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A CENTRAL LIMIT THEOREM FOR THE EULER CHARACTERISTIC OF A GAUSSIAN EXCURSION SET

ANNE ESTRADE AND JOSÉ RAFAEL LEÓN

We study the Euler characteristic of an excursion set of a stationary Gaussian random field.

Let $X : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a stationary isotropic Gaussian field having trajectories in $C^2(\mathbb{R}^d)$. Let us fix a level $u \in \mathbb{R}$ and consider the excursion set above u , $\{t \in \mathbb{R}^d : X(t) \geq u\}$. We take the restriction to a compact domain considering for any bounded rectangle $T \subset \mathbb{R}^d$, $A(T, u) = \{t \in T : X(t) \geq u\}$. The aim of this paper is to establish a central limit theorem for the Euler characteristic of $A(T, u)$ as T grows to \mathbb{R}^d , as conjectured by R. Adler more than ten years ago [3]. The required assumption on X is stronger than Geman's one in dimension one but weaker than having C^3 trajectories. Our result extends to higher dimension what is known in dimension one, since in that case, the Euler characteristic of $A(T, u)$ equals the number of up-crossings of X at level u .

INTRODUCTION

The Euler characteristic, also called Euler-Poincaré index, is one of the additive functionals that can be defined on the collection of all compact sets of \mathbb{R}^d . Intuitively, in dimension one, the Euler characteristic is the number of disjoint intervals constituting the compact set. In dimension two, the Euler characteristic equals the number of connected components minus the number of "holes" in the compact set. In dimension three, the Euler characteristic equals the number of connected components minus the number of "handles" plus the number of "interior hollows". We will use a precise definition later on but what is important is that the Euler characteristic contains information that describes (a part of) the morphology of a compact set in \mathbb{R}^d .

We are interested in the Euler characteristic of an excursion set $A(T, u) = \{t \in T : X(t) \geq u\}$ for a real valued smooth stationary isotropic Gaussian field $X = \{X(t) : t \in \mathbb{R}^d\}$, a compact set $T \subset \mathbb{R}^d$ and a level u . We denote it as $\chi(X, T, u)$ for a while. One should consider $\chi(X, T, u)$ as an extension in dimension greater than one of the very precious tool in dimension $d = 1$ that is $U^X(T, u)$ the number of up-crossings at the level u of X on the interval T . In 2000, Adler [3] conjectured that $\chi(X, T, u)$ satisfies a central limit theorem (CLT) as T grows to \mathbb{R}^d . We prove it in the present paper. In dimension one, a CLT result for $U^X(T, u)$ can be found in [10] Chapter 10.

Actually, twenty years ago, Worsley [28] discovered that, when T is a rectangle in \mathbb{R}^d , the expectation of $\chi(X, T, u)$ can be explicitly computed as a function of u depending on the covariance structure of X . When one wants to establish a CLT, the first step consists in estimating the asymptotic variance. Unfortunately

the tricky method that yields the expectation formula cannot operate to compute the variance. In [26], formulae for the higher moments of the differential topology characteristic of $A(T, u)$, which is not so far from $\chi(X, T, u)$, are obtained for fixed T under convenient assumptions. Our method allows us to obtain an L^2 approximation of $\chi(X, T, u)$. In order to prove that the asymptotic variance is finite, we use the Wiener-Itô chaos decomposition. The second step towards a CLT is to get the asymptotic normality. At this stage, the expansion of $\chi(X, T, u)$ into the Wiener-Itô chaos is used again as well as a continuous parameter version of the celebrated Breuer-Major theorem (see [9] for instance). Precisely, we follow Nourdin et al. [20] but with some modifications motivated by the fact that our process has parameter in \mathbb{R}^d instead of \mathbb{Z} as in the cited article.

In the present paper, two types of tools are mixed. One is concerned with level functionals and its Hermite expansion. This point of view allows us to use the revisited Stein method presented for instance in [23, 22, 20]. A CLT result for $\chi(X, T, u)$ appears then as a consequence of the asymptotic normality of each term in the Wiener-Itô decomposition. The second tool deals with the geometrical aspect of the work. Actually we have to consider the random vectorial field $\mathbf{X} = (X, \nabla X, \nabla^2 X)$ and it is convenient to see it as a random function from \mathbb{R}^d to \mathbb{R}^D with $D = 1 + d + d(d + 1)/2$. It implies tedious differential calculus in high dimension. We must admit that we are very debtful to Robert Adler and Mario Wschebor, many of their Gaussian and geometrical ideas inspired the present work.

Beyond our work, the study of the excursion sets of a stationary field is a very popular theme. Many authors were and still are interested in this domain as proved by the successful recent books of Adler & Taylor [6] and Azaïs & Wschebor [10]. On the one hand, the description of these excursion sets appears very powerful to characterize the field X . For instance, since the first Adler's book [2] one knows that the expectation of $\chi(X, T, u)$ is a good approximation for the probability of the maximum of X on T to be greater than u . Also the line integral with respect of the level curve at any level u provides information on the anisotropy property of X (see the works of Cabaña [12] and Iribarren [15]). On the other hand, at least in the Gaussian case, accurate methods such as the theory of crossings can be used to get explicit values for level functionals (see the seminal work of Slud [24] and also the paper of Kratz and León [16]).

Our study for establishing a CLT for level functionals has many precursors in the literature. These functionals are usually used to build statistical estimators or to construct statistical tests. The first one that we can cite is Adler's work [1] that uses the Euler characteristic of an excursion set to build a spectral moment estimator for two dimensional Gaussian fields. Moreover, this estimator was proved to satisfy asymptotical normality. Afterwards, the two cited works [12] and [15] established a CLT for studying the asymptotic behavior of estimators based on the level sets, actually the line integral with respect to a level curve. Following this direction, we have in mind statistical outcomes of our result. They could serve various fields of application such as brain exploration or representation of the universe following [27] or the nice introduction of the forthcoming book [7], as well as worn surfaces or more generally rough surfaces as proposed in [8, 25]. Our result could be used to get the asymptotic distribution of the statistic under the null hypothesis in a test of normality. Furthermore, by linearity, it should also give a CLT result for a family of $\chi(X, T, u)$'s corresponding to different levels as in [4, 25] where similar

questions are studied. We also have in mind extensions to non Gaussian fields or to Gaussian non stationary fields, starting from the recent results concerning the volume ([19, 11]) or the expected Euler characteristic ([13, 8]) of excursion sets in these cases.

The first section of the present paper is devoted to a formula, namely Morse's formula, which gives the Euler characteristic of an excursion set. The study of each term that is involved in this formula yield the structure of the paper. It is presented at the end of the section.

All over the paper we deal with a centered stationary isotropic Gaussian field $X = \{X(t) : t \in \mathbb{R}^d\}$ such that $\text{Var}(X(0)) = 1$. Furthermore we assume that almost every realization of X is of class C^2 on \mathbb{R}^d . We write X_i and X_{ij} the derivatives of X of first and second order

$$\frac{\partial X}{\partial t_i}(t) = X_i(t) \quad \text{and} \quad \frac{\partial^2 X}{\partial t_i \partial t_j}(t) = X_{ij}(t)$$

$\nabla X(t) = (X_i(t))_{1 \leq i \leq d}$ and $\nabla^2 X(t) = (X_{ij}(t))_{1 \leq i, j \leq d}$. Denoting by r the covariance function of X ,

$$r(t) = \text{Cov}(X(0), X(t)),$$

the assumption on X implies that $r \in C^4(\mathbb{R}^d)$ and for any multidimensional index $\mathbf{m} = (i_1, \dots, i_k)$ with $1 \leq k \leq 4$ and $1 \leq i_j \leq d$, we write

$$\frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t) = \frac{\partial^k r}{\partial t_{i_1} \dots \partial t_{i_k}}(t) = r_{i_1 \dots i_k}(t).$$

It is well know that for any fixed t , $X(t)$ and $\nabla X(t)$ are independent, as well as $\nabla X(t)$ and $\nabla^2 X(t)$. The next covariance identities are easy to obtain using the stationarity of X ,

$$\begin{aligned} \text{Cov}(X(0), X_i(t)) &= r_i(t) & ; & & \text{Cov}(X(0), X_{ij}(t)) &= r_{ij}(t) = -\text{Cov}(X_i(0), X_j(t)) \\ \text{Cov}(X_i(0), X_{jk}(t)) &= r_{ijk}(t) & ; & & \text{Cov}(X_{ij}(0), X_{kl}(t)) &= r_{ijkl}(t). \end{aligned}$$

Moreover, by isotropy of X , there exists a real $\lambda \geq 0$ such that $r_{ii}(0) = -\lambda$ for any $i = 1, \dots, d$. In order to avoid working with a degenerate field X , we assume that $\lambda > 0$.

1. THE EULER CHARACTERISTIC OF AN EXCURSION SET

We follow the presentation of Adler & Taylor's book [6] Section 9.4, inspired by Morse's theorem, to define the Euler characteristic of an excursion set.

Let us assume that $T = \prod_{1 \leq j \leq d} [a_j, b_j]$ is a bounded closed rectangle in \mathbb{R}^d and, for $\ell = 0, 1, \dots, d$, let us denote by $\partial_\ell T$ the collection of all the ℓ -dimensional faces of T . In particular, $\partial_d T$ only contains the interior $\overset{\circ}{T} = \prod_{1 \leq j \leq d}]a_j, b_j[$ of T and $\partial_0 T$ is the set of all the vertices of T . Each ℓ -dimensional face J of T is associated with a cardinal ℓ subset $\sigma(J)$ of $\{1, \dots, d\}$ and a sequence $(\varepsilon_j)_{j \in \{1, \dots, d\} \setminus \sigma(J)}$ in $\{-1, +1\}^{d-\ell}$ such that

$$\begin{aligned} J = \{v \in T : & \quad a_j < v_j < b_j \text{ for } j \in \sigma(J), \\ & \quad v_j = 1/2(1 - \varepsilon_j)a_j + 1/2(1 + \varepsilon_j)b_j \text{ for } j \notin \sigma(J)\}. \end{aligned}$$

We now deal with the Euler characteristic of the excursion set $A(T, u) = \{t \in T : X(t) \geq u\}$. Since the level u is fixed all over the paper, we will not mention it any more and write $\chi(T)$ for the Euler characteristic of $A(T, u)$. It can be computed as (see [6] pp.211-212)

$$(1) \quad \chi(T) = \sum_{0 \leq \ell \leq d} \sum_{J \in \partial_\ell T} \varphi(J)$$

where for any ℓ -dimensional face J of T

$$(2) \quad \varphi(J) = \sum_{k=0}^{\ell} (-1)^k \mu_k(J)$$

and the integers $\mu_k(J)$ are given by

$$(3) \quad \mu_k(J) = \# \{v \in J : X(v) \geq u, \\ X_j(v) = 0 \text{ for } j \in \sigma(J), \varepsilon_j X_j(v) > 0 \text{ for } j \notin \sigma(J), \\ \text{index}((X_{ij}(v))_{i,j \in \sigma(J)}) = \ell - k\}.$$

Let us write (1) again as $\chi(T) = \sum_{0 \leq \ell < d} \sum_{J \in \partial_\ell T} \varphi(J) + \varphi(\overset{\circ}{T})$. It is worth remarking that $\varphi(\overset{\circ}{T})$ actually reduces to $\varphi(T)$. Indeed, using (2) and (3) yields $\varphi(\overset{\circ}{T}) = \sum_{k=0}^d (-1)^k \mu_k(\overset{\circ}{T})$ with

$$\mu_k(\overset{\circ}{T}) = \#\{t \in \overset{\circ}{T} : X(t) \geq u, \nabla X(t) = 0, \text{index}(\nabla^2 X(t)) = d - k\}.$$

Bulinskaya Lemma (Lemma 11.2.10 of [6]) entails that with probability one there is no point t in the boundary set ∂T satisfying $\nabla X(t) = 0$, then $\mu_k(\overset{\circ}{T}) = \mu_k(T)$ and hence $\varphi(\overset{\circ}{T}) = \varphi(T)$. Therefore, from now on we will work with this last formula

$$(4) \quad \chi(T) = \sum_{0 \leq \ell < d} \sum_{J \in \partial_\ell T} \varphi(J) + \varphi(T).$$

Let us mention that $\varphi(T)$ is sometimes named as the differential topology characteristic (DT) of the excursion set $A(t, u)$. It is traditionally studied at the same time as the Euler characteristic.

Recall that we want to establish that $\chi(T)$ satisfies a central limit theorem (CLT). More precisely we will prove that

$$\frac{\chi(T) - \mathbb{E}\chi(T)}{|T|^{1/2}}$$

converges in distribution to a centered Gaussian random variable as T grows to \mathbb{R}^d , where $|T|$ stands for the Lebesgue volume of T .

In order to prove it, we first establish a CLT for $\varphi(T)$. This will be achieved in Section 3 using a Hermite expansion of $\varphi(T)$ obtained in Section 2. The main points of Section 3 consist first in the computation of the asymptotic variance V of $|T|^{-1/2}(\varphi(T) - \mathbb{E}\varphi(T))$ and second in the asymptotic normality. This last point is a consequence of a convergence result, namely Proposition 3.3, which is interesting for its own and should be used separately in further works.

Next in Section 4, we prove that for any ℓ -dimensional face J of T , with $\ell < d$, $\text{Var}(|T|^{-1/2}\varphi(J))$ vanishes as T grows to \mathbb{R}^d . This fact, together with (4), allows us

to state our main theorem: $|T|^{-1/2} (\chi(T) - \mathbb{E}\chi(T))$ converges to a $N(0, V)$ random variable.

At last, Section 5 is devoted to the computation of an explicit value of V in the two dimensional case.

2. HERMITE EXPANSION OF $\varphi(T)$

2.1. Approximation of $\varphi(T)$ via the area formula.

We introduce now the required assumption for the L^2 approximation of $\varphi(T)$.

Assumption (H1):

$$t \in \mathbb{R}^d \mapsto \frac{\Theta(t)}{\|t\|^2} \text{ is bounded near zero where } \Theta(t) = \nabla^2 r(t) + \lambda I_d.$$

Let us remark that **(H1)** is clearly satisfied as soon as X is supposed to have trajectories of class C^3 since in that case,

$$(\Theta_{ij}(t))_{1 \leq i, j \leq d} = \left(\sum_{m, n} r_{ijmn}(0) t_m t_n + o(\|t\|^2) \right)_{1 \leq i, j \leq d}.$$

Let $\varepsilon > 0$ and δ_ε be the function $(2\varepsilon)^{-d} 1_{[-\varepsilon, \varepsilon]^d}$ that satisfies $\int_{\mathbb{R}^d} \delta_\varepsilon(z) dz = 1$. The almost sure convergence in our next proposition is contained in Theorem 11.2.3 of [6].

Proposition 2.1. *The following convergence holds almost surely*

$$(5) \quad \varphi(T) = \lim_{\varepsilon \rightarrow 0} (-1)^d \int_T \det(\nabla^2 X(t)) 1_{[u, \infty)}(X(t)) \delta_\varepsilon(\nabla X(t)) dt.$$

Furthermore, under assumption **(H1)**, $\varphi(T) \in L^2(\Omega)$ and the previous convergence holds in $L^2(\Omega)$.

Proof. We first establish the a.s. convergence. We proceed as in the proof of Theorem 11.2.3 of [6] and we use the same notations. The functions appearing in the theorem are, $f(t) = \nabla X(t)$, $\nabla f(t) = \nabla^2 X(t)$, $g(t) = (\nabla^2 X(t), X(t))$ and for each $k = 0 \dots, d$, $B = D_k \times [u, \infty)$ with $D_k = \{\text{index} = d - k\}$.

As a consequence of Boulinskaya lemma, the following three statements hold

$$\begin{aligned} \mathbb{P}\{(\nabla X_{\partial T})^{-1}\{0\} = \emptyset\} &= 1, \\ \mathbb{P}\{\omega : \exists t \in T, \nabla X(t) = 0 \text{ and } \nabla^2 X(t) = 0\} &= 0, \\ \mathbb{P}\{\omega : \exists t \in T, \nabla X(t) = 0, g(t) \in \partial B\} &= 0. \end{aligned}$$

Hence with probability one there is no point t such that

- $t \in \partial T$ satisfying $\nabla X(t) = 0$.
- $t \in T$ satisfying both $\nabla X(t) = 0$ and either $g(t) \in \partial B$ or $\det(\nabla^2 X(t)) = 0$.

Under the above two conditions the set of points $\{t \in T : \nabla X(t) = 0\}$ is finite of cardinal equal to $M(\omega)$. Hence each of such points can be surrounded by an open ball of radius η , say $B(t_i, \eta)$, such that these balls do not intersect ∂T and are disjoint. Moreover η can be chosen small enough such that for t in these balls,

$g(t) \in B$ or in the interior of B^c , but never in both. Let $B_\infty(0, \varepsilon) = \{z : z \in \mathbb{R}^d, \sup_i |z_i| < \varepsilon\}$. We have

$$(\nabla X)^{-1}(B_\infty(0, \varepsilon)) \subset \bigcup_{i=1}^M B(t_i, \eta).$$

By the Inverse Function Theorem we can chose ε and η small enough such that $B_\infty(0, \varepsilon) \subset \nabla X(B(t_i, \eta))$ and the restriction of ∇X to this ball is a one-to-one. The Jacobian of this transformation is $|\det(\nabla^2 X(t))|$. Let us define

$$h_{k,\varepsilon}(t, z) = 1_{[u, \infty)}(X(t)) 1_{D_k}(\nabla^2 X(t)) \delta_\varepsilon(z).$$

Hence by the change of variable formula we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \sum_{t \in (\nabla X^{-1}(z)) \cap B(t_i, \eta)} h_{k,\varepsilon}(t, z) dz \\ = \int_{B(t_i, \eta)} |\det(\nabla^2 X(t))| 1_{[u, \infty)}(X(t)) 1_{D_k}(\nabla^2 X(t)) \delta_\varepsilon(\nabla X(t)) dt, \end{aligned}$$

yielding

$$\begin{aligned} \mu_k &= \int_{\mathbb{R}^d} \sum_{t \in (\nabla X^{-1}(z)) \cap T} h_{k,\varepsilon}(t, z) dz \\ &= \sum_{i=1}^M \int_{B(t_i, \eta)} |\det(\nabla^2 X(t))| 1_{[u, \infty)}(X(t)) 1_{D_k}(\nabla^2 X(t)) \delta_\varepsilon(\nabla X(t)) dt \\ &= (-1)^{d-k} \int_T \det(\nabla^2 X(t)) 1_{[u, \infty)}(X(t)) 1_{D_k}(\nabla^2 X(t)) \delta_\varepsilon(\nabla X(t)) dt. \end{aligned}$$

Since the left side of this equation is independent of ε , summing all the terms for $k = 0, \dots, d$, finally gives the a.s. convergence in (5).

We prove now that (5) also holds with $L^2(\Omega)$ convergence under assumption **(H1)**. Let us denote by $\varphi(\varepsilon, T)$ the right hand side of (5), ie

$$\varphi(\varepsilon, T) = (-1)^d \int_T \det(\nabla^2 X(t)) 1_{[u, \infty)}(X(t)) \delta_\varepsilon(\nabla X(t)) dt.$$

Actually we will establish that $(\varphi(\varepsilon, T))_{\varepsilon \rightarrow 0}$ is a Cauchy family in $L^2(\Omega)$. It will be achieved as soon as we prove that for any $\varepsilon > 0$, $\varphi(\varepsilon, T) \in L^2(\Omega)$ and that $\mathbb{E}(\varphi(\varepsilon, T) - \varphi(\eta, T))^2$ tends to 0 as ε, η tend simultaneously to 0. It is enough to establish that there exists a real number α such that

$$\mathbb{E}(|\varphi(\varepsilon, T)\varphi(\eta, T)|) < +\infty \text{ and } \mathbb{E}(\varphi(\varepsilon, T)\varphi(\eta, T)) \xrightarrow{\varepsilon, \eta \rightarrow 0} \alpha.$$

One can write

$$\begin{aligned} \mathbb{E}(|\varphi(\varepsilon, T)\varphi(\eta, T)|) \\ \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{T \times T} \delta_\varepsilon(x) \delta_\eta(x') \phi_{2d}(t-s, x, x') g_{s,t}(x, x') dx dx' dt ds \end{aligned}$$

where $\phi_{2d}(t, \cdot)$ denotes the probability density function of $(\nabla X(0), \nabla X(t))$ and $g_{s,t}$ denotes the conditional expectation,

$$g_{s,t}(x, x') = \mathbb{E} \left(1_{[u, +\infty)}(X(s)) 1_{[u, +\infty)}(X(t)) | \det(\nabla^2 X(s)) \det(\nabla^2 X(t)) | \right. \\ \left. / \nabla X(s) = x, \nabla X(t) = x' \right).$$

Using the stationarity of X , we get

$$(6) \quad \mathbb{E}(|\varphi(\varepsilon, T)\varphi(\eta, T)|) \\ \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{T_0} \delta_\varepsilon(x) \delta_\eta(x') \phi_{2d}(v, x, x') g_{0,v}(x, x') |T \cap (T - v)| dx dx' dv$$

where T_0 denotes the rectangle in \mathbb{R}^d around 0 obtained from $T = \Pi_{1 \leq j \leq d} [a_j, b_j]$ by prescribing $T_0 = \Pi_{1 \leq j \leq d} [a_j - b_j, b_j - a_j]$. Inverting the integrals on T_0 and on $\mathbb{R}^d \times \mathbb{R}^d$, we write the above right hand side as an integral $\int_{T_0} |T \cap (T - v)| G_{\varepsilon, \eta}(v) dv$.

We prove first that $G_{\varepsilon, \eta}$ is integrable near 0. We need an upper bound for $|g_{0,t}(x, x')|$ for t in a neighborhood of 0. Using Cauchy-Schwarz inequality as well as the fact that $\nabla^2 X(t)$ and $\nabla X(t)$ are independent for any t , we get

$$g_{0,t}(x, x')^2 \leq \mathbb{E}([\det(\nabla^2 X(0))]^2 | \nabla X(0) = x, \nabla X(t) = x') \\ \times \mathbb{E}([\det(\nabla^2 X(t))]^2 | \nabla X(0) = x, \nabla X(t) = x') \\ = \mathbb{E}([\det(\nabla^2 X(0))]^2 | \nabla X(t) = x') \mathbb{E}([\det(\nabla^2 X(t))]^2 | \nabla X(0) = x) \\ := h_t(x') h_{-t}(x).$$

Using again the independence between $\nabla^2 X(t)$ and $\nabla X(t)$, it is clear that $h_t(x)$ tends to $\mathbb{E}([\det(\nabla^2 X(0))]^2)$ as t goes to 0. Hence, for any t near 0 and any $x, x' \in \mathbb{R}^d$,

$$|g_{0,t}(x, x')| \leq 2\mathbb{E}([\det(\nabla^2 X(0))]^2).$$

We now study the behavior of $\phi_{2d}(t, \cdot)$ for t in a neighborhood of 0. The vector $(\nabla X(0), \nabla X(t))$ is a $2d$ centered Gaussian vector. Let us denote by $\Gamma(t)$ its covariance matrix. Since X is stationary and isotropic with covariance function r , we can write

$$\Gamma(t) = \begin{pmatrix} \lambda I_d & -\nabla^2 r(t) \\ -\nabla^2 r(t) & \lambda I_d \end{pmatrix} \text{ where } \lambda = -r_{ii}(0),$$

so that $\det(\Gamma(t)) = \lambda^{2d} - [\det(\nabla^2 r(t))]^2$. By hypothesis **(H1)** we can write $\nabla^2 r(t) = -\lambda I_d + \Theta(t)$ and we know that the function $L(t) = \frac{\Theta(t)}{\|t\|^2}$ is bounded. Using a Taylor expansion of the determinant, we obtain

$$\det(\nabla^2 r(t)) = (-\lambda)^d [1 - \frac{\|t\|^2}{\lambda} \text{Tr} L(t) + o(\|t\|^2)].$$

Hence, $\det \Gamma(t) = 2\lambda^{2d-1} \|t\|^2 \text{Tr} L(t) + o(\|t\|^2)$ and therefore, for a certain constant C ,

$$\phi_{2d}(t, x, x') \leq C \|t\|^{-1}.$$

Case $d \geq 1$. Since $t \mapsto \|t\|^{-1}$ is integrable near 0, we are now able to conclude that

$$\mathbb{E}(|\varphi(\varepsilon, T)\varphi(\eta, T)|) \leq 2C \mathbb{E}([\det(\nabla^2 X(0))]^2) \int_{T_0} |T \cap (T - t)| \|t\|^{-1} dt < +\infty$$

so that $\varphi(\varepsilon, T) \in L^2(\Omega)$.

We are also able to write again inequality (6) without absolute values, so that it becomes an equality where the function $g_{s,t}$ has been replaced by $\tilde{g}_{s,t}$ defined as

$g_{s,t}$ without absolute values around the ‘det’. It is clear that for any fixed $t \neq 0$, the map $(x, x') \mapsto \tilde{g}_{0,t}(x, x')\phi_{2d}(t, x, x')$ is continuous. Hence,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \delta_\varepsilon(x)\delta_\eta(x')\tilde{g}_{0,t}(x, x')\phi_{2d}(t, x, x') dx dx' \xrightarrow{\varepsilon, \eta \rightarrow 0} \tilde{g}_{0,t}(0, 0)\phi_{2d}(t, 0, 0) .$$

Finally, we use the dominated convergence theorem to get that

$$\mathbb{E}(\varphi(\varepsilon, T)\varphi(\eta, T)) \xrightarrow{\varepsilon, \eta \rightarrow 0} \alpha ,$$

with $\alpha = \int_{T_0} \tilde{g}_{0,t}(0, 0)\phi_{2d}(t, 0, 0)|T \cap (T - t)| dt$.

Case $d = 1$. In dimension $d = 1$, our proof is not valid since $\frac{1}{|t|}$ is not integrable near 0. Nevertheless, the fact that the number of up-crossings of X at level u , namely $U^X(T, u)$, can be approximated in $L^2(\Omega)$ has already been established (see for instance [17] or Th.10.10 in [10]). The usual condition for this result to hold is $\int_{|t| \leq 1} \Theta(t)/|t| dt < +\infty$, known as the Geman assumption. It is weaker than our assumption **(H1)**. \square

Remark: The first study in dimension $d > 1$ on the finiteness of the two order moment of the Euler characteristic was the article of Adler and Hasofer [5]. More recently the fact that $\varphi(T)$ belongs to $L^2(\Omega)$ has been implicitly established in [26] under convenient assumptions. Our upper bound proves that their assumptions hold under **(H1)**. Furthermore, following the proof of Proposition 2.1 with little changes yield that $N^{\nabla X}(T, 0)$ belongs to $L^2(\Omega)$, that $N_\varepsilon^{\nabla X}(T, 0)$ tends to $N^{\nabla X}(T, 0)$ in $L^2(\Omega)$ and that $v \mapsto N^{\nabla X}(T, v)$ is continuous on \mathbb{R}^d . As far as we know, this result cannot be found in the literature in the case $d > 1$.

2.2. Hermite type expansion of $\varphi(T)$.

In what follows, we use the Hermite polynomials $(H_n)_{n \in \mathbb{N}}$ defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{\partial^n}{\partial x^n}(e^{-x^2/2})$. They provide an orthonormal basis of $L^2(\mathbb{R}, \phi(x)dx)$ where ϕ denotes the standard Gaussian density on \mathbb{R} . We also denote by ϕ_m the standard Gaussian density on \mathbb{R}^m .

In order to get an Hermite type expansion of $\varphi(T)$, we establish as a first step the expansion of $\varphi(\varepsilon, T)$. Let us recall that

$$\varphi(\varepsilon, T) = (-1)^d \int_T \delta_\varepsilon(\nabla X(t)) \det(\nabla^2 X(t)) 1_{[u, \infty)}(X(t)) dt$$

and write it as

$$\varphi(\varepsilon, T) = (-1)^d \int_T G_\varepsilon(\nabla X(t), \nabla^2 X(t), X(t)) dt.$$

In the following we identify any symmetric matrix of size $d \times d$ with the $d(d+1)/2$ -dimensional vector containing the coefficients on and above the diagonal and write $\tilde{\det}$ the associated determinant map. Hence we consider the map G_ε as

$$\begin{aligned} G_\varepsilon : \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, \mathbf{y}, z) &\mapsto \delta_\varepsilon(x) \tilde{\det}(\mathbf{y}) 1_{[u, \infty)}(z) \end{aligned}$$

At this point, we need some notations. Let us define

$$D = d + d(d+1)/2 + 1$$

and write for any $z \in \mathbb{R}^D$,

$$z = (\underline{z}, \bar{z}) \text{ with } \underline{z} \in \mathbb{R}^d \text{ and } \bar{z} = (\mathbf{z}, z_D) \in \mathbb{R}^{d(d+1)/2} \times \mathbb{R}.$$

Then, one can remark that $G_\varepsilon(z)$ can be factorized as

$$(7) \quad G_\varepsilon(z) = \delta_\varepsilon(\underline{z})f(\bar{z}) \text{ with } f(\bar{z}) = \det(\mathbf{z})1_{[u, \infty)}(z_D).$$

On the other hand, let Λ be the $D \times D$ matrix such that

$$(\nabla X(t), \nabla^2 X(t), X(t)) = \Lambda Y(t)$$

with $Y(t)$ a $N(0, I_D)$ Gaussian vector. Given that the field X is stationary, it holds that $\nabla X(t)$ is independent from $(\nabla^2 X(t), X(t))$ for each fixed t . Hence, the matrix Λ factorizes into $\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$ and since X is isotropic, $\Lambda_1 = \lambda I_d$ where $\lambda = -r_{ii}(0)$ for any $i = 1, \dots, d$. We define

$$\tilde{G}_\varepsilon(y) = G_\varepsilon(\Lambda y) = \delta_\varepsilon(\Lambda_1 \underline{y})f(\Lambda_2 \bar{y}) = \delta_\varepsilon \circ \Lambda_1(\underline{y}) f \circ \Lambda_2(\bar{y}).$$

Since the map \tilde{G}_ε clearly belongs to $\mathbb{L}^2(\mathbb{R}^D, \phi_D(y)dy)$ the following expansion converges in this space

$$\tilde{G}_\varepsilon(y) = \sum_{q=0}^{\infty} \sum_{|\mathbf{n}|=q} c(\tilde{G}_\varepsilon, \mathbf{n}) \tilde{H}_{\mathbf{n}}(y)$$

where $\mathbf{n} = (n_1, n_2, \dots, n_D)$, $|\mathbf{n}| = n_1 + n_2 + \dots + n_D$ and $\tilde{H}_{\mathbf{n}}(y) = \prod_{1 \leq j \leq D} H_{n_j}(y_j)$.

The \mathbf{n} -th Hermite coefficient of \tilde{G}_ε is given by

$$c(\tilde{G}_\varepsilon, \mathbf{n}) = \frac{1}{\mathbf{n}!} \int_{\mathbb{R}^D} \tilde{G}_\varepsilon(y) \tilde{H}_{\mathbf{n}}(y) \phi_D(y) dy$$

with $\mathbf{n}! = n_1! n_2! \dots n_D!$. The factorization (7) induces a factorization of the Hermite coefficient into

$$c(\tilde{G}_\varepsilon, \mathbf{n}) = c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) c(f \circ \Lambda_2, \bar{n})$$

with self understanding notations concerning $\mathbf{n} = (\underline{n}, \bar{n})$ and the Hermite coefficients of the maps $\delta_\varepsilon \circ \Lambda_1$ and $f \circ \Lambda_2$.

Writing $\varphi(\varepsilon, T)$ as $\varphi(\varepsilon, T) = \int_T \tilde{G}_\varepsilon(Y(t)) dt$ yields the following expansion

$$(8) \quad \varphi(\varepsilon, T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \tilde{H}_{\mathbf{n}}(Y(t)) dt.$$

We will take the limit as ε goes to 0 in (8) to obtain the Hermite expansion of $\varphi(T)$. Let us first compute the limit of the coefficient $c(\delta_\varepsilon \circ \Lambda_1, \underline{n})$:

$$(9) \quad \begin{aligned} c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) &= \frac{1}{\underline{n}!} \int_{\mathbb{R}^d} \delta_\varepsilon(\lambda \underline{y}) \tilde{H}_{\underline{n}}(\underline{y}) \phi_d(\underline{y}) d\underline{y} \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{\lambda^{-d}}{(2\pi)^{d/2} \underline{n}!} \tilde{H}_{\underline{n}}(\underline{0}) := d(\underline{n}). \end{aligned}$$

Let us point out that this coefficient is zero if at least one of the indices n_j is odd and in the other cases $d(\underline{2p}) = \frac{\lambda^{-d} (-1)^{|\underline{p}|}}{(2\pi)^{d/2} 2^{|\underline{p}|} p!}$.

We are now able to establish the following proposition which gives the Hermite expansion of $\varphi(T)$.

Proposition 2.2. *Under Assumption (H1), the following expansion holds in $L^2(\Omega)$*

$$\varphi(T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} d(\underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H}_{\mathbf{n}}(Y(t)) dt$$

Proof. Let us take the formal limit of the rhs of (8) and define the random variable

$$\eta(T) := (-1)^d \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} d(\underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H}_{\mathbf{n}}(Y(t)) dt.$$

The first step consists in proving that it belongs to $L^2(\Omega)$.

Let us denote by $\eta_Q(T)$ the projection of $\eta(T)$ obtained with the finite expansion

$$\eta_Q(T) = (-1)^d \sum_{q=0}^Q \sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} d(\underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H}_{\mathbf{n}}(Y(t)) dt.$$

The orthogonality of the Hermite polynomials allows us to bound its second moment. So,

$$\begin{aligned} \mathbb{E}\eta_Q^2(T) &= \sum_{q=0}^Q \mathbb{E} \left[\sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} d(\underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H}_{\mathbf{n}}(Y(t)) dt \right]^2 \\ &\leq \sum_{q=0}^Q \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) c(f \circ \Lambda_2, \bar{n}) \int_T \widetilde{H}_{\mathbf{n}}(Y(t)) dt \right]^2 \\ &\leq \lim_{\varepsilon \rightarrow 0} \mathbb{E}\varphi^2(\varepsilon, T) = \mathbb{E}\varphi^2(T) < \infty, \end{aligned}$$

where we observe that in the second line we have used Fatou's Lemma. Thus $\mathbb{E}\eta^2(T) < \infty$.

We will now establish that $\varphi(T) = \eta(T)$.

We introduce π^Q the projection onto the first Q -order chaos $\oplus_0^{k=Q} \mathfrak{H}_k$ and π_Q the projection onto the chaos $\oplus_{k=Q+1}^{\infty} \mathfrak{H}_k$. So, $\eta_Q(T) = \pi^Q(\eta(T))$ and similarly we write $\varphi_Q(\varepsilon, T) = \pi^Q(\varphi(\varepsilon, T))$. For $\|\cdot\|$ being the norm in $L^2(\Omega)$, we have $\|\varphi(T) - \eta(T)\|$

$$\begin{aligned} &\leq \|\varphi(T) - \varphi(\varepsilon, T)\| + \|\pi_Q(\varphi(\varepsilon, T))\| + \|\pi_Q(\eta(T))\| + \|\varphi_Q(\varepsilon, T) - \eta_Q(T)\| \\ &\leq \|\varphi(T) - \varphi(\varepsilon, T)\| + \|\pi_Q(\varphi(\varepsilon, T) - \varphi(T))\| + \|\pi_Q(\varphi(T))\| + \|\pi_Q(\eta(T))\| \\ &\quad + \|\varphi_Q(\varepsilon, T) - \eta_Q(T)\| \\ &\leq 2\|\varphi(T) - \varphi(\varepsilon, T)\| + \|\pi_Q(\varphi(T))\| + \|\pi_Q(\eta(T))\| + \|\varphi_Q(\varepsilon, T) - \eta_Q(T)\|. \end{aligned}$$

It holds that $\|\pi_Q(\varphi(T))\| + \|\pi_Q(\eta(T))\|$ tends to zero whenever $Q \rightarrow \infty$ because both functionals $\varphi(T)$ and $\eta(T)$ are in $L^2(\Omega)$. Hence by taking limit when $\varepsilon \rightarrow 0$ and $Q \rightarrow \infty$, we get $\varphi(T) = \eta(T)$. \square

3. CENTRAL LIMIT THEOREM FOR $\varphi(T)$

In this section we concentrate on the normalized differential topology characteristic $\zeta(T) = \frac{\varphi(T) - \mathbb{E}\varphi(T)}{|T|^{1/2}}$. We are interested in the asymptotics of $\zeta(T)$ as T grows to \mathbb{R}^d . To make this precise, we assume that the compact domain T has the following shape $T = [-N, N]^d$ with N a positive integer, and we let N go to infinity. We will prove that, as $N \rightarrow +\infty$, the random variable

$$\zeta([-N, N]^d) = \frac{\varphi([-N, N]^d) - \mathbb{E}\varphi([-N, N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable. From now on, in order to simplify computations, we will drop the $(-1)^d$ terms, it has no consequence on the CLT result.

We need to introduce two more assumptions.

Assumption (H2): Introducing $\psi(t) = \max_{|\mathbf{m}| \leq 4} \left| \frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t) \right|$,

$$\psi \in L^1(\mathbb{R}^d, dt) \quad \text{and} \quad \psi(t) \rightarrow 0 \quad \text{when} \quad \|t\| \rightarrow +\infty$$

Note that **(H2)** implies that $r \in L^1(\mathbb{R}^d)$ and hence that X admits a spectral density f_X that is continuous. Note also that **(H2)** implies that $r \in L^q(\mathbb{R}^d)$ for all $q \geq 1$.

Assumption (H3): the spectral density f_X satisfies $f_X(0) > 0$.

We start with a crucial result that states that the asymptotic variance of $\zeta([-N, N]^d)$ does not degenerate as N goes to infinity. As expected, the asymptotic variance depends on the level u . Surprisingly, there are at most $d + 1$ values of u for which we are not able to establish that it does not vanish.

Proposition 3.1. *Under Assumptions (H1) and (H2), as $N \rightarrow +\infty$,*

$$\text{Var}(\zeta([-N, N]^d)) \rightarrow V(u) \quad \text{with} \quad V(u) < +\infty.$$

Moreover, under Assumption (H3), $V(u)$ is positive except at most $d + 1$ values of u where it may vanish.

In order to demonstrate this proposition, we need a ‘‘generalized Mehler’s formula’’ given in the next lemma. It is an extension of Lemma 10.7 in [10] and can be proved in the same way.

Lemma 3.1. *Let $(Y, Z) = (Y_1, \dots, Y_D, Z_1, \dots, Z_D)$ be a 2D centered Gaussian vector with covariance matrix given by*

$$\begin{pmatrix} I_D & K \\ K & I_D \end{pmatrix}$$

Then, for any $\mathbf{n} \in \mathbb{N}^D$ and $\mathbf{m} \in \mathbb{N}^D$,

- *if $|\mathbf{n}| \neq |\mathbf{m}|$ then $\mathbb{E}(\widetilde{H}_{\mathbf{n}}(Y)\widetilde{H}_{\mathbf{m}}(Z)) = 0$,*
- *if $|\mathbf{n}| = |\mathbf{m}|$ then*

$$\mathbb{E}(\widetilde{H}_{\mathbf{n}}(Y)\widetilde{H}_{\mathbf{m}}(Z)) = \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = n_j; \sum_j d_{ij} = m_i}} \mathbf{n!} \mathbf{m!} \prod_{1 \leq i, j \leq D} \frac{(K_{ij})^{d_{ij}}}{(d_{ij})!}$$

Proof of Proposition 3.1.

First step: we prove that $\text{Var } \zeta([-N, N]^d) \rightarrow V < +\infty$.

We write again the Hermite expansion in $L^2(\Omega)$ as it is given in Proposition 2.2 under Assumption **(H1)**.

$$(10) \quad \varphi([-N, N]^d) = \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n} \in \mathbb{N}^D \\ |\mathbf{n}|=q}} a(\mathbf{n}) \int_{[-N, N]^d} \widetilde{H}_{\mathbf{n}}(Y(t)) dt$$

with $a(\mathbf{n}) = d(\underline{n}) c(f \circ \Lambda_2, \bar{n})$ for $\mathbf{n} = (\underline{n}, \bar{n})$.

Hence, using the first item of Lemma 3.1 and the orthogonality properties for different chaos yield the following expression for $\text{Var } \zeta([-N, N]^d)$:

$$(2N)^{-d} \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n}) a(\mathbf{m}) \int_{[-N, N]^d} \int_{[-N, N]^d} \text{Cov}(\widetilde{H}_{\mathbf{n}}(Y(t)), \widetilde{H}_{\mathbf{m}}(Y(s))) ds dt.$$

Using the stationarity of Y , we get

$$\begin{aligned} & \int_{[-N, N]^d} \int_{[-N, N]^d} \text{Cov}(\widetilde{H}_{\mathbf{n}}(Y(t)), \widetilde{H}_{\mathbf{m}}(Y(s))) ds dt \\ &= (2N)^d \int_{[-2N, 2N]^d} \text{Cov}(\widetilde{H}_{\mathbf{n}}(Y(0)), \widetilde{H}_{\mathbf{m}}(Y(v))) \prod_{1 \leq k \leq d} \left(1 - \frac{|v_k|}{2N}\right) dv. \end{aligned}$$

Finally, with Lemma 3.1,

$$(11) \quad \text{Var}(\zeta([-N, N]^d)) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n}) a(\mathbf{m}) \mathbf{n}! \mathbf{m}! R^N(\mathbf{n}, \mathbf{m})$$

where

$$R^N(\mathbf{n}, \mathbf{m}) = \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = n_j; \sum_j d_{ij} = m_i}} \int_{[-2N, 2N]^d} \prod_{1 \leq i, j \leq D} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{(d_{ij})!} \prod_{1 \leq k \leq d} \left(1 - \frac{|v_k|}{2N}\right) dv$$

and

$$\Gamma_{ij}^Y(v) = \text{Cov}(Y_i(0), Y_j(v)) \quad \text{for } i, j = 1, \dots, D.$$

Since $\Gamma^Y(v) = \Lambda^{-1} \Gamma^{\mathbf{X}}(v)^t (\Lambda^{-1})$ where $\Gamma^{\mathbf{X}}$ is the covariance function of $(\nabla X, \nabla^2 X, X)$, we have for any $v \in \mathbb{R}^d$,

$$\sup_{1 \leq i, j \leq D} |\Gamma_{ij}^Y(v)| \leq K \psi(v)$$

where ψ has been introduced in Assumption **(H2)** and K equals the norm of the matrix $(\Lambda^{-1})^2$.

Hence, for any $\mathbf{n}, \mathbf{m} \in \mathbb{N}^D$ such that $|\mathbf{n}| = |\mathbf{m}| = q$ and any sequence $(d_{ij})_{ij}$ such that $d_{ij} \geq 0; \sum_i d_{ij} = n_j; \sum_j d_{ij} = m_i$, it holds

$$\left| \prod_{1 \leq i, j \leq D} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{(d_{ij})!} \prod_{1 \leq k \leq d} \left(1 - \frac{|v_k|}{2N}\right) \right| \leq \prod_{1 \leq i, j \leq D} \frac{1}{(d_{ij})!} K^q \psi^q(v).$$

By Assumption **(H2)**, $\psi \in L^q(\mathbb{R}^d, d\lambda)$. So we can apply the dominated convergence theorem and get

$$(12) \quad R^N(\mathbf{n}, \mathbf{m}) \xrightarrow{N \rightarrow \infty} R(\mathbf{n}, \mathbf{m}) = \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = n_j; \sum_j d_{ij} = m_i}} \int_{\mathbb{R}^d} \prod_{1 \leq i, j \leq D} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{(d_{ij})!} dv.$$

Equations (11) and (12) allow us to write

$$V_N := \text{Var}(\zeta([-N, N]^d)) = \sum_{q=1}^{\infty} u_q^N$$

with $u_q^N \geq 0$ and

$$(13) \quad u_q^N \xrightarrow{N \rightarrow \infty} u_q := \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^D \\ |\mathbf{n}| = |\mathbf{m}| = q}} a(\mathbf{n})a(\mathbf{m})\mathbf{n}!\mathbf{m}! R(\mathbf{n}, \mathbf{m}).$$

We will establish that

$$(14) \quad \sup_N \sum_{q=Q+1}^{\infty} u_q^N \xrightarrow{Q \rightarrow \infty} 0.$$

Using Fatou Lemma, it will prove that the series $V = \sum_{q=1}^{\infty} u_q$ is convergent and that V_N tends to V . And the first step of Proposition 3.1 will be achieved.

Let us remark that (14) is equivalent to

$$\text{Var}(\pi_Q(\zeta([-N, N]^d))) \xrightarrow{Q \rightarrow \infty} 0 \text{ uniformly w.r.t } N,$$

where π_Q is the projection onto the chaos of order $> Q$.

Let $\mathbf{s} \in \mathbb{R}^d$ and set $\theta_{\mathbf{s}}$ the shift operator associated with the field X , i.e. $\theta_{\mathbf{s}}X = X_{\mathbf{s}+}$. Introducing the set of indices

$$\mathcal{I}_N = [-N, N]^d \cap \mathbb{Z}^d$$

we can write

$$\zeta([-N, N]^d) = \zeta([-N, N]^d) = (2N)^{-d/2} \sum_{\mathbf{s} \in \mathcal{I}_N} \theta_{\mathbf{s}} \circ \zeta([0, 1]^d).$$

Then, denoting by $V_{N,Q}$ the variance $\text{Var}(\pi_Q(\zeta([-N, N]^d)))$ and using the stationarity of X , we obtain

$$V_{N,Q} = (2N)^{-d} \sum_{\mathbf{s} \in \mathcal{I}_{2N}} \alpha_{\mathbf{s}}(N) \mathbb{E}(\pi_Q(\zeta([0, 1]^d)) \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0, 1]^d)))) ,$$

where $\alpha_{\mathbf{s}}(N)$ denotes the cardinal of $\{\mathbf{t} \in \mathcal{I}_N : \mathbf{t} - \mathbf{s} \in \mathcal{I}_N\}$ which is certainly less than $(2N)^d$.

Let chose a such that $\psi(\mathbf{s}) \leq \rho < 1/K$ for $\|\mathbf{s}\|_{\infty} \geq a$. We split $V_{N,Q}$ into $V_{N,Q}^1 + V_{N,Q}^2$ where in $V_{N,Q}^1$ the sum runs for the indices $\mathbf{s} \in \{\mathbf{s} \in \mathcal{I}_{2N} : \|\mathbf{s}\|_{\infty} < a + 1\}$ and in $V_{N,Q}^2$ for $\{\mathbf{s} \in \mathcal{I}_{2N} : \|\mathbf{s}\|_{\infty} \geq a + 1\}$.

At first, it holds for $2N > a + 1$,

$$|V_{N,Q}^2| \leq (2N)^{-d} (2a + 2)^d (2N)^d \mathbb{E}(\pi_Q(\zeta([0, 1]^d)))^2$$

which goes to 0 as Q goes to ∞ uniformly with respect to N .

Next, for any $\mathbf{s} \in \mathcal{I}_{2N}$ such that $\|\mathbf{s}\|_\infty \geq a + 1$, we write

$$(15) \quad \mathbb{E} \left(\pi_Q(\zeta([0, 1]^d)) \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0, 1]^d))) \right) \\ = \sum_{q=Q+1}^{\infty} \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s} + u))] dt du,$$

where

$$\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s} + u))] = \mathbb{E} \left[\sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n}) \tilde{H}_{\mathbf{n}}(Y(t)) \sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n}) \tilde{H}_{\mathbf{n}}(Y(\mathbf{s} + u)) \right].$$

Arcones inequality ([9] Lemma 1), implies that

$$|\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s} + u))]| \leq K^q \psi^q(\mathbf{s} - (u - t)) \sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n})^2 \mathbf{n}!.$$

Let us remark that the series $\sum_{\mathbf{n} \in \mathbb{N}^D} a(\mathbf{n})^2 \mathbf{n}!$ diverges so that we have to handle carefully in what follows. Recall that $a(\mathbf{n}) = d(\underline{n}) c(f \circ \Lambda_2, \bar{n})$ with $d(\underline{n})$ given by (9). In Imkeller et al. [14] it is shown that $\sup_x |\frac{H_1(x)}{\sqrt{|x|}} \varphi(x)| \leq C$ for a universal constant C . This yields $d^2(\underline{n}) \mathbf{n}! \leq C^d$ and hence

$$\sum_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} a(\mathbf{n})^2 \mathbf{n}! \leq C^d q^d \sum_{|\bar{n}| \leq q} c(f \circ \Lambda_2, \bar{n})^2 \bar{n}! \leq C^d q^d \|f \circ \Lambda_2\|^2.$$

Therefore the absolute value of (15) can be bounded by

$$C^d \|f \circ \Lambda_2\|^2 \sum_{q=Q+1}^{\infty} q^d K^q \int_{[0,1]^d} \int_{[0,1]^d} \psi^q(\mathbf{s} - (u - t)) du dt.$$

Hence

$$|V_{N,Q}^2| \leq \sum_{\mathbf{s} \in \mathcal{I}_{2N}; \|\mathbf{s}\|_\infty \geq a+1} |\mathbb{E}(\pi_Q(\zeta([0, 1]^d)) \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0, 1]^d))))| \\ \leq C^d \|f \circ \Lambda_2\|^2 \sum_{q=Q+1}^{\infty} q^d K^q \rho^{q-1} \\ \times \sum_{\mathbf{s} \in \mathcal{I}_{2N}; \|\mathbf{s}\|_\infty \geq a+1} \int_{[0,1]^d} \int_{[0,1]^d} \psi(\mathbf{s} - (u - t)) du dt$$

where we have used that for any $\|\mathbf{s}\|_\infty \geq a + 1$ and any $u, t \in [0, 1]^d$, $\psi(\mathbf{s} - (u - t)) \leq \rho$.

On the one hand, since $\rho < 1/K$, $\sum_{q=Q+1}^{\infty} q^d K^q \rho^{q-1}$ is the tail of a convergent series.

On the other hand,

$$\sum_{\mathbf{s} \in \mathcal{I}_{2N}; \|\mathbf{s}\|_\infty \geq a+1} \int_{[0,1]^d} \int_{[0,1]^d} \psi(\mathbf{s} - (u - t)) du dt \\ \leq \sum_{\mathbf{s} \in \mathcal{I}_{2N}} \int_{[0,2]^d} \psi(\mathbf{s} + u) du \leq 2 \int_{\mathbb{R}^d} \psi(u) du < +\infty.$$

Hence $\sup_N |V_{N,Q}^2|$ goes to 0 as Q goes to infinity and we have proved that $\text{Var}\zeta([-N, N]^d)$ tends to

$$(16) \quad V = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\mathbf{n}!\mathbf{m}! R(\mathbf{n}, \mathbf{m}) < +\infty.$$

Second step: We assume now that **(H3)** is satisfied, i.e. that the spectral density f_X of X is such that $f_X(0) > 0$, and we prove that V is positive for all values of u except for at most $d+1$ of them.

We only need to prove that at least one of the terms in the sum (16) is positive. This will be achieved for the $q=1$ term

$$V_1 = \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=1}} a(\mathbf{n})a(\mathbf{m}) R(\mathbf{n}, \mathbf{m}).$$

Note that $|\mathbf{n}|=1$ implies that only one coordinate n_i is different from zero and equal to one. So we can write $\mathbf{n} = \mathbf{e}_i$ for one index $i = 1, \dots, D$ with the usual convention concerning the canonical basis $(\mathbf{e}_i)_{1 \leq i \leq D}$ of \mathbb{R}^D .

We recall that the following factorization occurs $a(\mathbf{n}) = d(\underline{n})c(f \circ \Lambda_2, \bar{n})$. Hence for $\mathbf{n} = \mathbf{e}_i$, due to the explicit form of $d(\underline{n})$ given in (9), we have

$$a(\mathbf{e}_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq d \\ d(0)c(f \circ \Lambda_2, \bar{e}_i) & \text{if } d+1 \leq i \leq D \end{cases}$$

with $d(0) = \lambda^{-d}(2\pi)^{-d/2}$. Hence,

$$V_1 = \sum_{d+1 \leq i, j \leq D} d(0)^2 c(f \circ \Lambda_2, \bar{e}_i) c(f \circ \Lambda_2, \bar{e}_j) R(\mathbf{e}_i, \mathbf{e}_j)$$

where we deduce from (12) that $R(\mathbf{e}_i, \mathbf{e}_j) = \int_{\mathbb{R}^d} \Gamma_{i,j}^Y(v) dv$.

Lemma 3.2. *We consider the D -dimensional Gaussian field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$ and denote by $\Gamma^{\mathbf{X}}$ its covariance function. For any $1 \leq i, j \leq D$,*

$$\int_{\mathbb{R}^d} \Gamma_{i,j}^{\mathbf{X}}(v) dv = (2\pi)^d f_X(0) \delta_{D,D}(i, j)$$

where δ stands for the Kronecker symbol.

Proof of Lemma 3.2.

For $(i, j) = (D, D)$ we have

$$\Gamma_{D,D}^{\mathbf{X}}(v) = \mathbb{E}(X(0)X(v)) = r(v) = \int_{\mathbb{R}^d} e^{i\lambda \cdot v} f_X(\lambda) d\lambda$$

and then, since f_X is supposed to be continuous,

$$\int_{\mathbb{R}^d} \Gamma_{D,D}^{\mathbf{X}}(v) dv = \widehat{f_X}(0) = (2\pi)^d f_X(0).$$

For $(i, j) \neq (D, D)$, recall that $\Gamma_{i,j}^{\mathbf{X}}$ equals a derivative of order between one and four of the function r . Since r and all its derivative tends to 0 at infinity due to assumption **(H2)**, we get $\int_{\mathbb{R}^d} \Gamma_{i,j}^{\mathbf{X}}(v) dv = 0$ for $(i, j) \neq (D, D)$. \square

We come back to the proof of Proposition 3.1. Let us recall that we wrote $\mathbf{X} = \Lambda Y$

with a matrix Λ that factorizes into $\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$. Then Lemma 3.2 implies that for any $d+1 \leq i, j \leq D$,

$$\int_{\mathbb{R}^d} \Gamma_{i,j}^Y(v) dv = (2\pi)^d f_X(0) (\Lambda_2^{-1})_{i,D} (\Lambda_2^{-1})_{j,D}.$$

Finally, we obtain

$$(17) \quad V_1 = (2\pi)^d f_X(0) d(\underline{0})^2 \left(\sum_{d+1 \leq i \leq D} c(f \circ \Lambda_2, \bar{e}_i) (\Lambda_2^{-1})_{i,D} \right)^2.$$

In order to conclude that $V_1 > 0$, under the assumption $f_X(0) > 0$, it remains to establish that

$$\sum_{d+1 \leq i \leq D} c(f \circ \Lambda_2, \bar{e}_i) (\Lambda_2^{-1})_{i,D} \text{ is not equal to } 0.$$

Let us denote by $H_\Lambda(u)$ this sum of Hermite coefficients which depends on the level u as we will see in the next computation.

$$\begin{aligned} H_\Lambda(u) &:= \sum_{d+1 \leq i \leq D} c(f \circ \Lambda_2, \bar{e}_i) (\Lambda_2^{-1})_{i,D} \\ &= \int_{\mathbb{R}^{D-d}} f \circ \Lambda_2(x) \left(\sum_{d+1 \leq i \leq D} x_i (\Lambda_2^{-1})_{i,D} \right) \phi_{D-d}(x) dx \end{aligned}$$

For convenience reason, we choose Λ , and so Λ_2 , to be upper triangular with $\Lambda_{D,D} = 1$. Then we can write the matrices Λ_2 and ${}^t(\Lambda_2^{-1})$ as

$$\Lambda_2 = \begin{pmatrix} U & v \\ \mathbf{0} & 1 \end{pmatrix} \quad \text{and} \quad {}^t(\Lambda_2^{-1}) = \begin{pmatrix} L & \mathbf{0} \\ h & 1 \end{pmatrix}$$

with U a $d(d+1)/2$ upper triangular matrix, L a $d(d+1)/2$ lower triangular matrix, v a $d(d+1)/2$ vertical vector and h a $d(d+1)/2$ horizontal vector. Hence, for $x = (y, z) \in \mathbb{R}^{D-d} = \mathbb{R}^{d(d+1)/2} \times \mathbb{R}$, we have

$$f \circ \Lambda_2(y, z) = 1_{[u, +\infty)}(z) \tilde{\det}(Uy + zv) \quad \text{and} \quad \sum_{d+1 \leq i \leq D} x_i (\Lambda_2^{-1})_{i,D} = \langle h, y \rangle + z.$$

Therefore,

$$\begin{aligned} H_\Lambda(u) &= \int_{\mathbb{R}^{d(d+1)/2} \times \mathbb{R}} 1_{[u, +\infty)}(z) \tilde{\det}(Uy + zv) (\langle h, y \rangle + z) \phi_{d(d+1)/2}(y) \phi(z) dy dz \\ &= \int_{[u, +\infty)} \phi(z) \left(\int_{\mathbb{R}^{d(d+1)/2}} \tilde{\det}(Uy + zv) (\langle h, y \rangle + z) \phi_{d(d+1)/2}(y) dy \right) dz. \end{aligned}$$

Note that the inner integral is nothing but a polynomial function of z with degree $d+1$. Let us denote it by P_Λ (its coefficients depend on Λ) and remark that it has the same parity as $d+1$ since $\det(-(Uy + zv)) = (-1)^d \tilde{\det}(Uy + zv)$. Hence

$$H_\Lambda(u) = \int_{[u, +\infty)} \phi(z) P_\Lambda(z) dz$$

and it is not difficult to establish that $H_\Lambda(u) \neq 0$ except for at most d values of u (depending on P_Λ) when d is even and $d+1$ values of u when d is odd. \square

Remark: The explicit value of V seems to be very difficult to obtain in the general case. Nevertheless, we tackle this question in the Appendix in the case $d = 2$. We also compute the exact value of $H_\Lambda(u)$ in the case $d = 2$ and exhibit the only two values of u for which $H_\Lambda(u)$ vanishes.

Since Proposition 3.1 guarantees a finite asymptotic variance, we are now able to state our main result.

Theorem 3.2. *Let X be a stationary isotropic Gaussian field indexed by \mathbb{R}^d with C^2 trajectories that satisfies Assumptions **(H1)** and **(H2)**. As $N \nearrow +\infty$,*

$$\zeta([-N, N]^d) = \frac{\varphi([-N, N]^d) - \mathbb{E}\varphi([-N, N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable with finite variance V given by (16).

*Moreover, V depends on the level u and under Assumption **(H3)** V does not vanish except for at most $d + 1$ values of u .*

Proof. By the proof of Proposition 3.1, we already know that

$$\sup_N \text{Var}(\pi_Q(\zeta([-N, N]^d))) \xrightarrow{Q \rightarrow \infty} 0.$$

So $\pi_Q(\zeta([-N, N]^d)) \rightarrow 0$ in $L^2(\Omega)$ when $N \rightarrow \infty$ and $Q \rightarrow \infty$ in this order. Hence in order to establish the CLT, it is enough to show the asymptotic normality, for a fixed Q as N goes to infinity, of the sequence

$$\begin{aligned} \pi^Q(\zeta([-N, N]^d)) &= \frac{1}{(2N)^{d/2}} \sum_{q=1}^Q \sum_{\substack{\mathbf{n} \in \mathbb{N}^D \\ |\mathbf{n}|=q}} a(\mathbf{n}) \int_{[-N, N]^d} \widetilde{H}_{\mathbf{n}}(Y(t)) dt \\ (18) \qquad \qquad \qquad &:= \frac{1}{(2N)^{d/2}} \int_{[-N, N]^d} \sum_{q=1}^Q G_q(Y(t)) dt, \end{aligned}$$

where we have defined

$$G_q(\mathbf{x}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^D \\ |\mathbf{n}|=q}} a(\mathbf{n}) \widetilde{H}_{\mathbf{n}}(\mathbf{x}).$$

Moreover Proposition 3.1 states that the asymptotic variance of $\pi^Q(\zeta([-N, N]^d))$ is finite. Then the result follows from the classical Breuer-Major Theorem. A reference for this can be found for instance in the paper of Arcones [9]. However, in this result the parameter set is the set of integers whereas in our setting the parameter set is \mathbb{R}^d . Therefore for completeness, we give a statement and a proof of this famous theorem, both adapted to our setting, namely Proposition 3.3 below. \square

Proposition 3.3. *Let X be a stationary isotropic Gaussian field indexed by \mathbb{R}^d with C^2 trajectories that satisfies Assumptions **(H1)** and **(H2)**. For any fixed positive integer Q , as $N \nearrow +\infty$, $\pi^Q(\zeta([-N, N]^d))$ converges in distribution to a centered Gaussian variable with finite variance $\sigma_Q^2 = \sum_{q=1}^Q u_q$, where the u_q 's are introduced in (13).*

Proof. Our proof follows very closely the method of proof of the CLT in Nourdin et al. [20], considering \mathbb{R}^d as parameter set instead of \mathbb{Z} . To set up our framework we must consider the isonormal process associated to some zero mean stationary Gaussian process $Y : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^D$. The following paragraph has for goal this objective.

Let W^j , $j = 1, \dots, D$ be a complex Wiener process defined on \mathbb{R}^d and $F(\lambda) := (f_{jl}(\lambda))$ a definite positive self-adjoint matrix of densities functions i.e. for any Borel set A of \mathbb{R}^d we have that $(\int_A f_{jl}(\lambda)d\lambda)$ is definite positive. Let $B(\lambda) = (b_{jl}(\lambda))$ a square root of F . In this form defining

$$W_F^j(A) = \sum_{l=1}^D \int_A b_{jl}(\lambda) dW^l(\lambda),$$

it holds

$$\mathbb{E}[W_F^j(A) \overline{W_F^k(C)}] = \sum_{l=1}^D \int_{A \cap C} b_{jl}(\lambda) b_{lk}(\lambda) d\lambda = \int_{A \cap C} f_{jk}(\lambda) d\lambda.$$

Consider now m_d the Lebesgue measure in \mathbb{R}^d and ν the counting measure $\nu(\{j\}) = 1$ for $j = 1, 2, \dots, D$. We can define the product measure $\tilde{\mu} : \mathcal{P}(\{1, 2, \dots, D\}) \otimes \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}^+$ as $\tilde{\mu}(A_1 \times A_2) = \#A_1 \times m_d(A_2)$, where $A_1 \subset \{1, 2, \dots, D\}$ and $A_2 \in \mathcal{B}(\mathbb{R}^d)$. If $h : \{1, 2, \dots, D\} \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a measurable function,

$$\int_{\mathcal{W}} h(\omega) d\tilde{\mu}(\omega) = \sum_{j=1}^D \int_{\mathbb{R}^d} h(j, \lambda) dm_d(\lambda) \text{ where } \mathcal{W} = \{1, 2, \dots, D\} \times \mathbb{R}^d.$$

Moreover

$$\int_{\mathcal{W}} h(\omega) \overline{g(\omega)} d\tilde{\mu}(\omega) = \sum_{j=1}^D \int_{\mathbb{R}^d} h(j, \lambda) \overline{g(j, \lambda)} dm_d(\lambda),$$

yielding that $\mathcal{H} = L^2(\mathcal{W}, \tilde{\mu}) \cong L^2(\mathbb{R}^d, \mathbb{R}^D, m_d)$. Thus we can define for $h \in \mathcal{H}$

$$W(h) = \sum_{j=1}^D \int_{\mathbb{R}^d} h(j, \lambda) dW^j(\lambda).$$

In the same form considering the space

$$\mathcal{H}_F := \{h : \|h\|_F^2 := \int_{\mathbb{R}^d} \langle h(\lambda), F(\lambda)h(\lambda) \rangle d\lambda < \infty\},$$

we can define

$$(19) \quad W_F(h) = \sum_{j=1}^D \sum_{l=1}^D \int_{\mathbb{R}^d} h(j, \lambda) b_{jl} dW^l(\lambda) = W({}^t B h).$$

This implies

$$\begin{aligned} \mathbb{E}[W_F(h) \overline{W_F(h)}] &= \sum_{j_1=1}^D \sum_{j_2=1}^D \sum_{l=1}^D \int_{\mathbb{R}^d} h(j_1, \lambda) b_{j_1 l}(\lambda) b_{l j_2}(\lambda) \overline{h(j_2, \lambda)} d\lambda \\ &= \sum_{j_1=1}^D \sum_{j_2=1}^D \int_{\mathbb{R}^d} h(j_1, \lambda) f_{j_1 j_2}(\lambda) \overline{h(j_2, \lambda)} d\lambda = \int_{\mathbb{R}^d} \langle h(\lambda), F(\lambda)h(\lambda) \rangle d\lambda. \end{aligned}$$

Hence W_F appears as the isonormal process associated to the space \mathcal{H}_F . In the same form, W results the isonormal space associated to $\mathcal{H} = L^2(\mathcal{W}, \tilde{\mu}) \cong L^2(\mathbb{R}^d, \mathbb{R}^D, m_d)$.

We assume now that $F = (f_{jl})_{1 \leq j, l \leq D}$ is the matrix of spectral densities associated with a zero mean stationary Gaussian process $Y : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^D$. In other words, we assume that $\text{Var}(Y(0)) = I_D$ and

$$\mathbb{E}[Y_j(0)Y_l(s)] = \int_{\mathbb{R}^d} e^{i\langle s, \lambda \rangle} f_{jl}(\lambda) d\lambda, \quad \forall s \in \mathbb{R}^d.$$

It yields

$$Y_j(t) = \sum_{l=1}^D \int_{\mathbb{R}^d} e^{i\langle t, \lambda \rangle} b_{jl}(\lambda) dW^l(\lambda) = W(\varphi_{t,j}),$$

where we consider the functions $\varphi_{t,j}(l, \lambda) = e^{i\langle t, \lambda \rangle} b_{jl}(\lambda) \mathbf{e}_l$, for $j = 1, \dots, D$ and $(\mathbf{e}_l)_{1 \leq l \leq D}$ the canonical basis of \mathbb{R}^D . Moreover

$$(20) \quad \langle \varphi_{t,j}, \varphi_{t,l} \rangle_{\mathcal{H}_F} = \int_{\mathbb{R}^d} f_{j,l}(\lambda) d\lambda = \mathbb{E}[Y_j(0)Y_l(0)] = \delta_{jl}.$$

Hence, the functions $(\varphi_{t,j})_{1 \leq j \leq D}$ are orthogonal in \mathcal{H}_F . Recall that the isonormal process associated to the Hilbert space \mathcal{H}_F is $W_F = \{W_F(h) : h \in \mathcal{H}\}$. For each q let us denote \mathcal{H}_q the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(W_F(h)), h \in \mathcal{H}_F, \|h\|_{\mathcal{H}_F} = 1\}$. It holds $L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q$, see [21] page 6. By the relation (19) the elements of \mathcal{H}_q can be represented as a multiple integral of Itô-Wiener. In fact any functional $G \in L^2(\Omega)$ admits the following expansion

$$G = \sum_{q=0}^{\infty} I_q(g_q), \quad \text{where the series converges in } L^2(\Omega),$$

with $I_0(g_0) = \mathbb{E}(G)$ and the kernels g_q are uniquely determined by G . In our particular case $\mathcal{H} = L^2(\mathcal{W}, \mathcal{A}, \tilde{\mu})$, one has that $\mathcal{H}^{\odot q} = L_s^2(\mathcal{W}^q, \mathcal{A}^{\otimes q}, \tilde{\mu}^{\otimes q})$ is the space of symmetric and square integrable functions on \mathcal{W}^q . Moreover $I_q(g)$ coincides with the multiple Wiener-Itô integral of order q , with respect to W . For a detailed account of these matters, see the classical Peter Major's book [18].

The Itô formula for multiple Wiener integrals (see [21] page 13 although the normalization is different. In this book the Hermite polynomials are the same of that we use but multiplied by $\frac{1}{m!}$. See also [18] page 37) allows us to write the next formula. For any $k = 1, \dots, D$ and any positive integer l ,

$$H_l(Y_k(s)) = H_l(W(e^{i\langle t, \cdot \rangle} \sum_{l=1}^D b_{kl}(\cdot) \mathbf{e}_l)) = I_l(\varphi_{s,k} \otimes \dots \otimes \varphi_{s,k}) = I_l(\varphi_{s,k}^{\otimes l}),$$

where the tensorial product has l terms.

Recall that we aim at computing $G_q(Y(s))$ with

$$\begin{aligned} G_q(Y(s)) &= \sum_{|\mathbf{n}|=q} a_{\mathbf{n}} \tilde{H}_{\mathbf{n}}(Y(s)) = \sum_{|\mathbf{n}|=q} a_{\mathbf{n}} H_{n_1}(Y_1(s)) \dots H_{n_D}(Y_D(s)) \\ (21) \quad &= \sum_{|\mathbf{n}|=q} a_{\mathbf{n}} I_{n_1}(\varphi_{s,1}^{\otimes n_1}) \dots I_{n_D}(\varphi_{s,D}^{\otimes n_D}). \end{aligned}$$

The multiplication formula (formula (3.33) of [20]) entails

$$I_{n_1}(\varphi_{s,1}^{\otimes n_1})I_{n_2}(\varphi_{s,2}^{\otimes n_2}) = \sum_{r=0}^{n_1 \wedge n_2} r! \binom{n_1}{r} \binom{n_2}{r} I_{n_1+n_2-2r}(\varphi_{s,1}^{\otimes n_1} \tilde{\otimes}_r \varphi_{s,2}^{\otimes n_2}).$$

The relation of orthogonality (20) yields that in the above sum the only non vanishing term is for $r = 0$. Thus

$$I_{n_1}(\varphi_{s,1}^{\otimes n_1})I_{n_2}(\varphi_{s,2}^{\otimes n_2}) = I_{n_1+n_2}(\varphi_{s,1}^{\otimes n_1} \tilde{\otimes} \varphi_{s,1}^{\otimes n_2}),$$

where the tilde tensorial product $\tilde{\otimes}$ denotes the symmetrization of $\varphi_{s,1}^{\otimes n_1} \otimes \varphi_{s,2}^{\otimes n_2}$, namely

$$\sum_{\mathbf{m} \in \{1,2\}^{n_1+n_2}} \varphi_{s,m_1} \otimes \dots \otimes \varphi_{s,m_{n_1+n_2}}.$$

Thus

$$I_{n_1}(\varphi_{s,1}^{\otimes n_1})I_{n_2}(\varphi_{s,2}^{\otimes n_2}) = I_{n_1+n_2} \left(\sum_{\mathbf{m} \in \{1,2\}^{n_1+n_2}} \varphi_{s,m_1} \otimes \dots \otimes \varphi_{s,m_{n_1+n_2}} \right).$$

We iterate the process in order to compute (21). For a more convenient form, we introduce before some notation. For $\mathbf{n} = (n_1, \dots, n_D) \in \mathbb{N}^D$ such that $|\mathbf{n}| = q$, we define

$$\mathcal{A}_{\mathbf{n}} = \{\mathbf{m} \in \{1, \dots, D\}^q : \sum_{j=1}^q 1_{\{i\}}(m_j) = n_i, i = 1, \dots, D\}$$

and remark that

$$\{1, 2, \dots, D\}^q = \cup_{\mathbf{n} \in \mathbb{N}^D; |\mathbf{n}|=q} \mathcal{A}_{\mathbf{n}}.$$

So we are able to introduce the following symmetric kernel

$$\sum_{\mathbf{m} \in \{1,2,\dots,D\}^q \cap \mathcal{A}_{\mathbf{n}}} \varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q},$$

which leads us to write

$$(22) \quad G_q(Y(s)) = \sum_{|\mathbf{n}|=q} a_{\mathbf{n}} \sum_{\mathbf{m} \in \mathcal{A}_{\mathbf{n}}} I_q(\varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q}).$$

Let set

$$b_{\mathbf{m}} = a_{\mathbf{n}} \text{ whenever } \mathbf{m} \in \mathcal{A}_{\mathbf{n}}$$

and remark that the sets $\mathcal{A}_{\mathbf{n}}$ are invariant by the permutations of $\{1, \dots, q\}$. Indeed if $\mathbf{m} = (m_1, \dots, m_q) \in \mathcal{A}_{\mathbf{n}}$ it holds also that $\mathbf{m}' = (m_{\sigma(1)}, \dots, m_{\sigma(q)}) \in \mathcal{A}_{\mathbf{n}}$, for any permutation σ . Thus the function $\mathbf{m} \mapsto b_{\mathbf{m}}$ is symmetric on $\{1, 2, \dots, D\}^q$.

Thus (22) writes

$$G_q(Y(s)) = \sum_{\mathbf{m} \in \{1,2,\dots,D\}^q} b_{\mathbf{m}} \sum_{\mathbf{m} \in \mathcal{A}_{\mathbf{n}}} I_q(\varphi_{s,m_1} \otimes \varphi_{s,m_2} \dots \otimes \varphi_{s,m_q})$$

and moreover

$$\mathbb{E} (G_q(Y(s))^2) = q! \sum_{|\mathbf{n}|=q} a_{\mathbf{n}}^2 = q! \sum_{\mathbf{m} \in \{1,2,\dots,D\}^q} b_{\mathbf{m}}^2.$$

We begin now to prove the CLT. We still follow the article [20] and when it is possible its notations. From now on, our field Y is the same that we have considered in the other sections, whose covariance function is Γ^Y .

The linearity of multiple Wiener integral and Fubini theorem for multiple Wiener integrals applied to formula (18) yield

$$\pi^Q \zeta([-N, N]^d) = \sum_{q=1}^Q I_q(g_q^N)$$

where

$$g_q^N = \frac{1}{(2N)^{d/2}} \int_{[-N, N]^d} \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}} \varphi_{s, m_1} \otimes \varphi_{s, m_2} \cdots \otimes \varphi_{s, m_q} ds,$$

as [20] pages 806-807.

Let us first get an upper bound for $q! \|g_q^N\|_{\mathcal{H}^{\otimes q}}^2$.

$$\begin{aligned} q! \|g_q^N\|_{\mathcal{H}^{\otimes q}}^2 &= \frac{q!}{(2N)^d} \int_{[-N, N]^d} \int_{[-N, N]^d} \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}} b_{\mathbf{l}} \prod_{j=1}^q \Gamma_{m_j l_j}^Y(s_1 - s_2) ds_1 ds_2 \\ &= q! \int_{[-2N, 2N]^d} \prod_{1 \leq k \leq d} \left(1 - \frac{|v_k|}{2N}\right) \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}} b_{\mathbf{l}} \prod_{j=1}^q \Gamma_{m_j l_j}^Y(v) dv. \end{aligned}$$

We divide the domain of integration in two parts $|v| < 2K$ for a constant K and its complement. Using Schwarz inequality for the first term and the definition of ψ for the second one we get

$$\begin{aligned} q! \|g_q^N\|_{\mathcal{H}^{\otimes q}}^2 &\leq q! (2K)^d \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}}^2 + q! \int_{|v| > K} \psi(v)^q dv \left(\sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} |b_{\mathbf{m}}| \right)^2 \\ &\leq q! (2K)^d \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}}^2 + q! \int_{|v| > K} (D\psi(v))^q dv \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}}^2 \\ &\leq q! \sum_{\mathbf{m} \in \{1, 2, \dots, D\}^q} b_{\mathbf{m}}^2 ((2K)^d + D^q \int_{\mathbb{R}^d} \psi(v)^q dv). \end{aligned}$$

The dominated convergence theorem allows us to obtain

$$q! \|g_q^N\|_{\mathcal{H}^{\otimes q}}^2 \rightarrow u_q.$$

Let us denote \mathbf{D} the Malliavin derivative (see [20] for the definition). We have $\mathbb{E}[\frac{1}{q} \|\mathbf{D}I_q(g_q^N)\|_{\mathcal{H}}^2] = q! \|g_q^N\|_{\mathcal{H}^{\otimes q}}^2$ so that

$$(23) \quad \sum_{q=1}^Q \|u_q - \frac{1}{q} \|\mathbf{D}I_q(g_q^N)\|_{\mathcal{H}}^2\|_{L^2(\Omega)} \rightarrow 0.$$

Next step consists in showing that for $q > p$

$$(24) \quad \|\frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}}\|_{L^2(\Omega)}^2 \rightarrow 0.$$

Formula (3.6) of [20] implies

$$\begin{aligned} & \left\| \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}} \right\|_{L^2(\Omega)}^2 \\ & \leq p! \binom{q-1}{p-1}^2 (q-p)! \mathbb{E}[I_p(g_p^N)]^2 \|g_q^N \otimes_{q-p} g_q^N\|_{\mathcal{H}^{\otimes 2p}} \\ & + \frac{p^2}{2} \sum_{l=1}^{p-1} (l-1)!^2 \binom{p-1}{l-1}^2 \binom{q-1}{l-1}^2 (p+q-2l)! (\|g_p^N \otimes_{p-l} g_p^N\|_{\mathcal{H}^{\otimes 2l}} + \|g_q^N \otimes_{q-l} g_q^N\|_{\mathcal{H}^{\otimes 2l}}), \end{aligned}$$

where

$$\begin{aligned} g_p^N \otimes_e g_p^N &= \frac{1}{(2N)^d} \int_{[-N, N]^d \times [-N, N]^d} \sum_{\mathbf{m} \in \{1, \dots, D\}^p} b_{\mathbf{m}} b_1 \prod_{j=1}^e \Gamma_{m_j l_j}^Y((s_1 - s_2) \\ & \quad \times u_{s_1, m_{e+1}} \otimes \dots \otimes u_{s_1, m_p} \otimes u_{s_2, l_{e+1}} \otimes \dots \otimes u_{s_2, l_p} ds_1 ds_2 \end{aligned}$$

In this form defining $I(N) = [-N, N]^d \times [-N, N]^d \times [-N, N]^d \times [-N, N]^d$ we get

$$\|g_p^N \otimes_e g_p^N\|_{\mathcal{H}^{\otimes 2(p-e)}}^2 \leq (D^p \sum_{\mathbf{m} \in \{1, \dots, D\}^p} |b_{\mathbf{m}}|^2) \mathcal{Z}(N),$$

with

$$\mathcal{Z}(N) = \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1 - s_2) \psi^e(s_3 - s_4) \psi^{p-e}(s_1 - s_3) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_3 ds_4.$$

Moreover we have $\psi^e(s_3 - s_4) \psi^{p-e}(s_1 - s_3) \leq \psi^p(s_3 - s_4) + \psi^p(s_1 - s_3)$. Thus we can write $\mathcal{Z}(N) \leq \mathcal{Z}_1(N) + \mathcal{Z}_2(N)$ where

$$\mathcal{Z}_1(N) \leq \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1 - s_2) \psi^p(s_3 - s_1) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_3 ds_4.$$

Let us look at the integral

$$\int_{[-N, N]^d} \psi^p(s_3 - s_1) ds_3 \leq \int_{\mathbb{R}^d} \psi^p(v) dv < \infty,$$

and for the remaining terms

$$\begin{aligned} & \frac{1}{(2N)^{2d}} \int_{[-N, N]^d \times [-N, N]^d \times [-N, N]^d} \psi^e(s_1 - s_2) \psi^{p-e}(s_2 - s_4) ds_1 ds_2 ds_4 \\ & \leq \frac{1}{(2N)^{2d}} \int_{[-N, N]^d \times [-N, N]^d} \psi^e \star \psi^{p-e}(s_1 - s_2) ds_2 ds_2 \leq \frac{1}{(2N)^d} \int_{\mathbb{R}^d} \psi^e \star \psi^{p-e}(s) ds \rightarrow 0. \end{aligned}$$

The term $\mathcal{Z}_2(N)$ can be treated similarly obtaining

$$\|g_p^N \otimes_e g_p^N\|_{\mathcal{H}^{\otimes 2(p-e)}}^2 \rightarrow 0.$$

Hence (24) holds in force. It implies that

$$\sum_{1 \leq p \neq q \leq Q} \left\| \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}} \right\|_{L^2(\Omega)} \rightarrow 0$$

since

$$\sum_{1 \leq p < q \leq Q} \left\| \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}} \right\|_{L^2(\Omega)} + \sum_{1 \leq q < p \leq Q} \left\| \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}} \right\|_{L^2(\Omega)}$$

$$= \sum_{1 \leq p < q \leq Q} \frac{p+q}{p} \left\| \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) \rangle_{\mathcal{H}} \right\|_{L^2(\Omega)}.$$

The Central Limit Theorem for $\pi^Q(\zeta([-N, N]^d))$ follows from the following bound. Let h be a two times differentiable and bounded map with bounded derivatives and let Z_Q be a centered Gaussian random variable with variance equal to $\sigma^2 = \sum_{q=1}^Q u_q$. Following (4.47) of [20], it yields

$$\begin{aligned} & |\mathbb{E}[h(Z_Q)] - \mathbb{E}[h(\pi^Q(\zeta([-N, N]^d)))]| \\ & \leq \frac{\|h''\|_{\infty}}{2} \sum_{p,q=1}^Q \left\| \delta_{pq} u_p - \frac{1}{q} \langle \mathbf{D}I_p(g_p^N), \mathbf{D}_q I_q(g_q^N) \rangle \right\|_{L(\Omega)}. \end{aligned}$$

The right hand term tends to zero as N goes to $+\infty$ as a consequence of the previous computations. \square

4. THE OTHER FACES

In this section we deal with the faces of T that have a dimension lower than d . Our goal is to prove that Theorem 3.2 provides a CLT for the Euler characteristic $\chi(T)$ through the identity (4). We still assume that $T = [-N, N]^d$ and we establish that for any $\ell = 0, 1, \dots, d-1$ and any face J in $\partial_{\ell} T$, the variance of $|T|^{-d/2} \varphi(J)$ tends to 0 as T grows to \mathbb{R}^d .

For $\ell = 0$, the previous statement is obvious since $\varphi(\{v\})$ is either 0 or 1 for any vertex v of T .

Let us now be concerned with $\ell \in \{1, \dots, d-1\}$. We deal with a fixed face $J \in \partial_{\ell} T$. We recall that J can be written as

$$J = \{v \in T : -N < v_j < N \text{ for } j \in \sigma(J), v_j = \varepsilon_j N \text{ for } j \notin \sigma(J)\}$$

where $\sigma(J) \subset \{1, \dots, d\}$ has cardinal ℓ and $\varepsilon(J) = (\varepsilon_j)_{j \in \{1, \dots, d\} \setminus \sigma(J)} \in \{-1, +1\}^{(d-\ell)}$. Let us introduce the following notations:

- With any $v \in \mathbb{R}^{\ell}$, we associate $v^{(J)} = (v_1^{(J)}, \dots, v_d^{(J)}) \in \mathbb{R}^d$ defined by

$$v_j^{(J)} = v_j \text{ if } j \in \sigma(J); v_j^{(J)} = \varepsilon_j N \text{ if } j \notin \sigma(J).$$

- A random field $X^{(J)}$ is defined on \mathbb{R}^{ℓ} by

$$X^{(J)}(v) = X(v^{(J)}) \text{ for any } v \in \mathbb{R}^{\ell}.$$

It clearly inherits the properties of X so that $X^{(J)}$ is Gaussian, stationary, centered and its trajectories are a.s. smooth.

With these notations, (2) and (3) can be written as

$$\varphi(J) = \sum_{k=0}^{\ell} (-1)^k \mu_k(J)$$

with

$$\begin{aligned} \mu_k(J) = \# \{ & v \in [-N, N]^{\ell} : X^{(J)}(v) \geq u, \\ & \nabla X^{(J)}(v) = 0, \text{index}(\nabla^2 X^{(J)}(v)) = \ell - k, \\ & \varepsilon_j X_j(v^{(J)}) > 0 \text{ for } j \notin \sigma(J)\}. \end{aligned}$$

Following the same arguments as in Section 2, one can get an analogous proposition of Proposition 2.1

$$\begin{aligned} \varphi(J) \stackrel{L^2(\Omega)}{=} \lim_{\varepsilon \rightarrow 0} (-1)^\ell \int_{[-N, N]^\ell} \det(\nabla^2 X^{(J)}(v)) 1_{[u, \infty)}(X^{(J)}(v)) \delta_\varepsilon(\nabla X^{(J)}(v)) \\ \times \prod_{j \notin \sigma(J)} 1_{(0, \infty)}(\varepsilon_j X_j(v^{(J)})) dv. \end{aligned}$$

One can get as well an analogous proposition of Proposition 2.2

$$\varphi(J) \stackrel{L^2(\Omega)}{=} (-1)^\ell \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n}, \bar{n}) \in D(\ell) \\ |\mathbf{n}|=q}} d^{(J)}(\underline{n}) c^{(J)}(\bar{n}) \int_{[-N, N]^\ell} \widetilde{H}_{\mathbf{n}}(Y^{(J)}(v)) dv$$

where

- $D(\ell) = d + \ell(\ell + 1)/2 + 1$
- $Y^{(J)}(v)$ is a Gaussian $N(0, I_{D(\ell)})$ vector such that

$$(\nabla X^{(J)}(v), \nabla X_{(J)}(v), \nabla^2 X^{(J)}(v), X^{(J)}(v)) = \Lambda^{(J)} Y^{(J)}(v),$$

where $\nabla X_{(J)}(v)$ denotes the vector $(X_k(v^{(J)}))_{k \notin \sigma(J)}$

and $\Lambda^{(J)}$ is a $D(\ell) \times D(\ell)$ matrix that can be factorized into $\begin{pmatrix} \Lambda_1^{(J)} & 0 \\ 0 & \Lambda_2^{(J)} \end{pmatrix}$

- $d^{(J)}(\underline{n}) = \lim_{\varepsilon \rightarrow 0} c((\delta_\varepsilon \otimes 1_{(0, +\infty)^{d-\ell}}) \circ \Lambda_1^{(J)}, \underline{n})$
- $c^{(J)}(\bar{n}) = c(f \circ \Lambda_2^{(J)}, \bar{n})$.

At last, a similar proposition as Proposition 3.1 can be formulated

$$\text{Var} \left(\frac{\varphi(J) - \mathbb{E}\varphi(J)}{(2N)^{\ell/2}} \right)_{N \rightarrow +\infty} V^{(J)} < +\infty$$

and hence

$$\text{Var} \left(\frac{\varphi(J) - \mathbb{E}\varphi(J)}{(2N)^{d/2}} \right)_{N \rightarrow +\infty} 0.$$

Finally, recalling (4) that gives $\chi(T) = \sum_{0 \leq \ell < d} \sum_{J \in \partial_\ell T} \varphi(J) + \varphi(T)$, we are able to state that Theorem 3.2 not only provides a central limit theorem for the differential topology characteristic $\varphi([-N, N]^d)$ but also for the Euler characteristic $\chi([-N, N]^d)$, with the same asymptotic variance.

5. APPENDIX: THE CASE $d = 2$

In this section, we assume $X : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ to be a stationary centered isotropic Gaussian field with the required conditions **(H1)**, **(H2)**, **(H3)**. We provide an explicit value for the asymptotic variance V (see (16)) as well as an explicit value for V_1 (see (17)) which ensures the positivity of V .

Since $d = 2$, we will deal with 6-dimensional Gaussian vectors $\mathbf{X}(v)$ and $Y(v)$ such that for any $v \in \mathbb{R}^2$,

- $\mathbf{X}(v) = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5, \mathbf{X}_6)(v)$ with $\nabla X = (\mathbf{X}_1, \mathbf{X}_2) = (X_1, X_2)$, $\mathbf{X}_3 = X_{11}$, $\mathbf{X}_4 = X_{22}$, $\mathbf{X}_5 = X_{12}$ and $\mathbf{X}_6 = X$.
- $Y(v)$ is a standard $N(0, I_6)$ Gaussian vector,
- $\mathbf{X}(v) = \Lambda Y(v)$ with Λ a 6×6 upper triangular matrix.

The usual independence properties between a stationary random field and its derivatives, as well as the isotropy property of X , yield the following shape for Λ

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \text{ with } \Lambda_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } \Lambda_2 = \begin{pmatrix} \sigma_3 & \gamma & 0 & \beta \\ 0 & \sigma_4 & 0 & \alpha \\ 0 & 0 & \sigma_5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that we use the following regression model to get Λ_2

$$\begin{aligned} \mathbf{X}_6 &= X = Y_6 \\ \mathbf{X}_5 &= X_{12} = \sigma_5 Y_5 \\ \mathbf{X}_4 &= X_{22} = \alpha X + \sigma_4 Y_4 \\ \mathbf{X}_3 &= X_{11} = \beta X + \gamma Y_4 + \sigma_3 Y_3 \end{aligned}$$

Denoting by f_X the spectral density of X and using the isotropy of X , we can write

$$f_X(x) = \bar{f}(|x|) \text{ for any } x \in \mathbb{R}^2, \text{ with } \bar{f}: [0, +\infty) \rightarrow [0, +\infty].$$

Then we are able to compute the covariance matrix of the vector $\mathbf{X}(v)$

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\zeta & \zeta & 0 & -\lambda \\ 0 & 0 & \zeta & 3\zeta & 0 & -\lambda \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & -\lambda & -\lambda & 0 & 1 \end{pmatrix} \text{ where } \begin{cases} \lambda &= \pi \int_0^\infty \rho^3 \bar{f}(\rho) d\rho \\ \zeta &= \frac{\pi}{4} \int_0^\infty \rho^5 \bar{f}(\rho) d\rho \end{cases}$$

At last, using the regression equations, the coefficients of Λ_2 can be computed

$$\begin{aligned} \alpha &= -\lambda & ; & & \beta &= -\lambda & ; & \gamma &= \frac{\zeta - \lambda^2}{\sqrt{3\zeta - \lambda^2}} \\ \sigma_5^2 &= \zeta & ; & \sigma_4^2 &= 3\zeta - \lambda^2 & ; & \sigma_3^2 &= \frac{4\zeta(2\zeta^2 - \lambda^2)}{3\zeta^2 - \lambda^2}. \end{aligned}$$

5.1. Exact computation of V .

Following (16), we have $V = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}^6 \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\mathbf{n}!\mathbf{m}! R(\mathbf{n}, \mathbf{m})$ and it remains to compute the $a(\mathbf{n})$'s and the $R(\mathbf{n}, \mathbf{m})$'s.

First step: we start with $a(\mathbf{n}) = d(\underline{n}) c(f \circ \Lambda_2, \bar{n})$ for $\mathbf{n} = (\underline{n}, \bar{n})$.

By (9), for $\underline{n} = (n_1, n_2)$,

$$d(\underline{n}) = \frac{\lambda^{-2}}{(2\pi)^{\underline{n}!}} H_{n_1}(0)H_{n_2}(0)$$

which equals zero if at least one of the indices n_j is odd.

For $\bar{n} = (n_3, n_4, n_5, n_6)$,

$$c(f \circ \Lambda_2, \bar{n}) = \frac{1}{\bar{n}!} \int_{\mathbb{R}^4} f \circ \Lambda_2(x) \widetilde{H}_{\bar{n}}(x) \phi_4(x) dx.$$

Using the specific form of f and Λ_2 and writing $x = (y, z) = (y_3, y_4, y_5, z) \in \mathbb{R}^3 \times \mathbb{R}$, we get

$$\begin{aligned} f \circ \Lambda_2(y, z) &= 1_{[u, +\infty)}(z) \tilde{\det}(\Lambda_2(y, z)) \\ &= 1_{[u, +\infty)}(z) [\sigma_3 \sigma_4 y_3 y_4 + \gamma \sigma_4 y_4^2 - \sigma_5^2 y_5^2 + (\alpha \sigma_3 y_3 + \beta \sigma_4 y_4 + \alpha \gamma y_4) z + \alpha \beta z^2]. \end{aligned}$$

We introduce the following quantity which will be computed later on:

$$(25) \quad \ell_k(n, u) = \int_{[u, +\infty)} H_n(z) z^k \phi(z) dz \text{ for any integers } k, n \geq 0$$

Hence, plugging the expression of $f \circ \Lambda_2(y, z)$ into the integral allows us to split the Hermite coefficient $c(f \circ \Lambda_2, \bar{n})$ into three terms

$$c(f \circ \Lambda_2, \bar{n}) = c_0(\bar{n}) + c_1(\bar{n}) + c_2(\bar{n}),$$

where we use the orthogonality of the Hermite polynomials to get

$$\begin{aligned} c_0(\bar{n}) &= \ell_0(n_6, u) [\sigma_3 \sigma_4 \delta_{(1,1,0)} + \gamma \sigma_4 (\delta_{(0,0,0)} + \delta_{(0,2,0)}) - \sigma_5^2 (\delta_{(0,0,0)} + \delta_{(0,0,2)})] (n_3, n_4, n_5), \\ c_1(\bar{n}) &= \ell_1(n_6, u) [\alpha \sigma_3 \delta_{(1,0,0)} + (\beta \sigma_4 + \alpha \gamma) \delta_{(0,1,0)}] (n_3, n_4, n_5), \\ c_2(\bar{n}) &= \ell_2(n_6, u) \alpha \beta \delta_{(0,0,0)} (n_3, n_4, n_5). \end{aligned}$$

Here above, $\delta_{(i,j,k)}(n_3, n_4, n_5)$ denotes as usual the Kronecker symbol.

At last, let us compute the $\ell_k(n, u)$'s given by (25). We recall the Hermite's expansion of the function $1_{[u, \infty)}(\cdot)$:

$$1_{[u, \infty)}(z) = \Psi(u) + \sum_{m=1}^{\infty} \frac{H_{m-1}(u) \phi(u)}{m!} H_m(z).$$

Hence, one can easily obtain

$$\ell_0(n, u) = \Psi(u) \delta_0(n) + \frac{1}{n!} \phi(u) H_n(u) \delta_{\geq 1}(n),$$

and with an integration by part, for $k \geq 1$,

$$\ell_k(n, u) = (k-1) \ell_{k-2}(n, u) + n \ell_{k-1}(n-1, u) + u^{k-1} \phi(u) H_n(u).$$

This concludes the computation of $a(\mathbf{n})$.

Second step: we focus now on $R(\mathbf{n}, \mathbf{m})$. By (12), we know that

$$R(\mathbf{n}, \mathbf{m}) = \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = n_j; \sum_j d_{ij} = m_i}} \int_{\mathbb{R}^2} \prod_{1 \leq i, j \leq 6} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{(d_{ij})!} dv.$$

Recall that the covariance function $\Gamma^Y(\cdot)$ is given by $\Gamma^Y(v) = \Lambda^{-1} \Gamma^{\mathbf{X}}(v) \Lambda^{-1}$ for any $v \in \mathbb{R}^2$. The covariance function $\Gamma^{\mathbf{X}}(\cdot)$ of \mathbf{X} , and hence $\Gamma^Y(\cdot)$, can be written in terms of integrals with respect to the radial part \tilde{f} of the spectral density f_X .

5.2. Exact computation of V_1 and $H_\Lambda(u)$.

Equation (17) gives

$$V_1 = \lambda^{-4} f_X(0) H_\Lambda(u)^2 \quad \text{with} \quad H_\Lambda(u) = \sum_{3 \leq i \leq 6} c(f \circ \Lambda_2, \bar{e}_i) (\Lambda_2^{-1})_{i,6}.$$

The last column of the matrix Λ_2^{-1} and the Hermite coefficients of $f \circ \Lambda_2$ are the following

$$\begin{cases} (\Lambda_2^{-1})_{3,6} &= -\frac{\beta}{\sigma_3} + \frac{\alpha\gamma}{\sigma_3\sigma_4} \\ (\Lambda_2^{-1})_{4,6} &= -\frac{\alpha}{\sigma_4} \\ (\Lambda_2^{-1})_{5,6} &= 0 \\ (\Lambda_2^{-1})_{6,6} &= 1 \end{cases} \quad \text{and} \quad \begin{cases} c(f \circ \Lambda_2, \bar{e}_3) &= \phi(u)\alpha\sigma_3 \\ c(f \circ \Lambda_2, \bar{e}_4) &= \phi(u)(\beta\sigma_4 + \alpha\gamma) \\ c(f \circ \Lambda_2, \bar{e}_5) &= 0 \\ c(f \circ \Lambda_2, \bar{e}_6) &= \phi(u)(u^2 + 1)\alpha\beta \end{cases}$$

so that $V_1 = f_X(0) \phi(u)^2 (u^2 - 1)^2$ which only vanishes for $u = \pm 1$.

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