}}{2}\,\tau\,(1+\circ(|\tau|)) \end{aligned} \tag{25}$$

where α_{kj} is defined by prescribing $r^{(3)}_{kj}(\tau e_1) = \mu\,\tau\,\alpha_{kj} + \circ(|\tau|^2)$.
Starting from (24), we compute now the variance of each coordinate of the Gaussian

vector $Z(\tau)$. For $1 \leq k \leq K$,

$$\begin{aligned}\mathrm{Var}(Z(\tau)_k) &= \mathrm{Var}(\nabla^2 X(0)_k) + \lambda \sum_{1\leq i\leq d} (A(\tau)_{ki})^2 + \lambda \sum_{1\leq i\leq d} (B(\tau)_{ki})^2 \\ &+ 2\sum_{1\leq i\leq d} B(\tau)_{ki} r^{(3)}_{ki}(\tau e_1) - 2\sum_{1\leq i\leq d} A(\tau)_{ki}B(\tau)_{ki}(-\lambda + \frac{\mu J_i}{3!}\,\tau^2 + \circ(|\tau|^3)) \\ &= r^{(4)}_{kk}(0) - 3\mu \sum_{1\leq i\leq d} \frac{(\alpha_{ki})^2}{J_i} + \circ(|\tau|)\,.\end{aligned}$$

We return to the evaluation of $h(v, \tau e_1)$ and write it as $\mathbb{E}([\det(M(\tau)]^2)$ where $M(\tau)$ is the symmetric $d \times d$ matrix corresponding to the K-dimensional vector $(A(\tau) + B(\tau))\, v + Z(\tau)$. Identifying the indices $k \in \{1, \ldots, K\}$ with the pairs (i, j) with $1 \leq i \leq j \leq d$, then

$$\mathrm{Var}(Z(\tau)_{(1,1)}) = \mu - 3\mu\,\frac{1}{3} + \circ(|\tau|) = \circ(|\tau|),$$

and for any $2 \leq j \leq d$,

$$\mathrm{Var}(Z(\tau)_{(1,j)}) = \frac{\mu}{3} - 3\mu\,\frac{(1/3)^2}{1} + \circ(|\tau|) = \circ(|\tau|)\,.$$

Together with (25), this shows that the first line of $M(\tau)$ consists in d Gaussian variables, denoted by $(M(\tau)_{1j})_{1\leq j\leq d}$ with

$$|\mathbb{E}(M(\tau)_{1j})| \leq C||v||\,|\tau| \quad \text{and} \quad \mathrm{Var}(M(\tau)_{1j}) = \circ(|\tau|)\,.$$

We write $\det(M(\tau)) = \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma)\, \prod_{1\leq i\leq d} M(\tau)_{i\sigma(i)}$ where S_d stands for the set of all permutations of $\{1, \ldots, d\}$ and sgn for the signature. Using Cauchy-Schwarz inequality as well as the upper bounds for the first two moments of $M(\tau)_{1j}$ allow us to get

$$h(v, \tau e_1) = \mathbb{E}([\det(M(\tau))]^2) = \circ(|\tau|),$$

uniformly with respect to v contained in any fixed compact of $\mathbb{R}^d$. Finally,

$$F(v, \tau e_1) = g(v, \tau e_1)\, p_{0,\tau e_1}(v, v) \;\leq\; C\,\frac{h(v, \tau e_1)}{|\tau|},$$

and this concludes the proof of Lemma 1.1. □

3.2. **Proof of Lemma 2.2.**

We recall (13) that yields the following expression for V_1

$$V_1 = \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=1}} a(\mathbf{n})a(\mathbf{m})\, R(\mathbf{n}, \mathbf{m}). \tag{26}$$

Although the notation does not mention it explicitely, the coefficients $a(\mathbf{n})$ depend on the level u. Actually, for $\mathbf{n} = (\underline{n}, \overline{n}) \in \mathbb{N}^D = \mathbb{N}^d \times \mathbb{N}^{d(d+1)/2+1}$,

$$a(\mathbf{n}) = a(\mathbf{n}, u) = d(\underline{n})c(f \circ \Lambda_2, \overline{n}) \text{ with } f = \tilde{\det} \otimes 1_{[u,+\infty)}$$

as given by (7) and (4).

Along the proof, K denotes the integer $d(d+1)/2$, so that $D - d = K + 1$. As well, we choose Λ, and so Λ_2, to be lower triangular. Then, we can write the matrix Λ_2 as

$$\Lambda_2 = \begin{pmatrix} L & \mathbf{0} \\ {}^t l & \alpha \end{pmatrix}, \tag{27}$$

with L a $K \times K$ lower triangular matrix, l a vector in $\mathbb{R}^K$ and $\alpha > 0$. Furthermore, the fact that $\mathrm{Cov}(X(t), X_{ii}(t)) = -\lambda \neq 0$ implies that the vector l does not vanish and the fact that $\mathrm{Var}(X(t)) = 1$ implies that $||l||^2 + \alpha^2 = 1$.
Hence, for $(y, z) \in \mathbb{R}^{D-d} = \mathbb{R}^K \times \mathbb{R}$, we have

$$f \circ \Lambda_2(y, z) = \tilde{\det}(Ly)\, 1_{[u,+\infty)}(\langle l, y\rangle + \alpha z)\ . \tag{28}$$

The computation of $\tilde{\det}(Ly)$ is solved in the next lemma. It states that the expansion of the map $y \in \mathbb{R}^K \mapsto \tilde{\det}(Ly)$ in the basis of Hermite polynomials on $\mathbb{R}^K$, as well as in the canonical basis of polynomials on $\mathbb{R}^K$, only involves polynomials of degree d.

Lemma 3.1. *Let L be the matrix introduced in* (27)*. There exists a family of real numbers $(\beta_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=d}$ such that*

$$\forall y \in \mathbb{R}^K\ ,\ \tilde{\det}(Ly) = \sum_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=d} \beta_{\Lambda,\mathbf{m}} \widetilde{H_{\mathbf{m}}}(y) = \sum_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=d} \beta_{\Lambda,\mathbf{m}} y^{(\mathbf{m})}$$

where $y^{(\mathbf{m})} = \prod_{1\leq k\leq K} (y_k)^{m_k}$.

Proof. The map $F : y \in \mathbb{R}^K \mapsto F(y) = \tilde{\det}(Ly)$ is a polynomial function of degree d. We expand it first in the basis of Hermite polynomials on $\mathbb{R}^K$ as follows

$$F(y) = \sum_{q=0}^{d} \sum_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=q} \beta_{\mathbf{m}} \widetilde{H_{\mathbf{m}}}(y)\,,$$

where the coefficients are given by

$$\beta_{\mathbf{m}} = \frac{1}{\mathbf{m}!} \int_{\mathbb{R}^K} F(y) \widetilde{H_{\mathbf{m}}}(y) \phi_K(y) dy = \frac{1}{\mathbf{m}!} F * \phi_K^{(\mathbf{m})}(0) \ = \ \frac{1}{\mathbf{m}!} F^{(\mathbf{m})} * \phi_K(0)\,.$$

Note that in the previous line, for $\mathbf{m} = (m_1, \ldots, m_K)$ with $|\mathbf{m}| = q$ and for any function G defined on $\mathbb{R}^K$, $G^{(\mathbf{m})}$ denotes the derivative $\frac{\partial^q G}{\partial y_{m_1} \cdots \partial y_{m_K}}$.
In order to compute $F^{(\mathbf{m})}$, we write $F(y)$ as $F(y) = \det(A(y))$ where for any $y \in \mathbb{R}^K$, $A(y)$ is a symmetric $d \times d$ matrix. The map $y \in \mathbb{R}^K \mapsto A(y)$ is linear, so that for any $1 \leq i, j \leq d$ we have $A(y)_{i,j} = \sum_{k=1}^K a_{ij}^k\, y_k$. Hence

$$F(y) = \sum_{\sigma\in\mathcal{S}_d} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} \left(\sum_{k=1}^{d} a_{i,\sigma(i)}^k y_k \right)$$

and then for $k = 1, \ldots, K$

$$\begin{aligned} \frac{\partial F}{\partial y_k}(y) &= \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{S}_d} \mathrm{sgn}(\sigma)\, A(y)_{1,\sigma(1)} \cdots A(y)_{i-1,\sigma(i-1)} a_{i,\sigma(i)}^k A(y)_{i+1,\sigma(i+1)} \cdots A(y)_{d,\sigma(d)} \\ &= \sum_{1\leq i\leq d} \det(\widehat{A(y)}^{ik}) \end{aligned}$$

where $\widehat{A(y)}^{ik}$ denotes the $d \times d$ matrix obtained from $A(y)$ replacing the i-th line with the line $(a_{i1}^k, \ldots, a_{id}^k)$.
More generally, for any $\mathbf{m} \in \mathbb{N}^K$ with $|\mathbf{m}| < d$, $F^{(\mathbf{m})}(y)$ writes as a sum of determinants of $d \times d$ matrices obtained from $A(y)$ replacing $|\mathbf{m}|$ lines by lines equal to $(a_{i1}^k, \ldots, a_{id}^k)$ with some $i \in \{1, \ldots, d\}$ and some $k \in \{1, \ldots, K\}$. Let us denote

$\widehat{A(y)}^{(\mathbf{m})}$ for any such matrix.
Coming back to the computation of the coefficients $\beta_{\mathbf{m}}$ we get

$$\beta_{\mathbf{m}} \;=\; \text{sum of terms as } \frac{1}{\mathbf{m}!}\int_{\mathbb{R}^K} \det(\widehat{A(y)}^{(\mathbf{m})})\phi_K(y)dy\,,$$

and we will prove that they vanish for all indices $\mathbf{m}$ such that $|\mathbf{m}| < d$.
Each of the above integrals is equal to $\mathbb{E}\left(\det(\widehat{A(Z)}^{(\mathbf{m})})\right)$ with Z a $N(0, I_K)$ Gaussian vector such that the $d \times d$ matrix $\nabla^2 X(0)$ is equal to $\nabla^2 X(0) = A(Z)$. Hence each integral reduces to the computation of

$$\sum_{\sigma\in\mathcal{S}_d} \mathrm{sgn}(\sigma)\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\dots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)a^{k_1}_{j_1,\sigma(j_1)}\dots a^{k_{|\mathbf{m}|}}_{j_{|\mathbf{m}|},\sigma(j_{|\mathbf{m}|})}$$

where $\{i_1, i_2, \dots, i_{d-|\mathbf{m}|}\} \cup \{j_1, i_2, \dots, j_{|\mathbf{m}|}\} = \{1, \dots, d\}$. By using Wick's formula which is recalled in Lemma 11.6.1 of [5], we get that if $d - |\mathbf{m}|$ is odd the above term is zero. Thus let assume that $d - |\mathbf{m}| = 2l$. We can paired the indices $i_1, i_2, \dots, i_{d-|\mathbf{m}|}$ and denoting them as $\{k_1, k_2\}, \dots, \{k_l, k_{l+1}\}$, we get

$$\begin{aligned}&\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\dots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)\\ &= \mathbb{E}(X_{k_1,\sigma(k_1)}X_{k_2,\sigma(k_2)})\dots\mathbb{E}(X_{k_l,\sigma(k_l)}X_{k_{l+1},\sigma(k_{l+1})})\,.\end{aligned}$$

With any fixed permutation σ, we associate a new permutation σ' as follows:

$$\sigma'(j_k) = \sigma(j_k) \text{ for } 1 \le k \le |\mathbf{m}|$$

$$\sigma'(k_1) = \sigma(k_2),\ \sigma'(k_2) = \sigma(k_1),\ \sigma'(k_m) = \sigma(k_m) \text{ for the others } k.$$

Then, $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$ because σ' is the composition of σ with a transposition. Moreover

$$\mathbb{E}(X_{k_1,\sigma(k_1)}X_{k_2,\sigma(k_2)}) = r^{(4)}_{k_1,\sigma(k_1),k_2,\sigma(k_2)}(0) = \mathbb{E}(X_{k_1,\sigma'(k_1)}X_{k_2,\sigma'(k_2)}).$$

This implies the cancelation of all pairs of two associated permutations and hence

$$\sum_{\sigma\in\mathcal{S}_d} \mathrm{sgn}(\sigma)\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\dots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)a^{k_1}_{j_1,\sigma(j_1)}\dots a^{k_{|\mathbf{m}|}}_{j_{|\mathbf{m}|},\sigma(j_{|\mathbf{m}|})} = 0.$$

We have proved that all the coefficients $\beta_{\mathbf{m}}$ with $|\mathbf{m}| < d$ equal 0 and so the first expansion of the lemma is established.

In order to get the second expansion, we remark that for any $x \in \mathbb{R}$ and any positive integer k, $\lim_{\alpha\to+\infty} \alpha^{-k}H_k(\alpha x) = x^k$. Then, using our first expansion, we get for any $y \in \mathbb{R}^K$,

$$\sum_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=d} \beta_{\mathbf{m}}\, y^{(\mathbf{m})} \;=\; \lim_{\alpha\to+\infty} \alpha^{-d}\,\tilde{\det}(\alpha\, Ly) \;=\; \tilde{\det}(Ly).$$

Lemma 3.1 is established. □

Let us come back to equation (26). Note that $|\mathbf{n}| = 1$ implies that we can write $\mathbf{n} = \mathbf{e}_i$ for one index $i = 1, \dots, D$ where $(\mathbf{e}_i)_{1\le i\le D}$ stands for the canonical basis of $\mathbb{R}^D$.
Then for $\mathbf{n} = \mathbf{e}_i$, due to the explicit form of $d(\underline{n})$ given in (6), we have

$$a(\mathbf{e}_i) = \begin{cases} 0 & \text{if} \quad 1 \le i \le d \\ \lambda^{-d}(2\pi)^{-d/2}\, c(f\circ\Lambda_2, \overline{e_i}) & \text{if} \quad d+1 \le i \le D \end{cases}$$

Hence,

$$V_1 = \lambda^{-2d}(2\pi)^{-d} \sum_{d+1\le i,j\le D} c(f\circ\Lambda_2, \overline{e_i})\, c(f\circ\Lambda_2, \overline{e_j})\, R(\mathbf{e}_i, \mathbf{e}_j),$$

where we deduce from (9) that $R(\mathbf{e}_i, \mathbf{e}_j) = \int_{\mathbb{R}^d} \mathrm{Cov}(Y_i(0), Y_j(v))\, dv$.

Remark 3.2. *Denoting by $\Gamma^{\mathbf{X}}$ the covariance function of the D-dimensional Gaussian field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$, for any $1 \le i, j \le D$,*

$$\int_{\mathbb{R}^d} \Gamma^{\mathbf{X}}_{i,j}(v)\, dv = (2\pi)^d\, f_X(0)\, \delta_{D,D}(i,j),$$

where δ stands for the Kronecker symbol.

Indeed, $\Gamma^{\mathbf{X}}_{D,D}(v) = \mathbb{E}(X(0)X(v)) = r(v) = \widehat{f_X}(v)$ and then, since f_X is supposed to be continuous,

$$\int_{\mathbb{R}^d} \Gamma^{\mathbf{X}}_{D,D}(v)\, dv = \widehat{\widehat{f_X}}(0) = (2\pi)^d\, f_X(0)\ .$$

For $(i,j) \neq (D,D)$, recall that $\Gamma^{\mathbf{X}}_{i,j}$ equals a derivative of order between one and four of the function r. Since r and all its derivative tends to 0 at infinity due to assumption **(H)**, we get $\int_{\mathbb{R}^d} \Gamma^{\mathbf{X}}_{i,j}(v)\, dv = 0$ for $(i,j) \neq (D,D)$.

With (27), Remark 3.2 implies that for any $d+1 \le i, j \le D$,

$$R(\mathbf{e}_i, \mathbf{e}_j) = \int_{\mathbb{R}^d} \mathrm{Cov}(Y_i(0), Y_j(v))\, dv = (2\pi)^d\, f_X(0)\, \alpha^{-2}\, \delta_{D,D}(i,j)$$

and therefore

$$V_1 = f_X(0)\, \lambda^{-d}\, \alpha^{-2}\, c(f\circ\Lambda_2, \overline{e_D})^2\,.$$

Using (28) we get

$$\begin{aligned} c(f\circ\Lambda_2, \overline{e_D}) &= \int_{\mathbb{R}^K\times\mathbb{R}} \tilde{\det}(Ly)\, 1_{[u,+\infty)}(\langle l, y\rangle + \alpha z)\, z\, \phi_K(y)\, \phi(z)\, dy\, dz \\ &= \int_{\mathbb{R}^K} \tilde{\det}(Ly)\, \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K(y)\, dy\ , \end{aligned}$$

where we have used $z\phi(z) = -\phi'(z)\,,\ \forall z \in \mathbb{R}$.

Hence, using Lemma 3.1, we have $c(f\circ\Lambda_2, \overline{e_D}) = \sum_{|\mathbf{m}|=d} \beta_{\mathbf{m}}\, I_{\mathbf{m}}$ where we introduce the next integral

$$\begin{aligned} I_{\mathbf{m}} &:= \int_{\mathbb{R}^K} \widetilde{H_{\mathbf{m}}}(y)\, \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K(y)\, dy \\ &= (-1)^d \int_{\mathbb{R}^K} \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K^{(\mathbf{m})}(y)\, dy \\ &= (-1)^d \varphi_0 * \phi_K^{(\mathbf{m})}(ul^*) \ =\ (-1)^d\, (\varphi_0 * \phi_K)^{(\mathbf{m})}\, (ul^*)\,. \end{aligned}$$

In the previous lines, φ_0 denotes the map $y \in \mathbb{R}^K \mapsto \varphi_0(y) = \phi(\frac{1}{\alpha}\langle l, y\rangle)$ and l^* is any vector in $\mathbb{R}^K$ such that $\langle l, l^*\rangle = 1$. The following lemma will help us for the computation of $I_{\mathbf{m}}$.

Lemma 3.3. *Let α and l be such that $\alpha^2 + ||l||^2 = 1$. Then, for any integer $k \geq 0$, denoting by φ_k the map $y \in \mathbb{R}^K \mapsto \varphi_k(y) = \phi^{(k)}(\frac{1}{\alpha}\langle l, y\rangle)$, we have*

$$\forall z \in \mathbb{R}^K \, , \, \varphi_k * \phi_K(z) = \alpha^{k+1} \, \phi^{(k)}(\langle l, z\rangle) \, .$$

Proof. For $k = 0$, we remark that the map $z \mapsto \frac{1}{\alpha}\int_{\mathbb{R}^K} \phi(\frac{1}{\alpha}(z - \langle l, y\rangle)) \, \phi_K(y) \, dy$ is the probability density function of a random variable $Z = \alpha N + \langle l, G\rangle$ where N is a standard Gaussian random variable and G is a standard Gaussian vector of dimension K independent of X. But Z is clearly Gaussian, centered, with variance $\alpha^2 + ||l||^2 = 1$ and so $\varphi_0 * \phi_K(.) = \alpha \, \phi(\langle l, .\rangle)$.
For $k > 0$, let us first note that for any $\mathbf{m} \in \mathbb{N}^K$, taking the $\mathbf{m}$ derivative of φ_0 yields

$$\varphi_0^{(\mathbf{m})}(y) = \alpha^{-|\mathbf{m}|} \, l^{(\mathbf{m})} \, \phi^{(|\mathbf{m}|)}(\frac{1}{\alpha}\langle l, y\rangle) = \alpha^{-|\mathbf{m}|} \, l^{(\mathbf{m})} \, \varphi_{|\mathbf{m}|}(y),$$

and hence $\varphi_{|\mathbf{m}|}(y) = \alpha^{|\mathbf{m}|} \, (l^{(\mathbf{m})})^{-1} \, \varphi_0^{(\mathbf{m})}(y)$. Then, choosing $\mathbf{m} \in \mathbb{N}^K$ such that $|\mathbf{m}| = k$ and $l^{(\mathbf{m})} \neq 0$ (which is possible since l is not the zero vector) and using the case $k = 0$, we obtain

$$\varphi_k * \phi_K = \alpha^{|\mathbf{m}|} \, (l^{(\mathbf{m})})^{-1} \, (\alpha\phi(\langle l, .\rangle))^{(\mathbf{m})} = \alpha^{k+1} \, \phi^{(k)}(\langle l, .\rangle) \, . \; \square$$

Using previous lemma, we get for any $y \in \mathbb{R}^K$,

$$(\varphi_0 * \phi_K)^{(\mathbf{m})}(y) = \alpha \, l^{(\mathbf{m})} \, \phi^{(d)}(\langle l, y\rangle) = (-1)^d \, \alpha \, l^{(\mathbf{m})} \, H_d(\langle l, y\rangle) \, \phi(\langle l, y\rangle) \, .$$

Coming back to the computation of $I_{\mathbf{m}}$, since $\langle l, ul^*\rangle = u$, we get $I_{\mathbf{m}} = \alpha \, l^{(\mathbf{m})} \, H_d(u) \, \phi(u)$. Then, the desired Hermite coefficient writes as

$$c(f \circ \Lambda_2, \overline{e_D}) = \alpha \, (\sum_{|\mathbf{m}|=d} \beta_{\mathbf{m}} \, l^{(\mathbf{m})}) \, H_d(u) \, \phi(u) = \alpha \, \tilde{\det}(Ll) \, H_d(u) \, \phi(u),$$

where we have used Lemma 3.1 to compute the sum inside the parenthesis. It remains to compute $\tilde{\det}(Ll) = \det(A(l))$.

Let us first compute the product $L\,l$. We write the coordinates of the $K+1$-dimensional Gaussian vector $(\nabla^2 X, X)$ in the following order

$$(\nabla^2 X, X) = ((X_{ij})_{1 \leq i < j \leq d} \; , \; (X_{ii})_{1 \leq i \leq d} \; , \; X)$$

so that the lower triangular matrix L can be written as $L = \begin{pmatrix} L^{(1)} & 0 \\ L^{(3)} & L^{(2)} \end{pmatrix}$ and the vector $l = \begin{pmatrix} l^{(1)} \\ l^{(2)} \end{pmatrix}$, where the top part has length $K - d$ whereas the bottom part has length d. With these notations, $(X_{ij})_{1 \leq i < j \leq d} = L^{(1)} \, (Y_k)_{d+1 \leq k \leq K-d}$. Since $\mathrm{Cov}(X, X_{ij}) = 0$ for all $i < j$, we deduce that $l^{(1)}$ vanishes and since $\mathrm{Cov}(X, X_{ii}) = r_{ii}(0) = -\lambda$ for any $i = 1, \ldots, d$, we deduce that all the coordinates of $L^{(2)} \, l^{(2)}$ are equal to λ. It implies that the symmetric $d \times d$ matrix $A(l)$ induced by the K dimensional vector $L\,l$ is diagonal and equal to $-\lambda \, I_d$. Hence, $\tilde{\det}(Ll) = \det(A(l)) = (-\lambda)^d$. So $V_1 = f_X(0) \, \lambda^d \, H_d(u)^2 \, \phi(u)^2$ and Lemma 2.2 is proved. $\square$

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